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PRECESSION OF INCLINED RINGS

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ABSTRACT

Differential precession due to the planet's quadrupole moment tends to destroy the alignment of particles in inclined rings. We propose that alignment is maintained by the self-gravity of the ring. This hypothesis predicts that $\delta i/\delta a > 0$ across the ring. If $\delta i/i_0 \ll 1$, $\delta e/e_0 \ll 1$, $a\delta i/\delta a \ll 1$, and $a\delta e/\delta a \ll 1$, a further prediction is that $\delta i/i_0 = \delta e/e_0$. The α and β rings of Uranus may be used to test these predictions.

I. INTRODUCTION

As data from stellar occultations by the rings of Uranus accumulate, the shapes and kinematics of the rings become better defined. A recent analysis by French *et al.* (1982) has uncovered yet another remarkable feature of the ring system; several of the rings are inclined to the planet's equator. The inclinations i (rad) are small, $\lesssim 10^{-3}$, and generally of the same order as the eccentricities. It is not known whether the narrow ringlets in the Saturn system possess substantial inclinations. Nonzero inclinations have only been established where bending waves are excited near the 5:3 and 8:5 inclination resonances of the satellite Mimas (Shu, Cuzzi, and Lissauer 1982).

The existence of inclined ringlets raises two questions. How are the inclinations produced? What maintains node alignment across an inclined ring? In similar discussions involving ring eccentricity (Goldreich and Tremaine 1978, 1979, 1981), it was found that the two analogous questions largely decoupled, the latter being much simpler. Accordingly, we only address the problem of node alignment here and consider the hypothesis that the ring's self-gravity cancels the tendency for differential node precession due to the planet's multipole moments.

We digress to discuss the evidence for self-gravity as the mechanism for maintaining apse alignment in eccentric rings. First, apse locking by self-gravity requires that the eccentricity e increase with semimajor axis a across a ringlet and all rings for which a width-radius relation is established show $\delta e/\delta a > 0$. Examples include the α, β, and ϵ rings of Uranus and the Saturn ringlets at $1.29R_s$, $1.45R_s$, and $1.96R_s$. Second, the self-gravity hypothesis yields predictions for the masses of eccentric ringlets with well-determined shapes. The masses of the eccentric Saturnian ringlets thus deduced are in accord with independent estimates based on the relation between optical depth and surface mass density established from both the radial wavelengths of density waves (Cuzzi *et al.* 1981; Holberg *et al.* 1982) and the scattering of the *Voyager 1* radio signal (Tyler 1982).

For simplicity we restrict our attention to circular rings. The circular approximation is self-consistent because the interactions among circular and inclined ringlets do not generate secular perturbations of eccentricity. It is straightforward although somewhat tedious to extend the analysis to eccentric and inclined rings.

II. DYNAMICS

We represent the ring as a collection of circular inclined wires. For a wire of mass m and radius a, the linear density is

$$\rho = m/2\pi a. \tag{1}$$

The force per unit mass exerted by the wire on a ring particle p with semimajor axis and inclination a_p and i_p is

$$\mathbf{F} = 2G\rho\mathbf{d}/d^2, \tag{2}$$

where $\mathbf{d}$ is the perpendicular from the particle to the wire, $d = |\mathbf{d}|$, and G is the gravitational constant. Since the rings under investigation are narrow, $\delta a/a \ll 1$, the wire may be locally approximated as straight.

The geometry is shown in Fig. 1. We assume that the nodes of the particle and the wire are aligned since we are only concerned with the maintenance of this alignment. The angle ϑ is measured from the common nodal line. A simple exercise in geometry shows that

$$\mathbf{d} \simeq \Delta a[\cos(\Omega + \vartheta)\hat{e}_x + \sin(\Omega + \vartheta)\hat{e}_y] + a\Delta i \sin\vartheta\hat{e}_z, \tag{3}$$

where

$$\Delta a = a - a_p, \tag{4a}$$

$$\Delta i = i - i_p, \tag{4b}$$

with the subscript p denoting the ring particle. In deriving Eq. (3), it is implicitly assumed that $O(\Delta a/a) = O(i) = O(\Delta i) \ll 1$; higher-order terms are discarded. Note that $\Delta\vartheta = \vartheta - \vartheta_p = O(i\delta i, i^2\Delta a/a)$, where ϑ_p locates the intersection of $\mathbf{d}$ with the wire.

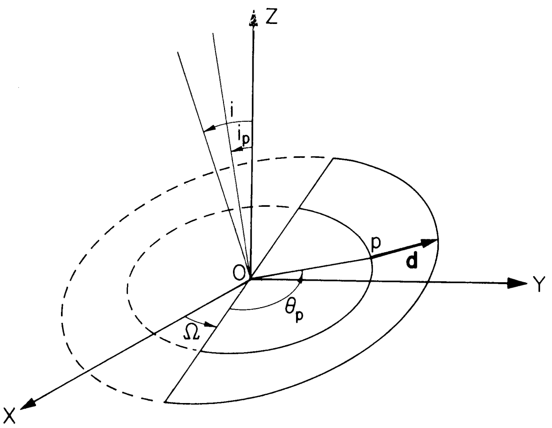

FIG. 1. The geometry of the particle-wire interaction. The X,Y,Z axes are inertial and the Z axis is aligned with the planet's spin sector.

The node precession rate due to **F** is (Brouwer and Clemence 1961)

$$\frac{d\Omega_p}{dt} \simeq \frac{\sin \vartheta_p F_z}{n_p a_p i_p}, \tag{5}$$

where n_p is the mean motion. Combining Eqs. (2), (3), and (5) yields

$$\frac{d\Omega_p}{dt} = \frac{mn}{\pi M}\left(\frac{a}{\Delta a}\right)^2 \frac{\Delta i}{i_p} \frac{\sin^2 \vartheta}{(1+q^2 \sin^2 \vartheta)}, \tag{6}$$

where

$$q \equiv a\Delta i/\Delta a, \tag{7}$$

and M is the planet's mass. In writing Eq. (6) we do not distinguish a from a_p and n from n_p because $\Delta a/a \ll 1$. We average the precession rate over the orbital period to obtain

$$\left\langle \frac{d\Omega_p}{dt} \right\rangle = \frac{mna \tanh \chi}{\pi M i_p \cosh 2\chi}, \tag{8}$$

where

$$2\chi \equiv \sinh^{-1} q. \tag{9}$$

Equation (8) for $\langle d\Omega_p/dt \rangle$ is the analog of Eq. (11) for $\langle d\tilde{\omega}_p/dt \rangle$ in Goldreich and Tremaine (1979).

Next we sum the contributions to the node precession from all parts of the ring. Let the semimajor axes of the inner and outer boundaries of the ring be $a_{\rm in}$ and $a_{\rm out} = a_{\rm in} + \delta a$. Divide the region $a_{\rm in}$ to $a_{\rm out}$ into N equal intervals of width $\delta a/N$, each of which contains a wire of mass m_k, semimajor axis $a_k = a_{\rm in} + (k - 1/2)\delta a/N$, and inclination i_k, $k = 1,\ldots,N$. We write $m_k = h_k m_r$, where m_r is the total ring mass and

$$\sum_{k=1}^{N} h_k = 1. \tag{10}$$

The nodal precession rate of wire j due to all other wires is

$$\left\langle \frac{d\Omega_j}{dt} \right\rangle_{SG} = + \frac{naN}{ni_j \delta a} \frac{m_r}{M} \sum_{k \neq j} \frac{n_k}{k-j} \frac{\tanh \chi_{jk}}{\cosh 2\chi_{jk}}, \tag{11}$$

where

$$\sinh 2\chi_{jk} = \frac{Na}{\delta a}\left(\frac{i_k - i_j}{k-j}\right). \tag{12}$$

The node precession rate due to the quadrupole moment J_2 of the planet is

$$\left\langle \frac{d\Omega_j}{dt} \right\rangle_Q = \text{const} + \frac{21 J_2}{4}\left(\frac{R}{a}\right)^2 \frac{nj}{N} \frac{\delta a}{a}, \tag{13}$$

where R is the planet's radius. The condition that the node precession rate be the same for all wires reads

$$j\frac{A}{N} + \frac{N}{i_j} \sum_{k \neq j} \frac{h_k}{k-j} \frac{\tanh \chi_{jk}}{\cosh 2\chi_{jk}} = B, \quad j = 1,\ldots,N, \tag{14}$$

where B is a constant and

$$A = \left(\frac{21\pi}{4}\right) J_2 \left(\frac{M}{m_r}\right)\left(\frac{R}{a}\right)^2 \left(\frac{\delta a}{a}\right)^2.$$

If the inclinations of the ring boundaries, i_1, and i_N, were known, these N equations could be solved for m_r, $\langle \Omega \rangle$ and $i_2,\ldots,i_{N-1}$. This procedure would complete the analogy with the treatment of the eccentric epsilon ring for which e_1 and e_N are well determined. Unfortunately, the quality of the observational data collected to date probably does not permit a reliable determination of δi across any ring.

Rather than solve Eq. (14) for assumed values of i_1 and i_N, we explore further the analogy between the node and apse precessions. To do so we note that for $a\delta i/\delta a \ll 1$ and $\delta i/i_0 \ll 1$, Eq. (14) reduces to

$$\frac{jC}{N} + N \sum_{k \neq j} \frac{h_k \chi_{jk}}{k-j} = D, \quad j = 1,\ldots,N, \tag{15}$$

where $C = i_0 A$, $D = i_0 B$, and i_0 is the mean inclination of the ring. For $a\delta e/\delta a \ll 1$, the corresponding apse precession Eq. (14) from Goldreich and Tremaine (1979) reduces to an expression equivalent to Eq. (15), except that i_k is replaced by e_k. These results prove that if $a\delta i/\delta a \ll 1$ and $a\delta e/\delta a \ll 1$, then $\delta i/i_0 = \delta e/e_0$.

There is an alternative way to interpret Eq. (14) which reveals a close relation between inclined rings and bending waves. An analogous relation exists between eccentric rings and density waves. Consider a ring of mass m_r. We expect that, subject to appropriate boundary conditions, Eq. (14) would yield N independent solutions for the $\{i_j\}$, each accompanied by a different B, or precession rate. The solutions are probably uniquely characterized by the number of sign changes of the i_j which range from 0 to $N-1$. A sign change in the i_j is equivalent to a reversal of the ascending and descending nodes. These solutions are standing bending waves; the B value gives the pattern speed; the number of sign changes of the i_j is the number of radial nodes. The nodeless solution is the appropriate one for an inclined ring. In the linear limit, $\chi_{jk} \ll 1$, Eq. (14) reduces to Eq. (15). For $h_k = 1/N$, or uniform surface mass density, the condition for the vanishing of the determinant of Eq. (15)

yields the WKB dispersion relation for linear bending waves (Shu, Cuzzi, and Lissauer 1982).

III. DISCUSSION

The self-gravity hypothesis predicts that $\delta i/\delta a > 0$. The hypothesis is as plausible for node alignment as for apse alignment. An observational determination of both δi and δe would provide an excellent test of the hypothesis. If $a\delta i/\delta a \ll 1$ and $a\delta e/\delta a \ll 1$, the hypothesis would predict that $\delta i/i_0 = \delta e/e_0$. The α and β rings of Uranus are the best-known candidates for such a test. For ring α, $e_0 \simeq 8\times10^{-4}$, $\delta e \simeq 4\times10^{-5}$, $i_0 = 3\times10^{-4}$, which implies $\delta i \simeq 1.5\times10^{-5}$. For ring β, $e_0 \simeq 4\times10^{-4}$, $\delta e \simeq 8\times10^{-5}$, $i_0 \simeq 1\times10^{-4}$, which implies $\delta i \simeq 2\times10^{-5}$. An inclination gradient would produce a variation of the projected width of the ring on the plane of the sky of magnitude $\simeq a\delta i$ which would be modulated at the node precession rate. The value of $a\delta i$ is of order 1 km for both the α and β rings.

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