Temporal and Spatial Plasma Wave Echoes

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It is shown that, if a longitudinal wave is excited in a collision-free plasma and Landau-damps away, and later a second wave is excited and also damps away, then a third wave will spontaneously appear in the plasma. This wave appears long after the first two waves have damped away at a time proportional to the interval between the first two waves, and is in that sense an echo. It is also shown that, if a wave is continuously excited at one point in a plasma and a second wave is continuously excited many Landau damping lengths from the first point, then a third wave will spontaneously appear many Landau damping lengths from the second point. Fundamentally, plasma wave echoes are possible because of the reversible nature of Landau damping. However, small-angle Coulomb collisions are very effective in destroying the echo.

I. INTRODUCTION

It has long been recognized that electron plasma waves can be damped, even in the absence of collisions.[1] Collisionless damping (Landau damping) has been the subject of extensive theoretical treatments in recent years and is now believed to play an important role in many related, but more complicated, oscillation and instability phenomena. Only recently has Landau damping been demonstrated experimentally.[2] Landau's treatment shows that macroscopic quantities such as the electric field and the charge density are damped exponentially, but that perturbations in the electron phase space distribution $f(x, v, t)$ oscillate indefinitely. Since the electron density is given by $n_e = \int f(x, v, t)\, dv$, one may think of the damping as arising out of the phase mixing of various parts of the distribution function. In a previous letter[3] we have outlined a method by which the direction of the phase evolution of the perturbed distribution function can be reversed by the application of a second electric field. This results in the subsequent reappearance of a macroscopic field (i.e., the echo), many Landau damping periods after the application of the second pulse. The plasma echo is related to other known echo phenomena[4] in that the decay of a macroscopic physical quantity of the system, caused by phase mixing of rapidly oscillating microscopic elements in the system, is reversed by reversing the direction of phase evolution of the microscopic elements. In this paper we give a more complete treatment of the plasma wave echo, extend the previous work to several important new situations, and consider some of its consequences.

The basic mechanism behind the plasma echo can easily be understood. When an electric field of spatial dependence $\exp(-ik_1x)$ is excited in a plasma and then Landau-damps away, it modulates the distribution function leaving a perturbation of the form[1] $f_1(v) \exp(-ik_1x + ik_1vt)$. For large t there is no electric field associated with this perturbation, since an integral over velocity will phase-mix to zero. If after a time τ a wave of spatial dependence $\exp(ik_2x)$ is excited and then damps away, it will modulate the unperturbed part of the distribution leaving a first-order term of the form $f_2(v) \exp[ik_2x - ik_2v(t - \tau)]$, but it also modulates the perturbation due to the first wave leaving a second-order term of the form $f_1(v)f_2(v) \exp[i(k_2 - k_1)x + ik_2v\tau - i(k_2 - k_1)vt]$. The coefficient of v in this exponential will vanish when $t = \tau[k_2/(k_2 - k_1)]$; so at this time an integral over velocity of this term will not phase-mix to zero, and an electric field reappears in the plasma. If τ is long compared to a collisionless

[1] L. Landau, J. Phys. (USSR) **10**, 45 (1946).
[2] A. Y. Wong, N. D'Angelo, and R. W. Motley, Phys. Rev. **133**, A436 (1964); J. H. Malmberg and C. B. Wharton, Phys. Rev. Letters **6**, 184 (1964); J. H. Malmberg, C. B. Wharton, and W. E. Drummond, in *Plasma Physics and Controlled Nuclear Fusion Research* (International Atomic Energy Agency, Vienna, 1966), Vol. I, p. 485.
[3] R. W. Gould, T. M. O'Neil, and J. H. Malmberg, Phys. Rev. Letters (to be published). See also, R. W. Gould, Phys. Letters (to be published).
[4] E. L. Hahn, Phys. Rev. **80**, 580 (1950); R. M. Hill, and D. E. Kaplan, Phys. Rev. Letters **14**, 1062 (1965); R. W. Gould, Phys. Letters **19**, 477 (1965); I. D. Abella, N. A. Kurnit, and S. R. Hartmann, Phys. Rev. **141**, 391 (1966).

damping period and $[k_2/(k_2 - k_1)]$ is of order unity, then this third electric field appears long after the first two waves have damped away (i.e., it will be an echo).

In addition to the second-order echo described above (i.e., second order in the perturbation amplitude), higher-order echoes are also possible. For example, a third-order echo is produced when the velocity space perturbation from the first pulse is modulated by the second spatial harmonic of the electric field from the second pulse. The echo then occurs at $t = \tau[2k_2/(2k_2 - k_1)]$ or $t = 2\tau$ when $k_1 = k_2$. This result is more closely related to echoes of other types,[4] which are also third order for small amplitudes.

It is also possible to have *spatial echoes*, and these are probably easier to observe experimentally than the *temporal echoes* described above. If an electric field of frequency ω_1 is continuously excited at one point in a plasma and an electric field of frequency $\omega_2 > \omega_1$ is continuously excited at a distance l from this point, then a second-order spatial echo of frequency $\omega_2 - \omega_1$ is produced at a distance $l[\omega_2/(\omega_2 - \omega_1)]$ from the point where the first field is excited. Of course, it is also possible to have higher-order spatial echoes.

The second-order temporal echo is treated in Sec. II of this paper. The calculation is based upon a perturbation expansion of the collisionless Boltzmann equation in powers of the applied fields. The self-consistent fields are taken into account through Poisson's equation and the combined system of equations is solved by the Laplace transform method used by Landau.[1] In Sec. III a similar calculation is presented for the second-order spatial echo. Higher-order echoes are treated in Sec. IV. This calculation is not a perturbation expansion but is valid to all orders. However, it does not include the effect of the self-consistent fields of the plasma. Section V discusses the effect of collisions and the possible use of echoes to study collision relaxation and weak turbulence in plasmas.

II. SECOND-ORDER TEMPORAL ECHO

The basic equations for the plasma wave echo are the collisionless Boltzmann equation and Poisson's equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0, \qquad (1)$$

$$\frac{\partial^2}{\partial x^2}[\phi - \phi_{\text{ext}}] = 4\pi n e\left[1 - \int dv f\right], \qquad (2)$$

where $f(x, v, t)$ is the electron distribution, $\phi(x, t)$ is the total electric potential, and ϕ_{ext} is the potential associated with the two externally applied pulses. For the sake of simplicity, we limit the presentation to one dimension and treat the ions as a uniform positive background charge. If we assume that the electron distribution is initially spatially homogeneous, $f(x, v, t = 0) = f_0(v)$, and that the two externally applied pulses are given by[5]

$$\phi_{\text{ext}} = \Phi_1 \cos(k_1 x)\,\delta[\omega_p t] + \Phi_2 \cos(k_2 x)\,\delta[\omega_p(t - \tau)],$$

then the Fourier transform of the spatial dependence and Laplace transform of the time dependence of both Eqs. (1) and (2) can be written as

$$(p + ikv)\tilde{f}_k(v, p) = \frac{e}{m}\,ik\tilde{\phi}_k(p)\frac{\partial f_0}{\partial v} + \frac{e}{m}\sum_q{}' \int \frac{dp'}{2\pi i}\,i(k - q)\tilde{\phi}_{k-q}(p - p')\frac{\partial \tilde{f}_q}{\partial v}(p'), \qquad (3)$$

$$k^2\tilde{\phi}_k(p) = 4\pi n e\int dv\,\tilde{f}_k(v, p) + \frac{k_1^2\Phi_1}{2\omega_p}(\delta_{k,k_1} + \delta_{k,-k_1}) + \frac{k_2^2\Phi_2}{2\omega_p}(\delta_{k,k_2} + \delta_{k,-k_2})e^{-p\tau}, \qquad (4)$$

where $\tilde{\phi}_k(p)$ and $\tilde{f}_k(v, p)$ are the transformed potential and distribution. The convolution (or Faltung) theorem has been used twice in expressing the nonlinear term $(\partial\phi/\partial x)\cdot(\partial f/\partial v)$, in Eq. (3), and the prime on the sigma in Eq. (3) indicates that the $q = 0$ term is being treated separately in the manner usual for mode-coupling calculations[6] (i.e., f_0 is recognized as having a zero-order part relative to the applied potentials).

To solve Eqs. (3) and (4), we expand in terms of the applied potentials Φ_1 and Φ_2. The first-order (or linear) solution can be written as

$$\tilde{\phi}_k^{(1)}(p) = \frac{\Phi_1(\delta_{k,k_1} + \delta_{k,-k_1})}{2\epsilon(k, p)\omega_p} + \frac{\Phi_2(\delta_{k,k_2} + \delta_{k,-k_2})e^{-p\tau}}{2\epsilon(k, p)\omega_p}, \qquad (5)$$

$$\tilde{f}_k^{(1)}(v, p) = \frac{(e/m)ik\tilde{\phi}_k^{(1)}(p)(\partial f_0/\partial v)}{p + ikv}, \qquad (6)$$

where

$$\epsilon(k, p) \equiv 1 - \frac{\omega_p^2}{k^2}\int dv\,\frac{\partial f_0}{\partial v}\left(v + \frac{p}{ik}\right)^{-1}.$$

The velocity integral in this dielectric function is to be evaluated along the contour originally prescribed by Landau.[1] If we assume that $|k_1 L_D| < 1$, where

[5] Φ_{k_1} and Φ_{k_2} have the dimensions of electric potential owing to our inclusion of ω_p in the arguments of the delta functions.

[6] W. E. Drummond, D. Pines, Nucl. Fusion Suppl., Pt. 3, 1049 (1962).

L_D is the Debye length, and retain only the Landau pole[1] while taking the Laplace inverse of Eq. (5), we find the following time-asymptotic solution (i.e., $\omega_p t \gg 1$)

$$\phi_{k_1}^{(1)}(t) \simeq -\frac{\phi_1}{2} e^{\gamma_1 t} \sin \omega_1 t, \tag{7}$$

where

$$\gamma_1 = \frac{\pi}{2} \frac{\omega_p^3}{k_1} \left[\frac{\partial f_0}{\partial v} \right]_{v=\omega_1/k_1}$$

is the Landau damping coefficient and $\omega_1 = \omega_p[1 + \frac{3}{2}(k_1 L_D)^2]$ is the frequency associated with wave number k_1. The second pulse produces a similar response, except that it is delayed by time τ and is associated with Fourier component ϕ_{k_2}.

For the second-order response we concentrate on the Fourier component $\phi_{k_2-k_1}$. Of course, $\phi_{k_1+k_2}$, $\phi_{k_1+k_1}$ and $\phi_{k_2+k_2}$ all have second-order terms, but there is no echo associated with these terms. If we let $k_3 = k_2 - k_1$ and use Eqs. (5) and (6) to iterate Eqs. (3) and (4) we find

$$\phi_{k_3}^{(2)}(t) = \frac{e}{m} \frac{\Phi_1 \Phi_2 k_1 k_2}{4k_3^2} \int_{-\infty}^{+\infty} dv \int_{-i\infty+\sigma}^{i\infty+\sigma} \frac{dp}{2\pi i} \cdot \int_{-i\infty+\sigma'}^{i\infty+\sigma'} \frac{dp'}{2\pi i} \frac{ik_3}{\epsilon(k_3, p)(p + ik_3 v)^2} \frac{\partial f_0}{\partial v} \cdot \left\{ \frac{e^{pt} e^{-p\tau'}}{\epsilon(k_2, p')\epsilon(-k_1, p - p')(p' + ik_2 v)} + \frac{e^{p(t-\tau)} e^{p'\tau}}{\epsilon(k_2, p - p')\epsilon(-k_1, p')(p' - ik_1 v)} \right\}, \tag{8}$$

where the contours for the p and p' integrations are defined by requiring that $0 < \sigma' < \sigma$. To carry out the p and p' integrations, we use the Cauchy residue method closing the contours on the side which produces vanishingly small exponentials in the numerator. If we assume that τ is long compared to a collisionless damping period and that the time between the second pulse and the echo is the same order as τ (i.e., that $|\gamma_1\tau|$, $|\gamma_2\tau|$, $|\gamma_3\tau| \gg 1$ and that $k_1/k_3 \simeq 1$) then the residues at the poles arising from the roots of the dielectric functions will all be exponentially small and we may neglect them.

The pole at $p' = -ik_2 v$ and the double pole at $p = -ik_3 v$ yield the contribution

$$\phi_{k_3}^{(2)} = \frac{e}{m} \frac{\Phi_1 \Phi_2 k_1 k_2}{4k_3^2} \int_{-\infty}^{+\infty} dv\, ik_3 \frac{\partial f_0}{\partial v} \frac{e^{ik_1 v\tau}}{\epsilon(-k_1, ik_1 v)} \cdot \frac{\partial}{\partial p} \left[\frac{e^{p(t-\tau)}}{\epsilon(k_3, p)\epsilon(k_2, p - ik_1 v)} \right]_{p=-ik_3 v}. \tag{9}$$

This integral does not phase mix to zero when $t - \tau \simeq \tau(k_1/k_3)$ [i.e., when $t \simeq \tau' \equiv \tau(k_2/k_3)$] and this results in the echo. At this time the largest of the three terms obtained by performing the implied p derivative comes from the derivative of the exponential $e^{p(t-\tau)}$. By setting the coefficient $(t - \tau)$ produced in this differentiation equal to $\tau(k_1/k_3)$ and by neglecting the other two derivative terms, we find

$$\phi_{k_3}^{(2)}(t) \simeq \frac{e}{m} \frac{\Phi_1 \Phi_2 k_1 k_2 i k_1 \tau}{4k_3^2} \int_{-\infty}^{+\infty} dv \frac{\partial f_0}{\partial v} \cdot \frac{\exp[ik_3 v(\tau' - t)]}{\epsilon(-k_1, ik_1 v)\epsilon(k_2, -ik_2 v)\epsilon(k_3, -ik_3 v)}. \tag{10}$$

The various dielectric functions in the denominator of this integral are due to the self-consistent fields associated with the two applied pulses and the echo, and by setting these functions equal to unity one recovers the result of Sec. IV for free streaming particles (i.e., the limit in which the self-consistent fields are negligible in comparison with the applied fields). The velocity integral then yields

$$\phi_{k_3}^{(2)}(t) = \frac{e}{m} \frac{\Phi_1 \Phi_2 k_1^2 k_2 \tau(\tau' - t)}{4k_3} \cdot \int_{-\infty}^{+\infty} dv \frac{\exp(-v^2/2\bar{v}_e^2)}{(2\pi \bar{v}_e^2)^{\frac{1}{2}}} e^{ik_3 v(\tau' - t)},$$
$$\phi_{k_3}^{(2)}(t) = \frac{e}{m} \frac{\Phi_1 \Phi_2 k_1^2 k_2 \tau(\tau' - t)}{4k_3} \cdot \exp[-k_3^2 \bar{v}_e^2 \tfrac{1}{2}(t - \tau')^2], \tag{11}$$

where it has been assumed that $f_0 = (2\pi\bar{v}_e^2)^{-\frac{1}{2}} \exp(-v^2/2\bar{v}_e^2)$. One sees that the entire distribution function contributes to the echo and that the echo duration is of order $\Delta t \sim 2/k_3\bar{v}_e$.

The dielectric factor $[\epsilon(-k_1, ik_1 v)]^{-1}$ in Eq. (10) contains the effect of the self-consistent field which immediately follows the first pulse. For $k_1 L_D < 1$, this factor has a sharply peaked maximum for that velocity v which causes $\epsilon(-k_1, ik_1 v)$ to nearly vanish, the velocity of plasma waves with wavenumber k_1. Physically, the slowly damped plasma wave excited by the first pulse, acts preferentially on electrons with velocities near the wave phase velocity, giving them a much greater perturbation than they received from the externally applied pulse. Thus the perturbation is primarily concentrated in a narrow range in velocity space. The factor $[\epsilon(k_2, -ik_2 v)]^{-1}$ represents the effect of the self-consistent field following the second pulse. Since $k_2 > k_1$, a broader range of electron velocities is affected by the second pulse. Finally, the factor $[\epsilon(k_3, -ik_3 v)]^{-1}$ represents the effect of the self-consistent field generated at the time of the echo. For $k_3L_D < 1$, this factor also has a sharply peaked maximum, and for $k_3 = k_1$ the maximum in this factor will occur at the same velocity as the maximum in $[\epsilon(-k_1, ik_1v)]^{-1}$, thus giving the integral in Eq. (10) a particularly large value. Physically, the echo response is particularly large when $k_1 = k_3$, because the echo build up then drives the plasma resonantly.

When $t < \tau'$, we can evaluate the integral in Eq. (10) by closing the contour in the upper half v plane. Although the $\partial f_0/\partial v$ term in the numerator appears to make the integrand diverge for large imaginary v [recall that $f_0 = (2\pi\bar{v}_e^2)^{-\frac{1}{2}} \exp(-v^2/2\bar{v}_e^2)$], this factor is actually cancelled by the similar behavior of the dielectric function $\epsilon(-k_1, ik_1v)$. In this region of the v plane, we obtain contributions from the poles of $[\epsilon(-k_1, ik_1v)]^{-1}$. By working in the time-asymptotic limit (i.e., $\omega_p(\tau' - t) \gg 1$) we can neglect all but the least damped poles (i.e., the Landau poles) and Eq. (10) yields

$$\phi_{k_3}^{(2)}(t) \simeq -\frac{e}{m}\Phi_1\Phi_2\frac{k_1^2k_2 2\pi\tau}{4k_3^2} \cdot \sum_{\pm}\left\{\frac{\partial f_0}{\partial v}\bigg|_{v=\pm\omega_1/k_1} \frac{\exp[(k_3/k_1)(\pm i\omega_1+\gamma_1)(\tau'-t)]}{\epsilon\left[k_2, \frac{k_2}{k_1}(\pm i\omega_1-\gamma_1)\right]\epsilon\left[k_3, \frac{k_3}{k_2}(\pm\omega_1-\gamma_1)\right]\frac{\partial}{\partial v}[\epsilon(-k_1, ik_1v)]_{v=\pm\omega_1/k_1}}\right\}, \tag{12}$$

where the sum is over the two Landau roots $(\pm i\omega_1 + \gamma_1)$ of $\epsilon(-k_1, i\omega) = 0$ [see Eq. (7)].

When $t > \tau'$, we must close the contour in the lower half v plane. In this region, we obtain contributions from the poles of $[\epsilon(k_2, -ik_2v)]^{-1}$ and $[\epsilon(k_3, -ik_3v)]^{-1}$. Retaining only the least damped poles yields the result

$$\phi_{k_3}^{(2)}(t) \simeq \frac{e}{m}\Phi_1\Phi_2\frac{k_1^2k_2 2\pi\tau}{4k_3^2} \cdot \sum_{\pm}\left\{\frac{\partial f_0}{\partial v}\bigg|_{v=\pm\omega_2/k_2} \frac{\exp[k_3/k_2(\pm i\omega_2-\gamma_2)(\tau'-t)]}{\epsilon\left[-k_1, \frac{k_1}{k_2}(\pm i\omega_2-\gamma_2)\right]\epsilon\left[k_3, \frac{k_3}{k_2}(\pm i\omega_2+\gamma_2)\right]\frac{\partial}{\partial v}[\epsilon(k_2, -ik_2v)]_{v=\pm\omega_2/k_2}} + \frac{\partial f_0}{\partial v}\bigg|_{v=\pm\omega_3/k_3} \frac{\exp[(\pm i\omega_3-\gamma_3)(\tau'-t)]}{\epsilon\left[-k_1, \frac{k_1}{k_3}(\pm i\omega_3-\gamma_3)\right]\epsilon\left[k_2, \frac{k_2}{k_3}(\pm i\omega_3+\gamma_3)\right]\frac{\partial}{\partial v}\left[\epsilon(k_3, -ik_3v)\right]_{v=\pm\omega_3/k_3}}\right\}. \tag{13}$$

It is interesting to note that the echo build-up and decay is not symmetric; the build-up is governed by the term $\exp[\gamma_1(k_3/k_1)(\tau' - t)]$ and the decay by the terms

$$\exp[\gamma_2(k_3/k_2)(t - \tau')]$$

and

$$\exp[\gamma_3(t - \tau')].$$

One can understand this result by recalling that the time until an echo occurs is proportional to the time between the excitation of the two fields causing the echo [i.e., $\tau' = \tau(k_2/k_3)$]. The time between the self-field following the first pulse and the application of the second pulse is less than the time between the two applied pulses (i.e., less than τ); so the echo due to the self-field of the first pulse should occur earlier than the main part of the echo; and, in fact, the echo builds up at the decay rate of the self-field of the first pulse—multiplied by a stretching factor k_3/k_1.

In a similar manner the time between the application of the first pulse and the self-field following the second pulse is greater than τ, so the echo due to the self-field of the second pulse appears after the main part of the echo. Of course, the self-field of the echo itself is masked as the echo builds up and appears only as the echo decays, much as a resonant circuit rings after it is excited.

It was mentioned earlier and can be seen directly from Eqs. (12) and (13) that the echo is largest when $k_1 \simeq k_3$ (i.e., when the echo drives the plasma resonantly). Specializing to this case allows us to neglect the first term in Eq. (13), since $k_1 \simeq k_3$ implies $k_2 \simeq 2k_3$ and $|\gamma_2| \gg |\gamma_3|$. Evaluating the dielectric functions for this case allows us to rewrite

Eqs. (12) and (13) as

$$\phi_{k_3}^{(2)}(t) = \Phi_1(\omega_p\tau)\left(\frac{e\Phi_2}{m\bar{v}_e^2}\right)\left(\frac{k_1^4k_2L_D^2}{k_3(k_1+k_3)^2}\right) \cdot \begin{cases} \dfrac{-(k_3/k_1)\gamma_1 e^{\gamma_1(k_3/k_1)(\tau'-t)}\cos[\omega_1(k_3/k_1)(\tau'-t)+\delta]}{[\omega_p^2(k_3-k_1)^2/(k_3+k_1)^2+\gamma_1^2]^{\frac{1}{2}}} & \text{for} \quad t<\tau', \\ \dfrac{\gamma_3 e^{\gamma_3(t-\tau')}\cos[\omega_3(t-\tau')+\delta']}{[\omega_p^2(k_3-k_1)^2/(k_3-k_1)^2+\gamma_3^2]^{\frac{1}{2}}} & \text{for} \quad t>\tau', \end{cases} \tag{14}$$

where $\tan\delta = [\gamma_1/\omega_p][(k_3+k_1)/(k_3-k_1)]$ and $\tan\delta' = [\gamma_3/\omega_p][(k_1+k_3)/(k_1-k_3)]$.

The results expressed in Eqs. (12), (13), and (14) take into account only the least damped poles and, consequently, are valid only in the time-asymptotic limit $\omega_p|t-\tau'| \gg 1$.

To obtain the shape of the echo near its peak (i.e., where $\omega_p|t-\tau'| \lesssim 1$), we have numerically evaluated the velocity integral in Eq. (10) for three representative values of k_1L_D and k_3L_D. Figure 1a shows the equal-wavenumber case, in which the build-up and decay is approximately symmetrical. Figures 1b and 1c show the cases $k_1 < k_3$ and $k_1 > k_3$, which exhibit asymmetrical build-up and decay discussed earlier.

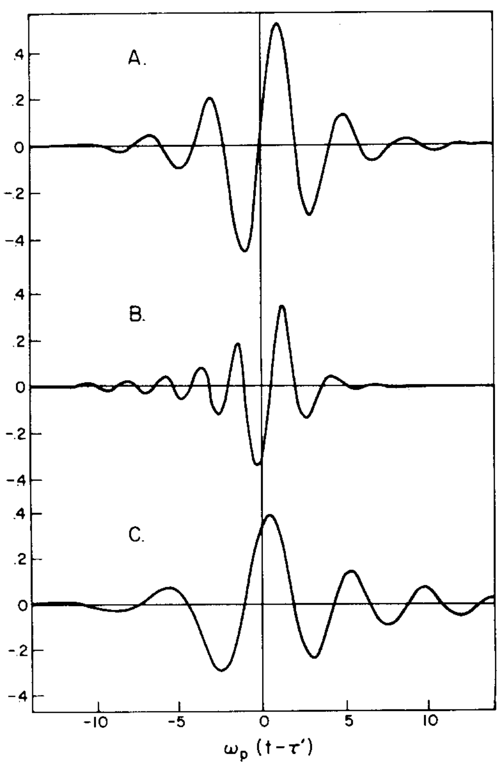

FIG. 1. Normalized echo wave shapes [$\bar{v}_e$ times integral in Eq. 10]. (A) $k_1L_D = k_3L_D = 1/\sqrt{2}$ $k_2L_D = \sqrt{2}$, (B) $k_1L_D = \frac{1}{2}$ $k_2L_D = \frac{3}{2}$ $k_3L_D = 1$, (C) $k_1L_D = 1$ $k_2L_D = \frac{3}{2}$ $k_3L_D = \frac{1}{2}$.

III. SECOND-ORDER SPATIAL ECHO

An idealized method of applying the two external fields for a spatial echo is with a set of two dipole grids (see Fig. 2), each dipole grid structure being composed of two infinitesimally separated single grids. If the two grids in the first dipole have peak to peak potential difference Φ_1 and are driven at frequency ω_1 and the second pair, separated from the first pair by distance l, have potential difference Φ_2 and are driven at frequency ω_2, then the externally applied field will be of the form

$$E_{\text{ext}} = \Phi_1\,\delta(x)\cos(\omega_1 t)e^{\delta t} + \Phi_2\,\delta(x-l)\cos(\omega_2 t)e^{\delta t}, \tag{15}$$

where the exponentials $e^{\delta t}$ are adiabatic switching factors introduced as a calculational convenience. We let δ/ω approach zero at the end of the calculation.

Of course, in any real grid system, the grids forming the dipole structure must be a finite distance apart, and the delta functions in Eq. (15) should correspondingly be replaced by functions of finite width. However, if this width is much less than a Debye length so that the electrons can pass between the grids in much less than one cycle, the external field may be approximated well by Eq. (15).

By Fourier transforming the spatial and temporal dependence of Eqs. (1), (2), and (15) and by iterating the resulting equations with respect to ω_1 and ω_2, we find the following expression for the echo electric field (i.e., the field associated with frequency $\omega_3 = \omega_2 - \omega_1$).

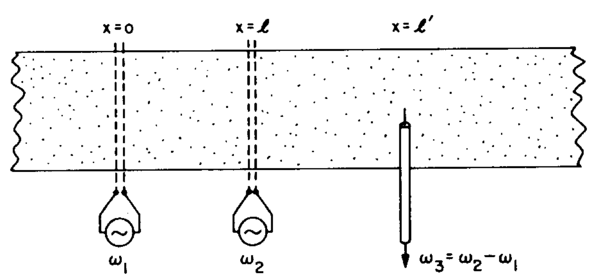

FIG. 2. Schematic drawing of transmitter-receiver arrangement for spatial echoes.

$$E_{\omega_3}(x) = \omega_p^2 \frac{e}{m} \frac{\Phi_1\Phi_2}{4} \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \int_{-\infty}^{+\infty} \frac{dk'}{2\pi} \frac{\partial f_0/\partial v}{[i(kv+\omega_3)+\delta]^2 \epsilon(k, \omega_3)}$$
$$\cdot \left\{ \frac{e^{ik(x-l)+ik'l}}{\epsilon(k', -\omega_1)\epsilon(k-k', \omega_2)[i(k'v-\omega_1)+\delta]} + \frac{e^{ikx-ik'l}}{\epsilon(k', \omega_2)\epsilon(k-k', -\omega_1)[i(k'v+\omega_2)+\delta]} \right\}, \quad (16)$$

where[7]

$\epsilon(k, \omega) \equiv 1 - (\omega_p^2/k^2) \int dv\, \partial f_0/\partial v (v + \omega/k - i\,\delta/k)^{-1}$.

Note that this expression is just the spatial analog of Eq. (8). Of course, there are also second-order terms associated with the frequencies $\omega_1 + \omega_2$, $2\omega_1$, and $2\omega_2$, but there is no echo associated with these terms.

The k and k' integrands in Eq. (16) are the product of a rapidly oscillating exponential and of several factors which have sharply peaked maxima. In the asymptotic limit of large l and $(x - l)$, phase mixing guarantees that the main contribution to the integrals comes from these maxima and that the contribution from any one maximum is small when the width of that maximum is large compared to an oscillation period of the exponential. Consequently, we may neglect the contribution from the maxima associated with the dielectric functions by working in the range where l and $x - l$ are large compared to a Landau damping length, the width of the maxima associated with the dielectric functions. There remains only the contribution from the maxima associated with the factors of the form $[i(kv + \omega) + \delta]^{-1}$ and, in the limit where δ approaches zero, these terms yield

$$E_{\omega_3}(x) = \omega_p^2 \frac{e}{m} \frac{\Phi_1\Phi_2}{4} \int_0^{\infty} dv \frac{\partial f_0}{\partial v} \frac{(-i)}{v^3} \frac{e^{il(\omega_1/v)}}{\epsilon[(\omega_1/v), -\omega_1]} \cdot \frac{\partial}{\partial k} \left\{ \frac{e^{ik(x-l)}}{\epsilon[k-(\omega_1/v), \omega_2]\epsilon(k, \omega_3)} \right\}_{k=-\omega_3/v}. \quad (17)$$

Note that only electrons with $v > 0$ drive the echo. However, electrons with $v < 0$ participate in the plasma response to this drive and are taken into account through $\epsilon[-(\omega_3/v), \omega_3]$.

This integral does not phase-mix to zero when $x - l = l[\omega_1/\omega_3]$ (i.e., when $x = l' \equiv l[\omega_2/\omega_3]$) and this results in the echo. Near this value of x, the largest of the three terms obtained by performing the implied k derivative comes from the derivative of the exponential. By setting the coefficient $(x - l)$ produced in this differentiation equal to $l[\omega_1/\omega_3]$ and by neglecting the other two terms, we find

$$E_{\omega_3}(x) = \omega_p^2 \frac{e}{m} \frac{\Phi_1\Phi_2}{4} \frac{\omega_1}{\omega_3} l \int_0^{\infty} dv \frac{1}{v^3} \frac{\partial f_0}{\partial v} \frac{e^{i(\omega_3/v)(l'-x)}}{\epsilon[(\omega_1/v), -\omega_1]\epsilon[-(\omega_2/v), \omega_2]\epsilon[-(\omega_3/v), \omega_3]}. \quad (18)$$

By making use of the definitions $f_0(v) \equiv \exp[-v^2/2\bar{v}_e^2](2\pi\bar{v}_e^2)^{-\frac{1}{2}}$ and $\zeta \equiv \bar{v}_e/v$. Eq. (18) can be rewritten as

$$E_{\omega_3}(x) = -\left(\frac{e\Phi_1}{m\bar{v}_e^2}\right)\left[\frac{\Phi_2(\omega_1/\omega_2)l}{4(2\pi)^{\frac{1}{2}}L_D^2}\right] \int_0^{\infty} d\zeta \frac{\exp[-(\frac{1}{2}\zeta^2) + i(\omega_3/\bar{v}_e)(l'-x)]}{\epsilon[(\omega_1/\bar{v}_e)\zeta, -\omega_1]\epsilon[-(\omega_2/\bar{v}_e)\zeta, \omega_2]\epsilon[-(\omega_3/v_e)\zeta, \omega_3]}. \quad (19)$$

This integral can be evaluated by the saddle point method. For $x < l'$ the argument of the exponential has a saddle point at $\zeta = e^{i\pi/6}[(\omega_3/\bar{v}_e)(l' - x)]^{-\frac{1}{3}}$; so we deform the contour through this point making sure that we still pass under the Landau pole of $1/\epsilon[(\omega_1/\bar{v}_e)\zeta, -\omega_1]$, labeled pole 1 in Fig. 3a. In the asymptotic limit (i.e., $|x - l'| \gg L_D$), the main contributions to the integral come from the saddle point integral and the Landau pole

$$E_{\omega_3}(x) = -\left(\frac{k_1\Phi_1}{4}\right)\left(\frac{e\Phi_2}{m\bar{v}^2}\right)\left(k_1 l \frac{\omega_1}{\omega_3}\right)$$
$$\cdot \left\{ \frac{e^{i(\pi/3)} \exp[\frac{3}{2}(\omega_3/\bar{v}_e)^{\frac{2}{3}}(l'-x)^{\frac{2}{3}}(-\frac{1}{2}+\frac{1}{2}i\sqrt{3})]}{27\sqrt{3}\ (k_1L_D)^4(k_2L_D)^2(k_3L_D)^2(\omega_3/v_e)^{\frac{1}{3}}(l'-x)^{\frac{1}{3}}} + \frac{-2iL_D^2k_1\Gamma_1 \exp[i(\omega_3/\omega_1)(k_1+i\Gamma_1)(l'-x)]}{3(k_2L_D)^2[1-\omega_1^2/\omega_3^2-(12ik_1\Gamma_1L_D^2)]} \right\}, \quad (20)$$

where

$$k_1 \equiv (\omega_1/\sqrt{3}\,\bar{v}_e)(\omega_1^2/\omega_p^2 - 1)^{\frac{1}{2}}$$

and

$$\Gamma_1 = [(\pi\omega_1^5/6\bar{v}_e^4k_1^4)] \exp[-\tfrac{1}{2}(\omega_1/k_1\bar{v}_e)^2]/(2\pi\bar{v}_e^2)^{\frac{1}{2}}$$

are the wavenumber and spatial decay constant associated with the wave of frequency ω_1.

[7] This definition differs from our earlier definition in that we write $-i\omega$ for p.

For $x > l'$ the appropriate saddle point is at

$$\zeta = e^{-i\pi/6}[\omega_3/\bar{v}_e(x - l')]^{-\frac{1}{3}};$$

so we deform the contour through this point, making sure that we still pass above the Landau poles of $[\epsilon(-\omega_2/v_e\zeta,\ \omega_2)]^{-1}$ and $[\epsilon(-\omega_3/v_e\zeta,\ \omega_3)]^{-1}$, labeled poles 2 and 3 in Fig. 3b. If we specialize to the case where $\omega_1 \simeq \omega_3$ (i.e., $\omega_2 \simeq 2\omega_p$) then Γ_2 will be much larger than Γ_1 and we may neglect the contribution from pole 2 compared with the corresponding contribution from pole 3. Taking into account the saddle point integral and pole 3 yields the result

$$E_{\omega_3}(x) = -k_3\Phi_1\left(\frac{e\Phi_2}{m\bar{v}_e^2}\right)\left(k_3 l\right) \cdot \left\{\frac{e^{-i\pi/3}\exp\left(\frac{3}{2}(\omega_3/\bar{v}_e)^{\frac{2}{3}}(x - l')^{\frac{2}{3}}[-\frac{1}{2} - i(\frac{1}{2}\sqrt{3})]\right)}{27\sqrt{3}\,(k_3L_D)^4(k_2L_D)^2(k_1L_D)^2(\omega_3/\bar{v}_e)^{\frac{4}{3}}(x - l')^{\frac{4}{3}}} - \frac{2ik_3\Gamma_3L_D^2e^{i(l'-x)(k_3-i\Gamma_3)}}{3(k_2L_D)^2[1 - \omega_1^2/\omega_3^2 - 12ik_3\Gamma_3L_D^2]}\right\}. \tag{21}$$

Both Eqs. (20) and (21) exhibit the familiar nonexponential build-up and decay far from the echo maximum (i.e., the first term in each brace) which is characteristic of the saddle point contribution. Furthermore, one expects the exponential contribution to be asymmetrical about $x = l'$ when $\omega_1 \neq \omega_3$ as for temporal echoes.

IV. HIGHER-ORDER ECHOES

In the previous sections we have discussed echoes in the lowest order in which they appear. The echo amplitude is proportional to the first pulse amplitude and to the second pulse amplitude. The appropriate small dimensionless parameters are $k_1u_1\tau$ and $k_2u_2\tau$

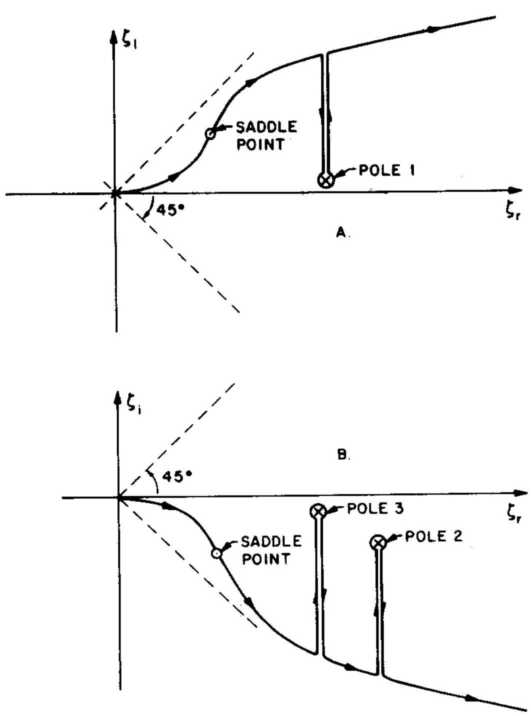

FIG. 3. Integration contour for Eq. (19). (A) $\omega_p(t - \tau') < 0$, (B) $\omega_p(t - \tau') > 0$.

where $u_1 = ek_1\Phi_1/m\omega_p$ and $u_2 = ek_2\Phi_2/m\omega_p$ are the velocities imparted to electrons by the electric fields of the first and second pulses, respectively, at points where field is a maximum. We now obtain a solution to the same problem, which is valid to all orders in the two pulses. The calculation is similar to Webster's[8] ballistic theory of the klystron in which particles are acted on impulsively by the external field and then free stream, bunching and debunching as they go. Since particle acceleration due to plasma supported fields is not taken into account, the calculation is only valid when these fields are negligible compared to the two externally applied fields. This can be insured by making the Debye length large enough.

The formulation of the problem in terms of particle orbits is particularly simple and allows for overtaking or crossover of particles of a given velocity class. The unperturbed particle orbit is $x = x_0 + v_0t$ where x_0 and v_0 are the initial ($t = 0$) position and velocity, respectively. Assuming the electric field to be given by

$$eE_x/m = u_1\,\delta(t)\cos k_1x + u_2\,\delta(t - \tau)\cos k_2x,$$

the perturbed orbit is readily found to be

$$x(t, x_0, v_0) = x_0 + t(v_0 + u_1\cos k_1x_0), \quad 0 \leq t \leq \tau \tag{22a}$$

$$= x(\tau, x_0, v_0) + u_2(t - \tau)\cos k_2[x(\tau, x_0\ v_0)] + u_1(t - \tau)\cos k_1x_0, \qquad \tau \leq t. \tag{22b}$$

The spatial Fourier components of the electric charge density is given by

$$\rho_k = n_0e\iint dx\,dv\,e^{-ikx}f(x, v, t) = n_0e\iint dx_0\,dv_0\,e^{-ikx(t,x_0,v_0)}f_0(v_0), \tag{23}$$

[8] D. L. Webster, J. Appl. Phys. 10, 501 and 864 (1939).

where the latter form follows from $f(x, v, t)\, dx\, dv = f_0(v_0)\, dx_0\, dv_0$.

Consider the response to the first pulse. Substituting Eq. 22a into Eq. (23) and expanding the exponential with the aid of the identity

$$e^{-i\alpha \cos\theta} = \sum_m (-i)^m J_m(\alpha) \exp(-im\theta)$$

we find that the charge density vanishes unless $k = -mk_1$, $m = 0, \pm 1, \cdots$ and that

$$\rho_{-mk_1} = n_0 e(-i)^m J_m(k_1 u_1 t) \exp[-(\tfrac{1}{2} m k_1 \bar{v}_e t)^2]. \qquad (24)$$

From this it follows that various Fourier components are present, rising initially as t^m, then falling rapidly. For $u_1 \ll \bar{v}$ only the fundamental components $m = \pm 1$ are substantial. The actual time behavior is not oscillatory (no plasma oscillations) since the self-consistent field of the plasma has been neglected. Nevertheless, one expects this to be approximately correct when $k_1 L_D \gg 1$.

To obtain the response to two pulses ($\tau < t$) we proceed in a similar manner using Eq. (22b) and Eq. (23) and find that the charge density vanishes unless

$$k_{mn} = -mk_1 + nk_2 \equiv k_{mn}. \qquad (25)$$

For these wave numbers Eq. (23) may be written as

$$\rho_k(t) = n_0 e(-i)^{n-m} J_n(\alpha) \cdot \sum_l J_{m+l}(\beta) J_l(\gamma) \int f_0(v_0) e^{-i\xi v_0/\bar{v}_e}\, dv_0, \qquad (26)$$

where $\alpha = k_{mn} u_2 (t - \tau)$, $\beta = k_{mn} u_1 t$, $\gamma = nk_2 u_1 \tau$, and $\xi = (k_{nm} t - nk_2 \tau)\bar{v}_e$. The l summation yields $J_m(\beta - \gamma)$ and the velocity integral yields exp $[-(\xi^2/2)]$ for a Maxwell distribution. We rewrite this result as

$$\rho_{mn}(t) = n_0 e A_{mn}(t) g_{mn}(t - \tau_{mn}), \qquad (27)$$

where

$$\tau_{mn} \equiv (nk_2/k_{mn})\tau \qquad (28)$$

is the echo time,

$$g_{mn}(t) = \frac{1}{m!}(k_{mn}\bar{v}_e t/\sqrt{2})^m \exp[-(k_{mn}\bar{v}_e t/\sqrt{2})^2] \qquad (29)$$

is the echo pulse shape factor, and

$$A_{mn}(t) = \frac{J_m[k_{mn} u_1 (t - \tau_{mn})]}{[k_{mn}\bar{v}_e (t - \tau_{mn})/\sqrt{2}]^m/m!} J_n[k_{mn} u_2 (t - \tau)] \qquad (30)$$

is the echo pulse amplitude. When u_1 and u_2 are small compared to $\bar{v}_e$ $A_{mn}(t)$ is slowly varying compared to $g_{mn}(t)$. Furthermore, only $A_{mn}(t)$ depends on the pulse amplitudes u_1 and u_2. In this limit $A_{mn}(\tau_{mn})$ gives the amplitude of the echo and $g_{mn}(t)$ gives its shape. Thus we see that, provided $\tau_{mn} > \tau$ each spatial Fourier component with combination wave number given by Eq. (25) exhibits an echo pulse at time τ_{mn} given by Eq. (28). For weak pulses Eq. (30) reduces to

$$A_{mn}(\tau_{mn}) = (-i)^{n-m} \left(\frac{u_1}{\sqrt{2}\,\bar{v}_e}\right)^m \left(\frac{u_2}{\sqrt{2}\,\bar{v}_e}\right)^n \frac{1}{n!} \left(\frac{mk_1\bar{v}_e}{\sqrt{2}}\right)^n \qquad (31)$$

which exhibits a power law dependence on the two-pulse amplitudes u_1 and u_2 and the pulse separation τ, as might be expected for a nonlinear process.

Higher-order spatial echoes have been treated in a similar manner assuming that two double grid structures provide localized oscillating electric fields at $z = 0$, with frequencies ω_1 and ω_2, respectively. A Fourier analysis of the beam *current* shows that the frequencies

$$\omega = -m\omega_1 + n\omega_2 \equiv \omega_{mn} \qquad (32)$$

are present, and that spatial echoes at frequency ω_{mn} occur at

$$z = \frac{n\omega_2}{\omega_{mn}} l \equiv l_{mn}, \qquad (33)$$

provided, of course, that $l_{mn} > l$. The separation of the current density expression into the product of an amplitude factor and a pulse shape factor as in (27) is, unfortunately, not possible in the spatial case because of the appearance of the integration variable v_0 in the Bessel functions. However, for *small* Φ_1 and Φ_2 one can show that the amplitude of the echo of frequency ω_{mn} is proportional to $\Phi_1^m \Phi_2^n l^{m+n}$, a result similar to Eq. (31).

V. DISCUSSION

In the previous sections we consistently treated the ions as a uniform positive background charge. By taking ion dynamics into account we could obviously extend the above work to include ion echoes. Since the typical time scales associated with ion dynamics is longer than that associated with electron dynamics by the factor $\omega_{p_i}/\omega_{p_e} = (m_i/m_e)^{\frac{1}{2}}$ temporal ion echoes may be easier to observe experimentally than temporal electron echoes.

In an experimental observation of echoes one must be sure that collisions do not destroy the echo. This occurs if collisions have enough time to smooth out the velocity space perturbations of the form e^{ikvt}. Since only small-angle collisions are required to smooth out such fine-scale velocity perturbations (the perturbation is fine-scale if $k\bar{v}_e t \gg 1$) and since there are many more small-angle Coulomb collisions than 90° collisions, the echo may be destroyed by Coulomb collisions even though the time between the first pulse and the echo is much less than the 90° deflection time.

This point can be expressed quantitatively by noting that when the $\partial^2/\partial v^2$ term in the Fokker–Planck collision operator acts on a perturbation of the form e^{ikvt} it brings down a factor t^2 and thereby produces an effective collision frequency much larger than the 90° collision frequency (ν_{90}),

$$\nu_{\text{eff}} \simeq \nu_{90}\bar{v}_e^2 \frac{\partial^2}{\partial v^2} e^{ikvt},$$
$$\nu_{\text{eff}} \simeq \nu_{90}\bar{v}_e^2 k^2 t^2 \simeq \nu_{90}(\omega_p t)^2. \tag{34}$$

As may be seen from heuristic arguments and as checked by the rigorous calculation of Karpman,[9] these collisions make the velocity space perturbation decay like $e^{-\nu_{\text{eff}} t} = e^{-\nu_{90}(\omega_p t)^2 t}$. Consequently, small angle Coulomb collisions will be important unless $\nu_{90}\omega_p^2\tau^3 < 1$. For spatial echoes, the equivalent condition is

$$\nu_{90}\omega_p^2(l/\bar{v}_e)^3 \equiv l^3/L_D^2\lambda_{90} < 1,$$

where $\lambda_{90} = \bar{v}_e/\nu_{90}$ is the mean free path for 90° scattering.

[9] V. I. Karpman, Zh. Eksp. Teor. Fiz. **51**, 907 (1966) [Sov. Phys.—JETP **24**, 603 (1967)].

By gradually increasing τ or l until collisions modify the echo, one might be able to use the plasma wave echo as a tool for studying the Coulomb collision rate. Since the echo enhances the effective Coulomb collision rate, such measurements would even be possible in a plasma where electron neutral collisions normally mask Coulomb collisions. By using a second-order echo with $k_1 = k_3$ (or $\omega_1 = \omega_3$) corresponding to weakly damped plasma waves, one could selectively study the collisions associated with the small group of electrons traveling at the wave phase velocity associated with $k_1 = k_3$ (or $\omega_1 = \omega_3$). This same technique might also be used to study the quasi-linear diffusion operator in a weakly turbulent plasma.

From a fundamental point of view, the experimental observation of plasma wave echoes would provide a demonstration of the reversible nature of collisionless damping and phenomena governed by the Vlasov equation.

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