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THE INTERACTION OF ACOUSTIC RADIATION WITH TURBULENCE

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ABSTRACT

We derive expressions for the spectral emissivity and absorptivity of acoustic radiation by low Mach number ($M \ll 1$) turbulent fluids. The emissivity and absorptivity depend on the manner in which the turbulence is excited. We consider three types of turbulence. The first is free turbulence, that is, turbulence which is not subject to external forces. The second and third examples are special cases of forced turbulence, turbulence maintained by stirring with spoons and turbulent pseudoconvection. Acoustic quadrupoles are the lowest order acoustic multipoles present in free turbulence, and they control both its emissivity and absorptivity. Acoustic dipoles are created in forced turbulence, and they enhance the acoustic emissivity by M^{-2} compared to that of free turbulence. The acoustic absorptivity of forced turbulence is quite subtle. The absorptivity of turbulence which is maintained by stirring is dominated by acoustic dipoles and exceeds that of free turbulence by M^{-2}. The dipole absorptivity of turbulent pseudoconvection is reduced by M^2 below that of turbulence maintained by stirring. Thus, the absorptivity of turbulent pseudoconvection is no larger than that of free turbulence.

We apply our results to estimate the equilibrium energies of the acoustic modes in a box filled with fluid some of which is turbulent. For both free turbulence and turbulence maintained by stirring, the most highly excited acoustic modes attain energies $E \sim \mathscr{M} v^2$, where $\mathscr{M}$ and v are the typical mass and velocity of an energy bearing eddy. The quality factors, or Q's, of the modes are larger by M^{-2} in the former case than in the latter. For turbulent pseudoconvection, the most energetic acoustic modes have equilibrium energies $E \sim \mathscr{M} c^2$, where c is the sound speed. Their Q's are comparable to those of modes in equilibrium with free turbulence.

We evaluate the scattering of acoustic radiation by turbulent fluids. For all types of turbulence, the scattering opacity is smaller by M^3 than the absorptive opacity for frequencies near the peak of the acoustic spectrum. Radiation scattered by free turbulence and turbulent pseudoconvection suffers frequency shifts $\Delta\omega \sim \omega$. The frequency shifts are much smaller, $\Delta\omega \sim M\omega$, for radiation scattered by turbulence maintained by stirring.

We investigate the rate at which nonlinear interactions transfer energy among the acoustic modes. If all of the fluid in the box is turbulent, this rate is slower, by M^3 for free turbulence, by M^5 for turbulence maintained by stirring, and by M for turbulent pseudoconvection, than the rate at which the individual acoustic modes exchange energy with the turbulence. If only a small portion of the fluid is turbulent, the nonlinear mode interactions can be significant, especially for modes in equilibrium with turbulent pseudoconvection.

Our results have potential applications to the acoustic radiation in regions of extended turbulence which often arise in nature. In particular, they should prove useful in understanding the excitation of solar oscillations.

Subject headings: hydrodynamics — Sun: oscillations — turbulence

I. INTRODUCTION

We evaluate all of the important interactions which acoustic radiation has with a turbulent fluid. These include the emission, absorption, and scattering of the radiation. Previous theoretical work along these lines has been dominated by the seminal papers of Lighthill (1952, 1954) which focus on the emission of radiation by free turbulence. An excellent review is presented by Crighton (1975).

Our ultimate goal is to relate the excitation of the Sun's acoustic modes to the turbulence in the solar convection zone. We take a first step in this direction and estimate the excitation of the acoustic modes in a box filled with fluid some of which is turbulent. Existing theory is inadequate for attaining even this more limited objective. It must be extended to include the effects of the forces which maintain the turbulence and to account for the reabsorption of the emitted radiation.

The plan of this paper is as follows. Section II is devoted to a discussion of some preliminary issues which must be addressed before we can proceed to develop formulae for the acoustic emissivity and absorptivity in §§ III and IV. In § V, we apply these formulae to estimate the equilibrium energies and quality factors of the acoustic modes in a box which contains turbulent fluid. The scattering of acoustic radiation by the turbulent velocity and pressure fluctuations is treated in § VI. In § VII, we evaluate the rate at which nonlinear interactions transfer energy among the acoustic modes. There are a number of subtle points involved in determining the acoustic emissivity, and especially the acoustic absorptivity, of a turbulent fluid. We buttress our heuristic arguments on these points by appeal to more rigorous calculations which can be performed on analogous electrodynamic systems in § VIII. The final section contains a discussion and generalization of our principal results. Detailed calculations of acoustic absorption are relegated to the Appendix.

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Our aim is to provide a collection of convenient formulae for later applications. Accordingly, our calculations are often rather crude and factors of order unity, or even 4π, are ruthlessly discarded.

II. PRELIMINARIES

a) Description of Turbulence

We adopt a naive picture of homogeneous, isotropic turbulence as a hierarchy of critically damped eddies. The largest, or energy bearing, eddies have linear sizes $\sim H$, velocity magnitudes $\sim v_H$, and lifetimes $\tau_H \sim H/v_H$. For smaller eddies of size $h \lesssim H$, the Kolmogoroff scaling implies $v_h \sim (h/H)^{1/3} v_H$ and $\tau_h \sim (h/H)^{2/3} \tau_H$. The Mach number of the eddies is $M_h \sim v_h/c$, where c is the adiabatic sound speed. We consider subsonic turbulence, $M \equiv M_H \ll 1$. The Reynolds number which characterizes the eddies is $\mathscr{R}e_h \sim v_h h/\nu$, where ν is the kinematic molecular viscosity. We assume that $\mathscr{R}e \equiv \mathscr{R}e_H \gg 1$ so that the effects of viscous dissipation on the acoustic modes may be neglected. The size of the smallest eddies, $h_\nu \sim H/\mathscr{R}e^{3/4}$, which is set by the viscous cutoff to the inertial range, is not pertinent here. The pressure fluctuations associated with eddies of size h have magnitudes $\sim \rho_0 v_h^2$ and vary on a time scale τ_h. The pressure fluctuations are accompanied by small density fluctuations whose magnitudes are $\sim \rho_0 M_h^2$. Thus, eddies of size $\sim h$ are strongly coupled to acoustic modes with wavelength $\lambda \gtrsim h/M_h$.

The picture given above is incomplete since it does not identify the manner in which the turbulence is maintained. We distinguish two general classes of turbulence, free turbulence and forced turbulence.

b) Free Turbulence

Free turbulence is turbulence which is unaffected by external forces. An isentropic turbulent jet provides an example of free turbulence. Energy is transferred from the bulk motion of the jet to the turbulent eddies by Kelvin-Helmholtz instabilities. The lowest order acoustic multipoles present in free turbulence are quadrupoles (Lighthill 1952).

c) Forced Turbulence

Our interest is principally in turbulence which is locally maintained by fluctuating external forces. We are particularly concerned with the external forces that are applied to the fluid since these are responsible for the creation of acoustic dipoles. Next we describe two types of forced turbulence in some detail.

i) Turbulence Maintained by Stirring

Turbulence may be excited by stirring a fluid with a spoon. The size and velocity of the spoon sets the characteristic size and velocity of the largest eddies. A single spoon would excite at most a few energy bearing eddies. Many spoons, with typical spacing H, would be required to maintain an extended region of homogeneous fluid turbulence. For the moment, we assume that the velocity of each spoon is prescribed. We avoid addressing the practical question of how the spoons' movements are coordinated so as to avoid collisions.

The force exerted on the fluid by each spoon has magnitude $\sim \rho_0 H^2 v_H^2$ and fluctuates on time scale $\sim \tau_H$. In addition, there are smaller, more rapidly fluctuating, components of the force which arise from the spoon's interactions with eddies of size $h \lesssim H$. A typical eddy of size $M^{3/2} H \lesssim h \leq H$ feels a force of magnitude

$$f_h \sim \rho_0 \frac{h^3}{H} v_h^2 , \tag{1}$$

which fluctuates on time scale $\sim \tau_h$ (Davies 1970). Equation (1) is easy to derive. We approximate the closest spoon by a planar surface and replace the eddy by a quadrupole source located a distance $d \lesssim H$ from it. In the limit that the planar surface is infinite in extent, the velocity potential follows immediately from the method of images. A simple integration of the perturbation pressure associated with the quadrupole source proves that the net force on the surface vanishes. For a surface of finite horizontal extent $\sim H$ the cancellation of the integrated pressure is incomplete, and results in the force given by equation (1). The lower limit, $M^{3/2} H \lesssim h$, insures that the closest spoon to each eddy is in its near field. The forces exerted by spoons on smaller eddies, $h \lesssim M^{3/2} H$, depend on the distances of the eddies from the nearest spoon. However, these forces are not important for the interaction of the fluid with acoustic radiation, and consequently we ignore them.

Note that f_h is smaller, by a factor $\sim h/H$, than the force associated with the Reynolds stress on eddies of size $\sim h$. Thus, the spoons do not affect the energy cascade in the inertial range, and the Kolmogoroff scaling applies to this type of forced turbulence.

ii) Turbulent Pseudoconvection

Turbulent convection is another type of forced turbulence. The coupling of gravity to the entropy fluctuations within the fluid gives rise to a fluctuating buoyancy force. Turbulent convection is anisotropic, at least on the scale of the energy bearing eddies, since the direction of the gravitational field is singled out. Furthermore, the gravitational field couples to the total density and not just to its fluctuations about the mean. This produces both pressure and density gradients in the fluid. In order to restrict our investigation to the simpler case of homogeneous isotropic turbulence, we imagine a slightly modified version of turbulent convection which we call turbulent pseudoconvection.

The excitation of turbulent pseudoconvection results from the coupling of a spatially uniform external vector field to the concentration of a nondiffusing, scalar contaminant which is randomly added and removed from the fluid on spatial scale $\sim H$ and time scale $\sim \tau_H$. To assure isotropy, the direction of the external force field is also assumed to vary randomly on time scale $\sim \tau_H$. In an obvious analogy with the notation for ordinary convection, we denote the external vector field by $\boldsymbol{g}(t)$ and the fluctuating component of the concentration of the scalar contaminant by s. The buoyancy force acting on an eddy of size $\sim h$ has a magnitude $\sim g\rho_0 h^3 s_h/c_p$, where c_p would be the specific heat per unit mass at constant pressure for true convection. The appropriate scaling law for a scalar contaminant is $s_h \sim (h/H)^{1/3} s_H$ (Tennekes and Lumley 1972). In order to maintain the turbulence, the magnitude of the external force which acts on an energy bearing eddy must be $\sim \rho_0 H^2 v_H^2$. Thus, $s_H \sim c_p v_H^2/(gH)$. The scaling relations for v_h and s_h imply that the external force felt by an eddy of size h is

$$f_h \sim \rho_0 \frac{h^{8/3}}{H^{2/3}} v_h^2 , \tag{2}$$

which is smaller by a factor $\sim (h/H)^{2/3}$ than the force due to the Reynolds stress. Thus, the Kolmogoroff scaling applies to turb-

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ulent pseudoconvection (Ledoux, Schwarzschild, and Spiegel 1962; Goldreich and Keeley 1977*a*).

d) Separation of Turbulent and Acoustic Velocity Fields

Both the turbulence and the acoustic radiation are associated with pressure and velocity fields which vary randomly in time and space. The velocity field can be decomposed, globally and uniquely, into a divergence-free and a curl-free part which we refer to as the shear and the compressive parts of the velocity field. The local values of the shear and compressive components of the velocity field may be expressed in terms of integrals over the distributions of the curl of the vorticity and the divergence of the velocity, respectively (Kraichnan 1953). In the vector Fourier expansion of the velocity field, the shear and compressive parts arise from the components of $\boldsymbol{v}(\boldsymbol{k})$ which are perpendicular and parallel to $\boldsymbol{k}$.

The turbulent and acoustic parts of the velocity field are included within the shear and compressive parts, respectively; there are shear flows which are not turbulent and compressive flows which are not composed of acoustic waves. The turbulent power is spread out over a wide range of k at fixed ω, whereas the acoustic power is concentrated at $\omega = ck$. We assume that the energy density in the acoustic radiation is small compared to that in the turbulence. This assumption is justified in § V.

e) Acoustic Emissivity and Absorptivity

The emission and absorption of acoustic radiation by a turbulent fluid is expressed in terms of the spectral emissivity, $\epsilon(\omega)$, and absorptivity, $\alpha(\omega)$. The former is energy emission rate, per unit volume, per unit frequency. The latter is the coefficient which relates the energy absorption rate, per unit volume, per unit frequency, to the spectral energy density.

Classical calculations of emissivity are easier than those of absorptivity because the latter involve the response of the radiator to incident waves. From a quantum mechanical perspective this might, at first, seem paradoxical. After all, given the value of the coefficient of spontaneous emission, the value of the coefficient of stimulated emission follows immediately from the Bose nature of the quanta. Furthermore, once the coefficient of stimulated emission is known, the absorption coefficient is determined by the invariance of the laws of mechanics under time reversal. The paradox is resolved by noting that classical absorption is net absorption, the difference between true absorption and stimulated emission. To calculate the net absorption, we must know how the quantum states of the system are populated. This is obvious for systems which are in thermodynamic equilibrium. However, turbulent fluids are definitely not in thermodynamic equilibrium although they may be in statistically steady states. Turbulence is inherently dissipative and requires a continuous supply of mechanical energy to be kept from decaying.

In our calculations of acoustic emissivity and absorptivity, we treat eddies of size h as though they are particles. Then v_h provides a measure of the relative velocities of neighboring particles of size h. Of course, these particles are advected by the energy bearing eddies so their total velocities are of order v_H.

f) Electrodynamic Analogs

The interaction of acoustic radiation with a turbulent fluid has many similarities to the interaction of electromagnetic radiation with a system of charged particles. We exploit these analogies to clarify subtle points in § VIII. We have identified eddies as the basic excitations in a turbulent fluid. Their electrodynamic analogs are charged particles in motion. The sound speed plays a role similar to that of the velocity of light. We use the same symbol, c, for both speeds. Turbulent pressure fluctuations are to be associated with fluctuating electrostatic fields. The counterpart of acoustic radiation is electromagnetic radiation.

Quadrupoles are the lowest order acoustic multipoles present in free turbulence of a homogeneous fluid; as a consequence of the fluid's homogeneity, all eddies behave as though they have the same effective acoustic charge to mass ratio. Thus, the analogous charged particles must all have identical charge to mass ratio. We assume that overall charge neutrality is maintained by a uniform background of opposite sign charge.

Radiative interactions are often easier to analyze in the analogous electrodynamic system than in the original turbulent one. The properties of a charged particle are conceptually simpler than those of an eddy. Moreover, there is a cleaner separation between the radiators and the medium through which the radiation propagates in electrodynamics than in acoustics.

g) The Acoustic Wave Equation

Following Lighthill (1952), we manipulate the continuity equation and the inviscid momentum equation to yield

$$\left(\frac{\partial^2}{\partial x_i^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\rho = \frac{1}{c^2}\left(\frac{\partial F_i}{\partial x_i} - \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}\right), \tag{3}$$

where F_i is the external force per unit volume and

$$T_{ij} \equiv \rho v_i v_j + (p - c^2 \rho)\delta_{ij} . \tag{4}$$

We adopt the adiabatic equation of state $p = \kappa \rho^\gamma$. Thus $c^2 \equiv \gamma p_0/\rho_0$ and

$$p - c^2\rho \approx p_0 - c^2\rho_0 + (\gamma - 1)c^2 \frac{(\rho - \rho_0)^2}{\rho_0} . \tag{5}$$

With different choices for the terms on the right-hand side, the wave equation (3) is used to evaluate the emission, the scattering, and the three-mode couplings of acoustic waves in the sections which follow.

III. EMISSIVITY

The starting point for the calculation of emissivity is the wave equation (3). For the moment, we do not specify the form of F_i since it depends upon the type of turbulence under consideration. In this application, the velocities which appear in the first term of the tensor T_{ij} come from the turbulent part of the total velocity field. The fluctuations of the second term in T_{ij} are much smaller than those of the first since the fractional density fluctuations are of order M^2. This allows us to make an immediate simplification and set

$$T_{ij} \approx \rho v_i v_j . \tag{6}$$

We refer to T_{ij} as the Reynolds stress tensor.

The radiation field, $r \gg H/M$, density perturbation produced by the eddies located near the coordinate origin is

$$\rho_{\mathrm{ff}}(\boldsymbol{r}, t) \approx \frac{1}{4\pi c^2 r}\int d\boldsymbol{x}\left[\frac{n_i}{c}\frac{\partial F_i}{\partial t}\left(\boldsymbol{x}, t - \frac{r}{c}\right) + \frac{n_i n_j}{c^2}\frac{\partial^2 T_{ij}}{\partial t^2}\left(\boldsymbol{x}, t - \frac{r}{c}\right)\right], \tag{7}$$

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where $\boldsymbol{n} = \boldsymbol{r}/r$. From the angular dependences of the terms in equation (7), we see that the external force and the Reynolds stress are sources of dipole and quadrupole radiation, respectively.

It follows from elementary fluid mechanics that the total radiated acoustic power is

$$P^T \sim \frac{c^3 r^2 \rho_{\rm ff}^2}{\rho_0} . \tag{8}$$

Since different eddies are uncorrelated, their contributions to $\rho_{\rm ff}$ add incoherently and their contributions to P^T simply sum. We separate the contributions to P^T according to eddy size, h. The total power radiated per unit volume by eddies of size $\sim h$, in other words, their emissivity is

$$\epsilon_h^T \sim \frac{v_h^2 h F_h^2}{\rho_0 c^3} + \frac{\rho_0 v_h^8}{h c^5} . \tag{9}$$

In writing equation (9), we substitute F_h for the magnitude of F_i and set the magnitude of $T_{ij} \sim \rho_0 v_h^2$.

In all of our examples, the spectral emissivity associated with eddies of size $\sim h$ has a power-law dependence on frequency, $\epsilon_h(\omega) \propto \omega^n$, below a cutoff at $\omega\tau_h \sim 1$. Thus, we relate $\epsilon_h(\omega)$ to ϵ_h^T by

$$\epsilon_h(\omega) \sim (\omega\tau_h)^n \tau_h \epsilon_h^T \quad \text{for } \omega\tau_h \lesssim 1 . \tag{10}$$

The emissivity of a turbulent fluid is obtained by summing the contributions from eddies in the size range $h_v \lesssim h \lesssim H$. Hence

$$\epsilon(\omega) \sim \int_{h_v}^{H} \frac{dh}{h} \epsilon_h(\omega) , \tag{11}$$

We are now in a position to evaluate the spectral emissivity of specific types of turbulent fluids explicitly.

a) *Free Turbulence*

As discussed previously, in the absence of external forces the emissivity is entirely due to the acoustic quadrupoles associated with the Reynolds stress. We identify two kinds of acoustic quadrupoles. They are easily characterized in limiting cases.

Consider a planar elastic collision between two identical spherical particles which reverses the component of their relative velocity along the line connecting their centers but leaves their tangential velocities unchanged. The collision gives rise to a time-dependent quadrupole with associated $T_{ij} \sim \rho_0 v_h^2 f_{ij}(t)$, where the f_{ij} are dimensionless functions which describe the time dependences of the tensor components. We distinguish longitudinal from lateral quadrupoles. This is conveniently accomplished using a rectangular coordinate system defined such that the z-axis is normal to the plane of motion and the x-axis points along the line of centers. The time-dependent longitudinal quadrupole moment is given by T_{xx}. Since the component of relative velocity along the line of centers reverses on a time scale $\sim\tau_h$ during the collision, $f_{xx}(t)$, which is a constant of order unity both before and after the collision, drops to zero within a time interval $\sim\tau_h$ around impact. The time-dependent lateral quadrupole moment has components $T_{xy} = T_{yx}$. Since the y components of the particles' velocities are unchanged during the collision, $f_{xy}(t) = f_{yx}(t)$ has the character of a step function of approximately unit magnitude spread out over time interval $\sim\tau_h$.

The second time derivative of T_{ij} is the source of the quadrupole radiation field. Thus, the contributions from the longitudinal and lateral quadrupoles to $\epsilon_h(\omega)$ vary as ω^4 and ω^2 below a cutoff at $\omega\tau_h \sim 1$. Since the two kinds of acoustic quadrupoles make comparable contributions to the total emissivity, we take $\epsilon_h(\omega) \propto \omega^2$ for $\omega\tau_h \lesssim 1$. Now we combine equations (9) and (10) to find

$$\epsilon_h(\omega) \sim \frac{\rho_0 h^2 v_h^5 \omega^2}{c^5} . \tag{12}$$

To evaluate $\epsilon(\omega)$, we substitute the expression for $\epsilon_h(\omega)$ given in equation (12) into equation (11) and use the Kolomorgoroff scaling for v_h. This procedure yields

$$\begin{aligned} &\epsilon(\omega) \sim \rho_0 v_H^2 M^5 (\omega\tau_H)^2 && \text{for } \omega\tau_H \lesssim 1 , \\ &\epsilon(\omega) \sim \rho_0 v_H^2 M^5 (\omega\tau_H)^{-7/2} && \text{for } 1 \lesssim \omega\tau_H \lesssim \mathscr{R}e^{3/4} , \\ &\epsilon(\omega) \sim 0 && \text{for } \mathscr{R}e^{3/4} \lesssim \omega\tau_H . \end{aligned} \tag{13}$$

The acoustic efficiency, η, the ratio of the total radiated power to the total power dissipated by the fluid, is given by

$$\eta \sim \frac{H \epsilon^T}{\rho_0 v_H^3} \sim M^5 . \tag{14}$$

b) *Forced Turbulence*

For both types of forced turbulence, the contribution to the external force per unit volume, F_i, associated with eddies of size $\sim h$ may be approximated by a sequence of independent pulses. Each pulse has magnitude $\sim F_h$, duration $\sim\tau_h$, and spatial correlation length $\sim h$. Since the first time derivative of F_i is the source of the dipole radiation field, the dipole emissivity must be proportional to ω^2 up to a cutoff at $\omega\tau_h \sim 1$.

i) *Turbulence Maintained by Stirring*

We obtain the external force per unit volume, F_h, by dividing the external force per eddy, f_h, given in equation (1) by the eddy volume, h^3. Thus

$$F_h \sim \frac{\rho_0 v_h^2}{H} . \tag{15}$$

for $M^{3/2} H \lesssim h \lesssim H$.

The emissivity due to eddies of size $\sim h$ follows from combining equations (9), (10), and (15). It reads

$$\epsilon_h(\omega) \sim \frac{\rho_0 h^4 v_h^3 \omega^2}{H^2 c^3} + \frac{\rho_0 h^2 v_h^5 \omega^2}{c^5} . \tag{16}$$

The above expression reveals that the dipole emission dominates for eddies with $h \gtrsim M^{3/2} H$.

We obtain the spectral emissivity, $\epsilon(\omega)$, by substituting equation (16) into equation (11):

$$\begin{aligned} &\epsilon(\omega) \sim \rho_0 v_H^2 M^3 (\omega\tau_H)^2 && \text{for } \omega\tau_H \lesssim 1 , \\ &\epsilon(\omega) \sim \rho_0 v_H^2 M^3 (\omega\tau_H)^{-11/2} && \text{for } 1 \lesssim \omega\tau_H \lesssim M^{-1} , \\ &\epsilon(\omega) \sim \rho_0 v_H^2 M^5 (\omega\tau_H)^{-7/2} && \text{for } M^{-1} \lesssim \omega\tau_H \lesssim \mathscr{R}e^{3/4} , \\ &\epsilon(\omega) \sim 0 && \text{for } \mathscr{R}e^{3/4} \lesssim \omega\tau_H . \end{aligned} \tag{17}$$

The energy bearing eddies are the principal contributors to the emissivity at the low-frequency end, $\omega\tau_H \lesssim 1$, of the spectrum. The emissivity is dominated by dipole emission for $\omega\tau_H \lesssim M^{-1}$ and by quadrupole emission at higher frequencies.

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The acoustic efficiency is given by

$$\eta \sim \frac{H\epsilon^T}{\rho_0 v_H^3} \sim M^3 . \tag{18}$$

It is larger, by a factor $M^{-2} \gg 1$, than that for free turbulence.

ii) *Turbulent Pseudoconvection*

The emissivity of this type of forced turbulence differs only slightly from that of the former case. The difference arises because the magnitude of the external force density,

$$F_h \sim \frac{\rho_0 v_h^2}{h^{1/3} H^{2/3}} , \tag{19}$$

obtained by dividing f_h given in equation (2) by h^3, is larger, by a factor $(H/h)^{1/3}$, than that in the previous case. Allowing for this minor difference, the emissivity of a fluid undergoing turbulent pseudoconvection is obtained by a small modification of equation (17). It reads

$$\begin{aligned}
&\epsilon(\omega) \sim \rho_0 v_H^2 M^3 (\omega\tau_H)^2 && \text{for } \omega\tau_H \lesssim 1 , \\
&\epsilon(\omega) \sim \rho_0 v_H^2 M^3 (\omega\tau_H)^{-9/2} && \text{for } 1 \lesssim \omega\tau_H \lesssim M^{-2} , \\
&\epsilon(\omega) \sim \rho_0 v_H^2 M^5 (\omega\tau_H)^{-7/2} && \text{for } M^{-2} \lesssim \omega\tau_H \lesssim \mathscr{R}e^{3/4} , \\
&\epsilon(\omega) \sim 0 && \text{for } \mathscr{R}e^{3/4} \lesssim \omega\tau_H .
\end{aligned} \tag{20}$$

The stronger dipole emission of the inertial range eddies leads to a shallower drop off of $\epsilon(\omega)$ on the high-frequency side of its peak than that given by equation (17) for turbulence maintained by stirring.

The acoustic efficiency,

$$\eta \sim M^3 , \tag{21}$$

is the same as that for turbulence maintained by stirring.

IV. ABSORPTIVITY

The time reversal invariance of the laws of mechanics guarantees that turbulent fluids have absorption processes which are the inverses of their emission processes. However, as discussed in § II, we are concerned with the net absorption rate, the difference between the rates of true absorption and stimulated emission. Even estimating the order of magnitude of this quantity requires considerable care since it depends in subtle ways upon the manner in which the turbulence is excited.

We consider heuristic models for acoustic absorption in free and forced turbulence. More detailed calculations, both classical and quantum mechanical, are presented in the Appendix. We identify three distinct absorption mechanisms. The first mechanism involves quadrupole absorption and operates in all types of turbulence both free and forced. The other two mechanisms involve dipole absorption and act only in the presence of an appropriate external force. Each of our examples of forced turbulence illustrates one of these dipole mechanisms.

As a first step, we concentrate on estimating the spectral absorptivity, $\alpha_h(\omega)$, due to eddies of a particular size, $\sim h$. Then we sum the contributions from eddies of all sizes to obtain the acoustic absorptivity, $\alpha(\omega)$, appropriate to each type of turbulence:

$$\alpha(\omega) \sim \int_{h_v}^{H} \frac{dh}{h}\, \alpha_h(\omega) . \tag{22}$$

a) *Free Turbulence*

To obtain an intuitive understanding of the quadrupole absorption of acoustic radiation, we consider the elastic scattering of two fluid particles of size $\sim h$ in the presence of an acoustic wave. The wave, with pressure amplitude, p_w, frequency, ω, and wavevector, $\boldsymbol{k}$, forces a periodic oscillation of the velocity of each particle,

$$\boldsymbol{v}_w = \frac{\boldsymbol{k} p_w}{(\omega - \boldsymbol{k} \cdot \boldsymbol{v})\rho_0} , \tag{23}$$

which is superposed on the mean velocity, $\boldsymbol{v}$. We view the scattering in the frame which corresponds to the center of mass computed using the colliding particles' mean velocities. At the moment of impact, each particle's velocity will differ from its mean value because of the perturbation by the wave. To a first approximation, at impact, the velocity perturbations of the two particles are equal because, at that time, their spacing is much smaller than one wavelength.

The quadrupole absorption is attributable to the small differential velocity perturbation, $\Delta\boldsymbol{v}_w$, at the moment of impact which has two separate causes. The mass centers of the particles are separated by a distance $\sim h$ at impact. This accounts for a differential velocity perturbation $\sim(\boldsymbol{k} \cdot \boldsymbol{h})\boldsymbol{v}_w$. Prior to impact, the particles are moving at a mean relative velocity $\Delta\boldsymbol{v}$, where $|\Delta\boldsymbol{v}| \sim v_h$. If $\boldsymbol{k} \cdot \Delta\boldsymbol{v} \neq 0$, they see the wave at different Doppler-shifted frequencies. This gives rise to a differential velocity perturbation $\sim(\boldsymbol{k} \cdot \Delta\boldsymbol{v})\boldsymbol{v}_w/\omega$.

The differential velocity perturbation at impact results in an increase of the mean energy of the particles following impact. The increase comes at the expense of the acoustic wave. This is the physical basis for quadrupole absorption. Part of the energy increase is associated with a random walk of the relative mean velocity with step size $\sim|\Delta\boldsymbol{v}_w|$. An additional and comparable contribution arises because the instantaneous relative velocity at impact is slightly larger, by an amount $\sim|\Delta\boldsymbol{v}_w|^2/|\Delta\boldsymbol{v}|$, than the mean relative velocity; the particles are more likely to collide when the differential velocity perturbation increases their velocity of approach. On average, the mean energy of the particles increases by $\Delta E_a \sim \rho_0 h^3 |\Delta\boldsymbol{v}_w|^2$ per scattering.

The duration of a collision is $\sim\tau_h$. For $\omega\tau_h \gtrsim 1$, the acoustic absorption is severely diminished because it depends upon the differential velocity perturbation averaged over the finite time of the collision. This implies that the absorption of acoustic energy is substantial only for $\omega\tau_h \lesssim 1$.

We have identified two components of the differential velocity perturbation at impact, the first due to the separation of the particles and the second to their relative mean velocity. The ratio of the magnitude of the first component to that of the second is $\sim\omega\tau_h$.[1] Thus, except at the very top of the frequency range $\omega\tau_h \lesssim 1$, it is the mean relative velocity of the colliding fluid particles rather than their separation which is responsible for the major share of the absorptivity. Accordingly, we set $|\Delta\boldsymbol{v}_w| \sim M_h|\boldsymbol{v}_w|$ and determine that the absorption of wave energy,

$$\Delta E_a \sim \rho_0 h^3 |\Delta\boldsymbol{v}_w|^2 \sim \frac{h^3 p_w^2 v_h^2}{\rho_0 c^4} , \tag{24}$$

[1] The two components of the differential velocity perturbation give rise to two components of absorptivity, the first proportional to ω^2 and the second independent of ω. These absorptivity components are intimately related to the two emissivity components identified with longitudinal and lateral quadrupoles in § III.

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per scattering. It is then a simple matter to verify that the absorptivity

$$\alpha_h \sim M_h^2 \frac{v_h}{h}, \tag{25}$$

for $\omega\tau_h \lesssim 1$.

The quadrupole contribution to the spectral absorptivity is obtained from equations (22) and (25). It reads

$$\begin{aligned} \alpha(\omega) &\sim \frac{M^2}{\tau_H} \quad \text{for } \omega\tau_H \lesssim \mathscr{R}e^{3/4}, \\ \alpha(\omega) &\sim 0 \quad \text{for } \mathscr{R}e^{3/4} \lesssim \omega\tau_H. \end{aligned} \tag{26}$$

b) *Forced Turbulence*

i) *Turbulence Maintained by Stirring*

The simplest type of absorption is that which occurs during the interaction of turbulent fluid with a spoon. We can understand the absorption process intuitively by analyzing the multiple elastic scattering of a fluid particle by stationary spoons in the presence of an acoustic wave. The restriction to stationary spoons is a convenience to assure that all changes in the particle's energy are attributable to the wave.

A single scattering of a particle which represents an eddy of size $\sim h$ lasts for a time $\sim \tau_h$, the lifetime of the eddy. Thus, the absorption of energy from the wave cuts off for $\omega\tau_h \gtrsim 1$. Furthermore, from the force, f_h, that the spoons exert on the eddy (see eq. [1]), it follows that the momentum transfer in a single scattering is limited to a fraction $\sim h/H$ of the typical momentum of the eddies.

During each scattering, the particle suffers an impulse which partially reverses the component of its incident momentum along $\boldsymbol{s}$, the unit normal to the plane of the spoon. Consequently, the magnitude of the particle's mean velocity changes by an amount $\sim (h/H)|\boldsymbol{s} \cdot \boldsymbol{v}_w|$, the precise value depending upon the phase of the wave at the time of the scattering. Multiple scatterings lead to a random walk of the particle's mean velocity with an associated increase, per scattering, in the mean kinetic energy. An additional and comparable increase in the mean kinetic energy per scattering arises because, on average, the instantaneous normal component of the velocity at impact exceeds the mean normal velocity. Thus, the average energy absorbed by an eddy during a single scattering is

$$\Delta E_a \sim \frac{h^5 p_w^2}{H^2 \rho_0 c^2}. \tag{27}$$

The angular dependence, $\alpha(\boldsymbol{s} \cdot \boldsymbol{k})^2$, of the absorbed energy makes manifest the dipole nature of the absorption process. The dipole absorptivity due to eddies of size $\sim h$ then turns out to be

$$\alpha_h(\omega) \sim \frac{\rho_0 c^2 \Delta E_a}{\tau_h h^3 p_w^2} \sim \left(\frac{h}{H}\right)^2 \frac{1}{\tau_h}, \tag{28}$$

for $\omega\tau_h \lesssim 1$ and $h \gtrsim M^{3/2}H$.

The dipole absorptivity due to eddies for which $h \gtrsim M^{3/2}H$ is obtained by combining equations (22) and (28). Adding this to the quadrupole absorptivity given by equation (26), we arrive at the total absorptivity of turbulence which is maintained by stirring:

$$\begin{aligned} \alpha(\omega) &\sim \frac{1}{\tau_H} \quad \text{for } \omega\tau_H \lesssim 1, \\ \alpha(\omega) &\sim \frac{1}{\omega^2\tau_H^3} \quad \text{for } 1 \lesssim \omega\tau_H \lesssim M^{-1}, \\ \alpha(\omega) &\sim \frac{M^2}{\tau_H} \quad \text{for } M^{-1} \lesssim \omega\tau_H \lesssim \mathscr{R}e^{3/4}, \\ \alpha(\omega) &\sim 0 \quad \text{for } \mathscr{R}e^{3/4} \lesssim \omega\tau_H. \end{aligned} \tag{29}$$

The dipole absorption is dominant at low frequencies, and the quadrupole absorption is dominant at high frequencies. They are of comparable importance for $\omega\tau_H \sim M^{-1}$.

ii) *Turbulent Pseudoconvection*

To investigate the nature of the acoustic absorption process in turbulent pseudoconvection, we consider the motion of a particle under the combined influence of an acoustic wave and the fluctuating external force. Acting alone, the acoustic wave produces a periodic perturbation velocity, $\boldsymbol{v}_w$, given by equation (23). The external force effects random changes of magnitude $|\Delta \boldsymbol{v}| \sim v_h(h/H)^{2/3}$ in the mean velocity over time scales $\sim \tau_h$. These translate into random variations, $\sim (h/H)^{2/3} M_h \omega$, of the Doppler-shifted frequency at which the particle feels the wave. If $\omega\tau_h \gg 1$, the particle's oscillation in the acoustic wave responds adiabatically to the changing Doppler-shifted frequency and there is no net acoustic energy absorbed from the wave. However, for $\omega\tau_h \lesssim 1$, the random variations of the Doppler shift result in a random walk of the particle's velocity with step size $\sim (h/H)^{2/3} v_h |\boldsymbol{v}_w|/c$.[2] The average amount of wave energy absorbed by the eddy per impulse is

$$\Delta E_a \sim \rho_0 h^3 |\Delta \boldsymbol{v}_w|^2 \left(\frac{h}{H}\right)^{4/3} \sim \frac{h^3 p_w^2 v_h^2}{\rho_0 c^4} \left(\frac{h}{H}\right)^{4/3}. \tag{30}$$

The magnitude of ΔE_a given by equation (30) is smaller, by the factor $(h/H)^{4/3}$, than ΔE_a given by equation (24) which results from the interactions among eddies. Also, interactions among eddies produce quadrupole absorption, whereas the present case involves dipole absorption as indicated by the proportionality of the velocity step to $\boldsymbol{k} \cdot \Delta \boldsymbol{v}$.

The preceding argument proves that the dipole absorptivity of eddies is small compared to their quadrupole absorptivity except for the energy bearing eddies where it is comparable. Thus, at all frequencies, the absorptivity of turbulent pseudoconvection is comparable to that of free turbulence and is given by

$$\begin{aligned} \alpha(\omega) &\sim \frac{M^2}{\tau_H} \quad \text{for } \omega\tau_H \lesssim \mathscr{R}e^{3/4}, \\ \alpha(\omega) &\sim 0 \quad \text{for } \mathscr{R}e^{3/4} \lesssim \omega\tau_H. \end{aligned} \tag{31}$$

We compare the results of the model calculations of acoustic absorption carried out in the Appendix with the heuristic results deduced in this section in order to better understand the net absorption in the two types of forced turbulence. In the interest of brevity, we focus the discussion on the energy bearing eddies. To connect the calculations carried out in the

[2] This contribution to the random walk of the velocity is a small addition to that produced directly by the external force.

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Appendix to those done in the main body of the text, we associate the particle mass with the eddy mass,

$$m \sim \rho_0 H^3 , \tag{32}$$

the plane wave potential with the acoustic wave pressure perturbation divided by the unperturbed density,

$$\Phi \sim \frac{p_w}{\rho_0} , \tag{33}$$

and the magnitude of the impulsive velocity change with the typical velocity of the energy bearing eddies,

$$|\Delta \boldsymbol{v}| \sim v_H . \tag{34}$$

c) *Classical Calculations*

i) *Turbulence Maintained by Stirring*

The average wave energy absorbed, ΔE_a, during the reflection of a particle by a wall is given by equation (A10). With the relations expressed in equations (32) and (33), ΔE_a converts to

$$\Delta E_a \sim \frac{H^3 p_w^2}{\rho_0 c^2} , \tag{35}$$

which agrees with equation (27) for $h = H$.

ii) *Turbulent Pseudoconvection*

The average wave energy absorbed due to the impulsive acceleration of a particle is given by equation (A24). Here ΔE_a contains a pair of terms, one of which is proportional to the first, and the other to the second, power of $\boldsymbol{k} \cdot \Delta \boldsymbol{v}$. Since the direction of $\Delta \boldsymbol{v}$ is random, only the latter term is relevant. Applying the conversion relations expressed in equations (32)–(34), we find

$$\Delta E_a \sim \frac{H^3 p_w^2 v_H^2}{\rho_0 c^4} , \tag{36}$$

which is equivalent to equation (30) for $h = H$.

d) *Quantum Mechanical Calculations*

We have emphasized that quantum mechanical calculations hold the key to understanding the difference of a factor v_H^2/c^2 between ΔE_a given by equations (27) and (30). This point is amplified further below.

i) *Turbulence Maintained by Stirring*

The probability of absorption or stimulated emission during a scattering is given by equation (A36). Application of the conversion factors yields

$$\mathscr{P}_\pm \sim \left(\frac{H^3 p_w v_H}{\hbar\omega c}\right)^2 \left(1 \pm \frac{\hbar\omega}{\rho_0 H^3 v_H^2}\right) . \tag{37}$$

ii) *Turbulent Pseudoconvection*

The absorption and stimulated emission probabilities during an impulse are given by equation (A49). Once again, this equation contains terms which are proportional to both the first and second powers of $\boldsymbol{k} \cdot \Delta \boldsymbol{v}$. Dropping the former, since the direction of $\Delta \boldsymbol{v}$ is random, and applying the conversion factors yields

$$\mathscr{P}_\pm \sim \left(\frac{H^3 p_w v_H}{\hbar\omega c}\right)^2 \left(1 \pm \frac{\hbar\omega}{\rho_0 H^3 c^2}\right) . \tag{38}$$

The absorption and stimulated emission transition probabilities for turbulence maintained by stirring and for turbulent pseudoconvection are given by equations (37) and (38). They are identical except for their second factors which express the fractional differences between the probabilities for absorption and stimulated emission. For macroscopic systems these fractional differences are extremely small for both types of forced turbulence. However, the fractional difference is larger, by a factor c^2/v_H^2, for turbulence maintained by stirring than for turbulent pseudoconvection. This accounts for factor c^2/v_H^2 by which the net dipole absorption of the former type of turbulence exceeds that of the latter.

V. AMPLITUDES AND QUALITY FACTORS OF ACOUSTIC MODES

We are concerned with the excitation of the acoustic modes in a rigid box of volume V which is filled with a homogeneous, isentropic fluid. The fluid in a patch of volume fV, $f \le 1$, is maintained in a steady state of homogeneous, isotropic turbulence. We assume that the acoustic wavelength at the peak of the acoustic spectrum, $\sim H/M_H$, is smaller than $V^{1/3}$. Thus, even the largest eddies interact with many modes. Also the energy density of a mode with $\omega\tau_H \gtrsim 1$ has the same average value in the turbulent region as it does in the box as a whole. The mode density, or number of modes per unit frequency interval, is given by

$$\frac{dN}{d\omega} \sim \frac{\omega^2}{c^3} V . \tag{39}$$

The energy in each acoustic mode is determined from

$$E(\omega) \sim \frac{c^3 \epsilon(\omega)}{\omega^2 \alpha(\omega)} , \tag{40}$$

and its quality factor is defined by

$$Q(\omega) \sim \frac{\omega}{\alpha(\omega) f} . \tag{41}$$

Another quantity of interest is the ratio, $R(\omega)$, of the spectral energy density in acoustic part of velocity field to that in the turbulent part. Of course, this ratio is meaningful only in the region where the fluid is turbulent. Using the Kolmogoroff scaling, which implies $v_h(\omega) \sim (\omega\tau_H)^{-1/2} v_H$, it is straightforward to show that

$$R(\omega) \sim \frac{\omega^2 \tau_H \epsilon(\omega)}{\rho_0 v_H^2 \alpha(\omega)} \sim \frac{\omega^4 \tau_H}{c^3 \rho_0 v_H^2} E(\omega) . \tag{42}$$

To evaluate $E(\omega)$, $Q(\omega)$, and $R(\omega)$, we apply the expressions deduced in §§ III and IV for $\epsilon(\omega)$ and $\alpha(\omega)$.

a) *Free Turbulence*

$$\begin{aligned} E(\omega) &\sim \rho_0 H^3 v_H^2 \quad \text{for } \omega\tau_H \lesssim 1 , \\ E(\omega) &\sim \frac{\rho_0 H^3 v_H^2}{(\omega\tau_H)^{11/2}} \quad \text{for } 1 \lesssim \omega\tau_H \lesssim \mathscr{R}e^{3/4} . \end{aligned} \tag{43}$$

Equation (43) states that modes with $\omega\tau_H \lesssim 1$ have energies equal to the kinetic energies of the energy bearing eddies and that modes with $1 \lesssim \omega\tau_H \lesssim \mathscr{R}e^{3/4}$ have energies equal to the

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kinetic energies of the eddies with which they resonate.

$$Q(\omega) \sim \frac{\omega\tau_H}{M^2 f} \quad \text{for } \omega\tau_H \lesssim \mathscr{R}e^{3/4} . \quad (44)$$

$$R(\omega) \sim \left(\frac{\omega H}{c}\right)^3 \frac{(\omega\tau_H)}{1+(\omega\tau_H)^{11/2}} \quad \text{for } \omega\tau_H \lesssim \mathscr{R}e^{3/4} . \quad (45)$$

Note that the peak value of $R(\omega)$ occurs at $\omega\tau_H \sim 1$ and has magnitude $\sim M^3$.

b) *Forced Turbulence*

i) *Turbulence Maintained by Stirring*

$$E(\omega) \sim \rho_0 H^3 v_H^2 \quad \text{for } \omega\tau_H \lesssim 1 ,$$
$$E(\omega) \sim \frac{\rho_0 H^3 v_H^2}{(\omega\tau_H)^{11/2}} \quad \text{for } 1 \lesssim \omega\tau_H \lesssim \mathscr{R}e^{3/4} . \quad (46)$$

The energies of acoustic modes in equilibrium with turbulence maintained by stirring are identical to those of modes in equilibrium with free turbulence:

$$Q(\omega) \sim \frac{\omega\tau_H}{f} \quad \text{for } \omega\tau_H \lesssim 1 ,$$
$$Q(\omega) \sim \frac{(\omega\tau_H)^3}{f} \quad \text{for } 1 \lesssim \omega\tau_H \lesssim M^{-1} , \quad (47)$$
$$Q(\omega) \sim \frac{\omega\tau_H}{M^2 f} \quad \text{for } M^{-1} \lesssim \omega\tau_H \lesssim \mathscr{R}e^{3/4} .$$

Note that for acoustic modes in equilibrium with turbulence maintained by stirring and $\omega\tau_H \lesssim M^{-1}$, $Q(\omega)$ is lower than it is for free turbulence by the factor $[1 + (\omega\tau_H)^2]M^2 \lesssim 1$.

$$R(\omega) \sim \left(\frac{\omega H}{c}\right)^2 \frac{(\omega\tau_H)}{1+(\omega\tau_H)^{11/2}} \quad \text{for } \omega\tau_H \lesssim \mathscr{R}e^{3/4} . \quad (48)$$

We see that $R(\omega)$ is identical to that for free turbulence.

ii) *Turbulent Pseudoconvection*

$$E(\omega) \sim \rho_0 H^3 c^2 \quad \text{for } \omega\tau_H \lesssim 1 ,$$
$$E(\omega) \sim \frac{\rho_0 H^3 c^2}{(\omega\tau_H)^{13/2}} \quad \text{for } 1 \lesssim \omega\tau_H \lesssim M^{-2} , \quad (49)$$
$$E(\omega) \sim \frac{\rho_0 H^3 v_H^2}{(\omega\tau_H)^{11/2}} \quad \text{for } M^{-2} \lesssim \omega\tau_H \lesssim \mathscr{R}e^{3/4} .$$

Equation (49) states that the equilibrium energies of modes with $\omega\tau_H \lesssim 1$ are equal to the thermal energies of the largest eddies rather than to their kinetic energies as was the case for free turbulence and turbulence maintained by stirring. The enhancement of mode energy persists up to $\omega\tau_H \sim M^{-2}$.

$$Q(\omega) \sim \frac{\omega\tau_H}{M^2 f} \quad \text{for } \omega\tau_H \lesssim \mathscr{R}e^{3/4} . \quad (50)$$

The value of $Q(\omega)$ for modes in equilibrium with turbulent pseudoconvection is identical to that for modes in equilibrium with free turbulence.

$$R(\omega) \sim \frac{M(\omega\tau_H)^4}{1+(\omega\tau_H)^{13/2}} \quad \text{for } \omega\tau_H \lesssim M^{-2} ,$$
$$R(\omega) \sim \left(\frac{\omega H}{c}\right)^3 \frac{1}{(\omega\tau_H)^{9/2}} \quad \text{for } M^{-2} \lesssim \omega\tau_H \lesssim \mathscr{R}e^{3/4} . \quad (51)$$

The maximum value attained by $R(\omega) \sim M$ at $\omega\tau_H \sim 1$. This value is larger, by a factor M^{-2}, than that for either free turbulence or for turbulence maintained by stirring. However, even in this case, the equilibrium spectral energy density of the acoustic part of the velocity field is smaller than that of the turbulent part.

c) *Comparison with Previous Results*

The literature on the energies of acoustic modes in equilibrium with turbulence is very sparse. We are aware of two earlier contributions. Crow (1967) estimated the acoustic absorptivity by modeling a turbulent fluid as a viscoelastic medium. He computed the emissivity from Lighthill's theory for free turbulence and reached the conclusion that the equilibrium acoustic energy density is equal to $\sim M^3$ times the turbulent kinetic energy density. Goldreich and Keeley (1977*b*) considered the excitation of the acoustic modes of the Sun by turbulence in the solar convection zone. They assumed an absorptivity due to turbulent viscosity which they modeled using mixing length theory. These authors failed to recognize the enhanced emissivity of turbulent convection and used Lighthill's emissivity. Consequently, they underestimated the equilibrium mode energies by a factor M^2.

VI. SCATTERING OF ACOUSTIC WAVES BY TURBULENCE

As acoustic waves propagate through turbulent fluids, they are scattered by inhomogeneities associated with velocity and pressure fluctuations. These interactions are described by the acoustic wave equation (3) with the nonlinear source term, S, on the right-hand side cast in the form

$$S = -\frac{1}{c^2}\nabla\cdot\left\{2\rho_0\nabla\cdot(\boldsymbol{v}_w\boldsymbol{v}) + \nabla\cdot(\rho_w\boldsymbol{v}\boldsymbol{v}) + \frac{(\gamma-1)c^2}{\rho_0}\nabla[\rho_w(\rho-\rho_0)]\right\} . \quad (52)$$

We have retained only those source terms which are linear in the acoustic field since we are treating the wave propagation as a perturbation with negligible back reaction. Quantities associated with the acoustic wave are given the subscript w. Because the turbulence is assumed to be of low Mach number and $|\boldsymbol{v}|/c = O(M)$, $(\rho-\rho_0)/\rho_0 = O(M^2)$, the largest contribution to the scattering is made by the source term which is linear in $\boldsymbol{v}$.

The inhomogeneous wave equation is transformed into an integral equation for the scattered density field, $\rho_s(\boldsymbol{x}, t)$, using the free space Green's function:

$$\rho_s(\boldsymbol{x}, t) = \frac{1}{4\pi}\int d\boldsymbol{x}_1\, dt_1 \frac{S(\boldsymbol{x}_1, t_1)}{|\boldsymbol{x}-\boldsymbol{x}_1|}\,\delta\left(t_1 - t + \frac{|\boldsymbol{x}-\boldsymbol{x}_1|}{c}\right) . \quad (53)$$

The first-order scattering is evaluated by substituting for ρ_w and $\boldsymbol{v}_w$ in S the fields corresponding to the incoming plane wave,

$$\rho_w = \rho_{w0}\exp(i\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) ,$$
$$\boldsymbol{v}_w = \frac{\boldsymbol{k}c^2}{\rho_0\omega}\rho_{w0}\exp(i\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) . \quad (54)$$

This procedure yields

$$\rho_s(\boldsymbol{x}, t) \approx -\frac{\rho_{w0}\exp(i|\boldsymbol{k}||\boldsymbol{x}|)}{2\pi|\boldsymbol{x}|c^2}\frac{\partial^2}{\partial t^2} \times \int d\boldsymbol{x}_1\left(\frac{\boldsymbol{n}\cdot\boldsymbol{k}}{\omega}\right)\boldsymbol{n}\cdot\boldsymbol{v}\left(\boldsymbol{x}_1, t - \frac{|\boldsymbol{x}-\boldsymbol{x}_1|}{c}\right) \times \exp[-i\omega t + i\boldsymbol{k}'\cdot\boldsymbol{x}_1] , \quad (55)$$

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for the largest source terms, where

$$k' \equiv k - kn \, , \quad n \equiv \frac{x - x_1}{|x - x_1|} \, . \tag{56}$$

We express the scattering efficiency of the turbulent fluid in terms of the scattering cross section associated with a single eddy. Since different eddies are uncorrelated, their scattered waves add incoherently. We restrict our investigation to waves for which $kH \ll 1$. This includes the main portion of the acoustic spectrum associated with the turbulent fluid. For these waves, the scattering is dominated by the velocity fields of the energy bearing eddies. Accordingly, we calculate the scattering by an eddy of size H. Since $kH \ll 1$, the term $\exp(ik' \cdot x_1)$ in the integrand is effectively a constant. Therefore, the scattering cross section is

$$\sigma = \int d\Omega \, \frac{|x|^2 \langle \rho_s^2 \rangle}{|\rho_w|^2} \sim M^6 H^2 [1 + (\omega \tau_H)^4] \, , \tag{57}$$

where the angle brackets denote an expectation value with respect to the turbulent fluctuations. The scattered flux has a broad angular distribution. The scattering is not elastic but involves a frequency shift, $\Delta\omega \sim v_H/H$, which arises from the finite lifetime of the eddies. While not explicitly stated, the scattering cross section given by equation (57) follows from the detailed treatments of the scattering of sound by turbulence in Kraichnan (1953) and Lighthill (1953).

The scattering length associated with σ is

$$L_s(\omega) \sim \frac{H^3}{\sigma} \sim \frac{H}{M^6 [1 + (\omega \tau_H)^4]} \, , \tag{58}$$

and holds for $\omega \tau_H \lesssim M^{-1}$. A comparison of L_s from equation (58) with the absorption length, $L_a \equiv c/\alpha(\omega)$, for frequencies near the peak of the acoustic spectrum of turbulent fluids reveals that $L_s \gg L_a$ for all types of turbulence. However, at much higher frequencies the scattering opacity exceeds the absorptive opacity. The precise frequency at which the two opacities are equal depends upon the type of turbulence.

The scattering length given by equation (58) applies to both free turbulence and turbulent pseudoconvection. For turbulence which is maintained by stirring, the spoons are the dominant scatterers. A standard calculation shows that, in this case,

$$L_s \sim \left(\frac{c}{\omega H}\right)^4 H \sim \frac{H}{(M \omega \tau_H)^4} \, , \tag{59}$$

which is shorter, by a factor $M^2[1 + (\omega \tau_H)^4]$, than L_s given in equation (58). Moreover, the scattering by spoons is nearly elastic, $\Delta\omega \sim M\omega$.

VII. THREE-MODE COUPLINGS

In addition to interacting with the turbulence, the acoustic modes interact with each other. These interactions are described by the nonlinear terms on the right-hand side of the acoustic wave equation (3). The forms of these terms imply that the lowest order nonlinear interactions involve triplets of modes. The frequencies and wavevectors of the three-modes involved in each interaction must satisfy the constraints

$$\omega_a \approx \omega_b + \omega_c \, , \quad k_a = k_b + k_c \, , \tag{60}$$

where we have arbitrarily chosen ω_a to be the highest frequency. These constraints express the conservation of energy and momentum. The frequency matching condition is only approximate since the modes have finite line widths as a consequence of their interactions with the turbulence and with each other. Taken together, the relations given in equation (60) imply that the three-modes must be nearly collinear, a consequence of the homogeneity of the fluid.

The density eigenfunctions of the acoustic modes in a box of volume V are

$$\frac{\rho_k}{\rho_0} = \frac{A(t)}{(2V)^{1/2}} \exp(ik \cdot x - i\omega t) + \mathrm{cc} \, , \tag{61}$$

where "cc" denotes complex conjugate. The velocity eigenfunction is $v_k/c = \hat{k}(\rho_k/\rho_0)$. The energy contained in a mode is related to its amplitude by

$$E(\omega) = \rho_0 c^2 |A|^2 \, . \tag{62}$$

To calculate the time evolution of A_a, we substitute the eigenfunctions into the wave equation (3) and use the approximate collinearity of the modes when evaluating the nonlinear terms. We obtain

$$\frac{dA_a}{dt} = \frac{-i\omega_a(\gamma + 1)}{2(2V)^{1/2}} A_b A_c \exp(i\Delta\omega t) \, , \tag{63}$$

where $\Delta\omega = \omega_a - \omega_b - \omega_c$. Similar equations hold for dA_b/dt and dA_c/dt. In arriving at equation (63), we have discarded the second time derivative of A_a which is valid for $\Delta\omega \ll \omega$.

We solve equation (63) perturbatively. The zeroth-order amplitude is taken to be the actual amplitude at some arbitrarily chosen time t, $A_a^{(0)} \equiv A_a(t)$. The correction of order n to A_a at time $t + \tau$, $A_a^{(n)}$, is computed with the right-hand side of the equation evaluated to order $n - 1$. This procedure yields first- and second-order corrections at time $t + \tau$:

$$A_a^{(1)} = -\frac{(\gamma + 1)}{2(2V)^{1/2}} \frac{\omega_a}{\Delta\omega} [\exp(i\Delta\omega\tau) - 1] \times A_b^{(0)} A_c^{(0)} \exp(i\Delta\omega t) \, , \tag{64}$$

$$A_a^{(2)} = \frac{(\gamma + 1)^2}{8V} \frac{\omega_a}{(\Delta\omega)^2} \{[\exp(-i\Delta\omega\tau) - 1] + i\Delta\omega\tau\} \times A_a^{(0)}(\omega_b |A_c^{(0)}|^2 + \omega_c |A_b^{(0)}|^2) \, . \tag{65}$$

We wish to determine the rate of change of the energy, or equivalently, the absolute square amplitude, $|A_a|^2$, in mode a. To this end, we calculate

$$|A_a(t + \tau)|^2 - |A_a(t)|^2 = A_a^{*(0)} A_a^{(1)} + A_a^{(0)} A_a^{*(1)} + |A_a^{(1)}|^2 + A_a^{*(0)} A_a^{(2)} + A_a^{(0)} A_a^{*(2)} \, . \tag{66}$$

Taking the expectation value (denoted by angle brackets) of the preceding equation yields

$$\langle |A_a(t + \tau)|^2 - |A_a(t)|^2 \rangle = \frac{(\gamma + 1)^2}{2V} \frac{\omega_a}{(\Delta\omega)^2} \sin^2\left(\frac{\Delta\omega\tau}{2}\right) \times (\omega_a |A_b^{(0)}|^2 |A_c^{(0)}|^2 - \omega_b |A_a^{(0)}|^2 |A_c^{(0)}|^2 - \omega_c |A_a^{(0)}|^2 |A_b^{(0)}|^2) \, . \tag{67}$$

The first-order terms do not contribute to equation (67) since they have zero expectation values. Similar relations govern the evolution of $|A_b|^2$ and $|A_c|^2$.

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It is easy to verify that three-mode couplings imply

$$\langle |A_a(t+\tau)|^2 + |A_b(t+\tau)|^2 + |A_c(t+\tau)|^2 - |A_a(t)|^2 - |A_b(t)|^2 - |A_c(t)|^2 \rangle = 0 \,, \quad (68)$$

and

$$\frac{\langle |A_a(t+\tau)|^2 - |A_a(t)|^2 \rangle}{\omega_a} = -\frac{\langle |A_b(t+\tau)|^2 - |A_b(t)|^2 \rangle}{\omega_b} = -\frac{\langle |A_c(t+\tau)|^2 - |A_c(t)|^2 \rangle}{\omega_c} \,. \quad (69)$$

The former relation expresses energy conservation and the latter states that one phonon of mode a may convert into, or be created from, one phonon from each of modes b and c.

We are now in a position to determine $\langle d|A_{\boldsymbol{k}}|^2/dt \rangle$ for a mode which is coupled to a continuum of other modes. To do so, we make a number of modifications to equation (67). We replace a, b, c by $\boldsymbol{k}$, $\boldsymbol{k}'$, $\boldsymbol{k} - \boldsymbol{k}'$. Then we multiply by the density of states, $V/(2\pi)^3$, and integrate over $d^2\boldsymbol{k}'$. We take the integral over $k' \equiv |\boldsymbol{k}'|$ to go from $k/2$ to k to avoid double counting the pairs which interact with mode $\boldsymbol{k}$. Since $\sin^2(x)/x^2$ is sharply peaked at $x = 0$, most of the contribution to the integral arises from triplets for which $\Delta\omega \lesssim \pi/\tau$. Taking into account that

$$\int_{-\infty}^{\infty} dx\, x^{-2} \sin^2 x = \pi \,,$$

we make the replacement

$$\frac{1}{(\Delta\omega)^2} \sin^2\left(\frac{\Delta\omega\tau}{2}\right) \approx \frac{\pi\tau}{2}\, \delta(\Delta\omega) \,, \quad (70)$$

where $\Delta\omega \equiv \omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'} - \omega_{\boldsymbol{k}-\boldsymbol{k}'}$. Finally, we integrate over solid angle and eliminate the δ-function. These operations determine the portion of $\langle d|A_{\boldsymbol{k}}|^2/dt \rangle$ which arises from pairs of modes that satisfy $\boldsymbol{k} = \boldsymbol{k}' + \boldsymbol{k}''$. With a few minor changes, we can add the contributions from pairs which satisfy $\boldsymbol{k} = \boldsymbol{k}' - \boldsymbol{k}''$. The resulting expression for $\langle d|A_{\boldsymbol{k}}|^2/dt \rangle$ reads

$$\left\langle \frac{d|A_{\boldsymbol{k}}|^2}{dt} \right\rangle = \frac{(\gamma+1)^2 c}{16\pi} \left\{ \int_{k/2}^{\infty} dk'\, k' |k' - k| \left[k |A_{k'}|^2 |A_{|k'-k|}|^2 - k' |A_k|^2 |A_{|k'-k|}|^2 + (k' - k) |A_k|^2 |A_{k'}|^2 \right] \right\} . \quad (71)$$

In deriving equation (71) we have implicitly assumed that each mode is involved in many triplets which satisfy the frequency matching condition to within the sum of the line widths of the three modes. In this case, the rate at which energy is transferred among the modes is independent of their line widths, and we can safely pass to the limit of an infinite size box.

The three-mode couplings tend to equalize the energies of the modes. In all of our examples, the equilibrium modal energies are independent of ω for $\omega\tau_H \lesssim 1$ and decline steeply with ω for $\omega\tau_H \gtrsim 1$. Thus, the nonlinear interactions transfer energy from the lower to the higher frequency modes. To assess the importance of this process, we compare the time scale, $\tau_{\rm NL}$, over which it depletes the energy of a mode having $\omega\tau_H \sim 1$ to $\tau_H Q(\tau_H^{-1})$, where Q given in § V is associated with turbulent absorption. From equation (71) we obtain

$$\frac{\tau_{\rm NL}}{\tau_H} \sim \frac{\rho_0 H^3 c^2}{M^3 E(\tau_H^{-1})} \,. \quad (72)$$

Thus, $(\tau_H/\tau_{\rm NL})Q \sim M^3/f$, M^5/f, M^1/f, for free turbulence, turbulence maintained by stirring, and turbulent pseudoconvection, respectively. If the fluid is turbulent throughout most of the box, the nonlinear interactions do not play a decisive role in determining the equilibrium mode energies. However, they could be important if the turbulence is confined to a small fraction of the volume. This is particularly so for turbulent pseudoconvection.

VIII. GEDANKAN EXPERIMENTS AND THEIR ELECTRODYNAMIC ANALOGS

We have taken several uncertain steps in arriving at our conclusions. Perhaps the most serious is our literal acceptance of the simple picture of turbulent fluid motion as a hierarchy of critically damped eddies which obey the Kolmogoroff scaling. To a great extent these oversimplications are unavoidable since there is no fundamental theory of turbulence on which to base an investigation of the interaction of turbulence with acoustic radiation. For this reason, as much as we believe the validity of our results, they must be regarded as tentative.

Below we describe three gedankan experiments which could, at least in principle, serve to test the theoretical conclusions which we have reached. Some day these experiments may be simulated by hydrodynamical computations performed with the aid of a supercomputer. However, this day is probably well in the future. Meanwhile, these gedankan experiments serve a very useful purpose. For each, we can propose an electrodynamic analog which can be subjected to detailed analysis. The more rigorous analysis of the analog electrodynamic experiment serves to enhance the plausibility of our hydrodynamical conclusions. Actually we never really bother to carry through the analysis of these electrodynamic experiments. Rather we assume that the results we state for each of them are sufficiently obvious that the reader needs no further demonstration of their validity.

In all our electrodynamic experiments, electrons play the role of turbulent eddies. Where needed, it is implicitly assumed that there is a smooth background of positive charge to maintain overall charge neutrality. For simplicity, we treat only the energy bearing eddies and neglect the smaller eddies of the inertial range. Eddies are critically damped in the Kolmogoroff picture of turbulence. Therefore, we assume that there is of order one electron per Debye sphere so that the electron-electron collision time is comparable to the time an electron takes to move the interparticle distance. Moreover, we imagine that collisions between electrons are substantially inelastic. In this manner, we simulate the inelastic interactions among eddies which are responsible for the turbulent cascade. The condition that the interelectron distance be of order the Debye length leads to a relation between the mass, m, the change, e, the number density, n, and the mean square velocity, v^2, of the electrons which we write in the form

$$e^2 \sim \frac{mv^2}{n^{1/3}} \,. \quad (73)$$

This relation is used to simplify the formulae which pertain to the analog electrodynamic experiments.

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There is one detail in which the properties of the acoustic and the electrodynamic radiators differ. Each multipole radiation field is proportional to one order higher time derivative of the relevant multipole moment in the acoustic case than in the electrodynamical case. Thus, at low frequency, the acoustic emissivity is proportional to ω^2, whereas the electrodynamic emissivity is independent of ω. By restricting our comparisons of spectral emissivity and absorptivity and energy per mode to frequencies near the peak of the equilibrium spectra, we avoid the complications introduced by this detail.

a) *Free Turbulence*

A region of turbulent fluid is maintained in an interaction region where laminar jets which enter through several small holes in the sides of the box intersect and become turbulent. The jets have diameters of order H and speeds of order v_H. The return flow exits through many other holes in the box at speeds $\ll v_H$. For modes whose wavelengths are large compared to the hole diameters, the loss of acoustic energy due to the holes is negligible and may be ignored. It is important that the interaction region of turbulent fluid be confined to the interior of the box since otherwise acoustic dipoles would be created by the forces exerted on eddies by the walls.

In the electrodynamic analog experiment, electron beams replace the laminar jets. The electrons suffer Coulomb scatterings in the interaction region and, when doing so, emit and absorb electromagnetic radiation by the quadrupole process. We assume that each electron is flushed out of the box after making at most a few collisions. A little thought convinces one that the electromagnetic modes come into equipartition with the electrons. That is, at equilibrium, the energy per mode $\sim mv^2$.

Application of the standard expressions for the emission and absorption of quadrupole radiation during electron-electron collisions yields

$$\epsilon(\omega_p) \sim n^{4/3}e^2\left(\frac{v}{c}\right)^5 \sim nmv^2\left(\frac{v}{c}\right)^5 ; \tag{74}$$

$$\alpha(\omega_p) \sim \frac{n^{2/3}e^2 v}{mc^2} \sim n^{1/3}v\left(\frac{v}{c}\right)^2 . \tag{75}$$

We use equation (73) to obtain the final forms of the expressions for $\epsilon(\omega_p)$ and $\alpha(\omega_p)$. With the replacement of $n^{1/3}$ by H^{-1}, these reduce to the corresponding acoustic results given in equations (13) and (26).

b) *Forced Turbulence*

i) *Turbulence Maintained by Stirring*

Here the turbulence is excited by stirring with spoons which transmit velocity-dependent impulses to the fluid. In a statistically steady state, the energy put into the fluid by the spoons balances that which is dissipated into heat at the bottom end of the turbulent cascade.

In the electrodynamic analog experiment, the spoons are replaced by heavy, positively charged, particles, for example, protons. The protons transmit velocity-dependent impulses to the electrons they scatter. Since each spoon is represented by a proton, there must be of order one proton per Debye sphere. Thus the proton density is equal to the electron density. We imagine that the protons maintain a constant temperature by interacting with some external heat bath. Since collisions between electrons are assumed to be inelastic,[3] the electron temperature is lower than the proton temperature. We assume that the protons are so heavy that they do not couple significantly to the electromagnetic radiation. The electrons emit and absorb electromagnetic radiation during their collisions with the protons principally by the dipole process. The quadrupole emission and absorption which accompanies electron-electron collisions is less important and may be neglected in comparison. It is clear that the equilibrium energies of the electromagnetic modes must again be $\sim mv^2$. However, energy is transferred between the electrons and the electromagnetic modes more rapidly in this case than in the previous one.

The calculation of $\epsilon(\omega_p)$ and $\alpha(\omega_p)$ again follows from standard formulae. We find

$$\epsilon(\omega_p) \sim n^{4/3}e^2\left(\frac{v}{c}\right)^3 \sim nmv^2\left(\frac{v}{c}\right)^3 ; \tag{76}$$

$$\alpha(\omega_p) \sim \frac{n^{2/3}e^2}{mv} \sim n^{1/3}v . \tag{77}$$

Once again, the final versions of the formulae for $\epsilon(\omega_p)$ and $\alpha(\omega_p)$ are in accord with their acoustic counterparts, here given by equations (17) and (29).

ii) *Turbulent Pseudoconvection*

In this case, the energy input to the turbulence comes from the external field which couples to a scalar contaminant whose concentration fluctuates in both space and time. In true convection, the external field has a fixed direction. We have chosen to restrict our attention to isotropic turbulence. Hence, we take the direction of the external field to vary in time. The crucial difference between this example and the previous one is that here the impulses transmitted by the external force are independent of the fluid velocity.

The electrodynamic analog experiment involves a box within which there is a region filled with electrons and a smooth compensating positive charge. It is assumed that each electron carries some scalar contaminant which is randomly added and removed from it on time scale τ. The scalar contaminant couples to a spatially uniform but time-varying external field resulting in the impulsive acceleration of the electrons. The external field supplies energy to the system which is dissipated by the inelastic electron-electron collisions. The electrons emit and absorb electromagnetic radiation by the dipole process while they are being accelerated by the external field. This emission is much greater than the quadrupole emission which occurs during collisions between electrons. However, the net dipole absorption which takes place during the acceleration is only comparable to the quadrupole absorption during electron-electron collisions. This is the result of the near cancellation of the true dipole absorption by stimulated dipole emission. Thus, in this case, the equilibrium energies of the electromagnetic modes are $\sim mc^2$ instead of $\sim mv^2$ as in the previous two examples.

Once again, the calculation of $\epsilon(\omega_p)$ is straightforward. The only new twist is that the external force must have magnitude $\sim n^{1/3}mv^2/e \sim n^{1/2}m^{1/2}v$ so that it can maintain the energies of the electrons against the losses suffered during inelastic collisions. The determination of $\alpha(\omega_p)$ is more difficult. It requires calculations comparable to those presented in the Appendix for the analogous acoustic case. We have carried out both classical and quantum mechanical calculations to satisfy ourselves that the electrodynamic derivations do indeed resemble the acoustic ones. In fact, the only small differences are

[3] We assume that proton-proton and proton-electron collisions are elastic.

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attributable to relativistic effects in the electrodynamic calculations. The upshot is that

$$\epsilon(\omega_p) \sim n^{4/3}e^2\left(\frac{v}{c}\right)^5 \sim nmv^2\left(\frac{v}{c}\right)^5 ; \quad (78)$$

$$\alpha(\omega_p) \sim \frac{n^{2/3}e^2}{mv} \sim n^{1/3}v . \quad (79)$$

A comparison of equations (78) and (79) with equations (20) and (31) confirms the agreement between the electrodynamic and the acoustic results.

IX. DISCUSSION

Our main results are the expressions for the energies of acoustic modes in equilibrium with turbulence of different kinds. For free turbulence and turbulence maintained by stirring, the most highly excited modes come into equipartition with the energy bearing turbulent eddies. The mode energies are of order

$$E \sim \mathscr{M}v^2 , \quad (80)$$

where $\mathscr{M} \sim \rho_0 H^3$ is the mass associated with an energy bearing eddy. For turbulent pseudoconvection, and by extension for turbulent convection, the most highly excited modes have energies comparable to the thermal energies contained in a volume equal to that of an energy bearing eddy. Thus

$$E \sim \mathscr{M}c^2 . \quad (81)$$

The latter is our most surprising and important result. At first sight, it appears to imply that, for turbulent pseudoconvection, the equilibrium energy of the most energetic modes is independent of the magnitude of the turbulent velocity. This is a false impression. For a given mode, that is, for fixed ω_p, the equilibrium energy per mode is proportional to v_H^3 since $\mathscr{M} \sim \rho_0(v_H/\omega_p)^3$. In making this statement, we are comparing a sequence of experiments for which τ_H is constant and v_H and hence H varies.

We have identified eddies as the basic excitations of turbulent fluids. Turbulence must be continuously excited or else it will decay. The energy supplied to the energy bearing eddies cascades to smaller eddies and is ultimately dissipated into heat by molecular viscosity. The kinetic energy per eddy is $E_h \sim \rho_0 h^3 v_h^2$ and, accordingly to the Kolmogoroff scaling, varies with h as $h^{11/3}$. Thus, there is no equipartition of energy between eddies of different sizes. However, in equilibrium, acoustic modes which interact with either free turbulence or turbulence maintained by stirring do reach equipartition of energy with the eddies to which they couple most strongly. This conclusion does not hold for acoustic modes in equilibrium with turbulent pseudoconvection. How are these results to be understood?

The conclusions regarding modes in equilibrium with either free turbulence or turbulence maintained by stirring are an unavoidable consequence of our model of eddies. They are taken to be particles which scatter each other in free turbulence and are scattered by spoons in turbulence maintained by stirring. The scattering events are treated as elastic. Elementary thermodynamic considerations then demand that

$$\frac{\epsilon_h(\omega)}{\alpha_h(\omega)} \sim \rho_0 h^3 v_h^2 \frac{\omega^2}{c^3} \quad \text{for } \omega\tau_h \lesssim 1 . \quad (82)$$

This relation states that acoustic modes in equilibrium with eddies of size $\sim h$ have a Rayleigh-Jeans spectrum with energy per mode equal to the kinetic energy of the eddies. It is easy to verify that the expressions for $\epsilon_h(\omega)$ and $\alpha_h(\omega)$ given in §§ III and IV for free turbulence and turbulence maintained by mixing do satisfy equation (82). We believe that the model we have adopted for the eddies captures the essence of their radiative interactions so that the conclusions regarding equipartition are not only an inevitable consequence of it but are also correct.

The preceding discussion serves to highlight the special character of the interaction of acoustic modes with turbulent pseudoconvection. In this case, the equilibrium mode energies exceed the equipartition values, at least near the peak of the acoustic spectrum. The calculations in §§ III and IV and the Appendix show that acoustic dipoles created by the external force enhance the emissivity but not the absorptivity of turbulent pseudoconvection relative to that of free turbulence. The quantum mechanical calculations provide additional insights. They establish that both the true absorption and the stimulated emission rates for turbulent pseudoconvection are nearly the same as those for turbulence maintained by mixing. However, the small fractional differences between the rates of true absorption and stimulated emission are of order $\hbar\omega/(\rho_0 H^3 c^2)$ in the former case compared to $\hbar\omega/(\rho_0 H^3 v_H^2)$ in the latter. This subtle difference accounts for the reduction, by a factor M^2, of the dipole absorptivity of turbulent pseudoconvection with respect to that of turbulence maintained by stirring.

The higher than equipartition equilibrium energies of acoustic modes which interact with turbulent pseudoconvection does not signal a conflict with thermodynamics. The external force acting on the eddies tend, on average, to increase their kinetic energies. Thus, we cannot appeal to thermodynamic arguments to relate the absorptivity to the emissivity. However, this is not quite the entire story. In turbulence maintained by mixing, the spoons also do net work on the eddies and, in this case, the equilibrium mode energies correspond to equipartition. The crucial distinction may be traced to the *explicit* position dependence of the external force.

In turbulence maintained by mixing, the external force which acts on a fluid particle depends on its position with respect to the spoons. In scattering a particle, a spoon transmits an impulse which depends upon the particle's velocity. In the presence of an acoustic wave, the impulse depends upon the velocity perturbation forced by the wave. The *explicit* position dependence of the external force gives rise to the velocity dependence of the impulse which is responsible for the dipole absorption of the wave energy as described heuristically in § IV.

In turbulent pseudoconvection, the external force which acts on a fluid particle is a function of the instantaneous value of the scalar contaminant which it carries. There is no explicit position dependence of the external force. The impulses received by the particle are independent of its velocity. Thus, the principal mechanism responsible for the dipole absorption of wave energy in turbulence maintained by mixing is absent. All that remains is the reduced dipole absorption associated with changes in the Doppler-shifted wave frequency as discussed in § IV.

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APPENDIX

I. CLASSICAL CALCULATIONS OF ACOUSTIC ABSORPTION

a) Case A: Turbulence Maintained by Stirring

To model the absorption of acoustic radiation due to the interaction of an eddy with a spoon, we consider the motion of a particle of mass m under the influence of a plane wave potential $\Phi = \Phi_0 \sin(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})$ in the presence of a stationary wall. The particle and wall represent the eddy and spoon. The choice of a stationary wall implies that our calculations are made in the rest frame of the spoon. The z-axis is taken to be perpendicular to the wall. We assume that the particle is specularly reflected by the wall.

We denote the particle's position at time $t = 0$ by $\boldsymbol{x}_0$ and the collision time by $t = \tau$, where $\omega\tau \gg 1$. For $t \neq \tau$, the equation of motion reads

$$\frac{d^2\boldsymbol{x}}{dt^2} = \boldsymbol{k}\Phi_0 \cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{x}) . \tag{A1}$$

The time-dependent potential, Φ, induces a periodic variation of the particle's energy. We define the mean energy to be the time average of the instantaneous energy over one wave cycle. Since the wall is stationary, it does no work on the particle. Thus, the change in the mean energy upon reflection is equal to the energy absorbed from the wave.

We solve equation (A1) perturbatively to first order with respect to the parameter $\Phi_0/c^2 \ll 1$, where $c \equiv \omega/k$. The mean velocity jumps from $\boldsymbol{v}_-$ before $t = \tau$ to $\boldsymbol{v}_+$ after $t = \tau$. We determine this jump by applying the reflection condition, $v_z(\tau_+) = -v_z(\tau_-)$, to the instantaneous velocity just before and just after $t = \tau$.

The first order change in velocity, $\boldsymbol{v}_1$, is obtained by integrating the equation of motion with $\boldsymbol{x}$ in the wave potential set equal to $\boldsymbol{x}(t) = \boldsymbol{x}_0 + \boldsymbol{v}_- t$ for $t < \tau$ and $\boldsymbol{x}(t) = \boldsymbol{x}_0 + \boldsymbol{v}_- \tau + \boldsymbol{v}_+(t - \tau)$ for $t > \tau$. This procedure yields

$$\boldsymbol{v}_1 = \frac{\boldsymbol{k}\Phi_0}{\omega_-} \sin\theta_- , \tag{A2}$$

for $t < \tau$ and

$$\boldsymbol{v}_1 = \frac{\boldsymbol{k}\Phi_0}{\omega_+} \sin\theta_+ , \tag{A3}$$

for $t > \tau$. Here we have defined the shorthand notation

$$\begin{aligned} \theta_0 &\equiv -\boldsymbol{k} \cdot \boldsymbol{x}_0 , \qquad & \theta_- &\equiv \omega_- t + \theta_0 , \\ \theta_\tau &\equiv \omega_- \tau + \theta_0 , \qquad & \theta_+ &\equiv \theta_\tau + \omega_+(t - \tau) , \end{aligned} \tag{A4}$$

with

$$\omega_\pm \equiv \omega - \boldsymbol{k} \cdot \boldsymbol{v}_\pm . \tag{A5}$$

The velocity matching condition at $t = \tau$ is expressed by

$$\boldsymbol{v}_+ + \frac{\boldsymbol{k}\Phi_0}{\omega_+} \sin\theta_\tau = (\boldsymbol{I} - 2\hat{z}\hat{z}) \cdot \left(\boldsymbol{v}_- + \frac{\boldsymbol{k}\Phi_0}{\omega_-} \sin\theta_\tau\right) . \tag{A6}$$

To lowest order in v/c, this yields

$$\boldsymbol{v}_+ = \boldsymbol{v}_- - 2\hat{z}\hat{z} \cdot \left(\boldsymbol{v}_- + \frac{\boldsymbol{k}\Phi_0}{\omega} \sin\theta_\tau\right) . \tag{A7}$$

Note that $\boldsymbol{v}_+$ depends upon the value of the wave's phase at the instant of reflection. This implies that the change in mean energy, and, hence, the energy absorbed from the wave also depends on θ_τ. Our interest is in the average energy absorbed per collision. To obtain this quantity, we must average the absorbed energy over z_0 or, equivalently, over θ_τ. The appropriate probability density to use in the latter average is

$$\frac{dP(\theta_\tau)}{d\theta_\tau} = \frac{1}{2\pi}\left(1 + \frac{k_z\Phi_0}{\omega|\hat{z} \cdot \boldsymbol{v}_-|} \sin\theta_\tau\right) \tag{A8}$$

because, per unit time, the collision probability is proportional to the instantaneous value of velocity component along the direction of the wall's normal. We denote these averages by angle brackets. To lowest order in Φ_0, we have

$$\langle \sin\theta_\tau \rangle = \frac{k_z\Phi_0}{2\omega v_z} , \qquad \langle \sin^2\theta_\tau \rangle = \frac{1}{2} . \tag{A9}$$

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Armed with these averages, it is a simple matter to determine the average change in the particle's mean energy and thus the average wave energy absorbed per collision, ΔE_a. We find

$$\Delta E_a = \frac{2mk_z^2\Phi_0^2}{\omega^2} . \tag{A10}$$

Only the change in kinetic energy contributes to ΔE_a. The change in potential energy is of higher order in $v/c \ll 1$.

b) Case B: Pseudoconvection

To model the absorption of acoustic radiation, we again consider particle motion in the plane wave potential Φ. However, in this case, the particle is acted upon by a position independent impulsive external acceleration $\boldsymbol{a}(t) = \Delta \boldsymbol{v}\, \delta(t - \tau)$, where $\Delta \boldsymbol{v}$ is constant in both space and in time. The new equation of motion reads

$$\frac{d^2\boldsymbol{x}}{dt^2} = \boldsymbol{k}\Phi_0 \cos(\omega t - \boldsymbol{k} \cdot \boldsymbol{x}) + \Delta \boldsymbol{v}\, \delta(t - \tau) , \tag{A11}$$

where here and throughout this subsection we adopt the notation defined in the previous subsection. We solve equation (A11) perturbatively to second order with respect to the parameter $\Phi_0/c^2 \ll 1$. We choose $\omega\tau \gg 1$ and denote the particle's position at time $t = 0$ by $\boldsymbol{x}_0$. To simplify the algebra, we set the mean velocity $\boldsymbol{v}_- = 0$. Following the impulse at $t = \tau$, the mean velocity changes to $\boldsymbol{v}_+$. We apply the jump condition to the instantaneous velocity just before and just after $t = \tau$ and then solve for the mean velocity $\boldsymbol{v}_+$. Once this task is completed, it is a straightforward matter to compute the absorption of wave energy by the particle.

For $t < \tau$, the first-order variations in velocity, $\boldsymbol{v}_1$, and displacement, $\boldsymbol{x}_1$, are obtained by successive integrations of the equation of motion with $\boldsymbol{x}$ set to $\boldsymbol{x}_0$ in the wave potential. We find

$$\boldsymbol{v}_1 = \frac{\boldsymbol{k}\Phi_0}{\omega} \sin \theta_- , \tag{A12}$$

$$\boldsymbol{x}_1 = -\frac{\boldsymbol{k}\Phi_0}{\omega^2} (\cos \theta_- - \cos \theta_0) . \tag{A13}$$

The second-order velocity follows from integrating equation (A11) after expanding the wave potential to second order with the aid of $\boldsymbol{x}_1$ given by equation (A13). It reads

$$\boldsymbol{v}_2 = \frac{\boldsymbol{k}\Phi_0^2}{4c^2\omega} (\cos 2\theta_- - 4 \cos \theta_- \cos \theta_0) . \tag{A14}$$

For $t > \tau$, a similar procedure yields

$$\boldsymbol{v}_1 = \frac{\boldsymbol{k}\Phi_0}{\omega_+} \sin \theta_+ , \tag{A15}$$

$$\boldsymbol{x}_1 = -\frac{\boldsymbol{k}\Phi_0}{\omega_+^2} (\cos \theta_+ - \cos \theta_\tau) - \frac{\boldsymbol{k}\Phi_0}{\omega^2} (\cos \theta_\tau - \cos \theta_0) , \tag{A16}$$

$$\boldsymbol{v}_2 = \frac{\boldsymbol{k}\Phi_0^2}{4c^2\omega_+} \left[\frac{\omega^2}{\omega_+^2} (\cos 2\theta_+ - 4 \cos \theta_+ \cos \theta_\tau) + 4 \cos \theta_+ (\cos \theta_\tau - \cos \theta_0) \right] . \tag{A17}$$

The velocity matching condition at $t = \tau$ takes the form

$$\boldsymbol{v}_+ + \boldsymbol{v}_1(\tau_+) + \boldsymbol{v}_2(\tau_+) = \Delta \boldsymbol{v} + \boldsymbol{v}_1(\tau_-) + \boldsymbol{v}_2(\tau_-) . \tag{A18}$$

We substitute the expressions given by equations (A12), (A14), (A15), and (A17) for the first- and second-order velocities in the above equation. After a little rearrangement, we arrive at

$$\boldsymbol{v}_+ = \Delta \boldsymbol{v} - \frac{\boldsymbol{k}\Phi_0(\boldsymbol{k} \cdot \boldsymbol{v}_+) \sin \theta_\tau}{\omega\omega_+} + \frac{\boldsymbol{k}\Phi_0^2 \omega(\boldsymbol{k} \cdot \boldsymbol{v}_+)}{4c^2\omega_+^3} \left[2 \cos^2 \theta_\tau + 3 + \frac{\boldsymbol{k} \cdot \boldsymbol{v}_+}{\omega} (2 \cos^2 \theta_\tau - 3) \right] + \frac{\boldsymbol{k}\Phi_0^2(\boldsymbol{k} \cdot \boldsymbol{v}_+)}{c^2\omega\omega_+} \cos \theta_\tau \cos \theta_0 . \tag{A19}$$

We solve equation (A19) recursively for $\boldsymbol{v}_+$ retaining terms up to second order in Φ_0/c^2 and $f/(mc)$. Thus,

$$\boldsymbol{v}_+ = \Delta \boldsymbol{v} - \frac{\boldsymbol{k}\Phi_0(\boldsymbol{k} \cdot \Delta \boldsymbol{v})}{\omega^2} \sin \theta_\tau + \frac{\boldsymbol{k}\Phi_0^2(\boldsymbol{k} \cdot \Delta \boldsymbol{v})}{4c^2\omega^2} (2 \sin^2 \theta_\tau + 5) - \frac{\boldsymbol{k}\Phi_0(\boldsymbol{k} \cdot \Delta \boldsymbol{v})^2}{\omega^3} \sin \theta_\tau$$
$$+ \frac{\boldsymbol{k}\Phi_0^2(\boldsymbol{k} \cdot \Delta \boldsymbol{v})}{c^2\omega^2} \cos \theta_\tau \cos \theta_0 + \frac{\boldsymbol{k}\Phi_0^2(\boldsymbol{k} \cdot \Delta \boldsymbol{v})^2}{4c^2\omega^3} (24 \cos \theta_\tau \cos \theta_0 + 11) . \tag{A20}$$

The changes in the particle's mean kinetic and mean potential energies and the work done by the external force all depend on θ_τ, the value of the wave's phase at the position of the particle when the impulse occurs. Accordingly, we average these quantities over the initial position $\boldsymbol{x}_0$ or, equivalently, over θ_τ. Since the impulse occurs at a random time, the probability distribution for θ_τ is

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constant. To obtain the average wave energy absorbed by the particle per impulse, we add the average change in the mean kinetic energy,

$$\Delta K = \frac{m(\Delta \boldsymbol{v})^2}{2} + \frac{m\Phi_0^2(\boldsymbol{k} \cdot \Delta \boldsymbol{v})}{2c^2\omega} + \frac{5m\Phi_0^2(\boldsymbol{k} \cdot \Delta \boldsymbol{v})^2}{2c^2\omega^2}, \tag{A21}$$

to the average change in the mean potential energy,

$$\Delta P = \frac{m\Phi_0^2(\boldsymbol{k} \cdot \Delta \boldsymbol{v})}{c^2\omega} + \frac{3m\Phi_0^2(\boldsymbol{k} \cdot \Delta \boldsymbol{v})^2}{2c^2\omega^2}, \tag{A22}$$

and then subtract the average work done by the external force,

$$W = \frac{m(\Delta \boldsymbol{v})^2}{2}. \tag{A23}$$

These steps lead to

$$\Delta E_a = \frac{3m\Phi_0^2(\boldsymbol{k} \cdot \Delta \boldsymbol{v})}{2c^2\omega} + \frac{4m\Phi_0^2(\boldsymbol{k} \cdot \Delta \boldsymbol{v})^2}{c^2\omega^2}, \tag{A24}$$

for the average wave energy absorbed per impulse.

II. QUANTUM MECHANICAL CALCULATIONS OF ACOUSTIC ABSORPTION

Here we redo the absorption calculations from a quantum mechanical perspective. The quantum mechanical calculations emphasize that the classical absorption is a net absorption, that is, the difference between the true absorption and the stimulated emission. Furthermore, they illustrate, in a way which the classical calculations cannot why the acoustic absorption is so sensitive to the mechanism by which the turbulence is maintained.

a) Case A: Turbulence Maintained by Stirring with Spoons

We consider a particle of mass m which is confined to the half-space $z \geq 0$ and moves under the influence of a plane wave potential $\Phi = \Phi_0 \sin(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})$. The evolution of the particle's wave function is governed by Schrödinger's equation which reads

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + m\Phi\right)\Psi. \tag{A25}$$

We separate the Hamiltonian into a zeroth- and a first-order piece,

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2, \tag{A26}$$

and

$$H_1 = m\Phi = m(\Phi^{(+)} + \Phi^{(-)}) = \frac{im\Phi_0}{2}\{\exp[-i(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})] - \exp[+i(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})]\}. \tag{A27}$$

We assume that the wave amplitude, Φ_0, decays smoothly to 0 at large z.

The eigenfunctions of H_0,

$$\Psi_{\boldsymbol{p}} = \frac{2}{(2\pi\hbar)^{3/2}} \exp\left[\frac{i}{\hbar}(\boldsymbol{p}_\perp \cdot \boldsymbol{x} - E_{\boldsymbol{p}} t)\right] \sin\left(\frac{p_z z}{\hbar}\right), \tag{A28}$$

are normalized such that

$$\int d\boldsymbol{x} \Psi_{\boldsymbol{p}}^* \Psi_{\boldsymbol{q}} = \delta^3(\boldsymbol{p} - \boldsymbol{q}). \tag{A29}$$

Here $p_z > 0$ and $\boldsymbol{p}_\perp$ are the components of the momentum parallel and perpendicular to the z-axis and $E_{\boldsymbol{p}} \equiv (\boldsymbol{p}_\perp^2 + p_z^2)/2m$. Each eigenfunction can be decomposed into two plane waves, one of which may be viewed as incident upon and the other reflected from the wall at $z = 0$.

The wave potential, Φ, modifies the solutions of Schrödinger's equation. However, there are still simple solutions for which the incident wave is that associated with one of the eigenfunctions of H_0. As $z \to \infty$, the outgoing part of each solution decomposes into three pieces, the reflected part of the unperturbed eigenfunction and two additional waves, $\Psi'_{\boldsymbol{p}\pm}$, which determine the amplitudes for absorption and stimulated emission. The latter satisfy the following inhomogeneous equation,

$$\frac{\hbar^2}{2m}\nabla^2\Psi'_{\boldsymbol{p}\pm} + i\hbar\frac{\partial \Psi'_{\boldsymbol{p}\pm}}{\partial t} = m\Phi^{(\pm)}\Psi_{\boldsymbol{p}}. \tag{A30}$$

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It is obvious from equation (A30) that the energy and the x and y components of the momentum of $\Psi'_{p\pm}$ must be the sum of the corresponding quantities for the incident particle and the plane wave potential. This reduces the equation to the inhomogeneous, one-dimensional Helmholtz equation,

$$\frac{d^2\Psi'_{p\pm}}{dz^2} + \kappa^2_{p\pm}\Psi'_{p\pm} = \pm\frac{2im^2\Phi_0}{\hbar^2(2\pi\hbar)^{3/2}}\exp(\pm ik_z z)\sin\left(\frac{p_z z}{\hbar}\right), \tag{A31}$$

where

$$\kappa_{p\pm} = \frac{1}{\hbar}\sqrt{p_z^2 - \hbar^2|\boldsymbol{k}_\perp|^2 \mp 2\hbar \boldsymbol{p}_\perp \cdot \boldsymbol{k}_\perp \pm 2m\hbar\omega} \approx \frac{p_z}{\hbar}\left[1 \pm \frac{m\hbar\omega}{p_z^2} - \frac{1}{2}\left(\frac{m\hbar\omega}{p_z^2}\right)^2\right]. \tag{A32}$$

The Green's function for equation (A31) which vanishes at $z = 0$ and reduces to an outgoing wave at large z is given by

$$G(z, z') = \begin{cases} -\dfrac{\sin\kappa_{p\pm} z \exp(i\kappa_{p\pm} z')}{\kappa_{p\pm}} & z < z'; \\ -\dfrac{\sin\kappa_{p\pm} z' \exp(i\kappa_{p\pm} z)}{\kappa_{p\pm}} & z > z'. \end{cases} \tag{A33}$$

Using (A33), we write $\Psi'_{p\pm}$ at large z as

$$\Psi'_{p\pm}(z) = \mp\frac{2im^2\Phi_0}{\hbar^2(2\pi\hbar)^{3/2}\kappa_{p\pm}}\int_0^\infty dz' \exp(i\kappa_{p\pm} z \pm ik_z z')\sin\kappa_{p\pm} z' \sin\left(\frac{p_z z'}{\hbar}\right),$$

which reduces to

$$\Psi'_{p\pm}(z) \approx -\frac{k_z p_z \Phi_0 \exp(i\kappa_{p\pm} z)}{(2\pi\hbar)^{3/2}\hbar\omega^2}, \tag{A34}$$

where, in the last step, we have used

$$\int_0^\infty dz \sin qz = 1/q$$

for $q \neq 0$.

Next we calculate the flux, $F_{p\pm}$, associated with $\Psi'_{p\pm}(z)$ from

$$F_{p\pm} \equiv \frac{\hbar}{2im}(\Psi'^*_{p\pm}\nabla\Psi'_{p\pm} - \Psi'_{p\pm}\nabla\Psi'^*_{p\pm}) = (\hbar\kappa_{p\pm}\hat{z} + \boldsymbol{p}_\perp \pm \hbar\boldsymbol{k}_\perp)\frac{k_z^2\Psi_0^2 p_z^2}{(2\pi\hbar)^3\hbar^2 m\omega^4}. \tag{A35}$$

The probability of absorption or stimulated emission per scattering, $\mathscr{P}_\pm$, is evaluated by dividing $\hat{z}\cdot F_{p\pm}$ by $p_z/[(2\pi\hbar^3)m]$, the z-component of the incident flux. We obtain

$$\mathscr{P}_\pm = \left(\frac{k_z p_z \Phi_0}{\hbar\omega^2}\right)^2\left(1 \pm \frac{m\hbar\omega}{p_z^2}\right). \tag{A36}$$

The net energy absorbed per scattering is given by

$$\Delta E_a \equiv \hbar\omega(\mathscr{P}_+ - \mathscr{P}_-) = \frac{2mk_z^2\Phi_0^2}{\omega^2}. \tag{A37}$$

This expression is identical to that derived classically and given in equation (A10).

b) Case B: Pseudoconvection

We consider the behavior of a particle of mass m which is subject to the combined influence of a plane wave potential, $\Phi = \Phi_0 \sin(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})$, and the potential associated with an impulsive external acceleration, $U(\boldsymbol{x}, t) = -m\boldsymbol{x}\cdot\boldsymbol{a}(t) = -m\boldsymbol{x}\cdot\Delta\boldsymbol{v}\,\delta(t-\tau)$. With these potentials, Schrödinger's equation reads

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 - m\Delta\boldsymbol{v}\cdot\boldsymbol{x}\,\delta(t-\tau) + m\Phi\right)\Psi. \tag{A38}$$

We separate the Hamiltonian into a zeroth- and a first-order piece,

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2 - m\Delta\boldsymbol{v}\cdot\boldsymbol{x}\,\delta(t-\tau), \tag{A39}$$

and

$$H_1 = m\Phi = m(\Phi^{(+)} + \Phi^{(-)}) = \frac{im\Phi_0}{2}\{\exp[-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})] - \exp[+i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})]\}. \tag{A40}$$

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The normalized plane wave solutions of H_0 are given by

$$\Psi_p = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left[\frac{i}{\hbar}\left(\boldsymbol{P}\cdot\boldsymbol{x} - \int^t dt' \frac{P^2}{2m}\right)\right], \tag{A41}$$

where

$$\boldsymbol{P}(t) = \boldsymbol{p} + m\int_0^t dt'\, \Delta\boldsymbol{v}\, \delta(t'-\tau) = \boldsymbol{p} + m\,\Delta\boldsymbol{v}\theta(t-\tau)\,. \tag{A42}$$

Note that $\boldsymbol{p} \equiv \boldsymbol{P}(0)$ is used to label the plane wave solutions of H_0.

We solve for the transition amplitude,

$$\mathscr{A}_{\pm}(t) = \frac{1}{i\hbar}\int_0^t dt' \int dx \Psi^*_{p_f} H_{1\pm}\Psi_{p_i}\,, \tag{A43}$$

from an initial state $\boldsymbol{p}_i$ to a final state $\boldsymbol{p}_f$ using time-dependent perturbation theory. The amplitudes for absorption and emission come from the positive and negative frequency parts of the wave potential. The integrand in equation (A43) is evaluated with the aid of equations (A40) and (A41). The transition amplitudes for absorption and emission are easily shown to be

$$\mathscr{A}_{\pm}(t>\tau) = \pm\frac{m\Phi_0}{2\hbar}\int_0^t dt'\,\delta^3(\boldsymbol{p}_f - \boldsymbol{p}_i \mp \hbar\boldsymbol{k})\exp\left\{i\left[\mp\omega t' + \frac{p_f^2 - p_i^2}{2\hbar m}t' + \frac{\Delta\boldsymbol{v}\cdot(\boldsymbol{p}_f - \boldsymbol{p}_i)}{\hbar}(t'-\tau)\theta(t'-\tau)\right]\right\}. \tag{A44}$$

Evaluating the above integral yields

$$\mathscr{A}_{\pm} = \frac{im\Phi_0}{2\hbar}\left[\frac{\exp(\mp i\omega_{1\pm}\tau) - 1}{\omega_{1\pm}} + \frac{\exp(\mp i\omega_{2\pm}t \mp i\boldsymbol{k}\cdot\Delta\boldsymbol{v}\tau) - \exp(\mp i\omega_{1\pm}\tau)}{\omega_{2\pm}}\right], \tag{A45}$$

where

$$\omega_{1\pm} = \omega - \frac{\boldsymbol{k}\cdot\boldsymbol{p}_i}{m} \mp \frac{\hbar k^2}{2m}, \tag{A46}$$

$$\omega_{2\pm} = \omega_{1\pm} - \boldsymbol{k}\cdot\Delta\boldsymbol{v}\,. \tag{A47}$$

To determine the average absorption probability, $\mathscr{P}_{\pm}$, we average the absolute square of the coefficient of the δ-function in the transition amplitude with respect to both τ and t and then discard the terms which are independent of the impulse $\Delta\boldsymbol{v}$.[4] This procedure yields

$$\mathscr{P}_{\pm} = \frac{m^2\Phi_0^2(\boldsymbol{k}\cdot\Delta\boldsymbol{v})}{2\hbar^2\omega_1\omega_2^2}\,. \tag{A48}$$

Expanding $1/\omega_1\omega_2^2$ using equations (A46) and (A47), we arrive at

$$\mathscr{P}_{\pm} = \frac{m^2\Phi_0^2(\boldsymbol{k}\cdot\Delta\boldsymbol{v})}{2\hbar^2\omega_d^3}\left[1 \pm \frac{3\hbar k^2}{2m\omega_d} + \frac{2\boldsymbol{k}\cdot\Delta\boldsymbol{v}}{\omega_d} \pm \frac{4\hbar k^2(\boldsymbol{k}\cdot\Delta\boldsymbol{v})}{m\omega_d^2}\cdots\right], \tag{A49}$$

where

$$\omega_d \equiv \omega - \frac{\boldsymbol{k}\cdot\boldsymbol{p}_i}{m}\,. \tag{A50}$$

The net energy absorption, ΔE_a, is just $\hbar\omega$ times the difference between the upward and downward transition probabilities from the initial state $\boldsymbol{p}_i$. Thus

$$\Delta E_a \equiv \hbar\omega(\mathscr{P}_+ - \mathscr{P}_-) = \frac{3m\Phi_0^2(\boldsymbol{k}\cdot\Delta\boldsymbol{v})}{2c^2\omega} + \frac{4m\Phi_0^2(\boldsymbol{k}\cdot\Delta\boldsymbol{v})^2}{c^2\omega^2}\,. \tag{A51}$$

We note that ΔE_a given above is in exact accord with the corresponding classical result.

[4] Without this averaging and subtraction, the transition probabilities would depend on the value of τ and would contain terms which are independent of $\Delta\boldsymbol{v}$. The root of this problem is that, even for $\Delta\boldsymbol{v} = 0$, Φ mixes the plane wave solutions of H_0.

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