THE ASTROPHYSICAL JOURNAL, 284:429–434, 1984 September 1

EXCITATION OF INCLINATIONS IN RING-SATELLITE SYSTEMS[1]

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Received 1984 January 9; accepted 1984 March 19

ABSTRACT

Resonant gravitational interactions between a ring and a satellite produce secular variations of their orbital inclinations. Interactions at vertical resonances, analogous to Lindblad resonances but involving inclinations instead of eccentricities, excite inclinations. There is no inclination analog of the corotation resonance. An equatorial ring changes the inclination of a nearby satellite in qualitatively the same way that a satellite in an equatorial orbit changes the inclination of a nearby ring. Viscous dissipation in a ring leads to an equilibrium value of its inclination. These results provide a basis for discussing the origins of the inclinations of planetary rings.

Subject headings: celestial mechanics — planets: satellites — planets: Saturn — planets: Uranus

I. INTRODUCTION

We investigate the long-term evolution of ring and satellite inclinations induced by gravitational interactions. In many cases, mutual perturbations produce a growth of the inclinations. We derive expressions describing the secular growth of the inclination of a satellite (ring) caused by a ring (satellite) in § II. In § III, we investigate possible damping mechanisms. We discuss the inclinations of satellites and rings in the Saturnian and Uranian systems in § IV.

II. GROWTH OF INCLINATIONS

We show that resonant gravitational interactions between a ring and a nearby satellite lead to: (i) the growth of the satellite's inclination if the ring lies in the equatorial plane; (ii) the growth of the ring's inclination if the satellite moves on an equatorial orbit.

These effects are closely related. Both depend upon mutual interactions among the ring particles. A rigorous description of the collective interactions among the ring particles involves several subtle issues. Accordingly, the derivation of expressions for the secular variations of inclination is not trivial. Fortunately, simple artifices permit us to evaluate the essential effects of the self-gravity and viscosity of the ring material on the long-term orbital evolution of inclinations. In the first subsection we give one such simple derivation. Alternate derivations, including a more rigorous treatment involving bending waves, are outlined in the second subsection.

a) *Detailed Derivation*

i) *Equations*

We employ a treatment of the planetary equations devised by Yoder (1973) and reviewed by Peale (1976). It reduces the description of the motion of a pair of satellites with near-commensurate mean motions to a Hamiltonian with one degree of freedom.

We consider the dynamical system consisting of a ring particle and a satellite. Quantities relating to the ring particle are primed, and those relating to the satellite are unprimed: M is the mass; a, e, I, Ω, ϖ, λ are the standard orbital elements; n is the mean motion. Near resonance, the disturbing function for the action of the ring particle on the satellite can be written as $A \cos \phi$, with

$$\phi = j\lambda + j'\lambda' + k\varpi + k'\varpi' + i\Omega + i'\Omega' , \tag{1}$$

where i, j, k, i', j', k' are integers, $i + i'$ is even, and $i + j + k + i' + j' + k' = 0$.

Yoder shows that the state of the system is described by only two variables, the coordinate ϕ and its conjugate momentum x; x is related to the modified Delaunay variables (Brown and Shook 1933)

$$\begin{aligned} L &= (GM_p a)^{1/2} , \\ \Gamma &= L[(1 - e^2)^{1/2} - 1] , \\ Z &= L(1 - e^2)^{1/2}(\cos I - 1) , \end{aligned} \tag{2}$$

by

$$\begin{aligned} L &= jx + L_0 , \\ \Gamma &= kx + \Gamma_0 , \\ Z &= ix + Z_0 . \end{aligned} \tag{3}$$

Here G is the gravitational constant, M_p is the planet mass, and L_0, Γ_0, Z_0 are constants. Similar relations can be written with primed variables and indices. If the perturbation contains no contribution from the indirect term of the mutual gravitational potential of the two bodies (as is true for all the resonances considered in this paper), one has

$$x' = \frac{M}{M'} x . \tag{4}$$

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The evolution of the system is governed by the canonical equations

$$\frac{dx}{dt} = \frac{\partial K}{\partial \phi}, \quad \frac{d\phi}{dt} = -\frac{\partial K}{\partial x}, \tag{5}$$

with the Hamiltonian

$$K(x, \phi) = K_{\rm sec}(x) + A(x) \cos \phi \,. \tag{6}$$

The coefficient A is a sum of terms of the form

$$A(x) = \frac{GM'}{a} F(L, L')|\Gamma|^{\alpha}|\Gamma'|^{\alpha'}|Z|^{\gamma}|Z'|^{\gamma'}, \tag{7}$$

where $\alpha, \alpha', \gamma, \gamma'$ are half-integers of the form

$$\alpha = |k/2| + p_1, \quad \alpha' = |k'/2| + p_2, \quad \gamma = |i/2| + p_3, \quad \gamma' = |i'/2| + p_4, \tag{8}$$

and $p_1, \ldots, p_4$ are nonnegative integers. As a consequence of equations (2)–(4), A is a function of one variable, the canonical momentum x.

From equations (2) and (3), the rate of inclination growth is:

$$\frac{dI}{dt} = \left\{\frac{(j+k)Z - i(L+\Gamma)}{(L+\Gamma)[-Z^2 - 2Z(L+\Gamma)]^{1/2}}\right\}\frac{dx}{dt}. \tag{9}$$

ii) *Second-Order Perturbations*

Since the second term in the Hamiltonian K is smaller by a factor of order $\eta = M'/M_p$ than the first term, the equations of motion may be integrated by successive approximations. This method is appropriate if collective effects prevent the buildup of nonlinear perturbations in the ring.

We follow a procedure similar to the one used by Lynden-Bell and Kalnajs (1972) to work out the second order perturbations of a star in the gravitational field of a spiral density wave. We replace $A(x)$ by $\exp(st)B(x)$, and we look for a solution of the form

$$x = x_0 + x_1 + x_2 + \cdots, \tag{10a}$$

$$\phi = \phi_0 + \phi_1 + \phi_2 + \cdots, \tag{10b}$$

where $x_n = O(\eta^n)$ and $\phi_n = O(\eta^n)$. This procedure removes the singularity at resonance and yields finite perturbations.

There are no secular changes in x_0 or x_1. The second order equation for x_2 contains the secular term

$$\frac{dx_2}{dt} = \left[\frac{1}{2}\frac{d}{dx}\left(\frac{A^2 s}{s^2 + \dot{\phi}_0^{\,2}}\right)\right]_{x=x_0}, \tag{11}$$

where

$$\dot{\phi}_0 = -\frac{dK_{\rm sec}}{dx}(x_0)\,. \tag{12}$$

In the limit $s \rightarrow 0$

$$\frac{dx_2}{dt} = \left\{\frac{\pi}{2}\frac{d}{dx}[A^2\delta(\dot{\phi}_0)]\right\}_{x=x_0}. \tag{13}$$

We continue to use the expansion (10) even though it fails to converge as $s \rightarrow 0$. Our justification for this procedure is given in the following subsection.

We substitute the expression given by equation (7) for A into equation (13). Since the mass of the satellite is much larger than the mass of the ring particle, we replace d/dx by $(M/M')\partial/\partial x'$, where x and x' are now to be regarded as independent variables. We assume that $\dot{\phi}_0$ depends only on L and L', which is approximately correct for a nearly Keplerian potential. Equation (13) then becomes:

$$\frac{dx_2}{dt} = \frac{\pi G^2 M M'}{2a^2}\sum_{\alpha,\alpha',\gamma,\gamma'}\sum_{\tilde\alpha,\tilde\alpha',\tilde\gamma,\tilde\gamma'}|\Gamma|^{\alpha+\tilde\alpha}|\Gamma'|^{\alpha'+\tilde\alpha'}|Z|^{\gamma+\tilde\gamma}|Z'|^{\gamma'+\tilde\gamma'} \times\left\{j'\frac{\partial}{\partial L'}[F\tilde F\delta(\dot\phi_0)] - \left[\frac{(\alpha'+\tilde\alpha')k'}{|\Gamma'|} + \frac{(\gamma'+\tilde\gamma')i'}{|Z'|}\right]F\tilde F\delta(\dot\phi_0)\right\}. \tag{14}$$

iii) *Inclination Evolution of a Satellite*

Our aim is to calculate the changes produced in the satellite orbit by a continuous ring of particles of surface mass density $\Sigma(a')$ at radius a'. Thus we replace M' by $2\pi a'\Sigma(a')da'$ and integrate over the ring.[3] This procedure yields

$$\frac{dx_2}{dt} = \frac{-\pi^2 G^2 M F\tilde F|\Gamma|^{\alpha+\tilde\alpha}|Z|^{\gamma+\tilde\gamma}}{a^2|d\dot\phi_0/da'|} \times\left\{2j'\frac{d}{da'}\left(\frac{\Sigma}{n'}\right)|\Gamma'|^{\alpha'+\tilde\alpha'}|Z'|^{\gamma'+\tilde\gamma'} + [(\alpha'+\tilde\alpha')k'|\Gamma'|^{\alpha'+\tilde\alpha'-1}|Z'|^{\gamma'+\tilde\gamma'} + (\gamma'+\tilde\gamma')i'|\Gamma'|^{\alpha'+\tilde\alpha'}|Z'|^{\gamma'+\tilde\gamma'-1}]\Sigma a'\right\}, \tag{15}$$

where the summation over $\alpha, \alpha', \gamma, \gamma', \tilde\alpha, \tilde\alpha', \tilde\gamma, \tilde\gamma'$ is understood.

Formula (15) is quite general and allows us to recover results previously obtained by Goldreich and Tremaine (1981, Paper I). The terms of the potential with $(\alpha, \alpha', \gamma, \gamma') = (\tilde\alpha, \tilde\alpha', \tilde\gamma, \tilde\gamma')$ equal to (0, 0, 0, 0) and (0, $\frac{1}{2}$, 0, 0) lead, respectively, to their expressions (32) and (36) for the torques at corotation and Linblad resonances.

We evaluate dI_2/dt ($=dI/dt$ to second order with respect to the masses) from equations (9) and (15). The terms proportional to $x_1 dx_1/dt$ in dI/dt represent second order perturbations of the satellite by the unperturbed ring and are suppressed for the reason given in note 3. By examining equations (9) and (15) we find that dI_2/dt is of order 1 with respect to the inclinations and eccentricities, and that the terms of equation (15) which contribute to this order are the first one with $(|k|, |k'|, |i|, |i'|) = (0, 0, 0, 0)$, the second one with $(|k|, |k'|, |i|, |i'|) = (0, 1, 0, 0)$, and the third one with $(|k|, |k'|, |i|, |i'|) = (0, 0, 1, 1)$; in both cases the coefficients $\alpha, \alpha', \gamma, \gamma', \tilde\alpha, \tilde\alpha', \tilde\gamma, \tilde\gamma'$ are set to their smallest possible values. The resonance with $|k| = |k'| = |i| = |i'| = 0$ occurs only when the satellite is embedded in the ring and will not concern us here. The values of ϕ and F for the other resonances are given in Table 1, where the quantity $b_k^{(m)}$ is a Laplace coefficient defined by the integral:

$$b_k^{(m)}(r) = \frac{2}{\pi}\int_0^{\pi}\frac{\cos m\theta d\theta}{(1 + r^2 - 2r\cos\theta)^k}. \tag{16}$$

[3] We have neglected terms proportional to $M_1'M_2'$, where M_1' and M_2' are the masses of different ring particles. These terms represent second-order perturbations of the satellite by the unperturbed ring, and can be absorbed in the unperturbed axisymmetric potential.

TABLE 1

EXPRESSIONS FOR THE TERMS OF THE POTENTIAL GIVEN BY EQUATION (6) FROM WHICH FOLLOW THE SECULAR VARIATION OF I TO THE LOWEST ORDER WITH RESPECT TO THE INCLINATIONS AND ECCENTRICITIES

k	k'	i	i'	α	F	ϕ
0	1	0	0	a/a'	$\frac{\alpha}{(2n')^{1/2}a'}\left[(2m+1)b_{1/2}^{(m)}(\alpha)+\alpha\frac{db_{1/2}^{(m)}(\alpha)}{d\alpha}\right]$	$m\lambda-(m+1)\lambda'+\varpi'$
0	-1	0	0	a'/a	$\frac{-1}{(2n')^{1/2}a'}\left[2mb_{1/2}^{(m)}(\alpha)+\alpha\frac{db_{1/2}^{(m)}(\alpha)}{d\alpha}\right]$	$m\lambda-(m-1)\lambda'-\varpi'$
0	0	1	1	a/a'	$\frac{-\alpha^{7/4}b_{3/2}^{(m)}(\alpha)}{2n'a'^2}$	$(m-1)\lambda-(m+1)\lambda'+\Omega+\Omega'$
0	0	-1	-1	a'/a	$\frac{-\alpha^{5/4}b_{3/2}^{(m)}(\alpha)}{2n'a'^2}$	$(m+1)\lambda-(m-1)\lambda'-\Omega-\Omega'$

We find that if the satellite is exterior to the ring ($a > a'$):

$$\frac{1}{I}\frac{dI}{dt}=\frac{1}{(m-1)}\frac{\pi^2}{6}\frac{M}{M_p}\left\{\frac{1}{2}\frac{n_1'\Sigma_1 a_1'^2}{M_p}\alpha_1^{9/2}[b_{3/2}^{(m)}(\alpha_1)]^2-\frac{mn_2'\Sigma_2 a_2'^2}{M_p}\alpha_2^{5/2}\left[\left(2m+\alpha_2\frac{d}{d\alpha_2}\right)b_{1/2}^{(m)}(\alpha_2)\right]^2\right\}, \tag{17}$$

where $\alpha_i = a_i'/a$, $n_i' = n'(a_i')$ and $\Sigma_i = \Sigma(a_i')$ are evaluated at the resonance radii a_i, defined by $(m+1)n-(m-1)n_1'-\dot{\Omega}-\dot{\Omega}'=0$ or $mn-(m-1)n_2'-\dot{\varpi}'=0$ for the first and second terms, respectively. If the satellite is interior to the ring ($a' > a$), we find

$$\frac{1}{I}\frac{dI}{dt}=\frac{1}{(m+1)}\frac{\pi^2}{6}\frac{M}{M_p}\left\{\frac{1}{2}\frac{n_1'\Sigma_1 a_1'^2}{M_p}\alpha_1^{3/2}[b_{3/2}^{(m)}(\alpha_1)]^2+\frac{mn_2'\Sigma_2 a_2'^2}{M_p}\alpha_2^{-1/2}\left[\left(2m+1+\alpha_2\frac{d}{d\alpha_2}\right)b_{1/2}^{(m)}(\alpha_2)\right]^2\right\}, \tag{18}$$

where $\alpha_i = a/a_i'$ and the resonance radii a_i' are defined by $(m-1)n-(m+1)n_1'+\dot{\Omega}+\dot{\Omega}'=0$ or $mn-(m+1)n_2'+\dot{\varpi}'=0$ for the first and second terms, respectively.

The inclination of the satellite always grows if it is inside the ring. The inclination may grow or decay if the satellite is outside the ring.

If the satellite orbits close to a narrow ring, $m \approx 4a/(3|a-a_1'|) \approx 2a/(3|a-a_2'|)$. In this case we approximate the Laplace coefficients and their derivatives by

$$b_{3/2}^{(m)}(\alpha_1)\approx\frac{3m^2}{2\pi}K_1\left(\frac{4}{3}\right),\quad b_{1/2}^{(m)}(\alpha_2)\approx\frac{2}{\pi}K_0\left(\frac{2}{3}\right),\quad \alpha_2\frac{d}{d\alpha_2}b_{1/2}^{(m)}(\alpha_2)=\frac{2m}{\pi}K_1\left(\frac{2}{3}\right), \tag{19}$$

where K_i denotes the modified Bessel function of order i. For $m \gg 1$ the second term in equations (17) and (18) is smaller than the first by $O(m)$ and can be dropped. Let $\Delta a'$ be the radial width of the ring; there are $\Delta m \approx 4a'\Delta a'/[3(a-a')^2]$ resonances in the ring. By adding up the effect of this sequence of resonances we find

$$\frac{1}{I}\frac{dI}{dt}=u_I n\frac{MM_r}{M_p^2}\left|\frac{a}{a'-a}\right|^5, \tag{20}$$

where M_r is the mass of the ring and $u_I = +0.0118$.

iv) *Inclination Evolution of a Ring*

Goldreich and Tremaine (1981) investigated the evolution of the eccentricity of a ring perturbed by a nearby satellite on a circular orbit. The evolution of the ring inclination can be derived by an analogous procedure. Here we only sketch the steps and give the final answer (eq. [24]), referring the reader to Paper I for details. We assume throughout this section that the total ring mass M_r is much less than the satellite mass M.

We define the inclination vector of a ring particle to be $\boldsymbol{I}' = I'\cos\Omega'\hat{x} + I'\sin\Omega'\hat{y}$, where $\hat{x}$ and $\hat{y}$ are unit vectors in the equatorial plane and $\hat{x}$ points to the zero of longitude. The mass-weighted mean inclination vector is

$$\boldsymbol{I}_r \equiv \langle \boldsymbol{I}' \rangle \equiv \frac{\sum_i M_i'\boldsymbol{I}_i'}{\sum_i M_i'}, \tag{21}$$

where M_i' is the mass of particle i and the sum is over all particles in the ring. In a ring with small random velocities, $\boldsymbol{I}_r$ is approximately conserved in collisions (cf. Paper I), so we need only consider the evolution due to gravitational perturbations from the satellite. If we write $I_r = |\boldsymbol{I}_r|$, then the rate of evolution of the ring inclination dI_r/dt can be written as a power series in M and M'. Terms of order M/M_p represent the first-order perturbations by the satellite; the azimuthal average of dI_r/dt for these terms is zero. Terms of order M'/M_p represent first-order perturbations from the other ring particles; the only effect of these perturbations is to enforce uniform nodal precession (Borderies, Goldreich, and Tremaine 1983*a*). Second-order terms of order $M'M/M_p^2$ represent perturbations from the perturbed ring. The dominant contribution to these perturbations are from the arc of ring within $\sim|a-a'|$ of the particle in question. If the satellite is nearby ($|a-a'| \ll a$), these perturbations are less important than perturbations of order

$(M/M_p)^2$. Hence $dI_r/dt = O(M/M_p)^2$ and we may write (cf. eq. [54] of Paper I)[4]

$$\frac{dI_r}{dt} = \left\langle \frac{dI_2'}{dt} \right\rangle - I_0 \left\langle \Omega_1' \frac{d\Omega_1'}{dt} \right\rangle + I_0 \langle \Omega_1' \rangle \left\langle \frac{d\Omega_1'}{dt} \right\rangle , \quad (22)$$

where I_0 is the unperturbed value of I_r.

To evaluate dI_2'/dt we use equations (4), (14), and the primed analog of equation (9). Since the ring is close to the satellite, the resonances are closely spaced; for given i, i', k, k' there are $\Delta j = 2a'\Delta a'(i + i' + k + k')/[3(a - a')^2]$ resonances in a ring of width $\Delta a'$. We assume that there are many resonances in the ring ($\Delta j \gg 1$). In this case, the average over the ring can be replaced by an integration over j. To evaluate the second term in equation (22) we use the relation

$$\left\langle \Omega_1' \frac{d\Omega_1'}{dt} \right\rangle = \frac{\pi}{2} \left(\frac{\partial A}{\partial Z'} \right)^2 \delta(\dot{\phi}_0) , \quad (23)$$

whose derivation closely follows the derivation of equation (13). The third term in equation (22) is zero since $\langle d\Omega_1'/dt \rangle = 0$.

Thus, we arrive at an expression for the rate of growth of the inclination of a ring perturbed by a nearby satellite on an equatorial orbit:

$$\frac{1}{I_0} \frac{dI_r}{dt} = u_I n' \left(\frac{M}{M_p} \right)^2 \left| \frac{a}{a' - a} \right|^5 , \quad (24)$$

which is the same as equation (20) except that the ring mass M_r is replaced by the satellite mass M.

b) *Other Methods*

We give here another proof of equations (17) and (18) using the theory of bending waves in an equatorial ring developed by Shu, Cuzzi, and Lissauer (1983). We start with some definitions.

Vertical resonances occur where the magnitude of the synodic forcing frequency $m(n' - n_p)$ is equal to the natural vertical oscillation frequency $\mu' = n' - \dot{\Omega}'$. The quantity n_p is the pattern speed:

$$n_p = n + \left(\frac{l - m}{m} \right) \mu . \quad (25)$$

The perturbing potential is proportional to $I^{|l-m|}$.

At a vertical resonance where a linear steady-state bending wave is excited, the torque on the ring is perpendicular to the ring and of magnitude (Bertin and Mark 1980)

$$T = (\pi \Sigma h)^2 G a' m \operatorname{sgn}(n_p - n') , \quad (26)$$

where h is the amplitude of the vertical displacement of the disk. All quantities in equation (26) must be evaluated close enough to the resonance so that the unperturbed parameters of the ring are the same as at resonance and significant viscous damping has not occurred, but far enough that the wave is excited to its full amplitude. To derive this result, Bertin and Mark use a procedure from plasma physics; they suppose that the bending wave is excited by an external distribution of mass, slowly turned on at $t = -\infty$, with a shape similar to that of the wave, and rotating with the pattern speed; they compute the torque exerted by the disk on this mass. We use a similar procedure to prove that a bending wave transfers energy to the disk at the rate $n_p T$.

The calculations of dI/dt follows from the relation:

$$H_\perp = Mna^2 \cos I , \quad (27)$$

the component of satellite's angular momentum perpendicular to the equator plane;

$$E = \frac{-GM_p M}{2a} , \quad (28)$$

the satellite's energy; the dynamical equations,

$$\frac{dH_\perp}{dt} = -T , \quad (29)$$

and

$$\frac{dE}{dt} = -n_p T . \quad (30)$$

Combining equations (27)–(30), we obtain

$$\sin I \frac{dI}{dt} = \left[-(n_p - n) + 2n_p \sin^2 \frac{I}{2} \right] \frac{T}{M(na)^2} . \quad (31)$$

For $I \ll |a' - a|/a$, the largest contributions to dI/dt come from vertical resonances with $|l - m| = 1$ and from Lindblad resonances with $|l - m| = 0$.

To evaluate the effect of the vertical resonances, we use the expression for the amplitude of vertical oscillations for $|l - m| = 1$ derived by Shu, Cuzzi, and Lissauer (1983),

$$|h| = \left(\frac{a'}{|\mathscr{D}| G\Sigma} \right)^{1/2} \frac{a}{a_>^3} \frac{GMI}{2} b_{3/2}^{(m)} \left(\frac{a_<}{a_>} \right) , \quad (32)$$

where

$$\mathscr{D} = a' \frac{d}{da'} [\mu'^2 - m^2(n' - n_p)^2] . \quad (33)$$

All quantities in equations (32) and (33) are evaluated at the resonance. The first terms in equations (17) and (18) can be deduced from equations (25), (26), (31), (32), and (33). These terms always cause the satellite inclination to increase.

The second terms in equations (17) and (18) are due to the torques associated with $|l - m| = 0$ Lindblad resonances. A derivation of these torques using the density wave method is given by Goldreich and Tremaine (1980). Of course, the magnitudes of these torques are independent of I. Their primary effect is to change a. The change in I is a consequence of the adiabatic invariant $I^2 a^{1/2}$ associated with small oscillations perpendicular to the equatorial plane. If the satellite is inside the ring, the Lindblad torques cause a to decrease and I to increase. If the satellite is outside the ring, the Lindblad torques cause I to decrease. Since the torques from the vertical resonances with $|l - m| = 1$ always cause I to increase, the inclination always grows if the satellite is inside the ring, and may grow or decay if the satellite is outside the ring.

At least two alternate methods can be used to derive the rate of growth of inclinations in close ring-satellite pairs. They are the inclination analogs of methods described by Goldreich and Tremaine in two papers (1980, 1981). In the first paper, they use the equations which govern the interactions of two particles during a close encounter, and assume that successive encoun-

[4] There are several errors in eq. (54) of Paper I. The term proportional to $\langle e_1 \rangle$ should not be present. The term proportional to $\langle \varpi_1 \rangle$ should be preceded by a plus, not a minus, sign. All unsubscripted e's should be replaced by e_0's.

ters are independent; in the second paper, they introduce an artificial damping term of the form $-sx$ in the planetary equations for dx/dt. With both methods, the formulae describing the secular changes of eccentricities are obtained by means of a second order perturbation calculation.

The bending wave treatment is the only method described in this paper which faithfully represents the physics of the collective behavior of the ring particles. The reason why other techniques give the same coherent answer is that they retain the crucial effect of the collective interactions, the limiting of the disturbance near the resonance to a linear perturbation. In planetary rings, this can be accomplished in two ways: (i) self-gravity allows the formation of a wave which carries away the angular momentum and energy deposited by the satellite; (ii) collisions damp the orbital perturbations of the ring particles caused by the satellite.

As long as the perturbations are linear, it does not matter which one of these effects is acting. If the perturbations were not linear (nonlinear density waves have been observed in the rings of Saturn and similarly, nonlinear bending waves may exist), the formulae for the growth of inclination should be modified.

III. DECAY OF INCLINATIONS

There is a close analogy between the behavior of eccentricities and inclinations in planetary rings. The evolution of the satellite eccentricity due to a nearby narrow ring is (Goldreich and Tremaine 1980, 1981):

$$\frac{1}{e}\frac{de}{dt} = (u_{\rm L} + u_c)n' \frac{MM_r}{M_p^2} \left|\frac{a}{a'-a}\right|^5 , \qquad (34)$$

where $u_{\rm L}$ and u_c corresponds to Lindblad and corotation resonances and have the values $+1.5226$ and -1.5965, respectively. If all perturbations are linear, the net effect of the two types of resonance is given by equation (34) with $u_{\rm L} + u_c = -0.0739$, and the eccentricity decreases. If the corotation resonances are weakened by saturation and the Lindblad resonances are not, the eccentricity grows.

In contrast to equation (34), equation (20) for the inclination growth contains no corotation term. This has a simple explanation. At a corotation resonance in inclination, a ring particle feels a constant force in the vertical direction. This force adds to the restoring force due to the gravitational field of the planet but produces no other effect. There is no singularity in the perturbation equations at a corotation resonance in inclination. Consequently, no torque is exerted on the disk there.

The question arises as to what physical processes limit the growth of inclinations. It is clear that viscous dissipation damps ring inclinations. We have not identified any process which damps satellite inclinations.

a) Limit of Inclination Growth for Rings

Recently, Borderies, Goldreich, and Tremaine (1983*b*) investigated the dynamics of narrow eccentric rings. They considered the influence of: the quadrupole moment of the planet, self-gravity, viscosity, and shepherd satellites. They showed that, on the viscous spreading time scale, an eccentric ring evolves toward an equilibrium configuration in which self-gravity balances differential precession due to the oblateness of the planet; this configuration is characterized by a near alignment of apsidal lines and by a positive difference of eccentricity $\Delta e'$ between the outer and inner edges of the ring; $\Delta e'$ is proportional to the mean eccentricity of the ring. On a time scale longer by the factor $(e'/\Delta e')^2$, the mean eccentricity would decay if not excited by shepherd satellites. Since the damping rate of the mean eccentricity is an increasing function of $q_e' \approx a'\Delta e'/\Delta a'$, on the longer time scale, q_e' adjusts to an equilibrium value at which viscous damping and satellite excitation balance.

A comparable model of the dynamics of inclined rings has yet to be developed, although Borderies, Goldreich, and Tremaine (1983*a*) have studied the equilibrium configuration in the limit of zero viscosity. We expect that the role of viscosity in the dynamics of inclined rings is similar to its role in the dynamics of eccentric rings. If this expectation is justified, there are two possible scenarios:

i) If the viscous damping of inclination for $q_I' \approx a'\Delta I'/\Delta a' \approx 0$ is faster than the growth due to shepherd satellites, the ring remains circular.

ii) Otherwise, the mean inclination increases; as a consequence q_I' becomes bigger and viscous damping is enhanced. Ultimately q_I' reaches a value at which viscous damping compensates satellite excitation; this balance determines the equilibrium value of the mean inclination of narrow rings.

b) Limit of Inclination Growth for Satellites

We wondered if the inclination of a shepherd satellite might be limited by a feedback mechanism involving viscous damping of the ring inclination. We explored this idea and obtained a negative result.

When I is large, higher order terms and higher order resonances become important. It is possible that for $I \sim |a' - a|/a$, dI/dt changes sign. Goldreich and Tremaine (1981) have demonstrated that a similar effect occurs in the eccentricity case. We plan to develop such a general model in a subsequent paper.

IV. DISCUSSION

a) Mimas

Note that the bending waves in Saturn's rings associated with Mimas (Shu, Cuzzi, and Lissauer 1983) are not responsible for that satellite's inclination. The time scale for growth of Mimas's inclination due to its strongest, 5:3, inclination resonance is $I(dI/dt)^{-1} = 2.4\ 10^{11} \times (50 \text{ g cm}^{-2}/\Sigma)$ years.

b) 1980S27 and 1980S26

These satellites, which shepherd Saturn's F ring, have eccentric orbits ($e_{27} = 0.0024 \pm 0.0006$, $e_{26} = 0.0042 \pm 0.002$) and no measurable inclinations ($I < 2 \times 10^{-3}$ radians, Synnott *et al.* 1983).

Gravitational interactions with the narrow F ring and the wide A ring act to pump up the inclinations of the shepherd satellites. The rate of change of inclination and eccentricity of a shepherd due to the F ring is given by equations (20) and (34). Interaction with the ring drives the shepherd away at the rate:

$$\frac{1}{|a'-a|}\frac{d|a'-a|}{dt} = u_a n' \frac{MM_r}{M_p^2}\left|\frac{a}{a'-a}\right|^5 , \qquad (35)$$

with $u_a = +0.798$. There is an additional term in the rate of separation of shepherd and ring which is due to the changing semimajor axis of the ring which we are neglecting. Equations (20), (34), and (35) have the same form and differ only in their numerical coefficients u_I, $u_{\rm L}$, u_c, u_a. In particular, note that $u_a \approx 70u_I$.

Variations due to the A ring are given by the formulae

$$\frac{1}{I}\frac{dI}{dt} = n\frac{\pi u_I}{2}\frac{M_s}{M_p}\frac{\Sigma a^2}{M_p}\left(\frac{a}{\delta a}\right)^4, \tag{36}$$

$$\frac{1}{e}\frac{de}{dt} = n\frac{\pi(u_L + u_c)}{2}\frac{M_s}{M_p}\frac{\Sigma a^2}{M_p}\left(\frac{a}{\delta a}\right)^4, \tag{37}$$

$$\frac{1}{\delta a}\frac{d\delta a}{dt} = n\frac{2\pi u_a}{3}\frac{M_s}{M_p}\frac{\Sigma a^2}{M_p}\left(\frac{a}{\delta a}\right)^4, \tag{38}$$

where δa is the difference of the semimajor axes of the satellite orbit and the outer edge of the A ring, and we assume $\delta a \ll a$ ($\delta a/a \approx 0.035$ for 1980S26).

For instance, with $\Sigma = 25$ g cm^{-2}, the e-folding time for δa_{27} and I_{27} due to the A ring are, respectively, 1.7×10^6 and 1.5×10^8 years. One possible explanation for the small upper limit of I_{27} is that 1980S27 is young and has spent less than one inclination e-folding time close to the A ring. This hypothesis is not very satisfying because it seems unlikely that we would observe such an ephemeral state unless the whole ring system is very young. This question of the short time scale for satellite evolution is an unsolved problem which has been discussed elsewhere by Borderies, Goldreich, and Tremaine (1983*c*).

c) *Rings of Uranus*

Except for the ϵ ring, the inclinations of the Uranian rings are equal, to within a factor of 5, to their eccentricities (French, Elliot, and Allen 1982). The ϵ ring has no measurable inclination, but it has by far the largest eccentricity of all the Uranian rings. Of course, a zero inclination is not in itself so surprising. All it requires is that viscous damping exceeds satellite pumping for $q_I' \approx 0$ (cf. § III*a*[i]). What could account for the anomalous character of the ϵ ring? We suspect that the answer must be related to ring mass since the ϵ ring is much wider than the other rings of Uranus. Unfortunately, we have not been able to invent any convincing justification for this belief.

N. B. acknowledges an NSF US-France exchange postdoctoral fellowship and a grant from ATP Planetologie 1982 and 1983 and also a consortium arrangement with Ames. P. G. acknowledges support from NSF grant 80-20005 and NASA grant NGL-05-002-003. S. T. acknowledges support from NSF grant AST-8210463 and NASA grants NSG-7643 and NGL-22-009-638.

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