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Non-axisymmetric instability in thin discs

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Summary. Thin discs of arbitrary specific angular momentum are shown to have unstable non-axisymmetric modes provided there is at least one good reflecting edge.

1 Introduction

In a recent important paper, Papaloizou & Pringle (1984) have investigated the dynamical stability of a thick, differentially rotating disc of uniform entropy and uniform specific angular momentum. They find a strong global non-axisymmetric instability which grows on a dynamical time-scale. Here we show that a similar instability exists even in thin discs of arbitrary specific angular momentum. We find that a crucial ingredient for the existence of this instability is a good reflecting boundary at either the inner or outer edge of the disc (or both). We make simple estimates of the growth rate of the instability as a function of the azimuthal wavenumber of the mode, the angular momentum profile, the radial width and thickness of the disc, and the reflectivity of the boundaries. The importance of reflecting boundary conditions to the stability of *thick* discs/tori is however not completely clear because of the possibility that a Kelvin–Helmholtz-like instability, independent of boundary conditions, may be common (Papaloizou & Pringle, preprint).

2 Theory

Following Goldreich & Lynden-Bell (1965), Goldreich & Tremaine (1978) and Drury (1980), we model a two-dimensional disc by parallel shear flow of a thin, compressible, uniform-density gas sheet with a constant velocity gradient and an artificial Coriolis force. Let $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ be unit Cartesian vectors along the x, y, z axes, oriented so that the unperturbed flow is parallel to the y-axis in the xy plane. The unperturbed velocity field is given by $\mathbf{V}_0=2Ax\hat{\mathbf{j}}$. The 'Coriolis force' is taken in the form $-2\Omega_p\hat{\mathbf{k}}\times\mathbf{V}$ and its effect on the unperturbed flow is cancelled by a background 'gravitational field'. We neglect self-gravity in this paper and take the equation of state of the gas in the form

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$p=a^2\sigma$, where a is the sound speed and p and σ are the perturbations in the 2-D pressure and surface density.

Let the perturbed velocity be of the form $u\hat{\mathbf{i}}+v\hat{\mathbf{j}}$. Without loss of generality we write each perturbation variable Q in the form $Q=Q(x)\exp[i(k_y y-\omega t)]$. Drury (1980) has shown, following Tremaine (1980, private communication), that the perturbation equations can be expressed exactly in terms of the parabolic-cylinder differential equation

$$\frac{d^2v}{dX^2}+(\tfrac{1}{4}X^2-C)v=0 \tag{1}$$

$$X=\left(\frac{4|Ak_y|}{a}\right)^{1/2}\left(x-\frac{\omega}{2Ak_y}\right) \tag{2}$$

$$C=\frac{a|k_y|}{4|A|}+\frac{B\Omega_p}{a|Ak_y|}, \quad B=A+\Omega_p. \tag{3}$$

The 'corotation resonance' is at $X=0$ and the 'Lindblad resonances' are at $X=\pm 2(B\Omega_p/a|Ak_y|)^{1/2}$, but the differential equation is not singular at these points. The turning points are at $X=\pm 2C^{1/2}$ leading to a forbidden region in between. Provided the boundaries are sufficiently far from the forbidden region, the asymptotic form of the solution to the differential equation (1) (e.g. Abramowitz & Stegun 1970) may be used to impose the boundary conditions. Let the boundaries be perfect reflectors situated at a distance w on either side of the corotation resonance. Then it can be shown that there are eigenmodes with growing amplitude of the form $\exp(\omega_I t)$, where ω_I is given by

$$\omega_I \sim \frac{a}{2w}\ln[\tau+(1+\tau^2)^{1/2}] \tag{4}$$

$$\tau=\exp(-\pi C). \tag{5}$$

The quantity τ measures the tunnelling amplitude across the forbidden region and crucially influences the growth rate, as discussed in Section 3. From equations (4) and (5) we have the limiting results

$$\omega_I \sim \frac{a}{2\omega}\tau, \quad C \gg 1 \tag{6}$$

$$\omega_I \sim 0.88\,\frac{a}{2w}, \quad C \ll 1. \tag{7}$$

To relate the above discussion to a thin differentially rotating disc, we make the following identifications which follow from a WKB treatment: $x \to r$ the radius, $k_y \to m/r$ where m is the azimuthal wavenumber in a solution of the form $Q=Q(r)\exp[i(m\theta-\omega t)]$, $\Omega_p \to \Omega(r)$ the angular velocity, $A \to A(r) \equiv (r/2)(d\Omega/dr)$, $B \to B(r) \equiv (1/2r)[d(r^2\Omega)/dr]$ the Oort parameters and $\varkappa(r)=2(B\Omega)^{1/2}$=the epicyclic frequency. We then have

$$C=\frac{a}{2|m(d\Omega/dr)|}\left(\frac{m^2}{r^2}+\frac{\varkappa^2}{a^2}\right) \tag{8}$$

where all the quantities are to be evaluated at the corotation radius r_c. The maximum growth rate occurs for $|m|=\varkappa r/a$ when C attains its minimum value of $\varkappa/2|A|$. Let us parametrize the disc by means of a power-law angular-velocity profile

$$\Omega(r)=\Omega_0 r^{-\beta}, \quad 0\leqslant\beta\leqslant 2 \tag{9}$$

where $\beta=0$ corresponds to a rigidly rotating disc and $\beta=2$ to a constant-angular-momentum disc. Then the optimum value of $|m|$ for maximum growth is

$$m_*=(4-2\beta)^{1/2}\Omega r/a\sim(4-2\beta)^{1/2}(r/h) \tag{10}$$

where h is the disc thickness. For thin discs with a non-constant angular momentum, $m_*\sim r/h$ and the corresponding maximum value of the transmission coefficient τ is 0.12 for $\beta=3/2$ (Kepler disc), 0.012 for $\beta=1$ (flat rotation curve), and obviously 0 for $\beta=0$ (rigid rotation). For a constant-angular-momentum thick disc, $\varkappa=0$, and we find a constant growth rate on a sound-crossing time-scale $2d/a$, where d is the radial distance from corotation to the two reflecting boundaries, for all $|m|$ up to $r/h\sim r/d$, with a fall-off for higher $|m|$. Papaloizou & Pringle (1984) actually find a *maximum* growth rate at $|m|\sim r/h$; the difference is because the asymptotic solution used to estimate equation (7) is inappropriate when $|m|<r/d$ since the radial wavelength of the mode is larger than d.

An analysis similar to the one leading to (6), i.e. when $\tau\ll 1$, shows that for an asymmetric mode, where corotation is at distances d_1 and d_2 from the two reflecting edges, d should be replaced by $(d_1 d_2)^{1/2}$ if d_1 and d_2 are not very different. If only one edge (at distance d from corotation) is reflecting while the other has a radiating boundary condition, then the growth rate is

$$\omega_I\sim\frac{a}{4d}\tau^2, \qquad \tau\ll 1. \tag{11}$$

It might appear that the growth rate can be increased arbitrarily by reducing d. However, there is a limit to this imposed by the range of validity of the asymptotic solution on which the present analysis is based. Finally, if the two boundaries are not perfect reflectors but transmit fractional amplitudes T_1 and T_2, the growth rate is reduced from the value (6) but remains positive so long as $T_1 T_2<\tau^2$. Thus, strongly amplified modes can tolerate substantial leakage at the boundaries while weakly amplified ones need near-perfect reflection at least at one boundary.

3 Discussion

The physics of the growing WKB modes we have considered is most transparent in terms of the discussion by Mark (1976). The WKB waves have negative energy for $r<r_c$ and positive energy for $r>r_c$. Consider a wave packet of unit amplitude incident on the corotation region. It is partly transmitted through the forbidden region (with amplitude τ) and partly reflected. Since the energy in the transmitted wave has the opposite sign to that in the incident wave, energy conservation requires the reflected wave to have an enhanced amplitude ($\sim 1+\tfrac{1}{2}\tau^2$ when $\tau\ll 1$). To convert this amplifier into a self-sustaining oscillator we need feedback, and this is provided by having a reflecting boundary. Clearly, the wave amplitude grows by a factor $\sim 1+\tfrac{1}{2}\tau^2$ in each sound-crossing time $2d/a$, thus explaining the result in (11). Equations (6) and (7) refer to the case when both boundaries reflect and therefore correspond to correlated amplification from both sides of corotation.

The analysis in this paper shows that the instability found by Papaloizou & Pringle (1984) in the specific case of a thick, constant-angular-momentum disc is quite general (as in fact they had suspected) and can be understood in a thin disc using a WKB analysis, or the equivalent parallel-shear-flow model. An important conclusion is that a reflecting boundary is necessary for the modes we consider (which explains why no hint of this instability is found from a local analysis, Toomre 1969). A reflecting boundary will be obtained if the density at the inner (more likely) or outer (less likely) edge of the disc cuts off sharply on a scale shorter than the radial wavelength λ_r of the mode. It is not clear if this will be possible in a thin disc where $\lambda_r \sim h \ll r$, but it can certainly happen in a thick disc where $\lambda_r \sim r$. It is possible that in a thick disc the modes boil off material into the inner funnel, thus reducing the reflection coefficient and self-consistently saturating the amplitude of the mode at a finite level. Also, if there is significant viscosity, the growth rate is severely reduced and the modes may become localized to the vicinity of sharp reflecting edges. The role of the present instabilities in generating/modulating the α-viscosity in discs is not clear.

Papaloizou & Pringle have recently extended their investigation to the dynamical stability of thick tori with non-constant angular momentum (preprint). They find that a Kelvin–Holmholtz-like instability may occur when there is a stationary point in the ratio of vorticity to surface density. This instability can persist in the absence of reflecting boundaries and does not appear in our simple disc model which has a constant vorticity-to-density ratio.

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