DISTRIBUTION FUNCTIONS FOR THE TIME-AVERAGED ENERGIES OF STOCHASTICALLY EXCITED SOLAR p-MODES

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ABSTRACT

We study the excitation of a damped harmonic oscillator by a random force as a model for the stochastic excitation of a solar p-mode by turbulent convection. An extended sequence of observations is required to separate different p-modes and thus determine the energies of individual modes. Therefore, the observations yield time-averaged values of the energy. We apply the theory of random differential equations to calculate distribution functions for the time-averaged energy of the oscillator. The instantaneous energy satisfies a Boltzmann distribution. With increasing averaging time the distribution function narrows, and its peak shifts toward the mean energy. We also perform numerical integrations to generate finite sequences of time-averaged energies. These are treated as simulated data from which we obtain approximate probability distributions for the time-averaged energy. A comparison of our calculated distributions with those determined observationally should help to resolve whether the solar p-modes are stochastically excited. If they are, modes of the same frequency with degree $l \lesssim 200$ should have identical values for the products of their mean energies, line-widths, and masses. If, in addition, turbulence or radiative dissipation provides the principal damping mechanism, the mean energies should be independent of angular order, l.

Subject headings: convection — Sun: interior — Sun: oscillations — turbulence

I. INTRODUCTION

The frequency, line width, and photospheric velocity amplitude are the primary observables associated with each p-mode. The energy in a mode is proportional to the square of its velocity amplitude. If the mode is stochastically excited, its energy, E, will fluctuate on the damping time scale. Unfortunately, observations of solar p-modes cannot determine the instantaneous energy, E, but only, E_T, the energy averaged over some interval of time, say $-T \le t \le T$.

A damped harmonic oscillator excited by a random force provides a simple model for the stochastic excitation of a solar p-mode by turbulent convection (Goldreich and Keeley 1977; Goldreich and Kumar 1988). The current investigation is devoted to calculating distributions for E_T appropriate to the model harmonic oscillator. It is obvious that the distribution function is a decaying exponential, or Boltzmann, distribution for $T = 0$ and that it approaches a δ-function at the mean energy as $T \to \infty$. We determine distribution functions appropriate to arbitrary averaging times by applying the theory of random differential equations. In addition, we provide examples of approximate distribution functions obtained from finite strings of numerically simulated data. The latter are intended as a guide for interpreting observationally determined distribution functions of E_T for solar p-modes.

The organization of this paper is as follows. In § II we describe the model harmonic oscillator and outline the procedure for calculating distribution functions for its time-averaged energy. We develop the method used to compute the eigenvalues required to determine the distribution functions in § III. In § IV we present our theoretical distribution functions. We solve the harmonic oscillator equation numerically in § V to generate finite sequences of values for E_T which are treated as simulated data to determine approximate distribution functions. We briefly discuss some observational implications of our results for the solar p-modes in § VI.

II. THE STOCHASTICALLY EXCITED OSCILLATOR

The differential equation governing the time evolution of the coordinate, q, of a damped harmonic oscillator of frequency, ω_0, which interacts with a random force, $F(t)$, reads

$$\frac{d^2q}{dt^2} + 2\Gamma \frac{dq}{dt} + \omega_0^2 q = \frac{F(t)}{M} , \qquad (1)$$

where M is the mass and Γ is a positive damping constant. We assume that $F(t)$ is a Gaussian random process. This assumption is not very restrictive because the central limit theorem implies that other processes, such as the Poisson process, reduce to a Gaussian process in the limit of large N.

The energy, E, consists of a kinetic and a potential part;

$$E = \frac{M}{2}\left[\left(\frac{dq}{dt}\right)^2 + \omega_0^2 q^2\right] . \qquad (2)$$

Our basic strategy is to expand $x_1 \equiv \omega_0 q$ and $x_2 \equiv dq/dt$ in terms of a set of functions which are orthonormal over the time interval $-T \le t \le T$. We define x to be the column vector with components x_1 and x_2. Then, we have

$$x(t) = \sum_n A_n f_n \phi_n(t) \qquad \text{for} \qquad -T \le t \le T , \qquad (3)$$

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where the ϕ_n are two-component functions which satisfy

$$\int_{-T}^{T} dt \, \phi_m^\dagger(t) \cdot \phi_n(t) = \delta_{m,n} , \tag{4}$$

the f_n are uncorrelated, random, Gaussian variables with unit variance, and the A_n are constants. Equation (3) is an extension of the Karhunen-Loéve expansion (Davenport and Root 1958) to two stochastic processes.

To determine $A_m f_m$, we take the dot product of both sides of equation (3) by $\phi_m^\dagger$, then integrate over $-T \le t \le T$.[1] Next, we multiply $A_m f_m$ by $A_n^* f_n^*$ and take the expectation value of the resulting expression.[2] This procedure yields

$$\langle A_m A_n^* f_m f_n^* \rangle = \int_{-T}^{T} dt_1 \int_{-T}^{T} dt_2 \, [\phi_m^\dagger(t_1) \cdot \langle x(t_1) x^\dagger(t_2) \rangle \cdot \phi_n(t_2)] . \tag{5}$$

Since $\langle f_m f_n^* \rangle = \delta_{n,m}$, the ϕ_n must satisfy the integral equation

$$\int_{-T}^{T} dt_2 \mathscr{R}(t_1, t_2) \cdot \phi_n(t_2) = \lambda_n \phi_n(t_1) , \tag{6}$$

where

$$\mathscr{R}(t_1, t_2) = \langle x(t_1) x^\dagger(t_2) \rangle \tag{7}$$

is a 2×2 matrix. Thus, ϕ_n is an eigenfunction of the integral operator defined by equation (6) and A_n is the square root of the corresponding eigenvalue, λ_n. We may now rewrite equation (3) as

$$x(t) = \sum_n (\lambda_n)^{1/2} f_n \phi_n(t) . \tag{8}$$

We note that ϕ_2 is the time derivative of ϕ_1. To prove this we multiply equation (8) by f_n^* and take the expectation value of both sides. This yields

$$\langle f_n^* x(t) \rangle = (\lambda_n)^{1/2} \phi_n(t) \quad \text{or} \quad (\phi_n)_1 = \langle f_n^* x_1(t) \rangle / (\lambda_n)^{1/2} \quad \text{and} \quad (\phi_n)_2 = \langle f_n^* x_2(t) \rangle / (\lambda_n)^{1/2} . \tag{9}$$

But by definition $x_2 = \omega_0^{-1} dx_1/dt$. Therefore, the above equation implies that $(\phi_n)_2 = \omega_0^{-1} d(\phi_n)_1/dt$.

The operator defined by equation (6) is easily shown to be hermitian. Thus, the eigenvalues, λ_n, are real. Furthermore, rewriting equation (5) using $|A_n|^2 = \lambda_n$ yields

$$\lambda_n = \left\langle \left| \int_{-T}^{T} dt_1 \, \phi_n^\dagger(t_1) \cdot x(t_1) \right|^2 \right\rangle , \tag{10}$$

which proves that the λ_n are positive. Because the matrix $\mathscr{R}$ is real, the eigenfunctions can be chosen to be real. Also, since x is real the f_n must be real.

To evaluate the average energy, E_T, we integrate $|x|^2$ using equation (8) and the orthonormality property of the ϕ_n;

$$E_T \equiv \frac{1}{2T} \int_{-T}^{T} dt \, E = \frac{M}{4T} \int_{-T}^{T} dt \, x^\dagger \cdot x = \frac{M}{4T} \sum_n \lambda_n f_n^2 \equiv \sum_n y_n . \tag{11}$$

Since f_n is a Gaussian random variable with unit variance and zero mean the probability distribution for y_n, $P(y_n)$, is

$$P(y_n) = \left(\frac{4T}{\pi M y_n \lambda_n} \right)^{1/2} \exp \left(\frac{-4T y_n}{M \lambda_n} \right) , \tag{12}$$

and the characteristic function, $\mathscr{C}_{y_n}(\omega_E)$, is

$$\mathscr{C}_{y_n}(\omega_E) = 1/(1 - iM\omega_E \lambda_n/4T)^{1/2} . \tag{13}$$

The characteristic function for E_T is just the product of the $\mathscr{C}_{y_n}(\omega_E)$.

$$\mathscr{C}_{E_T}(\omega_E) = \prod_n \frac{1}{(1 - iM\omega_E \lambda_n/4T)^{1/2}} . \tag{14}$$

The inverse Fourier transform of equation (14) gives the probability distribution, $P(E_T)$, for E_T. Thus, our main task is to determine the eigenvalues of equation (6). We develop a procedure for doing this in the next section.

III. DETERMINING THE EIGENVALUES

This section is divided into three parts. In § IIIa we calculate the correlation matrix $\mathscr{R}$ by solving equation (1) for x. Next, in § IIIb, we transform the integral equation (6) into a differential equation for ϕ. Equipped with this we write down the most general form for ϕ. Finally, in § IIIc we substitute the general form for ϕ into the integral equation (6) and reduce the problem of finding the eigenvalues to that of finding the roots of an ordinary algebraic equation.

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[1] A dagger, $\dagger$, denotes the complex conjugate and transpose of a matrix.
[2] Angular brackets, $\langle \ \rangle$, denote an expectation value.

a) Determining The Correlation Matrix

Throughout this section we take $F(t)$ to be a Gaussian, white noise, random process. The solution of equation (1) is obtained in the terms of the causal Green's function, G, for a damped harmonic oscillator.

$$q(t) = \frac{1}{M} \int_{-\infty}^{t} dt' G(t, t') F(t') , \tag{15}$$

where

$$G(t, t') = \frac{1}{\omega_1} \theta(t - t') \exp\left[-\Gamma(t - t')\right] \sin \omega_1(t - t') . \tag{16}$$

Here $\theta(t)$ is the Heavyside function, $\theta(t) = 0$ for $t < 0$ and $\theta(t) = 1$ for $t > 0$.

Using equations (15) and (16) we determine the correlation matrix, $\mathscr{R}$, defined by equation (7), to be

$$\mathscr{R}_{11} \equiv \langle x_1(t_1) x_1(t_2) \rangle = \frac{g\omega_0}{4M^2\Gamma\omega_1} \exp\left(-\Gamma|t_1 - t_2|\right) \cos\left(\omega_1|t_1 - t_2| - \alpha\right) \equiv \frac{g\omega_0}{4M^2\Gamma\omega_1} R_0 , \tag{17}$$

where

$$\omega_1^2 = \omega_0^2 - \Gamma^2 , \qquad \cos\alpha = \frac{\omega_1}{\omega_0} , \qquad \sin\alpha = \frac{\Gamma}{\omega_0} . \tag{18}$$

In deriving equation (17) we have used $\langle F(t_1)F(t_2) \rangle = g\,\delta(t_1 - t_2)$. The other components of $\mathscr{R}$ may be expressed in terms of R_0. We find

$$\mathscr{R}(t_1, t_2) = \frac{g\omega_0}{4M^2\omega_1\Gamma} \begin{pmatrix} R_0 & \dfrac{dR_0}{\omega_0\, dt_2} \\[2ex] -\dfrac{dR_0}{\omega_0\, dt_2} & -\dfrac{d^2 R_0}{\omega_0^2\, dt_2^2} \end{pmatrix} . \tag{19}$$

Using equation (17)

$$\mathscr{R}(t_1, t_2) = \frac{g\omega_0}{4M^2\omega_1\Gamma} \exp\left(-\Gamma|t_1 - t_2|\right) \begin{bmatrix} \cos\left(\omega_1|t_1 - t_2| - \alpha\right) & \sin\omega_1(t_1 - t_2) \\[1ex] -\sin\omega_1(t_1 - t_2) & \cos\left(\omega_1|t_1 - t_2| + \alpha\right) \end{bmatrix} . \tag{20}$$

b) Differential Equation For Eigenvectors

In order to derive the general form of an eigenvector it is useful to convert the integral equation (6) into a differential equation. To achieve this we exploit a special property of the Fourier transform of the kernel $\mathscr{R}$.

Let

$$\mathscr{R}(\tau) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} d\omega\, \tilde{\mathscr{R}}(\omega) \exp(i\omega\tau) , \tag{21}$$

where $\tau \equiv t_1 - t_2$. Then $\tilde{\mathscr{R}}(\omega)$ is determined by taking the Fourier transform of equation (20):

$$\tilde{\mathscr{R}}(\omega) = \frac{g\omega_0^2}{(2\pi)^{1/2}M^2[\omega^4 + \omega_0^4 - 2\omega^2\omega_1^2\cos(2\alpha)]} \begin{pmatrix} 1 & \dfrac{-i\omega}{\omega_0} \\[2ex] \dfrac{i\omega}{\omega_0} & \dfrac{\omega^2}{\omega_0^2} \end{pmatrix} . \tag{22}$$

We note that $\tilde{\mathscr{R}}(\omega)$ is the ratio of rational functions. This property enables us to convert the integral equation into a differential equation. Rewriting equation (6) in terms of $\tilde{\mathscr{R}}(\omega)$ we arrive at

$$\frac{1}{(2\pi)^{1/2}} \int_{-T}^{T} dt' \int_{-\infty}^{\infty} d\omega \exp i\omega(t - t') \frac{\mathscr{P}(i\omega) \cdot \phi(t')}{Q(i\omega)} = \lambda\phi(t) , \tag{23}$$

where

$$\tilde{\mathscr{R}}(\omega) \equiv \frac{\mathscr{P}(i\omega)}{Q(i\omega)} , \qquad Q(i\omega) \equiv \omega^4 + \omega_0^4 - 2\omega_0^2\omega^2\cos 2\alpha , \tag{24}$$

and $\mathscr{P}(i\omega)$ is a 2×2 matrix equal to $Q(i\omega)\tilde{\mathscr{R}}(\omega)$. Operating on equation (23) by $Q(d/dt)$ we get

$$\lambda Q\left(\frac{d}{dt}\right)\phi(t) = (2\pi)^{1/2}\mathscr{P}\left(\frac{d}{dt}\right) \cdot \phi(t) . \tag{25}$$

Then, combining equations (22) and (25) we obtain a pair of differential equations for the components of ϕ:

$$\lambda' Q(D)\phi_1 = \omega_0(\omega_0\,\phi_1 - D\phi_2)\,, \qquad \lambda' Q(D)\phi_2 = \omega_0 D\phi_1 - D^2\phi_2\,, \qquad (26)$$

where

$$\lambda' = \frac{\lambda M^2}{g}\,, \qquad D \equiv \frac{d}{dt}\,, \qquad Q(D) = D^4 + 2\omega_0^2 \cos 2\alpha\, D^2 + \omega_0^4\,. \qquad (27)$$

Substituting $\phi_2 = D\phi_1/\omega_0$ into equations (26) yields uncoupled differential equations for ϕ_1 and ϕ_2:

$$Q_1(D)\phi_1 = 0 \qquad \text{and} \qquad Q_1(D)\phi_2 = 0\,, \qquad (28)$$

where

$$Q_1(D) = \lambda' Q(D) + D^2 - \omega_0^2\,. \qquad (29)$$

We see from equations (28) and (29) that the general form for the eigenfunctions of equation (6) is a sum of four exponentials, each of which satisfies $Q_1(D) \exp \beta_k t = 0$. Furthermore, the β_k cannot be equal to either $\Gamma \pm i\omega_1$ or $-\Gamma \pm i\omega_1$ because the four factors of the operator $Q(D)$ are $D \pm \Gamma \pm i\omega_1$. Therefore, if $Q(D) \exp (\beta_k t) = 0$, $Q_1(D) \exp (\beta_k t) = (D^2 - \omega_0^2) \exp (\beta_k t) \neq 0$, since $\omega_0^2 \neq (\Gamma \pm i\omega_1)^2$.

We conclude this subsection by summarizing the properties of the eigenfunctions.

1. $\phi = \begin{pmatrix} \phi_1 \\ D\phi_1/\omega_0 \end{pmatrix}\,.$

2. $\phi_1 = \sum_{k=1}^{4} a_k \exp (\beta_k t)\,,$ where β_k is solution of $Q_1(i\beta_k) = 0\,.$ (30)

3. $\beta_k \neq \Gamma \pm i\omega_1$ and $\beta_k \neq -\Gamma \pm i\omega_1\,.$

c) Algebraic Equation For Eigenvalues

We substitute the general form for ϕ given by equation (30) into the integral eigenvalue equation (6). Equating the coefficients of the exponentials on both sides of the resulting equation yields a set of algebraic equations which is solved to give the eigenvalues.

The evaluation of the left side of equation (6) is simplified by noting that all of the integrals there can be expressed in terms of the integral of $R_0 \exp (\beta t)$ which is given below:

$$2\int_{-T}^{T} dt'\, R_0(t, t') \exp (\beta t') = \frac{8\Gamma\omega_0\,\omega_1 \exp (\beta t)}{(\beta^2 + \omega_0^2)^2 - 4\Gamma^2\beta^2} - \frac{\exp [(t + T)(i\omega_1 - \Gamma) - \beta T - i\alpha]}{\Gamma + \beta - i\omega_1}$$

$$- \frac{\exp [(T - t)(i\omega_1 - \Gamma) + \beta T - i\alpha]}{\Gamma - \beta - i\omega_1} - \frac{\exp [-(t + T)(i\omega_1 + \Gamma) - \beta T + i\alpha]}{\Gamma + \beta + i\omega_1}$$

$$- \frac{\exp [(t - T)(i\omega_1 + \Gamma) + \beta T + i\alpha]}{\Gamma - \beta + i\omega_1}\,. \qquad (31)$$

The denominators in this equation do not vanish because $\beta \neq \Gamma \pm i\omega_1$ and $\beta \neq -\Gamma \pm i\omega_1$.

Armed with the above expression and equation (19), we are ready to evaluate the integral in equation (6) for the general ϕ given by equation (30). The first component of equation (6) yields

$$2\int_{-T}^{T} dt'\left[R_0(t, t')\phi_1(t') + \frac{1}{\omega_0^2}\frac{dR_0}{dt'}\frac{d\phi_1}{dt'} \right] = \sum_{k=1}^{4} \frac{8\Gamma(\omega_0^2 - \beta_k^2)a_k \exp (\beta_k t) \cos \alpha}{(\beta_k^2 + \omega_0^2)^2 - 4\Gamma^2\beta_k^2}$$

$$- \sum_{k=1}^{4} \frac{a_k[\omega_0^2 - \beta_k(i\omega_1 - \Gamma)]}{\omega_0^2(\Gamma + \beta_k - i\omega_1)} \exp [(t + T)(i\omega_1 - \Gamma) - \beta_k T - i\alpha]$$

$$- \sum_{k=1}^{4} \frac{a_k[\omega_0^2 + \beta_k(i\omega_1 - \Gamma)]}{\omega_0^2(\Gamma - \beta_k - i\omega_1)} \exp [(T - t)(i\omega_1 - \Gamma) + \beta_k T - i\alpha]$$

$$- \sum_{k=1}^{4} \frac{a_k[\omega_0^2 + \beta_k(i\omega_1 + \Gamma)]}{\omega_0^2(\Gamma + \beta_k + i\omega_1)} \exp [-(t + T)(i\omega_1 + \Gamma) - \beta_k T + i\alpha]$$

$$- \sum_{k=1}^{4} \frac{a_k[\omega_0^2 - \beta_k(i\omega_1 + \Gamma)]}{\omega_0^2(\Gamma - \beta_k + i\omega_1)} \exp [(t - T)(i\omega_1 + \Gamma) + \beta_k T + i\alpha]$$

$$= \frac{8\Gamma\omega_1\lambda'}{\omega_0} \sum_{k=1}^{4} a_k \exp (\beta_k t)\,. \qquad (32)$$

Comparing the coefficients of the exponential functions, $\exp \beta_k t$, on the two sides of the last equation, and recalling that $\cos \alpha = \omega_1/\omega_0$, we find

$$\lambda' = \frac{\omega_0^2 - \beta_k^2}{(\beta_k^2 + \omega_0^2)^2 - 4\Gamma^2 \beta_k^2} \qquad \text{for} \qquad k = 1, 2, 3, 4 \ . \tag{33}$$

Furthermore, the coefficients of $\exp t(\pm \Gamma \pm i\omega_1)$ must be zero because $\beta_k \neq \Gamma \pm i\omega_1$ and $\beta_k \neq -\Gamma \pm i\omega_1$. This yields four equations of constraint on the a_k:

$$\sum_{k=1}^{4} a_k \left(\frac{\Gamma + \beta_k + i\omega_1}{\Gamma + \beta_k - i\omega_1} \right) \exp(-\beta_k T) = 0 \ , \qquad \sum_{k=1}^{4} a_k \left(\frac{\Gamma - \beta_k + i\omega_1}{\Gamma - \beta_k - i\omega_1} \right) \exp(+\beta_k T) = 0 \ ,$$

$$\sum_{k=1}^{4} a_k \left(\frac{\Gamma + \beta_k - i\omega_1}{\Gamma + \beta_k + i\omega_1} \right) \exp(-\beta_k T) = 0 \ , \qquad \sum_{k=1}^{4} a_k \left(\frac{\Gamma - \beta_k - i\omega_1}{\Gamma - \beta_k + i\omega_1} \right) \exp(+\beta_k T) = 0 \tag{34}$$

The existence of a nonzero solution for the a_k requires that the determinant of the 4×4 matrix of the coefficients given by equations (34) vanish. This condition gives one equation for the β_k. Equations (33) provide three more independent constraints on the β_k. These four equations can be solved simultaneously to yield the four β_k. Next, we reduce the four β_k to a single unknown and also reduce the 4×4 matrix equation to 2×2 block diagonal form.

Consider two of the four β_k, say β_1 and β_2, such that $\beta_1^2 \neq \beta_2^2$. Equation (33) is solved to provide a relation between β_1 and β_2,

$$\beta_1^2 + \beta_2^2 = 4\Gamma^2 - 2\omega_0^2 - 1/\lambda' \ . \tag{35}$$

It is easy to see from equation (35) that β_2, and hence each of the β_k, must be either real or imaginary. We have shown in § II that the λ are positive. It follows that if $\Gamma < \omega_0/2$ equation (35) implies that at least one of the β_k must be imaginary. We name the imaginary β_k, β_1. Since Q is an even polynomial, the β_k must come in pairs of opposite sign. Thus, without loss of generality, the four β_k may be taken to be $\beta_1, \beta_2, -\beta_1$ and $-\beta_2$

The first and second constraint equations (34) now take the form

$$\sum_{k=1}^{2} \left[a_k \left(\frac{\Omega_+ + \beta_k}{\Omega_- + \beta_k} \right) \exp(-\beta_k T) + b_k \left(\frac{\Omega_+ - \beta_k}{\Omega_- - \beta_k} \right) \exp(+\beta_k T) \right] = 0 \ ,$$

$$\sum_{k=1}^{2} \left[a_k \left(\frac{\Omega_+ - \beta_k}{\Omega_- - \beta_k} \right) \exp(+\beta_k T) + b_k \left(\frac{\Omega_+ + \beta_k}{\Omega_- + \beta_k} \right) \exp(-\beta_k T) \right] = 0 \ , \tag{36}$$

where $\Omega_\pm = \Gamma \pm i\omega_1$, $b_1 = a_3$ and $b_2 = a_4$. Adding and subtracting these two equations, we obtain

$$\sum_{k=1}^{2} z_k \left[\left(\frac{\Omega_+ + \beta_k}{\Omega_- + \beta_k} \right) \exp(-\beta_k T) + \left(\frac{\Omega_+ - \beta_k}{\Omega_- - \beta_k} \right) \exp(\beta_k T) \right] = 0 \ ,$$

$$\sum_{k=1}^{2} z_k' \left[\left(\frac{\Omega_+ + \beta_k}{\Omega_- + \beta_k} \right) \exp(-\beta_k T) - \left(\frac{\Omega_+ - \beta_k}{\Omega_- - \beta_k} \right) \exp(\beta_k T) \right] = 0 \ , \tag{37}$$

where

$$z_1 = a_1 + b_1, \qquad z_2 = a_2 + b_2, \qquad z_1' = a_1 - b_1, \qquad z_2' = a_2 - b_2 \ . \tag{38}$$

Two other equations in the z_k and the z_k' are obtained by interchanging Ω_+ and Ω_- in equation (37).

The advantage of these new variables over the a_k is that they reduce the 4×4 matrix which expresses the equations of constraint to 2×2 block diagonal form. For convenience, we define $\beta_j^{(1)} = \beta_j$ and $\beta_j^{(2)} = -\beta_j$ for $j = 1, 2$. Then, the requirement that the determinant of the 4×4 matrix vanish reduces to the vanishing of the determinant of either one of the following 2×2 matrices:

$$\left| \begin{array}{cc} \displaystyle\sum_{k=1}^{2} \left(\frac{\Omega_+ + \beta_1^{(k)}}{\Omega_- + \beta_1^{(k)}} \right) \exp(-\beta_1^{(k)}T) \ ; & \displaystyle\sum_{k=1}^{2} \left(\frac{\Omega_+ + \beta_2^{(k)}}{\Omega_- + \beta_2^{(k)}} \right) \exp(-\beta_2^{(k)}T) \\[3ex] \displaystyle\sum_{k=1}^{2} \left(\frac{\Omega_- + \beta_1^{(k)}}{\Omega_+ + \beta_1^{(k)}} \right) \exp(-\beta_1^{(k)}T) \ ; & \displaystyle\sum_{k=1}^{2} \left(\frac{\Omega_- + \beta_2^{(k)}}{\Omega_+ + \beta_2^{(k)}} \right) \exp(-\beta_2^{(k)}T) \end{array} \right| = 0 \ , \tag{39}$$

or

$$\left| \begin{array}{cc} \displaystyle\sum_{k=1}^{2} i^{2k} \left(\frac{\Omega_+ + \beta_1^{(k)}}{\Omega_- + \beta_1^{(k)}} \right) \exp(-\beta_1^{(k)}T) \ ; & \displaystyle\sum_{k=1}^{2} i^{2k} \left(\frac{\Omega_+ + \beta_2^{(k)}}{\Omega_- + \beta_2^{(k)}} \right) \exp(-\beta_2^{(k)}T) \\[3ex] \displaystyle\sum_{k=1}^{2} i^{2k} \left(\frac{\Omega_- + \beta_1^{(k)}}{\Omega_+ + \beta_1^{(k)}} \right) \exp(-\beta_1^{(k)}T) \ ; & \displaystyle\sum_{k=1}^{2} i^{2k} \left(\frac{\Omega_- + \beta_2^{(k)}}{\Omega_+ + \beta_2^{(k)}} \right) \exp(-\beta_2^{(k)}T) \end{array} \right| = 0 \ . \tag{40}$$

Here, β_2 is determined in terms of β_1 from equation (35). This completes our reduction of the integral eigenvalue equation (6) to an algebraic equation.

IV. THE DISTRIBUTION FUNCTIONS

We solve the independent transcendental equations (39) and (40) numerically to obtain two infinite sequences of values for β_1. Given β_1 the eigenvalue λ' is obtained from equation (33). We arrange the eigenvalues in each sequence in order of decreasing size and label them by consecutive positive and negative integers, respectively. The sum of all the eigenvalues, λ'_n, is equal to T/Γ. This result follows from equating the expectation value of the integral over $(-T, T)$ of the absolute square of equation (8) to $2T$ times the trace of $\mathscr{R}(t, t)$ obtained from equation (20). The largest eigenvalues, divided by T/Γ, are displayed in Figure 1 for several values of ΓT. Note that the width of the peak decreases with increasing ΓT and that there is approximately unit area under each curve.

To obtain an accurate expression for the characteristic function $\mathscr{C}_{E_T}$, the sum of the eigenvalues used in equation (14) must be close to T/Γ. As equation (33) indicates, $\lambda(\beta_1)$ peaks for β_1 close to $i\omega_0$ and is independent of T. In our determinations of $\mathscr{C}_{E_T}$ we use only the few largest λ_n for $\Gamma T \lesssim 1$ but more than a hundred for $\Gamma T \gtrsim 10$.

The probability distribution function is obtained by taking the inverse Fourier transform of $\mathscr{C}_{E_T}$. It is a function of two dimensionless parameters, ΓT and Γ/ω_0. For $\Gamma/\omega_0 \ll 1$, the case of interest to us, the dependence on Γ/ω_0 is very weak and may be ignored. Figure 2 displays the distribution function for several values of ΓT. For $\Gamma T \ll 1$ the distribution function has the Boltzmann form; that is, it is a decreasing exponential. With increasing ΓT the peak of the distribution functions narrows and it shifts toward the mean energy of the oscillator. For $\Gamma T \gg 1$ the distribution function tends to a δ-function. The limiting distributions can be obtained analytically from equations (6), (19), and (30).

Our calculations for the distribution functions successfully passed the following three tests. Both the integrated distribution functions and the mean energy are independent of T and the sum of the λ'_n are equal to T/Γ. There is one additional test of our calculations. For $\Gamma T \gtrsim 1$ the orthogonal series expansion of x is approximately the same as the Fourier series expansion over the interval $(-T, T)$, as expected.

V. NUMERICAL SIMULATIONS

We can model the random force $F(t)$ as a Poisson process by setting it equal to a random sequence of δ-function impulses which occur at times t_n. The time intervals

$$\Delta t_n \equiv t_{n+1} - t_n \tag{41}$$

are independent random variables. The probability density for the waiting time between impulses, Δt, is given by

$$P(\Delta t) = r \exp(-r\Delta t), \tag{42}$$

where r denotes the average number of impulses per unit time. The random variable Δt may be generated from a random variable y which is uniformly distributed on the interval $(0,1)$ by using the relation

$$\Delta t = -\frac{1}{r} \ln(y). \tag{43}$$

Between impulses $q(t)$ satisfies the homogeneous version of equation (1) and can be written in the form

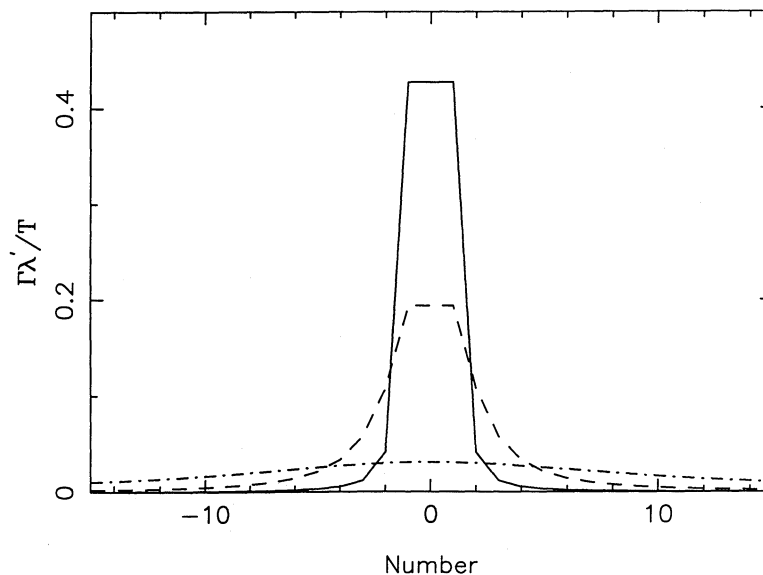$$q_n(t) = A_n \exp[-(\Gamma - i\omega_1)(t - t_n)] + A_n^* \exp[-(\Gamma + i\omega_1)(t - t_n)], \tag{44}$$

FIG. 1.—Plot of the largest eigenvalues (λ') times Γ/T for the two families of eigenfunctions for three cases of averaging times. The eigenvalues have been arranged in a descending series and labeled by positive and negative integers for the two families. Solid curve is for $\Gamma T = 0.5$, dashed curve is for $\Gamma T = 4$, and the dash-dot curve is for $\Gamma T = 32$.

for $t_n \leq t < t_{n+1}$. Inserting $F(t) = p\,\delta(t - t_{n+1})$, where p is a random sign (± 1), into equation (15) we obtain

$$q_{n+1}(t) = q_n(t) + \frac{p}{M}\,G(t, t_{n+1}) . \tag{45}$$

Substituting for the Green's function, G, from equation (16), equations (44) and (45) define a mapping from A_n and A_{n+1}, namely,

$$A_{n+1} = A_n \exp\left[-\Delta t_n(\Gamma - i\omega_1)\right] - \frac{ip}{2\omega_1} . \tag{46}$$

We define E_n to be the energy immediately following the nth impulse at $t = t_n$. With the aid of equations (2) and (44) it may be expressed as

$$E_n = M\omega_0^2[4\,|A_n|^2 + A_n^2 + A_n^{*2} - A_n^2 \exp(2i\alpha) - A_n^{*2} \exp(-2i\alpha)] , \tag{47}$$

where $\sin\alpha = \Gamma/\omega_0$.

We have now assembled all the ingredients needed to calculate a string of values of E_n. The initial conditions are specified by choosing a value for A_0. The time intervals between consecutive impulses are determined from equation (43) with the values of y obtained from a uniform random number generator. The new amplitude and the energy of the oscillator immediately after an impulse are calculated using equations (46) and (47), respectively. Taking a running mean over time $2T$ of the sequence of values for E yields a sequence of values for E_T. It is then a straightforward matter to compute approximate distribution functions for E_T and to demonstrate empirically that they converge to the distribution functions derived theoretically.

A theoretically computed distribution function for E_T corresponds to an infinite data sample, whereas observations inevitably produce a finite data sample. Figure 2 illustrates how observationally determined distribution functions might appear in comparison to our theoretically predicted ones. These plots display approximate distribution functions computed from data simulated for
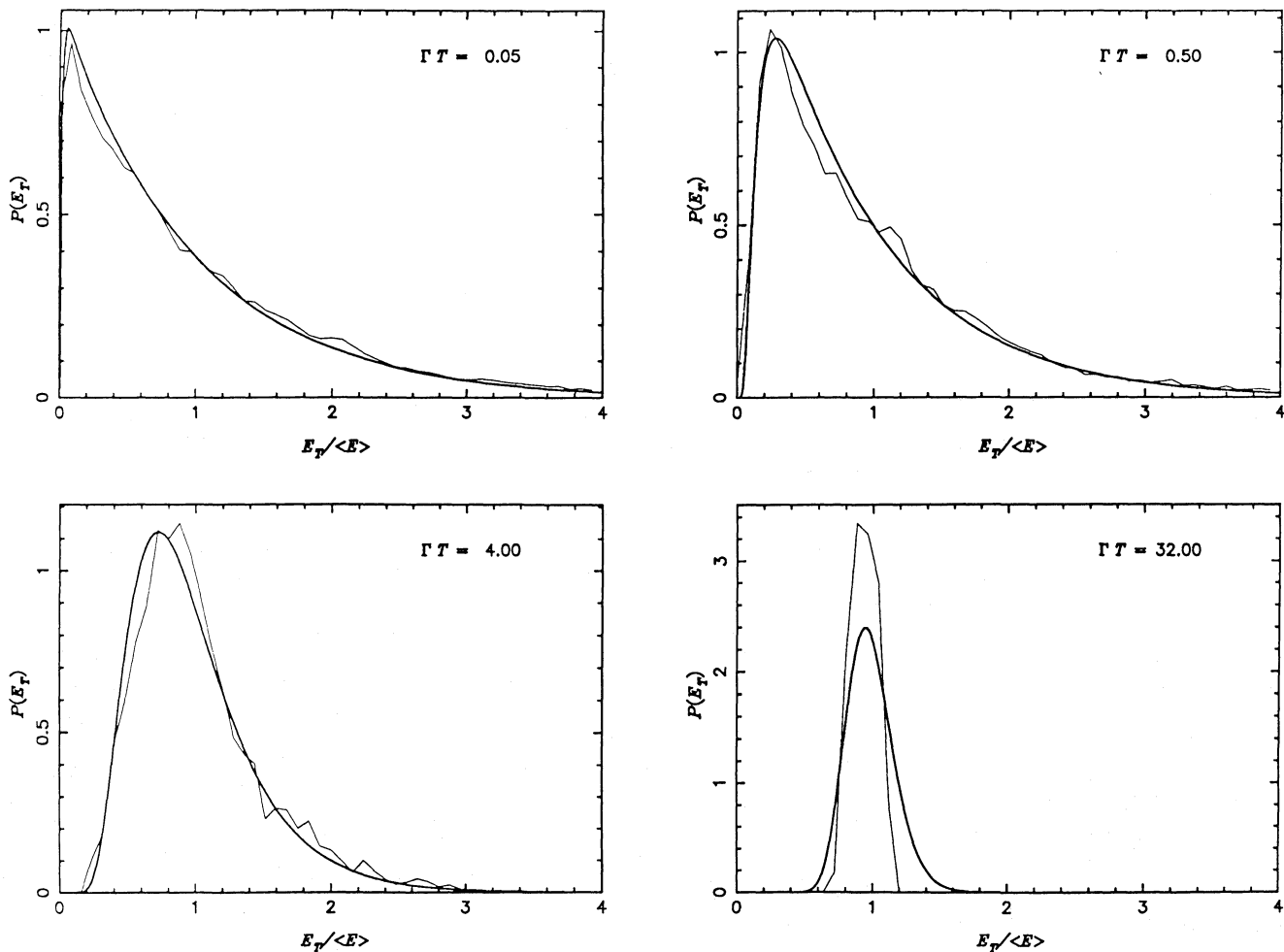

Fig. 2.—The heavy solid curves are theoretically determined distribution functions for four different values of ΓT, and in each case are same in the range $10 \leq \omega_0/\Gamma \leq \infty$. The light solid curves are distribution functions obtained from a string of simulated data 100 damping times long, with r equal to five hits per period and $\omega/\Gamma = 10^3$.
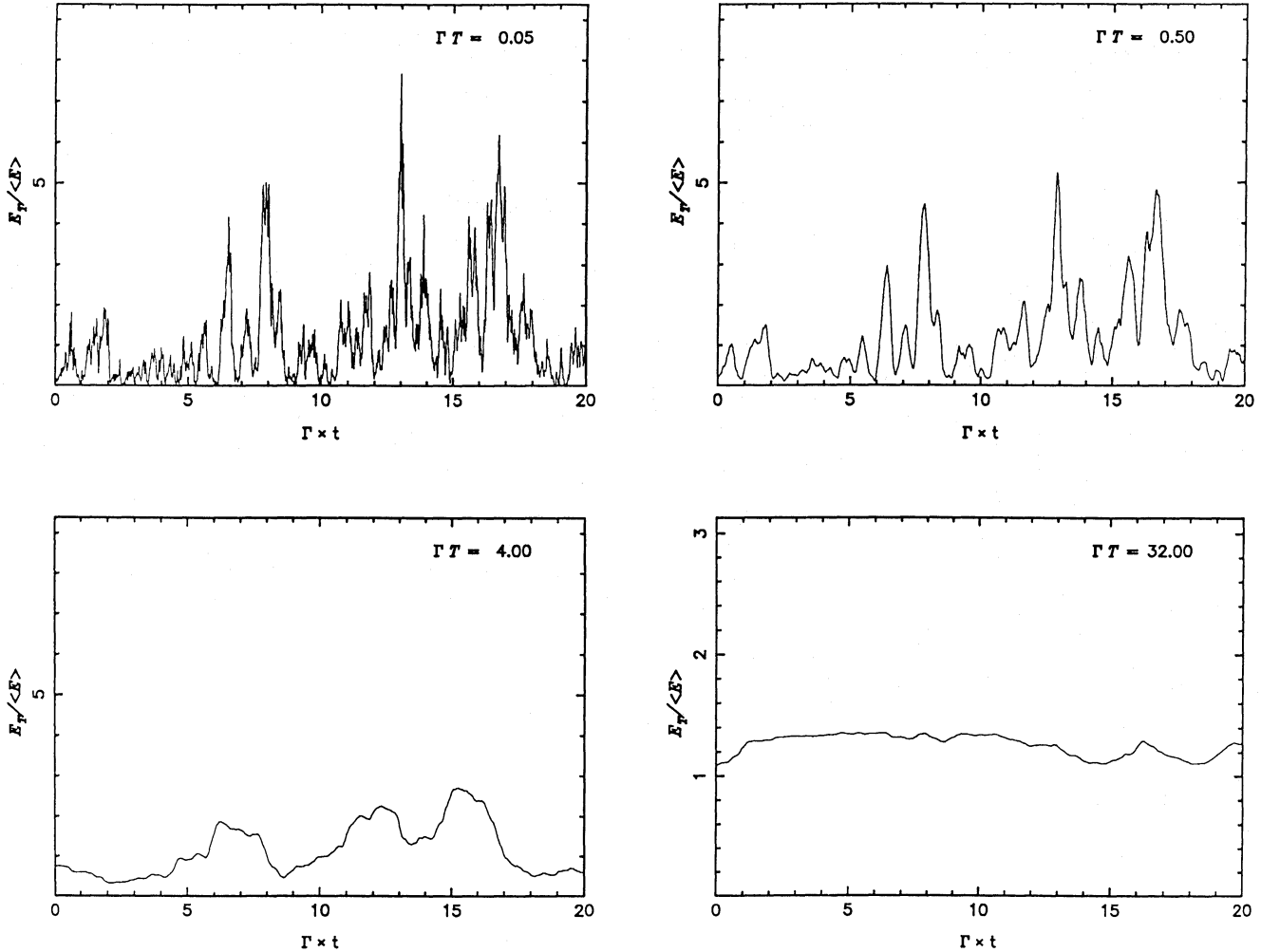
Fig. 3.—A sample of the average energy as a function of time for the four averaging times considered above

100 damping times, along with the theoretical distribution functions. The distribution function obtained from a simulated data of 1000 damping times is quite indistinguishable from the theoretical distribution. In addition, we show in Figure 3 a portion of the strings of values of E_T from which each of these distribution functions is obtained.

VI. OBSERVATIONAL IMPLICATIONS

One of our goals is to determine, with the help of the observations, if the solar p-modes are randomly excited. So far our discussion has been rather abstract, and consequently its relevance to the excitation of the solar oscillations may be a bit obscure. Here, we briefly indicate some ways in which the results of this investigation might aid in the understanding of the solar oscillations.

The most obvious application of our results would be to compare the theoretically calculated distribution functions for E_T with those determined from observations of solar p-modes. The observational determination of the distribution function requires many independent samples of E_T for a single mode. Since the lifetimes of the modes in the 5 minute band are of order days, rather lengthy data strings will be required to obtain accurate distribution functions for single modes (see Fig. 2). A more economical approach might be to treat modes with the same n and l but different m as copies of a single mode. It might be still better to further enlarge the set of similar modes to include those with a range of l values at a fixed value of n. For instance, lumping together all modes with fixed n having $90 < l < 110$ and including all values of m gives a set of ~ 4000 individual modes.

The spectral density of a random process is the expectation value of the time-averaged, absolute square of the finite Fourier transform in the limit of infinite averaging time. The Wiener-Khinchin theorem (Goodman 1985) states that the spectral density is equal to the Fourier transform of the autocorrelation function. Thus, the spectral density for q, $P_q(\omega)$, is equal to $\tilde{\mathcal{R}}_{11}(\omega)/\omega_0^2$, which from equation (22) is given by

$$P_q(\omega) = \frac{g}{(2\pi)^{1/2}M^2} \frac{1}{(\omega^2 - \omega_0^2)^2 + 4\omega^2\Gamma^2} . \tag{48}$$

To a very good approximation $P_q(\omega)$ is Lorentzian, with line width, Γ, near the peak of the power spectrum which is located at

$(\omega_0^2 - 2\Gamma^2)^{1/2}$. Thus, the drag force responsible for the damping produces a small frequency shift,

$$\Delta\omega \approx -\frac{\Gamma^2}{\omega_0}. \tag{49}$$

For solar p-modes in the 5 minute band $\Gamma \approx 1$ μHz (Libbrecht and Zirin 1986, Isaak 1986). Thus, the frequency shift caused by the decay is $\lesssim 1$ nHz, which is much smaller than typical errors in the frequency measurements, ~ 0.1 μHz, and is insignificant compared to the discrepancies between the observed and theoretically computed frequencies, ~ 15 μHz.

The mean energy, $\bar{E}$, is equal to $M/2$ times the trace of $\mathscr{R}(t, t)$. Using equation (20) we obtain

$$\bar{E} = \frac{g}{4M\Gamma}. \tag{50}$$

We may generalize the expression for $\bar{E}$ so that it applies to an arbitrary random force by replacing g by the spectral density of the random force evaluated at frequency ω_0. Thus, different oscillators interacting with random forces which have the same $g(\omega_0)$ should have identical values for the products of their mean energies, line widths, and masses.

The excitation and damping of solar oscillations by turbulent convection are described by effective random forces and damping constants for which $g(\omega_0)$ and Γ are given by overlap integrals of particular combinations of modal eigenfunctions and turbulent fields (Goldreich and Keeley 1977; Goldreich and Kumar 1988). Since the excitation and damping by turbulence depend sensitively upon the turbulent Mach number, it is likely that both are dominated by contributions from the upper few scale heights of the solar convection zone.

The radial displacement dominates the horizontal displacement for p-modes in the 5 minute band with $l \lesssim 200$. Moreover, the shape of the radial displacement component of the eigenfunction in the upper convection zone depends only on frequency and is independent of l. Thus, we would expect modes which have the same frequencies but different values of l and m to have identical values of $M\bar{E}\Gamma$ if they are stochastically excited by turbulent convection. If, in addition, either turbulent convection or radiative dissipation near the photosphere is the major source of damping, Γ should be inversely proportional to the mass of the mode.[3] This would lead to a stronger prediction, namely, that all modes of a given frequency having $l \lesssim 200$ should have the same value of $\bar{E}$.

The observational results of Libbrecht *et al.* (1986) suggest that the energies of p-modes in the 5 minute band are approximately independent of l, at least for $l \lesssim 100$.

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[3] The mass of a mode is the ratio of its total energy to the square of its rms, photospheric, radial velocity.

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