Mon. Not. R. astr. Soc. (1987) **225**, 695–711

The stability of accretion tori – II. Non-linear evolution to discrete planets

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Accepted 1986 October 30. Received 1986 October 2

Summary. Hawley has shown through two-dimensional computer simulations that a slender torus in which a linear Papaloizou–Pringle (PP) instability with azimuthal wavenumber m, is excited evolves non-linearly to a configuration with m nearly disconnected 'planets'. We present an analytical fluid equilibrium that we believe represents his numerical planets. The fluid has an ellipsoidal figure and is held together by the Coriolis force associated with the retrograde fluid motion. There is a bifurcation between the torus and planet configurations at precisely the vorticity below which the PP instability switches on. Although the solution is three-dimensional, there is perfect hydrostatic equilibrium and the motion is entirely two-dimensional. We analyse the linear modes of the analytical planet and find that there are numerous instabilities, though they are not as violent as the PP instability in the torus. We also discuss the energy and vorticity of neutral modes, and we argue that when the torus breaks up into planets, neutral modes with negative energy and non-zero vorticity are excited in order to conserve total energy and specific vorticity. We speculate that the fluid in Hawley's simulations may be approaching two-dimensional turbulence.

1 Introduction

Papaloizou & Pringle (1984, 1985; henceforth PP) made the important discovery that isentropic thick accretion tori are dynamically unstable to linear non-axisymmetric modes. Parameterizing the angular velocity profile of a torus in the form

$$\Omega(r) \propto r^{-q}, \tag{1.1}$$

where r is the orbital radius, they showed that the principal mode of instability occurs for all $q>\sqrt{3}$. Thus, a thin Kepler disc, with $q=3/2$, is stable, whereas a thick constant-angular-momentum torus, with $q=2$, is violently unstable.

The nature of the principal branch of the linear PP instability has been clarified by the work of

Blaes (1985), Blaes & Glatzel (1986), and by the present authors in Paper I of this series (Goldreich, Goodman & Narayan 1986). We showed in Paper I that, in the principal branch, the fluid maintains vertical hydrostatic equilibrium to a very good approximation, and so its motion is essentially two-dimensional, i.e. restricted to the orbital plane. Consequently, the modal analysis is greatly simplified and is described by an eigenvalue problem with an ordinary differential equation. Using this, we have computed the growth rate of the PP instability for different q and azimuthal wavenumber m, for a variety of fluid compressibilities.

In view of the violent growth rate of the PP instability, the non-linear fate of a thick torus is an interesting question. Zurek & Benz (1986) numerically studied constant-angular-momentum tori using a smooth-particle hydrodynamics code with 1000 particles. They confirm the existence of the PP instability, and find that the initial $q=2$ torus settles down finally to a somewhat disturbed and lumpy configuration that is describable on the average as a $q \sim 1.75$ torus. Thus the instability redistributes specific angular momentum on a dynamical time-scale.

Recent numerical work by Hawley (1987) has provided new intriguing clues regarding the non-linear evolution of unstable thick tori. Using the results of Paper I as a guide, Hawley assumed vertical hydrostatic equilibrium and studied a *two-dimensional* compressible fluid in an equilibrium configuration that simulated a slender* three-dimensional $q=2$ torus. He introduced a small multiple of one of the unstable modes calculated in Paper I, and studied the numerical evolution of the system using an accurate finite-difference hydrodynamics code. Two interesting results emerge from his work. First, the growth rate calculated in Paper I on the basis of linear perturbation theory is confirmed to be accurate, and to be valid well into the non-linear regime, even for surface density perturbations $\Delta\Sigma/\Sigma \sim 1$. Secondly, a torus in which a single mode with azimuthal wavenumber m is excited breaks up into m coherent blobs that are only tenuously connected to one another in the azimuthal direction. These orbits around the central mass with essentially the same angular velocity Ω as the original torus.

The blobs that Hawley (1987) finds, which he calls 'planets', seem to persist for a relatively long time, and they may signal a new underlying equilibrium configuration for the fluid. In Section 2 we present an exact analytical solution for a three-dimensional equilibrium with constant vorticity that, in two dimensions, looks qualitatively similar to Hawley's blobs. Our solution is valid for a polytropic fluid of arbitrary polytropic index, and corresponds in the incompressible limit to a non-self-gravitating Roche–Riemann ellipsoid. We consider in Section 3 whether the PP instability could represent a non-linear transition from a torus to a configuration related to our analytical planet solution. The strongest evidence in favour of this hypothesis is that the maximum vorticity attainable in the planet configuration is $(2-\sqrt{3})\Omega$, which is exactly the vorticity of the marginally stable torus with $q=\sqrt{3}$. This implies that there is a bifurcation between the torus and planet configurations at this vorticity. In Section 4 we show that the planets themselves suffer from a number of linear instabilities, both in two and three dimensions, but with a somewhat smaller growth rate than the PP instability in the torus. Section 5 is devoted to the neutral modes of the planet. We show that these could have vorticity, in contrast to the unstable modes, and also that they could have either sign of energy. We argue that differences in energy and specific vorticity between the initial torus and final planet configurations could be absorbed by finite-amplitude neutral modes. We end the paper with a discussion in Section 6 of the ultimate fate of a thick torus. The instabilities in the planets may saturate at a modest amplitude, leading to long-lived blobs, but this seems unlikely in view of more recent numerical work by Hawley (private communication). Alternatively, the planets may evolve further. An intriguing possibility is that the fluid may go two-dimensionally turbulent. In this case, the planets may coalesce into a single large blob, with very fine-scale vortical structure superposed on it.

*We call a torus 'thick' if the vertical and radial widths are comparable, and 'slender' or 'narrow' if the radial width is small compared to the orbital radius. The tori of Paper I are both slender and thick.

2 Planet equilibria

As in Paper I, we consider a non-self-gravitating, non-viscous, polytropic fluid in orbit around a central mass, and we assume that the fluid occupies a range Δr in radius very much smaller than the mean orbital radius, r_0. Then, in a frame corotating with the fluid, the Euler equation is approximately

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\Omega^2(3x\hat{\mathbf{i}}-z\hat{\mathbf{k}})-2\Omega\hat{\mathbf{k}}\times\mathbf{v}-\nabla Q. \tag{2.1}$$

Here, x, y, and z are $r-r_0$, $r_0\theta$, and z in terms of the natural cylindrical coordinate system, and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are corresponding unit vectors. We neglect the curvilinearity of the xyz system in this local approximation.

The first term on the right-hand side of (2.1) is the tidal gravity, the second is the Coriolis acceleration, and the last term is the acceleration due to the pressure gradient, where

$$Q\equiv(n+1)\frac{p}{\varrho}=(n+1)K\varrho^{1/n} \tag{2.2}$$

is the enthalpy for pressure p, density ϱ, and polytropic index n (K is a constant). It is convenient to express the continuity equation in terms of Q rather than ϱ:

$$n\left(\frac{\partial Q}{\partial t}+\mathbf{v}\cdot\nabla Q\right)+Q\nabla\cdot\mathbf{v}=0. \tag{2.3}$$

An exact equilibrium (i.e. time-independent) solution of equations (2.1) and (2.3) is

$$\mathbf{v}=(\varepsilon\alpha\Omega y)\hat{\mathbf{i}}-(\alpha\Omega x/\varepsilon)\hat{\mathbf{j}}, \tag{2.4}$$

$$Q=\tfrac{1}{2}\gamma^2\Omega^2(b^2-x^2-\varepsilon^2y^2-\gamma^{-2}x^2), \tag{2.5}$$

if

$$\alpha=\frac{\sqrt{3}\varepsilon}{\sqrt{1-\varepsilon^2}}, \tag{2.6}$$

$$\gamma^2=-\frac{3}{1-\varepsilon^2}+\frac{2\sqrt{3}}{\sqrt{1-\varepsilon^2}}. \tag{2.7}$$

Naturally, the solution applies only where $Q\geq 0$.

This solution has some remarkable properties.

(i) The planet has an ellipsoidal figure. Its principal axes are along x, y, and z, with semi-axes b, $b\varepsilon^{-1}$, and $b\gamma$. Solutions exist only for $0\leq\varepsilon\leq 1/2$. At $\varepsilon=0$, the y-axis is infinite, and the ratio of z- to x-axes is $[\sqrt{3}(2-\sqrt{3})]^{1/2}$; in the limit the planet coincides with the narrow torus for $q=\sqrt{3}$ (Section 3). At $\varepsilon=1/2$, the solution is pressureless, the vertical axis vanishes, and the fluid follows free epicycles in the equatorial plane.

(ii) It seems at first surprising that such solutions should exist at all without self-gravity, but the outward tidal and pressure forces are successfully balanced by the Coriolis force acting on the retrograde velocity field. In this sense, the planet is similar to the narrow tori of Paper I.

(iii) Every fluid element circulates around the centre of the planet with the same retrograde angular velocity $-\alpha\Omega$. The flow is similar to solid-body rotation but has shear since $\varepsilon\neq 1$. The total vorticity (vorticity measured in an inertial frame) lies along z and has magnitude.

$$\zeta\Omega=2\Omega+\hat{\mathbf{k}}\cdot(\nabla\times\mathbf{v})=\left[2-\frac{\sqrt{3}(1+\varepsilon^2)}{\sqrt{1-\varepsilon^2}}\right]\Omega. \tag{2.8}$$

(Note that the vorticity is positive for planets with $\varepsilon<0.3117$ despite the retrograde velocity. Thus, if there were sufficient self-gravity to compress such planets significantly, they would in fact spin in a prograde sense.)

(iv) It follows from (2.4) that $\nabla\cdot\mathbf{v}=0$, whence the polytropic index divides out of the continuity equation (2.3). Consequently, v and Q are independent of the compressibility of the fluid.

(v) The vertical velocity vanishes, and the horizontal velocities and accelerations are independent of z. Hence, each fluid element maintains perfect vertical hydrostatic equilibrium. It was shown in Paper I that a three-dimensional fluid of polytropic index n, in vertical hydrostatic equilibrium satisfies the equations of a two-dimensional polytrope of index

$N\equiv n+\frac{1}{2}$:

$$\frac{\partial\mathbf{v}_2}{\partial t}+(\mathbf{v}_2\cdot\nabla_2)\mathbf{v}_2=\Omega^2(3x\hat{\mathbf{i}})-2\Omega\hat{\mathbf{k}}\times\mathbf{v}_2-\nabla_2 Q_2, \tag{2.10}$$

$$N\left(\frac{\partial Q_2}{\partial t}+\mathbf{v}_2\cdot\nabla_2 Q_2\right)+Q_2\nabla_2\cdot\mathbf{v}_2=0, \tag{2.11}$$

where the subscript 2 marks a two-dimensional quantity. Equations (2.10)–(2.11) can be obtained by integrating (2.1) and (2.3) with respect to z. The enthalpy Q_2 in (2.10)–(2.11) is the $z=0$ section of the three-dimensional Q in equations (2.1)–(2.3), and Σ is the surface density, i.e.

$$Q_2(x,y)=Q(x,y,0) \tag{2.12}$$

$$\Sigma(x,y)=\int_{-h}^{h}\varrho(x,y,z)\,dz, \tag{2.13}$$

where the height, h, is given by

$$h(x,y)=\gamma(b^2-x^2-\varepsilon^2y^2)^{1/2}. \tag{2.14}$$

However, instead of $\varrho\propto Q^n$, now in two dimensions we have

$$\Sigma\propto Q_2^N. \tag{2.15}$$

(vi) For $n=0$ the planet reduces to a non-self-gravitating case of the Roche–Riemann ellipsoids (e.g. Chandrasekhar 1969), whose equilibria and stability have been studied by Aizenman (1968). Riemann discs, the two-dimensional analogue of the Roche–Riemann ellipsoids in the absence of the tidal potential, have been explored by Weinberg & Tremaine (1983) and Weinberg (1983). In the presence of self-gravity, however, the Roche–Riemann ellipsoids exist only for a constant-density fluid ($n=0$), and the Riemann discs only for $N=1/2$, whereas our solutions are valid for any polytrope. As we shall see later, the energy and stability of our solutions depend upon the compressibility.

In view of the fact that we now have two equilibria for the fluid, namely the torus and the planet, it is interesting to inquire whether there are other time-dependent equilibria relevant to the problem. We have searched for equilibrium figures that consist of sinusoidally perturbed tori; but have failed to find any analytically. Results presented in Section 4 suggest that there exist neighbouring equilibria differing from the planet by zero-frequency modes of small amplitude.

3 Torus versus planet

As noted above, the planet equilibria of Section 2 exactly solve the two-dimensional fluid equations as well as the three-dimensional ones. The numerical simulations of Hawley (1987), in which planet-like blobs evolved from the unstable torus, were based on the height-averaged equations (2.10)–(2.11). In the present section, therefore, we compare the height-averaged torus with the height-averaged planet (the planet disc).

When restricted to two dimensions, Kelvin's theorem on the conservation of circulation (*cf.* Landau & Lifshitz 1959) implies that every fluid element conserves its specific vorticity:

$$\frac{d}{dt}\left(\frac{\zeta\Omega}{\Sigma}\right)=0, \tag{3.1}$$

where $d/dt \equiv \partial/\partial t + \mathbf{v}_2 \cdot \nabla_2$. [The result (equation 3.1) can be obtained by taking the curl of equation (2.10) and replacing $\Sigma\nabla_2 \cdot \mathbf{v}_2$ with $-d\Sigma/dt$.] Therefore, the unstable torus can evolve only to final states having the same distribution of specific vorticity.

The vorticity of the torus measured in an inertial frame is

$$\zeta=(2-q)\Omega. \tag{3.2}$$

Let us consider first the so-called 'thin ribbon' of Paper I, which is the two-dimensionally incompressible ($N=0$), constant-Σ torus. This system has the same violent instabilities for $q>\sqrt{3}$ that the three-dimensional tori have. Because Σ is constant, equation (3.1) implies that in this case the vorticity itself is conserved. Thus, equating equations (3.2) and (2.8), we find

$$q=\frac{\sqrt{3}(1+\varepsilon^2)}{\sqrt{1-\varepsilon^2}}. \tag{3.3}$$

The planets exist only for $0 \leqslant \varepsilon \leqslant 1/2$, which corresponds by equation (3.3) to the range $\sqrt{3} \leqslant q \leqslant 5/2$. Note in comparison that the principal branch of the torus is unstable only for $q>\sqrt{3}$. We thus have the significant result that the planet solution is possible only when the torus is unstable. This alone suggests that the planet configurations are somehow related to the PP instability. In fact, the $\varepsilon=0$ planet is infinitely elongated along y and is identical in shape to the thin ribbon. Thus for $N=0$, the two sequences of planet and torus equilibria bifurcate at $q=\sqrt{3}$, $\varepsilon=0$.

The constant angular momentum ribbon, for $q=2$, has vanishing vorticity. The planet that is formed in this case has

$$\varepsilon=\left(\frac{(\sqrt{28}-5)}{3}\right)^{1/2}\approx 0.3117. \tag{3.4}$$

Because the specific vorticity vanishes, equation (3.1) would permit a $q=2$ torus with any N to evolve into a planet with this value of ε. These planets have axial ratios 1:3.208:0.5682, and 'rotate' with angular velocity -0.5682Ω.

In the case of the general $q \neq 2$, $N \neq 0$ torus, it is not straightforward to identify the final planet configuration that is formed. This is because it is the specific vorticity that is conserved in going from the torus to the planet, and hence the constant-vorticity torus will evolve into a non-constant-vorticity blob. Nevertheless, the fluid will probably oscillate around a planet configuration with the same *mass-averaged* specific vorticity as the torus. (The nature of the oscillations around the mean configuration is discussed in Section 5.) If so, then the identification (equation 3.3) is valid in general. In any case, since the $\varepsilon=0$ planet is identical in shape to the $q=\sqrt{3}$ torus, the specific vorticity distribution is also the same for this special case, and therefore the bifurcation between the two sequences of configurations at $q=\sqrt{3}$ still holds.

We have so far discussed only the vorticity constraint, but mass energy and angular momentum must also be conserved. The conservation of angular momentum can be arranged in a straightforward manner by locating the centre of mass of the planets at the same radius as the pressure maximum of the initial slender torus. To see the consequence of mass conservation, consider a torus of radial width $2a$ and orbital radius r that goes unstable with azimuthal wavenumber m. Using the equations of Paper I, the mass of the torus is

$$M_t=2\pi r\int_{-a}^{a}dx\left[\frac{(2q-3)\Omega^2}{2(N+1)K}\right]^N(a^2-x^2)^N=2\pi rC_N\left[\frac{(2q-3)\Omega^2}{2(N+1)K}\right]^N a^{2N+1}, \tag{3.5}$$

where

$$C_N=\int_{-1}^{1}dp\,(1-p^2)^N=\frac{\Gamma(1/2)\Gamma(N+1)}{\Gamma(N+3/2)}. \tag{3.6}$$

The mass of m planets with dimensions $2b\times 2b/\varepsilon$ in the xy-plane is similarly

$$M_p=m\int_{-b/\varepsilon}^{b/\varepsilon}dy\int_{-\sqrt{b^2-\varepsilon^2y^2}}^{\sqrt{b^2-\varepsilon^2y^2}}dx\left[\frac{\gamma^2\Omega^2}{2(N+1)K}\right]^N(b^2-x^2-\varepsilon^2y^2)^N$$

$$=\frac{m\pi}{\varepsilon(N+1)}\left[\frac{\gamma^2\Omega^2}{2(N+1)K}\right]^N b^{2N+2}. \tag{3.7}$$

Equating equations (3.5) and (3.7), we thus determine the dimensions of the planets that would be formed if the given torus were to split into m planets.

An obvious condition we require for the PP instability to go all the way to the non-linear planet stage is that there must be enough space for the planets to form. Let us assume that a single row of planets is formed. Since $\Sigma\propto Q^N$, most of the mass of a planet with $N\geq 1$ lies inside an ellipse with y semi-axis of order $Y\equiv b/\varepsilon\sqrt{N+1}$. (This is an approximate size, that is clearly valid in the incompressible limit, $N=0$; in the isothermal limit, $N\to\infty$, Σ drops to $1/e$ of its central value on this ellipse.) In order that there not be a significant overlap between adjacent planets, we require that the distance between their centres be greater than $2Y$, whence we require $2mY<2\pi r$. This places the following limit on m:

$$\beta\equiv\frac{ma}{r}<\pi\varepsilon\sqrt{N+1}\left(\frac{C_{N+1/2}\sqrt{N+1}}{2}\right)^{1/(2N+1)}\left(\frac{\gamma^2}{2q-3}\right)^{N/(2N+1)}\equiv\beta_{\max}. \tag{3.8}$$

Fig. 1 compares the shape of the curve $\beta=\beta_{\max}$ in the βq-plane with the region of PP-instability obtained in Paper I. There is excellent agreement at a qualitative level, confirming once again that the planet has an important role to play in the instability.

We may also require the planets to be energetically favoured over the torus. There are three contributions to the energy – internal energy, kinetic energy, and tidal-potential energy. The total energy of the torus is

$$E_t=2\pi r\int_{-a}^{a}dx\left[\frac{(2q-3)\Omega^2}{2(N+1)K}\right]^N(a^2-x^2)^N\Omega^2\left[\frac{N(2q-3)}{2(N+1)}(a^2-x^2)+\frac{q^2x^2}{2}-\frac{3x^2}{2}\right]$$

$$=\frac{2\pi rC_N}{(2N+3)}\left[\frac{(2q-3)\Omega^2}{2(N+1)K}\right]^N\Omega^2a^{2N+3}[N(2q-3)+\tfrac{1}{2}(q^2-3)]. \tag{3.9}$$

In the planet, the internal energy per unit mass is constant on streamlines, and so is the sum of the kinetic and potential energies. Hence the energy of the planet is

$$E_p=\frac{2\pi m}{\varepsilon}\int_0^b db'b'\left[\frac{\gamma^2\Omega^2}{2(N+1)K}\right]^N(b^2-b'^2)^N\left[\frac{N\gamma^2\Omega^2}{2(N+1)}(b^2-b'^2)+\frac{3\varepsilon^2\Omega^2}{2(1-\varepsilon^2)}b'^2\right]$$

$$=\frac{m\pi}{2\varepsilon(N+1)(N+2)}\left[\frac{\gamma^2\Omega^2}{2(N+1)K}\right]^N\Omega^2b^{2N+4}\left[N\gamma^2+\frac{3\varepsilon^2}{1-\varepsilon^2}\right]. \tag{3.10}$$

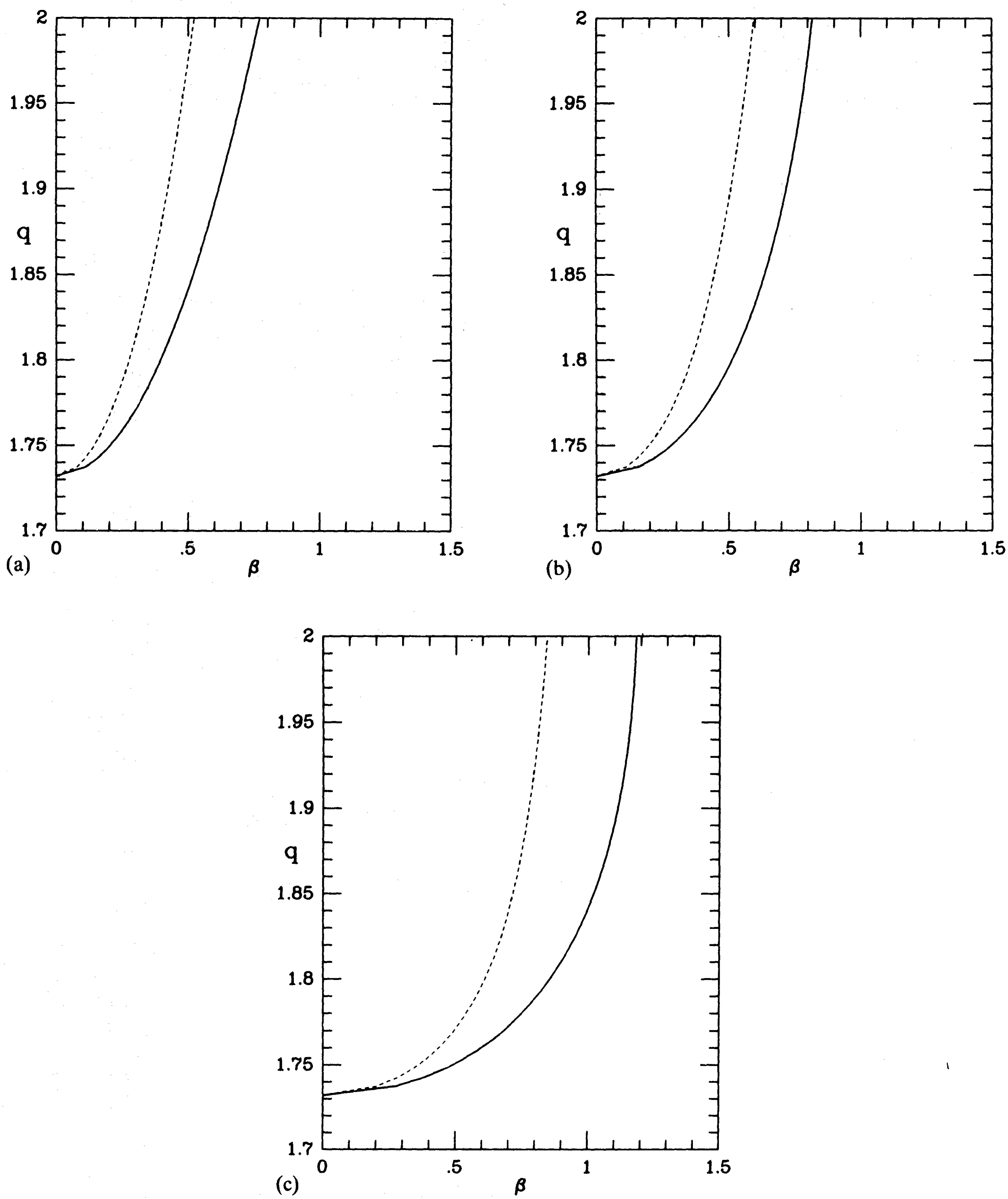

Figure 1. The solid curves correspond to $\beta=\beta_{\max}$ in the βq-plane. The dashed curves mark the boundary to the left of which the principal branch of the PP instability operates (see Paper I). (a) $N=0$, (b) $N=0.5$, (c) $N=3$. Note the qualitative agreement between the two sets of curves.

The requirement that $E_p<E_t$ translates to

$$\beta>\frac{2\pi\varepsilon}{C_{N+1/2}}\left(\frac{2q-3}{\gamma^2}\right)^N\left\{\frac{2(N+2)[N(2q-3)+(q^2-3)/2]}{(2N+3)[N\gamma^2+3\varepsilon^2/(1-\varepsilon^2)]}\right\}^{-(N+1)}\equiv\beta_{\min}. \tag{3.11}$$

The existence of a lower limit on β is puzzling. We have seen in Paper I that the PP instability extends all the way down to $\beta=0$. Moreover, all the numerical examples studied by Hawley (1987) correspond to $\beta<\beta_{\min}$. The question then arises: how can a higher energy planet configuration form from a lower energy torus configuration? The resolution of this paradox is discussed in Section 5, where we suggest that what is formed is a distorted planet with a finite-amplitude negative-energy neutral mode. Thus, because of the possibility of negative-energy distortions, the condition (equation 3.11) obtained from the energy criterion need not be satisfied.

4 Stability of planet discs

In this section, we analyse the linear modes of the two-dimensional planets with a view to understanding their stability. Our method is similar to that used by Weinberg (1983) in his study of the Riemann discs, except that we ignore self-gravity, and consequently we are able to carry out the analysis for arbitrary N, whereas he was limited to $N=1/2$. The linearization of equations (2.10) and (2.11) gives

$$N(sQ_2'+\mathbf{v}_2'\cdot\nabla_2Q_2+\mathbf{v}_2\cdot\nabla_2Q_2')+Q_2\nabla_2\cdot\mathbf{v}_2'=0, \tag{4.1}$$

$$s\mathbf{v}_2'+(\mathbf{v}_2'\cdot\nabla_2)\mathbf{v}_2+(\mathbf{v}_2\cdot\nabla_2)\mathbf{v}_2'+\nabla_2Q_2'+2\Omega\hat{\mathbf{k}}\times\mathbf{v}_2'=0. \tag{4.2}$$

We have indicated first-order quantities with a prime and assumed that they have the time dependence $\exp(st)$. Growing modes have $\mathrm{Re}\,(s)>0$.

The perturbation in the pressure must vanish on the perturbed boundary, since we assume that the planet is surrounded by vacuum. The time derivative of this condition, when expressed in terms of Eulerian quantities at the unperturbed boundary, is

$$(s+\mathbf{v}_2\cdot\nabla_2)Q_2'+\mathbf{v}_2'\cdot\nabla_2Q_2=0. \tag{4.3}$$

Since $Q_2=0$ at the edge of the planet, equation (4.3) follows directly from equation (4.1) when $N\neq0$. Therefore the boundary condition does not have to be imposed explicitly. (The case that $N=0$ will be considered separately.)

We write $\mathbf{v}_2'\equiv(u, v)$ and seek a solution to equations (4.1)–(4.2) in polynomials:

$$Q_2'(x,y)=\sum_{m=0}^{d}\sum_{j=0}^{m}Q_{j,m}'x^jy^{m-j},$$

$$u(x,y)=\sum_{m=0}^{d}\sum_{j=0}^{m-1}u_{j,m}x^jy^{m-1-j},$$

$$v(x,y)=\sum_{m=0}^{d}\sum_{j=0}^{m-1}v_{j,m}x^jy^{m-1-j}. \tag{4.4}$$

Because the unperturbed velocity field and enthalpy are linear and quadratic in x and y (see equations 2.4, 2.5), therefore equation (4.4) produces terms of degree d, or less in equation (4.1) and terms of degree $d-1$ or less in equation (4.2). (The degree of the monomial x^iy^j is defined to be $i+j$.) For these equations to be satisfied, the coefficents of independent monomials must separately vanish, and we obtain a system of homogeneous linear equations of the form

$$A_m(s)w_m+B_mw_{m+2}=0, \qquad m=0,1,\ldots,d. \tag{4.5}$$

Here w_m is a column vector containing the coefficients of the $m+1$ terms of degree m in Q' and of the m terms of degree $m-1$ in each of u and v; $A_m(s)$ is a $(3m+1)\times(3m+1)$ matrix with diagonal entries depending on s, and B_m is an s-independent $(3m+1)\times(3m+7)$ matrix. The latter derives from a term of the form $Q_2(x=0, y=0)\nabla\cdot\mathbf{v}'$ in equation (4.1).

The form (4.4) assumes that $w_{d+2}=0$, and without loss of generality we can suppose that $w_d\neq0$.

Therefore a necessary condition for an eigenmode is that

$$det[A_d(s)]=0, \tag{4.6}$$

which defines a polynomial of degree $3d+1$ in s. For each root s, we can find a non-zero eigenvector w_d, and we can go on to solve for w_{d-2}, w_{d-4}, etc. provided that s is not also a root of $det[A_m(s)]$ for any $m<d$ with $d-m$ even; we then have a true eigenmode. If a root should recur at some $m<d$, however, then equation (4.5) cannot be solved for w_m unless $B_m w_{m+1}$ lies within the range of $A_m(s)$.

Certain roots do recur – in particular, $s=0$. If s is a root of equation (4.6), then so is $-s$. (The physical reason for this is the invariance of the planet equilibrium when we change the sign of time and simultaneously change the sign of the y-coordinate.) Hence when d is even, one of the $3d+1$ roots of equation (4.6) is $s=0$. The $d=0$ mode corresponds to a small change in the constant term of the unperturbed enthalpy, and hence to a change in the mass of the planet. The $d=2$ root corresponds to the difference between two planets with slightly different values of ε, and hence to a change in the vorticity. In the latter case, $\nabla\cdot\mathbf{v}_2'=0$, so that $B_0=0$. For higher even degrees, we find that the divergence does not vanish when $N>0$, but that $B_m w_{m+2}$ nevertheless lies within the range of $A_m(0)$. We have not proved this, but we have checked it numerically in several cases and never found an exception. Furthermore, when $q=2$ the roots $s=\pm ik\alpha\Omega$ recur at $d=k+2$, $k+4$, $k+6, \ldots$, and the same 'miracle' makes it possible to solve (4.5) despite the singularity of $A_m(s)$. (These modes apparently represent k vortices spaced around the unperturbed streamlines.)

Given that equation (4.5) can always be solved for the lower order terms, it follows that the polynomial modes equations (4.4) with time-dependences $\exp(st)$ are complete (this will be important to us in the next section). The proof of this is as follows: if one integrates the time-dependent forms of equations (4.1)–(4.2) (obtained by replacing s with $\partial/\partial t$) with polynomial initial data, then, by arguments directly analogous to those given after equation (4.4), the fluid variables will remain polynomial in x and y for all time and of the same degree or less. The subspace S_d of initial data in the form of polynomials of degree $\lesssim d$ has a finite dimension; and, given the assumption above, there is a distinct eigenmode for every such polynomial. Hence the modes are complete in S_d. But a continuous function defined on the planet can be approximated arbitrarily closely by a polynomial of sufficiently high degree, hence as we take $d\to\infty$ we 'capture' all possible continuous initial conditions.

Using equations (4.4)–(4.6), we have solved numerically for the eigenfrequencies of polynomial modes having degrees $2\leqslant d\leqslant 7$. For each degree, we have sampled the εN plane on a 31×30 grid. The results for the unstable modes are summarized in Fig. 2, which shows ridges of instability divided by stable valleys. There is a tendency for larger values of ε to be more unstable, but at the higher degrees, unstable modes are found even for very small ε. Also at the higher degrees, instability extends to small N; in fact some of the largest growth rates occur there. However, as we now show, at $N=0$ the planets are stable throughout the interesting range of ε. (The growth rate does $\to 0$ continuously before $N\to 0$, but sometimes quite reluctantly.)

When $N=0$, equation (4.1) reduces to $\nabla_2\cdot\mathbf{v}_2'=0$. Furthermore, the surface density and vorticity are constant both in equilibrium and in perturbation, and it follows from equations (3.1) and (2.8) that $\nabla_2\times\mathbf{v}_2'=0$. Hence we can write

$$\mathbf{v}_2'=\nabla_2\phi,$$

$$\nabla_2^2\phi=0. \tag{4.7}$$

Because of equation (4.7), ϕ is the real part of an analytical function of x and y. It is useful to introduce the harmonic conjugate, $\bar{\phi}$. which is the corresponding imaginary part, so that

$$\frac{\partial\bar{\phi}}{\partial y}=\frac{\partial\phi}{\partial x},\qquad \frac{\partial\bar{\phi}}{\partial x}=-\frac{\partial\phi}{\partial y}. \tag{4.8}$$

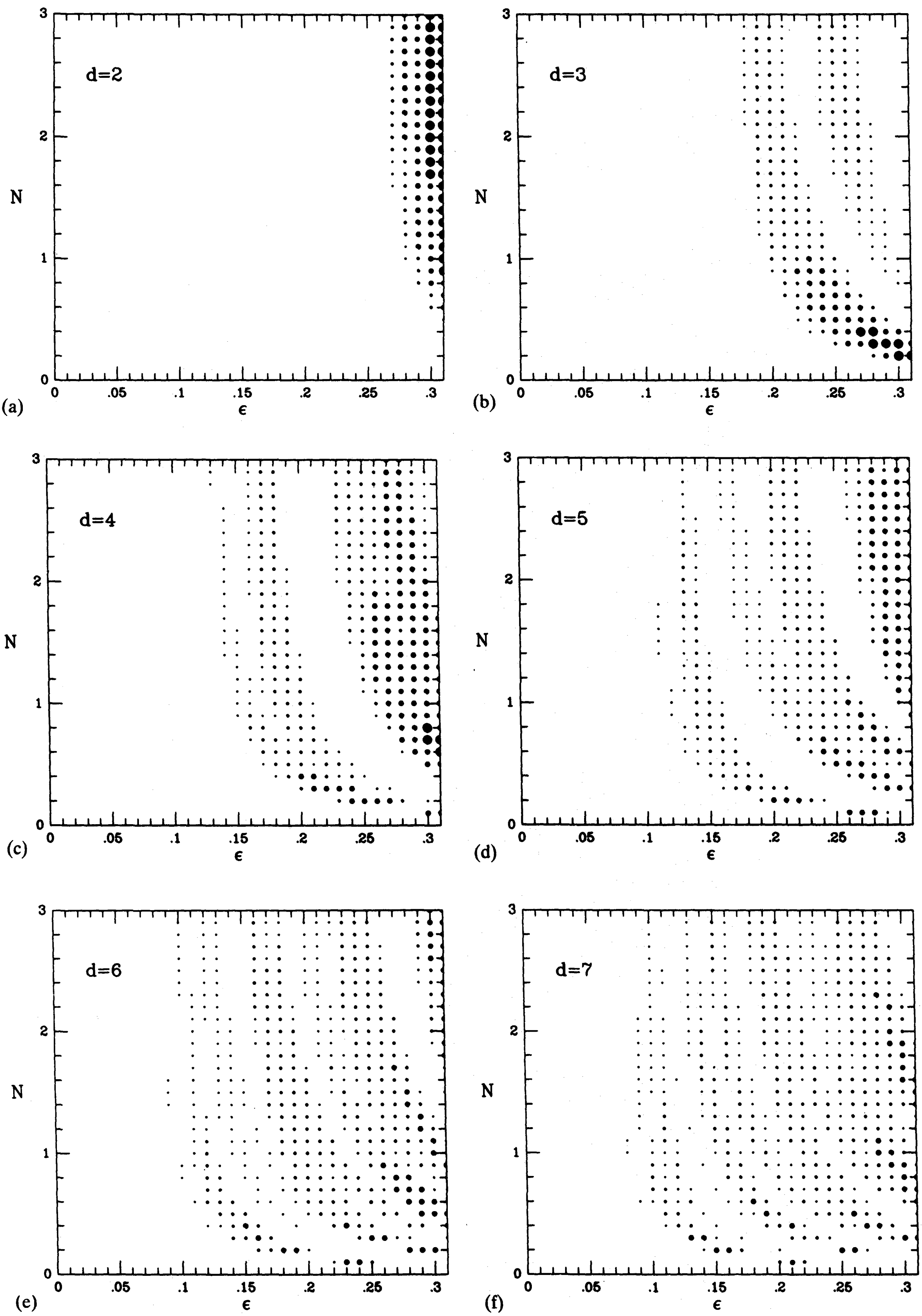

Figure 2. Stability of two-dimensional planet discs for degrees 2–7. From smallest to largest, the dots correspond to growth rates, $\mathrm{Re}(s)/\Omega$, in the ranges (0, 0.05], (0.05, 0,10], (0.10, 0.15], and (0.15, 0.20]. Where more than one unstable mode occurs at one point, the largest growth rate is shown.

Using equations (4.7) and (4.8), equation (4.2) can be spatially integrated to give

$$Q_2' = -\left(\frac{d}{dt}\right)_0 \phi - \zeta\Omega\bar{\phi}, \tag{4.9}$$

where

$$\left(\frac{d}{dt}\right)_0 \equiv \frac{\partial}{\partial t} + \mathbf{v}_2 \cdot \nabla_2 \tag{4.10}$$

is the advective derivative along the unperturbed flow. The boundary condition (4.3) becomes

$$\left(\frac{d}{dt}\right)_0^2 \phi + \zeta\Omega\left(\frac{d}{dt}\right)_0 \bar{\phi} - \nabla_2\phi \cdot \nabla_2 Q_2 = 0. \tag{4.11}$$

We seek a solution of the form

$$\phi = \exp(st)\,\mathrm{Re}\left(\sum_{k=0}^{d} a_k z^k\right), \qquad z \equiv (x+iy). \tag{4.12}$$

After equation (4.12) is substituted into equation (4.11), the even powers of x can be eliminated by using the equation for the shape of the boundary,

$$x^2 = b^2 - \varepsilon^2 y^2. \tag{4.13}$$

By requiring the coefficients of y^d and of xy^{d-1} to vanish, we then get a 2×2 homogeneous linear system. The determinant of this system yields a quadratic in s^2.

We have verified numerically that the roots for s^2 are real and non-positive for all ε between 0 and 0.3117... and for all d between 0 and 100. Hence there are just two neutral, non-vortical modes for each degree, and the $N=0$ planet appears to be completely stable. [We suspect that it is possible to prove this result directly from equation (4.11). Any such proof must use the explicit form of the unperturbed solution, however, since equation (4.11) is also valid for the $N=0$ torus, which was shown to be unstable in Paper I.]

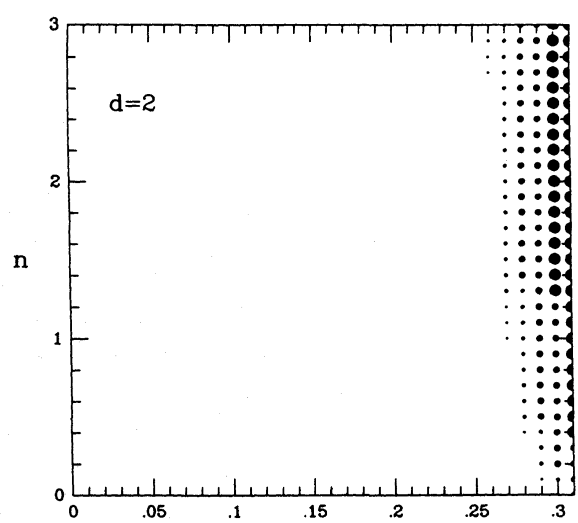

Figure 3. Stability of three-dimensional planet ellipsoids for degree 2. The dots are as in Fig. 2. If vertical hydrostatic equilibrium were exactly valid for the modes, as it is for the equilibrium, then the results should be identical to those for the planet disc with degree 2, provided the identification, $N=n+1/2$, is made. The agreement is, however, not perfect, reflecting the breakdown of the two-dimensional approximation for the motions involved in the modes.

The method of analysis used for $N\neq 0$ can be extended to three dimensions, but there are then many more coefficients. We have calculated only the second-degree modes that are even in z. Fig. 3 shows the results. As expected, the eigenfrequencies of a three-dimensional planet with polytropic index n are similar to those of the planet disc with $N=n+1/2$ and the same ε, but for the agreement between two and three dimensions is less precise than it was for the narrow tori of Paper I.

In summary, we have found that the planet discs are generally unstable to a large number of modes, except for $N=0$. Therefore, compressibility must be essential for these instabilities.

We suspect that the instabilities arise by a parametric resonance. We have analysed the stability of a planet disc with $\zeta=0$ and $\delta\equiv|\varepsilon-1|\ll 1$, using equation (5.15) (see below). Of course such an equilibrium requires a modification of the tidal field in equation (2.11). Thus, instead of the tidal potential $U(x)=-3\Omega^2x^2/2$ that leads to equation (2.10), we consider a problem where the tidal potential has the form

$$U(x)=-\frac{2(1-\varepsilon^2)\Omega^2x^2}{(1+\varepsilon^2)^2}. \tag{4.14}$$

This leads to

$$\alpha=2\varepsilon/(1+\varepsilon^2)=1+O(\delta^2). \tag{4.15}$$

If $\delta=0$, then the planet is circular and the fluid is stationary in a frame rotating at angular velocity $-\Omega$ with respect to the frame adopted in equations (2.10)–(2.11) (i.e. inertial space); in this case the planet is obviously stable and has a purely neutral spectrum of pressure modes. But for small, non-zero δ, the tidal force in the fluid frame has a time-varying component proportional to δ and with frequency 2α. If this frequency is nearly equal to twice the frequency of one of the pressure modes of the $\delta=0$ system, the tidal force parametrically excites the mode, which acquires a growth rate $\propto\delta$.

Of course it is a big step from this small-δ artificial model to the actual planets considered here. But parametric resonance would explain why the growing modes generally take the form of overstabilities. It would explain why the planet discs stabilize as $N\to 0$, since the frequencies of the pressure modes tend to infinity as $N^{-1/2}$.

5 Energy and vorticity of neutral modes

We noted in Section 3 that Hawley's lumps could not correspond exactly to planet equilibria because, for the values of m in his simulations, the planets have a higher energy per unit mass than the narrow torus. In the first part of this section, we shall show that the energy of the planets is actually decreased when certain neutral modes are excited. Negative energy modes are common in differentially rotating fluids (*cf.* Narayan, Goldreich & Goodman 1987, in preparation). Thus, although the narrow torus cannot evolve to a stationary planet, it can (at least on energetic grounds) evolve to an appropriately oscillating planet.

To obtain the energy associated with an excited mode, we start from the second order* Lagrangian density, $\mathscr{L}_2$. The appropriate expression is

$$\mathscr{L}_2=\frac{\varrho}{2}\left\{\left(\frac{d\xi}{dt}\right)^2-\frac{p}{\varrho}\left[(\Gamma-1)(\nabla\cdot\xi)^2+\nabla\xi:\nabla\xi\right]-\xi\xi:\nabla\nabla U+\left[(2\Omega\times\xi)\cdot\frac{d\xi}{dt}\right]\right\}, \tag{5.1}$$

*It is convenient to allow the linear, i.e. first-order, perturbation variables to be complex and to vary as $\exp(-i\omega t)$. When evaluating second-order quantities, we use the real parts of the first-order variables.

where

$$U=-\frac{3\Omega^2x^2}{2}. \tag{5.2}$$

Here $\boldsymbol{\xi}(\mathbf{x}, t)$ is the first-order displacement of a fluid element that would be at position $\mathbf{x}$ at time t in the unperturbed flow; $\boldsymbol{\xi}$ and the Eulerian velocity perturbation, $\mathbf{v}'$, are related by

$$\frac{d\boldsymbol{\xi}}{dt}-(\boldsymbol{\xi}\cdot\nabla)\mathbf{v}=\mathbf{v}. \tag{5.3}$$

In equations (5.1) and (5.2), d/dt is the time derivative following the unperturbed flow. The Euler–Lagrange equations that follow from equation (5.1) can be shown to be equivalent to equation (2.1).

To obtain the energy density $\mathscr{E}_2$ from $\mathscr{L}_2$. we use

$$\mathscr{E}_2=\frac{\partial\mathscr{L}_2}{\partial(\partial\boldsymbol{\xi}/\partial t)}\cdot\frac{\partial\boldsymbol{\xi}}{\partial t}-\mathscr{L}_2. \tag{5.4}$$

Carrying out these operations yields

$$\mathscr{E}_2=\frac{\varrho}{2}\left\{\left(\frac{\partial\boldsymbol{\xi}}{\partial t}\right)^2-[(\mathbf{v}\cdot\nabla)\boldsymbol{\xi}]^2-3\Omega^2(\hat{\mathbf{i}}\cdot\boldsymbol{\xi})^2\right.$$
$$\left.+\frac{p}{\varrho}[(\Gamma-1)(\nabla\cdot\boldsymbol{\xi})^2+\nabla\boldsymbol{\xi}:\nabla\boldsymbol{\xi}]-2(\Omega\times\boldsymbol{\xi})\cdot(\mathbf{v}\cdot\nabla)\boldsymbol{\xi}\right\}. \tag{5.5}$$

The above expressions for $\mathscr{L}_2$ and $\mathscr{E}_2$ apply to a three-dimensional gas of arbitrary polytropic index Γ. In the two-dimensional limit, we replace ϱ by the surface density Σ and p by the two-dimensional pressure P. In terms of the two-dimensional $\mathscr{E}_2$, the total energy E_2, associated with a linear mode of the planet disc is given by

$$E_2=\int\int dx\,dy\,\mathscr{E}_2, \tag{5.6}$$

where the domain of integration is the interior of the unperturbed disc. It follows from the Euler–Lagrange equations that $\dot{E}_2=0$.

We shall now discuss the simplest non-trivial case: the $d=2$ perturbations of the $N=0$ planet disc. Now, as $N\to 0$, $\nabla\cdot\boldsymbol{\xi}\propto\Gamma^{-1}\to 0$, so that the terms proportional to $(\Gamma-1)(\nabla\cdot\boldsymbol{\xi})^2\to 0$. These terms can thus be neglected in this limit. We saw in Section 4 that all of the modes of the $N=0$ system are stable. Therefore, although we take the time-dependence of all linear perturbation variables to be $\exp(-i\omega t)$ [instead of $\exp(st)$ as in Section 4], it is both possible and convenient to end up with real coefficients. This is accomplished by the judicious insertion of the factor i in the equations below.

For $d=2$, we can write equation (4.12) as

$$\phi=\frac{\tilde{D}\Omega^2}{2}[(x^2-y^2)+i2Rxy], \tag{5.7}$$

where $\tilde{D}\equiv D\exp(-i\omega t)$ with D real. Following the procedure outlined in equations (4.7)–(4.13), we obtain the dispersion relation

$$A\left(\frac{\omega}{\Omega}\right)^4+B\left(\frac{\omega}{\Omega}\right)^2+C=0. \tag{5.8}$$

where

$$A=1+\varepsilon^2$$

$$B=-4(1+\varepsilon^2)(1+\alpha^2)-[4\varepsilon^2+(1+\varepsilon^2)^2]\gamma^2+\frac{2(1-\varepsilon^2)^2\alpha}{\varepsilon}$$

$$C=4\left\{(1+\varepsilon^2)[4\alpha^2+\varepsilon^2\gamma^4]-2\,\frac{(1-\varepsilon^2)^2\alpha^3}{\varepsilon}-[4\varepsilon(1+\varepsilon^2)\alpha-(1-\varepsilon^2)^2\alpha^2]\gamma^2\right\}. \tag{5.9}$$

Solving for the eigenmode, we find

$$R=2\frac{\Omega}{\omega}\,\frac{[2\varepsilon\alpha\gamma^2\Omega^2-4\alpha^2\Omega^2+\omega^2]}{[(1+\varepsilon^2)\gamma^2\Omega^2+4\alpha^2\Omega^2-\omega^2]}. \tag{5.10}$$

Writing the displacements as

$$\xi_x=ic_1x+c_2y$$

$$\xi_y=d_1x+id_2y, \tag{5.11}$$

we have from equations (5.3) and (4.7) that

$$c_1=\tilde{D}\Omega\left[\frac{(1+\varepsilon^2)\alpha\Omega R-\varepsilon\omega}{\varepsilon(4\alpha^2\Omega^2-\omega^2)}\right]$$

$$c_2=\Omega\left(\frac{2\varepsilon\alpha c_1-\tilde{D}R}{\omega}\right)$$

$$d_1=\Omega\left(\frac{2\alpha c_1-\varepsilon\tilde{D}R}{\varepsilon\omega}\right)$$

$$d_2=-c_1. \tag{5.12}$$

Substituting equations (5.5) and (5.11) into equations (5.6), we find for the total energy of the mode

$$E_2=\frac{\pi\varepsilon}{16}\,\varrho d^4\Omega^2\Bigg[\Bigg\{\left[\left(\frac{\omega}{\Omega}\right)^2-\alpha^2\right][\varepsilon^2(c_1^2+d_1^2)+(c_2^2+d_2^2)]-3(\varepsilon^2c_1^2+c_2^2)$$
$$+\varepsilon^2\gamma^2(c_1^2+d_2^2+2c_2d_1)+4\alpha\varepsilon(c_1d_2-c_2d_1\Bigg\}$$
$$+\Bigg\{\left[\left(\frac{\omega}{\Omega}\right)^2+\alpha^2\right][\varepsilon^2(c_1^2-d_1^2)-(c_2^2-d_2^2)]+3(\varepsilon^2c_1^2-c_2^2)$$
$$-\varepsilon^2\gamma^2(c_1^2+d_2^2-2c_2d_1)-4\alpha\varepsilon(c_1d_2+c_2d_1)\Bigg\}\cos(2\omega t)\Bigg]. \tag{5.13}$$

We have solved equation (5.8) and evaluated equations (5.13) numerically. The coefficient of $\cos(2\omega t)$ vanishes in every case, as it should. For a given choice of ε, there are two solutions for ω^2: the larger corresponds to a 'fast', and the smaller to a 'slow' mode. Throughout the entire range $0<\varepsilon<0.5$ for which the planet discs exist, the energy of the fast mode is positive, but the energy of the slow mode is *negative*. Thus, it would be possible to accommodate the negative energy difference between the torus and the planet by exciting an appropriate amplitude of the negative energy mode. At $q=2$ and $\beta=0.4$, which corresponds to a typical simulation by Hawley (1987), the displacements must be (suppressing the time dependence)

$$\xi_x=(-i0.2909x+0.0841y),$$

$$\xi_y=(-1.4705x+i0.2909y). \tag{5.14}$$

The frequency of the corresponding slow mode is 0.6474Ω. The displacements are not very small, but neither are they large: max $(|\xi_x|/b, |\xi_y|/(\varepsilon^{-1}b)\approx 0.458<1$, so it is reasonable to suppose that linear theory is approximately correct.

As discussed in Section 3, a second obstacle preventing the general narrow torus from evolving into a planet is that the distribution of specific vorticity differs between the two configurations, unless $q=2$ or $N=0$. We shall now argue, however, that small but otherwise arbitrary variations in the vorticity distribution of the planet can be made by combining vortical neutral modes. Of course, we cannot justify the use of linear modes unless the variation is small, but we can suppose that they have non-linear extensions that would lead some finite distance from the original planet.

It is easy to see, by counting, that many of the modes calculated from equations (4.1)–(4.6) must be vortical. The simplest case is $q=2$ or $\varepsilon=0.3117$, where by equation (3.2) there is no vorticity in the equilibrium configuration. If we restrict our attention to non-vortical modes, then $\nabla_2\times\mathbf{v}_2'=0$, and this implies that the modes are describable by a velocity potential, as in the first part of equation (4.7). If the fluid is compressible, ϕ obeys

$$\frac{d^2\phi}{dt^2}-\frac{c^2}{\Sigma}\nabla\cdot(\Sigma\nabla\phi)=0, \tag{5.15}$$

where $c^2=Q_2/N$ is the square of the sound speed. This equation is again solvable by polynomials in x and y. The total number of modes of degree d governed by equation (5.15) is $2d+2$, because there are $d+1$ monomials of this degree, and the eigenfrequency enters the equation quadratically. The total number of modes of degree d, of the planet is, however, $3d+1$, as determined by equation (4.6). (We are assuming that the modes are complete.) Therefore the remaining $d-1$ modes must be vortical, i.e. they must involve a change in the initial vorticity distribution. Now, consider an initial disturbance of the planet that corresponds to Q_2 being described by an arbitrary polynomial of degree d, and u and v being described by polynomials of degree $d-1$. We need $3d+1$ modes to describe this, and since the initial perturbed vorticity is of degree $d-2$, we need $d-1$ of these modes to be vortical. Thus, there are exactly as many vortical and non-vortical modes at each degree in a $q=2$ planet disc as are needed to fit an arbitrary initial condition. The same numbers of vortical and non-vortical modes exist in the case of a general $q<2$ as well, but classifying the modes is more difficult, so we shall not discuss this case here. Note that, since specific vorticity must be conserved in two-dimensional flow, all the vortical modes must be neutral. Thus, the unstable modes are constrained to be non-vortical.

Most vortical perturbations of the planet will lead to time-dependent solutions. But it was shown in Section 4 that there exists at least one zero-frequency mode for every even degree. These zero-frequency modes, which are vortical for all $d\geqslant 2$, might point to neighbouring equilibria. We conjecture that there exists an infinite number of equilibrium sequences bifurcating from the planet discs! If so, then the planets are merely the simplest examples of an enormous class of 'lumpy' equilibria, some of which may well be more stable than the planets themselves.

6 Discussion

An interesting and obvious question raised by Hawley's (1987) work and our results is: what will the fluid do next, after the formation of planets? There are several possibilities:

(i) Perhaps the linear overstabilities of the planets will saturate at finite amplitudes, a somewhat unlikely possibility in view of more recent numerical work by Hawley (private communication), which show the instabilities growing to rather large amplitudes.

(ii) Perhaps one of the other blob-like equilibria, for which we found evidence above, is more stable than the planet, and perhaps the fluid is tending towards such a configuration.

(iii) Perhaps the overstabilities will continue to grow until shocks develop in the fluid and dissipation becomes important. This is likely in view of the rather steep density gradients found by Hawley (1987) at late times in his numerical evolutions.

(iv) Perhaps the planet will fission into smaller blobs, and these blobs will further subdivide, and so on until the fluid dissolves into a spray of droplets separated by near-vacuum.

(v) A most interesting possibility is that the fluid will evolve into fully developed turbulence. In two-dimensional turbulence, energy cascades to larger scales and vorticity to smaller scales (e.g. Hasegawa 1985). In our problem, this would mean that the m blobs would merge into a single blob that would represent most of the ordered large-scale motion. (The macroscopic structure could take the form of a single non-axisymmetric planet, or could be an axisymmetric torus with lower shear than the initial torus, i.e. a lower effective value of q.) Superposed on the large-scale structure would be very fine-scale velocity fluctuations, which would represent the vorticity fluctuations. The merger of eddies has been seen by Marcus (1986) in numerical two-dimensional simulations of the Jovian atmosphere and the Red Spot, and something similar could happen in the torus as well, so long as the motion remains two-dimensional. Ever since the work of Shakura & Sunyaev (1973), turbulence has been a prime candidate for the source of viscosity in accretion discs. If it could be demonstrated that the PP instability leads to turbulence, it would clearly be an important step forward in understanding viscosity in discs, at least of the thick variety.

A serious concern is that results obtained for the two-dimensional system may not be very relevant to three-dimensional tori. We make two remarks. First, we have shown in Paper I that the height-averaged equations very accurately reproduce the linear growth rates of the three-dimensional torus. Also, Hawley has found in his two-dimensional simulations that the linear growth continues until the blobs are well formed. Moreover, the planet equilibrium itself satisfies vertical hydrostatic equilibrium exactly, and is therefore essentially two-dimensional. Secondly, accurate three-dimensional simulations may not be available for some time, as Hawley has found the details of the evolution to be sensitive to the resolution of the grid, to the location of the boundaries, and to the choice of difference scheme. Therefore it may be wise to try to understand the two-dimensional case thoroughly, where we have the best 'experimental' results to guide us, before turning to three dimensions.

Because of the limitation to two dimensions, the astronomical relevance of the results so far may be somewhat limited. For the reasons cited above, we feel that a two-dimensional theory is adequate to describe the behaviour of a *slender* torus, certainly in the linear regime, and possibly also in the non-linear regime. However, real astronomical tori are likely to be *wide*, and the case for two dimensions is much less clear. We would guess that the two-dimensional results provide a reasonable guide to the qualitative features of violent instabilities even in wide tori. If that is the case, then the results presented by Hawley (1987) and in this paper may already have some observational applications. For instance, the presence of accretion in real tori implies that each fluid element spends only a finite time in the torus. It could happen in certain situations that the fluid has just enough time to go through the first stage of planet formation before being swallowed up by the central mass. The presence of the planets would then be reflected in some way in the light curves of these systems. We thus have the interesting possibility that some of the variability observed in accretion tori, in particular the quasi-periodic oscillations seen in cataclysmic variables and accreting neutron stars, may be due to the presence of planet-like structures arising from the action of the Papaloizou–Pringle instability.

Acknowledgments

We thank John Hawley for sharing with us his exciting numerical results well before publication, and James Binney, Roger Blandford, Sterl Phinney, and Scott Tremaine for valuable discussions. JG was supported by a W. M. Keck Foundation grant and by NSF grant PHY8217352, RN by NSF grant AST-8611121 and PG was supported by NSF grant AST83-13725.

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