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Unsteady Effects in Flow Rate Measurement at the Entrance of a Pipe

A. J. Acosta[1]

Introduction: Unsteady flow in pipes and nozzles occur frequently in engineering applications and they pose special problems of measurement and calibration. When the Reynolds number is high the entrance region of a pipe (following a smooth contraction) is characterized by a thin boundary layer and the unsteady effects are then bound up in the unsteady behavior of the boundary layer. Noblesse and Farrell [1][2] have recently considered unsteady effects in laminar pipe entrance flows that start from rest by an integral method. Periodic disturbances also arise which require a different treatment. The primary interest of the present work is for thin entrance boundary layers subject to peridodic disturbances. In either case the ratio of the average velocity to the velocity in the potential core is

$$V_{avg}/V_{core} = 1 - 2\delta^*/R \qquad (1)$$

where δ^* is the usual displacement thickness and R is the pipe radius. In steady flow this ratio is just the "discharge coefficient", c_d. In unsteady flow it is very desirable to know how this ratio changes with time because many of the presently available experimental methods enable one to measure V_{core} but not V_{avg} readily. In this brief note we will estimate the unsteady effects of a periodic, fluctuating main flow on the displacement thickness of a laminar, flat plate boundary layer. It is assumed that the boundary layer is sufficiently thin compared to the radius of a pipe so that the pressure gradient caused by this effect in a pipe can be neglected; the results should then be directly applicable to equation (1).

The problem then becomes one of determining the effect on δ^* of a core or main stream velocity given by

$$V_{core} \equiv U_0(1 + \epsilon e^{j\omega t}) \qquad (2)$$

where ϵ is a small parameter, ω is the angular frequency of excitation, U_0 is a constant reference velocity, and j is the imaginary time factor. Recently Miller and Han [2] have analyzed the flat plate boundary layer equations with the mainstream oscillations given by equation (2) by an integral technique. They present numerical results for the frequency parameter $\omega X/U_0$ running from zero to about 2.5. Previously Lighthill [3] and Gibellato [4] have considered the same problem: Lighthill provided both low frequency and high frequency approximations for shear stress and displacement thickness with Pohlhausen boundary layer profiles and Gibellato a two-term low frequency approximation but "exact" solution of the boundary layer equations.

Approximate Solution: It would be useful to have simple if less exact expressions for wall shear stress and displacement thickness as continuous functions of the frequency parameter $\omega X/U_0$ for the present purpose. Such an estimate is readily provided by replacing the non-linear boundary layer equations by the linearized equation

$$\frac{\partial u}{\partial t} + \beta U_0(1 + \epsilon e^{j\omega t}) \frac{\partial u}{\partial X} = -\frac{1}{\rho}\frac{\partial p}{\partial X} + \nu \frac{\partial^2 u}{\partial Y^2} \qquad (3)$$

as was done by Langhaar (5) in his analysis of the steady but developing flow in a tube. Here Y is the distance from the wall X along the wall and β is a constant to be chosen. The velocity function is separated into a steady and an unsteady part, i.e.,

$$u(X, Y, t) = U_0 f_s(X, Y) + \epsilon U_0 e^{j\omega t} f(X, Y).$$

The substitutions

$$YU_0/\nu = y \quad , \quad XU_0/\nu = x\beta$$

and elimination of the pressure term with the equation of motion result in

$$\frac{\partial f_s}{\partial x} = \frac{\partial^2 f_s}{\partial y^2} \qquad (4a)$$

and

$$j\frac{\omega\nu}{U_0^2}(f - 1) + \frac{\partial f_s}{\partial x} + \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y^2} \qquad (4b)$$

with both f_s and f satisfying the conditions

$$f(x, 0) = 0$$

$$f(x, \infty) = 1$$

$$f(0, y) = 1 \quad , \quad y \neq 0.$$

The solution of equation $4(a)$ is well-known; it is

$$f_s = \text{erf}(\eta) \qquad (5a)$$

where $\eta = y/\sqrt{4x}$.

The solution to equation $4(b)$ is obtained by use of the Laplace transform. After some manipulation and with the help of reference [6] we find that

$$f = 1 - \frac{1}{2}[e^{2\eta\sqrt{jq}}(1 - \text{erf}(\eta + \sqrt{jq})) + e^{-2\eta\sqrt{jq}}(1 - \text{erf}(\eta - \sqrt{jq}))] - \frac{j}{q\sqrt{\pi}}\eta e^{-\eta^2}(1 - e^{-jq}) \qquad (5b)$$

where q is the frequency parameter

$$q \equiv \frac{\omega\nu}{U_0^2} x.$$

Choice of β: We arbitrarily choose the value of β to make the steady state skin friction coefficient, c_f, agree with the exact value. Thus,

$$c_f \equiv \mu \frac{\partial u}{\partial Y}(X, 0)/\frac{\rho}{2}U_0^2 = \frac{2}{\sqrt{\pi}}\sqrt{\beta} = 0.664$$

or $\beta = 0.346$. (This value is only a little less than the value of the steady state velocity ratio at the momentum thickness of the Blasius profile.)

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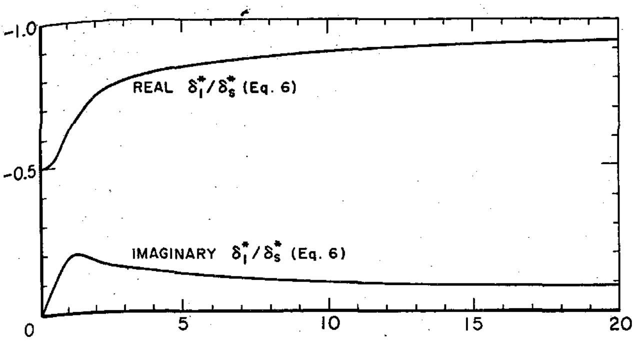

Fig. 1 Fluctuating displacement thickness function δ_1^* divided by the steady state displacement thickness σ_s^* on a flat plate boundary layer with the imposed free stream fluctuation $U_0\,(1+\epsilon e^{j\omega t})$ versus the frequency parameter $\omega\, X/U_0$

Two limiting cases are of interest. When $\omega \to 0$, we obtain the quasi-steady profile

$$f \to \text{erf}\,(\eta) + \frac{1}{\sqrt{\pi}}\,\eta e^{-\eta^2}\;(\omega = 0)$$

which is close to the profile of Miller and Han. The high frequency limit obtained as $q \to \infty$ becomes

$$f \to 1 - e^{-2\eta\sqrt{jq}} = 1 - e^{-y\sqrt{\frac{j\omega}{\nu}}}$$

which is the standard Stokes solution.

The Displacement Thickness: The displacement thickness δ^* is computed from the formula

$$\delta^* = \int_0^\infty dy\left(1 - \frac{u}{U_0(1+\epsilon e^{j\omega t})}\right) = \int_0^\infty (1-f_s)dy + \epsilon e^{j\omega t}\int_0^\infty (f_s - f)dy$$

of which the first term is the steady state term, δ_s^*. The *fluctuating* displacement thickness is given by the second integral and if we put

$$\delta^* = \delta_s^* + \epsilon e^{j\omega t}\delta_1^*$$

we find that the ratio δ_1^*/δ_s^* can be expressed in the closed form

$$\frac{\delta_1^*}{\delta_s^*} = -1 - \frac{\sin q}{q} + \sqrt{\pi}\,\frac{C_2(q)}{\sqrt{2q}} + j\left[\frac{1-\cos q}{2q} - \sqrt{\pi}\,\frac{S_2(q)}{\sqrt{2q}}\right] \quad (6)$$

where C_2, S_2 are Fresnel integrals (see reference [6]).

Equation (6) is plotted in Fig. 1 from which it may be seen that both real and imaginary parts are negative[3] for all values of frequency parameter $\omega X/U_0$. But, as Lighthill observed, the main point is that the ratio, equation (6), does not exceed unity. This has the important practical result that the difference between the fractional values of the fluctuating average and fluctuating core velocities of equation (1) is proportional to the fluctuation amplitude ϵ multiplied by $2\delta^*/R$. *Fractional* errors in the measurement of *fluctuating average* velocity itself will therefore not exceed $2\delta^*/R$ and for well designed contractions this should not be more than a few percent. This can be an important reassurance to experimenters who rely on fluctuating core velocity measurements for the determination of average velocity measurements in unsteady flow. *The shear stress:* Equation 5(*b*) can be used to evaluate the shear stress at the wall. Following Miller and Han we express this as the ratio of the fluctuating value to the steady value, i.e.,

$$C_r = \frac{\partial f/\partial y|_{y=0}}{\partial f_s/\partial y|_{y=0}},$$

and this ratio too can be expressed in closed form after noting that error functions of complex numbers whose arguments are $\pm\pi/4$ are easily expressed in the C_2 and S_2 functions. The result is

$$C_r = \cos q + \frac{\sin q}{2q} + 2\sqrt{\frac{\pi q}{2}}\,S_2(q) + j\left\{2\sqrt{\frac{\pi q}{2}}\,C_2(q) - \frac{1-\cos q}{2q} - \sin q\right\} \quad (7)$$

The phase is always positive and is within a few degrees of the Miller and Han calculations; actually equation (7) agrees more closely with the results of Lighthill. Gibellato's two term series agrees well also although only up to $\omega X/U_0 \simeq 1$. Similarly the real part of equation (7) agrees well with the results of references [2, 3].

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[3]This result means that the fluctuating displacement thickness is in the third quadrant relative to the fluctuating free stream velocity.

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Calculation of Velocity Profiles in Drag-Reducing Flows

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A calculation procedure for predicting mean velocity profiles in drag-reducing flows is presented. The procedure is based upon the eddy diffusivity model of Cess and it requires only pressure drop, flow rate and geometry information. The predictions show excellent agreement with experimentally measured profiles in both Newtonian and drag-reducing flows.

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