

Non-abelian Wilson surfaces

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ABSTRACT: A definition of non-abelian genus zero open Wilson surfaces is proposed. The ambiguity in surface-ordering is compensated by the gauge transformations.

KEYWORDS: p-branes, Gauge Symmetry, Differential and Algebraic Geometry.

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1. Introduction

A higher dimensional generalization of the non-abelian Wilson line is not known. Only recently the notion of a connection on a non-abelian 1-gerbe was introduced in the work of Breen and Messing [1].

A motivation for defining the Non-abelian Wilson Surfaces comes from the string theory. NWS are relevant to six dimensional theories on the world volumes of coincident five branes [2].

The main problem in defining NWS is the lack of a natural order on a 2-dimensional surface. A naive guess for the NWS is

$$P \exp \left(\int_{\Sigma} B \right), \tag{1.1}$$

where B is a non-abelian 2-form. The choice of a surface-ordering P involves a time-slicing of the 2-surface Σ . A no-go theorem of Teitelboim [3] states that no such a choice is compatible with the reparametrization invariance.

Let us recall the notion of a connection on a non-abelian 1-gerbe [1]. A connection on a principal bundle (0-gerbe) can be thought of as follows. Let x_0 and x_1 be two infinitesimally close points. The fibers S_{x_0} and S_{x_1} over these points are sets and the connection is a function

$$f_{01} : S_{x_1} \rightarrow S_{x_0}. \tag{1.2}$$

The connection on a non-abelian 1-gerbe is defined by analogy with the 0-gerbe case [1]. The fibers are categories C_{x_0} and C_{x_1} , and the connection is a functor

$$\varepsilon_{01} : C_{x_1} \rightarrow C_{x_0}. \tag{1.3}$$

Let x_0, x_1 and x_2 be three infinitesimally close points. A diagram of functors and natural transformations is shown in figure 1. Let $\text{Aut}(G)$ be the group of automorphisms of a non-abelian group G . Let $\text{Lie}(G)$ be the Lie algebra of G . It is shown in [1] that 2-arrow K , 1-arrow κ and 1-arrow ε in the diagram correspond to a $\text{Lie}(G)$ -valued 2-form B , a $\text{Lie}(\text{Aut}(G))$ -valued 2-form ν and a $\text{Lie}(\text{Aut}(G))$ -valued 1-form μ respectively.

The paper is organized as follows. In section 2 a definition of NWS is proposed. Section 3 is devoted to gauge transformations. Some comments are listed in section 4.

2. Definition

We interpret the infinitesimal 2-simplex in figure 1 as a transmuted form of an infinitesimal Wilson surface expressed in the language of category theory. The fibered category in the formulation of [1] can be thought of as an ‘internal symmetry space’ of a non-abelian string. Let Σ be a 2-dimensional surface with the disk topology. Let C be a clockwise oriented boundary of Σ and P a marked point on it (see figure 2). We associate group elements

$$W[\Sigma, C, P] \in G$$

and

$$V[\Sigma, C, P] \in \text{Aut}(G)$$

with the data (Σ, C, P) . We write $W[\Sigma]$ and $V[\Sigma]$ when the omitted arguments are obvious from the context. With an open curve C we associate an element of $\text{Aut}(G)$:

$$M[C] \in \text{Aut}(G). \quad (2.1)$$

Let $C = C_2 \circ C_1$ be a composition of curves C_2 and C_1 . We assume

$$M[C] = M[C_2 \circ C_1] = M[C_2]M[C_1]. \quad (2.2)$$

We now propose an equation relating $M[C]$, $W[\Sigma, C]$ and $V[\Sigma, C]$. For a group element $g \in G$ we denote by i_g the inner automorphism

$$i_g(h) = ghg^{-1}, \quad \forall h \in G. \quad (2.3)$$

The conjectural equation reads

$$M[C] = i_{W[\Sigma]}V[\Sigma]. \quad (2.4)$$

An infinitesimal version of this equation was first derived in [1] from the requirement that K in figure 1 is a natural transformation. We regard eq. (2.4) as a fundamental equation relating bulk and boundary of the non-abelian string world-sheet.

Eq. (2.4) can be used to find a composition rule for two NWS. Consider the 2-surfaces in figure 2. The identity

$$\begin{aligned} i_{W[\Sigma_2 \circ \Sigma_1, P_1]}V[\Sigma_2 \circ \Sigma_1] &= M[C \circ C_4 \circ C_3] \\ &= M[C]M[C_4 \circ C_5^{-1}]M[C^{-1}]M[C \circ C_5 \circ C_3] \\ &= M[C]i_{W[\Sigma_2, P_2]}V[\Sigma_2, P_2]M[C^{-1}]i_{W[\Sigma_1, P_1]}V[\Sigma_1, P_1] \end{aligned} \quad (2.5)$$

suggests the following composition rule for Wilson surfaces:

$$\begin{aligned} W[\Sigma_2 \circ \Sigma_1] &= M[C](W[\Sigma_2])M[C]V[\Sigma_2]M[C^{-1}](W[\Sigma_1]), \\ V[\Sigma_2 \circ \Sigma_1] &= M[C]V[\Sigma_2]M[C^{-1}]V[\Sigma_1]. \end{aligned} \quad (2.6)$$

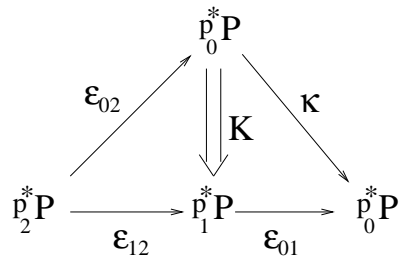


Figure 1: ε_{ij} is a cartesian functor from the fibered category p_j^*P to p_i^*P , κ is a cartesian functor from p_0^*P to p_0^*P , and K is a 2-arrow from $\kappa \circ \varepsilon_{02}$ to $\varepsilon_{01} \circ \varepsilon_{12}$.

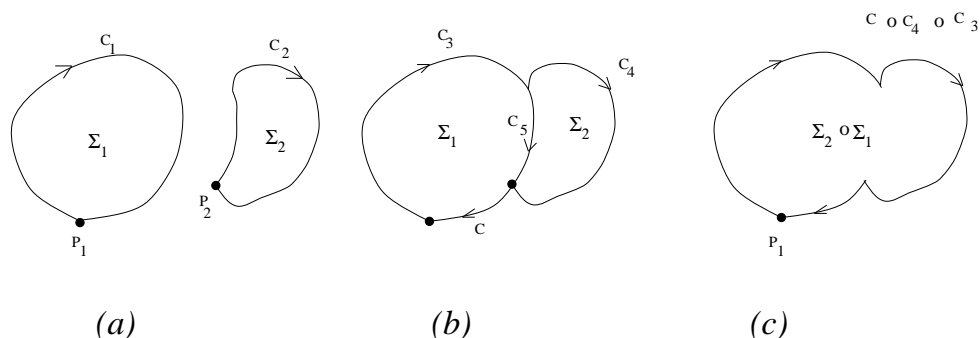


Figure 2: Composition of surfaces with the disk topology. (a) Surfaces Σ_i with the marked points P_i and the clockwise oriented boundaries C_i . (b) Surfaces are joined along the common boundary segment C_5 . (c) The resulting surface $\Sigma_2 \circ \Sigma_1$ with the marked point P_1 and the clockwise oriented boundary $C \circ C_4 \circ C_3$.

An infinitesimal version of eq. (2.6) appeared implicitly in the category-theoretic definition of the curvature in [1].

Eq. (2.6) can be understood as follows. When the curve C is absent, i.e. when the marked points of Σ_1 and Σ_2 coincide, eq. (2.6) simplifies to

$$\begin{aligned} W[\Sigma_2 \circ \Sigma_1] &= W[\Sigma_2]V[\Sigma_2](W[\Sigma_1]), \\ V[\Sigma_2 \circ \Sigma_1] &= V[\Sigma_2]V[\Sigma_1]. \end{aligned} \tag{2.7}$$

Thus when the marked points of the two surfaces coincide, the Wilson surfaces are composed as in eq. (2.7). If we think of $V[\Sigma, P]$ as an operator which acts on the objects with the marked point P and assume that only the objects with the same marked points can be multiplied, then the meaning of eq. (2.6) becomes clear. The role of $M[C]$ in eq. (2.6) is to transform the objects with the marked point P_2 to the objects with the marked point P_1 .

Composition of three or more surfaces is in general ambiguous. Consider figure 3. Using the composition rule (2.6) it can be shown that



Figure 3: $\Sigma_3 \circ (\Sigma_2 \circ \Sigma_1) \neq \Sigma_2 \circ (\Sigma_3 \circ \Sigma_1)$.

$$\begin{aligned} W[\Sigma_3 \circ (\Sigma_2 \circ \Sigma_1)] &\neq W[\Sigma_2 \circ (\Sigma_3 \circ \Sigma_1)], \\ V[\Sigma_3 \circ (\Sigma_2 \circ \Sigma_1)] &\neq V[\Sigma_2 \circ (\Sigma_3 \circ \Sigma_1)]. \end{aligned} \tag{2.8}$$

Given

$$V[\delta\Sigma] \approx 1 + v[P] \equiv 1 + v_{\mu\nu}[P]\sigma^{\mu\nu} \tag{2.9}$$

for an infinitesimal surface $\delta\Sigma$ with the area element $\sigma^{\mu\nu}$, we want to find $V[\Sigma]$ for a finite-size surface Σ . This can be done using a trick similar to the one used in the context of the non-abelian Stokes formula [4]. Consider the contour C' in figure 4. From the relation

$$M[C'] = M[C_P^{-1}]M[\delta C]M[C_P]M[C] \tag{2.10}$$

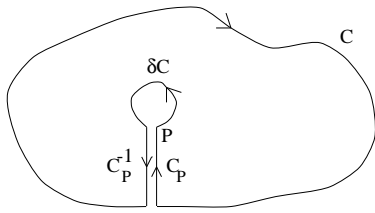


Figure 4: Contour $C' = C_P^{-1} \circ \delta C \circ C_P \circ C$.

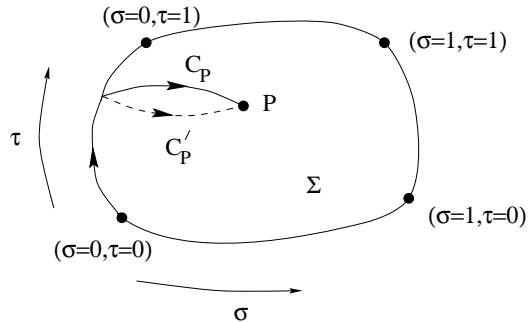


Figure 5: A parametrized surface Σ . The path C_P consists of two segments: the first segment ($\sigma = 0 = \text{const.}, \tau$) is from $\tau = 0$ to τ and the second segment ($\sigma, \tau = \text{const.}$) is from $\sigma = 0$ to σ .

and eq. (2.4) one finds

$$V[\Sigma'] = M[C_P^{-1}]V^{-1}[\delta\Sigma]M[C_P]V[\Sigma]. \quad (2.11)$$

Thus we have

$$\delta V[\Sigma] = M[C_P^{-1}]v[P]M[C_P]V[\Sigma]. \quad (2.12)$$

A solution of this equation involves a choice of ordering and it is given by

$$V[\Sigma] = \hat{P}_\tau \exp \left(\int_\Sigma M[C_P^{-1}]v[P]M[C_P] \right), \quad (2.13)$$

where \hat{P}_τ is the ordering in τ and the curve C_P is defined in figure 5. Note that the expression eq. (2.13) depends on the parametrization $x^\mu = x^\mu(\sigma, \tau)$ of the surface Σ . For example a boundary-preserving reparametrization will change C_P to a C'_P (see figure 5). Thus $V[\Sigma]$ and $W[\Sigma]$ depend on the parametrization of Σ :

$$V = V[\Sigma, x^\mu(\sigma, \tau)], \quad W = W[\Sigma, x^\mu(\sigma, \tau)]. \quad (2.14)$$

In section 3 we will see that if (σ, τ) and $(\tilde{\sigma}, \tilde{\tau})$ are two different parametrizations of a surface Σ , then

$$(V[\Sigma, x^\mu(\sigma, \tau)], W[\Sigma, x^\mu(\sigma, \tau)])$$

and

$$(V[\Sigma, x^\mu(\tilde{\sigma}, \tilde{\tau})], W[\Sigma, x^\mu(\tilde{\sigma}, \tilde{\tau})])$$

are related by the gauge transformation. In other words, the non-abelian internal symmetry and the reparametrization symmetry mix.

3. Gauge transformations

In this section we introduce the gauge transformations which compensate the ambiguity in the composition of NWS. Suppose that a surface Σ is composed out of three or more smaller

surfaces. Let $(W[\Sigma], V[\Sigma])$ and $(\tilde{W}[\Sigma], \tilde{V}[\Sigma])$ correspond to two different compositions resulting in the surface Σ . We have

$$M[C] = i_{W[\Sigma]}V[\Sigma] = i_{\tilde{W}[\Sigma]}\tilde{V}[\Sigma]. \quad (3.1)$$

Since W and \tilde{W} are elements of a group G , there is a group element $R[\Sigma] \in G$ such that

$$\tilde{W}[\Sigma] = W[\Sigma](R[\Sigma])^{-1}. \quad (3.2)$$

Let us decompose W and \tilde{W} into the abelian and non-abelian factors:

$$W = W_{\text{ab}} \cdot W_{\text{nonab}}, \quad \tilde{W} = \tilde{W}_{\text{ab}} \cdot \tilde{W}_{\text{nonab}}. \quad (3.3)$$

It is clear that the ambiguity in the composition does not affect the abelian part. Thus we have

$$\tilde{W}_{\text{ab}}[\Sigma] = W_{\text{ab}}[\Sigma]. \quad (3.4)$$

Combining this equation with eq. (3.2) we find

$$\tilde{W}_{\text{nonab}}[\Sigma] = W_{\text{nonab}}[\Sigma](R[\Sigma])^{-1}. \quad (3.5)$$

We propose that eq. (3.4) and eq. (3.5) define the gauge transformation of W . In order for this gauge transformation of W to be compatible with eq. (3.1), V should transform as

$$\tilde{V}[\Sigma] = i_{R[\Sigma]}V[\Sigma]. \quad (3.6)$$

It can be checked that the gauge transformations (3.4)–(3.6) are compatible with the composition rule (2.6) provided that the composition rule for R is the same as that of W , namely

$$R[\Sigma_2 \circ \Sigma_1] = M[C](R[\Sigma_2])M[C]V[\Sigma_2]M[C^{-1}](R[\Sigma_1]). \quad (3.7)$$

More generally, consider a surface Σ divided into n smaller surfaces $\Sigma_1, \dots, \Sigma_n$. Let C be the boundary of Σ . Repeating the reasoning leading to eq. (2.6) we have

$$M[C] = M[\mathcal{C}_1]i_{W[\Sigma_1]}V[\Sigma_1]M[\mathcal{C}_2]i_{W[\Sigma_2]}V[\Sigma_2]M[\mathcal{C}_3] \cdots \quad (3.8)$$

for some curves $\mathcal{C}_1, \mathcal{C}_2, \dots$. From this equation we find

$$\begin{aligned} W[\Sigma] &= M[\mathcal{C}_1](W[\Sigma_1])M[\mathcal{C}_1]V[\Sigma_1]M[\mathcal{C}_2](W[\Sigma_2]) \cdots, \\ V[\Sigma] &= M[\mathcal{C}_1]V[\Sigma_1]M[\mathcal{C}_2]V[\Sigma_2]M[\mathcal{C}_3] \cdots. \end{aligned} \quad (3.9)$$

It is easy to see that the gauge transformations (3.4)–(3.6) are compatible with eq. (3.9) provided that $R[\Sigma]$ is composed out of $R[\Sigma_i]$ as follows:

$$R[\Sigma] = M[\mathcal{C}_1](R[\Sigma_1])M[\mathcal{C}_1]V[\Sigma_1]M[\mathcal{C}_2](R[\Sigma_2]) \cdots. \quad (3.10)$$

Thus R should be composed by the rule of composition of W .

We now introduce new gauge transformations. These are the transformations of M , V and W compatible with eq. (2.4).

Let $\Lambda[P]$ be an $\text{Aut}(G)$ -valued function of point P . Let C be a directed path from P_1 to P_2 . The gauge transformation of $M[C]$ reads

$$\tilde{M}[C] = \Lambda[P_2]M[C]\Lambda[P_1]^{-1}. \quad (3.11)$$

When $P_1 = P_2 = P$ this equation becomes

$$\tilde{M}[C] = \Lambda[P]M[C]\Lambda[P]^{-1}. \quad (3.12)$$

From this equation and

$$\tilde{M}[C] = i_{\tilde{W}}\tilde{V} \quad (3.13)$$

one finds

$$i_W V = \Lambda^{-1}i_{\tilde{W}}\tilde{V}\Lambda = i_{\Lambda^{-1}(\tilde{W})}\Lambda^{-1}\tilde{V}\Lambda. \quad (3.14)$$

Thus we propose the gauge transformations:

$$\begin{aligned} \tilde{V}[\Sigma, P] &= \Lambda[P]V[\Sigma, P]\Lambda[P]^{-1}, \\ \tilde{W}[\Sigma, P] &= \Lambda[P](W[\Sigma, P]). \end{aligned} \quad (3.15)$$

We now consider a new gauge transformation which is a finite generalization of the infinitesimal transformation considered in [1]. The transformation reads

$$\tilde{M}[C] = i_{\mathcal{Z}[C]}M[C], \quad (3.16)$$

where $\mathcal{Z}[C]$ is a G -valued functional of C . The composition rule for \mathcal{Z} can be inferred from the following chain of equations:

$$\begin{aligned} i_{\mathcal{Z}[C_2 \circ C_1]}M[C_2 \circ C_1] &= \tilde{M}[C_2 \circ C_1] \\ &= \tilde{M}[C_2]\tilde{M}[C_1] \\ &= i_{\mathcal{Z}[C_2]}M[C_2]i_{\mathcal{Z}[C_1]}M[C_1] \\ &= i_{\mathcal{Z}[C_2]}i_{M[C_2](\mathcal{Z}[C_1])}M[C_2 \circ C_1]. \end{aligned} \quad (3.17)$$

This equation suggests the following composition rule for \mathcal{Z} :

$$\mathcal{Z}[C_2 \circ C_1] = \mathcal{Z}[C_2]M[C_2](\mathcal{Z}[C_1]). \quad (3.18)$$

If a $\text{Lie}(G)$ -valued 1-form ζ is given, $\mathcal{Z}[C]$ for an open path C can be constructed as follows. Let us divide C into n small subpaths as in figure 6a. Applying eq. (3.18) we find

$$\begin{aligned} \mathcal{Z}[C] &= \mathcal{Z}[C_n] \cdot M[C_n](\mathcal{Z}[C_{n-1}]) \cdot M[C_n \circ C_{n-1}](\mathcal{Z}[C_{n-2}]) \times \dots \\ &\quad \times M[C_n \circ C_{n-1} \dots C_2](\mathcal{Z}[C_1]) \\ &\approx (1 + \zeta_\mu[P_n]dx^\mu)(1 + M[C_n](\zeta_\mu[P_{n-1}]dx^\mu) \times \dots \\ &\quad \times (1 + M[C_n \circ C_{n-1} \dots C_2](\zeta_\mu[P_1]dx^\mu)). \end{aligned} \quad (3.19)$$

In the large n limit we thus find

$$\mathcal{Z}[C] = \hat{P} \exp \left(\int_C M[C''](\zeta_\mu[P])dx^\mu \right), \quad (3.20)$$

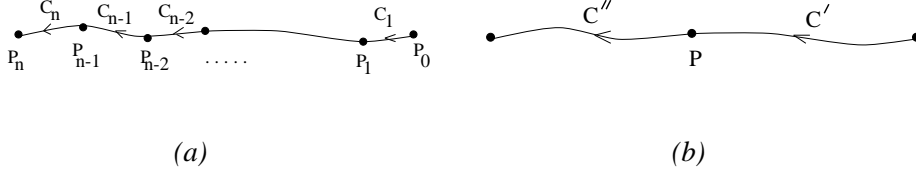


Figure 6: (a) The path C is divided into n small subpaths: $C = C_n \circ C_{n-1} \cdots \circ C_1$. (b) The point P divides $C = C'' \circ C'$.

where C'' and P are as in figure 6b, and \hat{P} is the path ordering operator.

A choice of transformation of V and W compatible with eq. (2.4) and eq. (3.16) is

$$\begin{aligned}\tilde{V}[\Sigma, C] &= V[\Sigma, C], \\ \tilde{W}[\Sigma, C] &= \mathcal{Z}[C]W[\Sigma, C].\end{aligned}\tag{3.21}$$

Infinitesimal versions of these transformations agree with the transformations that can be derived from [1]. Let us consider an infinitesimal surface $\delta\Sigma$ with the area element $\sigma^{\mu\nu}$. Assume that $M[C] \in \text{Aut}(G)$ is an inner automorphism given by

$$\begin{aligned}M[C](g) &= \hat{P}\exp\left(\int_C \mu\right) g \hat{P}\exp\left(-\int_C \mu\right) \\ &= \hat{P}\exp\left(\int_C \mu_{\text{adjoint}}\right)(g), \quad \forall g \in G,\end{aligned}\tag{3.22}$$

where μ is a $\text{Lie}(G)$ -valued 1-form. From eq. (3.21) and

$$W[\delta\Sigma] \approx 1 + B_{\mu\nu}\sigma^{\mu\nu},\tag{3.23}$$

one can find the transformation of the 2-form B :

$$\tilde{B} = B + d\zeta - \frac{1}{2}[\zeta, \zeta] - [\mu, \zeta].\tag{3.24}$$

The transformation of B corresponding to eqs.(3.4,3.5) reads

$$\tilde{B}_{\text{ab}} = B_{\text{ab}}, \quad \tilde{B}_{\text{nonab}} = B_{\text{nonab}} - \rho,\tag{3.25}$$

where ρ is a $\text{Lie}(G)$ -valued 2-form defined in

$$R[\delta\Sigma] \approx 1 + \rho_{\mu\nu}\sigma^{\mu\nu}.\tag{3.26}$$

Eq. (3.25) agrees with the transformations that can be derived from [1].

Unlike the gauge transformations (3.4)–(3.6), (3.15), the transformation (3.21) is not compatible with the composition rule (2.6). To find the correct transformation, $\mathcal{Z}[C]$ in eq. (3.21) should be ‘smeared’ over the surface Σ . We give an explicit formula for the gauge transformation of $V[\Sigma]$. It reads

$$\tilde{V}[\Sigma] = \hat{P}_\tau \exp\left(\int_\Sigma i_{\mathcal{Z}[C_P]} M[C_P] v[P] M[C_P^{-1}] i_{\mathcal{Z}[C_P]^{-1}}\right).\tag{3.27}$$

4. Comments

- We found three kinds of gauge transformations of M , V and W . These are $\Lambda[P]$ -transformations (3.11), (3.15), $R[\Sigma]$ -transformations (3.4)–(3.6) and $\mathcal{Z}[C]$ -transformations (3.16), (3.21). Eq. (3.21) is valid only for infinitesimal surfaces and should be replaced by a ‘smeared’ version eq. (3.27).
- The ambiguity in surface-ordering necessitates the introduction of gauge transformations which compensate the ambiguity. Locally this amounts to the transformation eq. (3.25). The number of gauge degrees of freedom present in a NWS is enormous. Thus NWS may be relevant to a topological string theory describing topological sectors of the non-abelian string of [2].
- Infinitesimal version of eq. (2.6) can be derived from the composition rule for the natural transformation K in figure 1.
- We defined NWS on a local trivial patch. To define NWS globally one should cover the manifold with an atlas $\{U_\alpha\}$ and introduce $W_\alpha, V_\alpha, M_\alpha$ for each patch U_α . As usual the quantities on the overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$ are related by the gauge transformations. An analysis of global issues will be carried out elsewhere.
- We defined NWS with the disk topology. A generalization to higher-genus surfaces will be discussed elsewhere.

Note added. After submitting the original version of this paper to hep-th, the work [5] was brought to our attention. In [5] an equation similar to eq. (2.13) was taken as a definition of Wilson surface. The case considered in [5] corresponds, in our notation, to the C -independent $M[C]$. The surface-ordering ambiguities are absent in this case. For a list of miscellaneous work on non-abelian 2-form theories, see [6].

Acknowledgments

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