EXCITATION OF SOLAR p -MODES

PETER GOLDREICH AND NORMAN MURRAY
 California Institute of Technology, Pasadena, CA 91125

AND

PAWAN KUMAR
 Massachusetts Institute of Technology, Cambridge, MA 02139

Received 1993 March 31; accepted 1993 September 16

ABSTRACT

We investigate the rates at which energy is supplied to individual p -modes as a function of their frequencies ν and angular degrees ℓ . The observationally determined rates are compared with those calculated on the hypothesis that the modes are stochastically excited by turbulent convection.

The observationally determined excitation rate is assumed to be equal to the product of the mode's energy E and its (radian) line width Γ . We obtain E from the mode's mean square surface velocity with the aid of its velocity eigenfunction. We assume that Γ measures the mode's energy decay rate, even though quasi-elastic scattering may dominate true absorption. At fixed ℓ , $E\Gamma$ rises as ν^7 at low ν , reaches a peak at $\nu \approx 3.5$ mHz, and then declines as $\nu^{-4.4}$ at higher ν . At fixed ν , $E\Gamma$ exhibits a slow decline with increasing ℓ .

To calculate energy input rates, $\mathcal{P}_x$, we rely on the mixing-length model of turbulent convection. We find entropy fluctuations to be about an order of magnitude more effective than the Reynolds stress in exciting p -modes. The calculated $\mathcal{P}_x$ mimic the ν^7 dependence of $E\Gamma$ at low ν and the $\nu^{-4.4}$ dependence at high ν . The break of 11.4 powers in the ν -dependence of $E\Gamma$ across its peak is attributed to a combination of (1) the reflection of high-frequency acoustic waves just below the photosphere where the scale height drops precipitously and (2) the absence of energy-bearing eddies with short enough correlation times to excite high-frequency modes. Two parameters associated with the eddy correlation time are required to match the location and shape of the break. The appropriate values of these parameters, while not unnatural, are poorly constrained by theory. The calculated $\mathcal{P}_x$ can also be made to fit the magnitude of $E\Gamma$ with a reasonable value for the eddy aspect ratio.

Our results suggest a possible explanation for the decline of mode energy with increasing ℓ at fixed ν . Entropy fluctuations couple to changes in volume associated with the oscillation mode. These decrease with decreasing n at fixed ν , becoming almost zero for the f -mode.

Subject headings: convection — Sun: interior — Sun: oscillations

1. INTRODUCTION

Our goal is to estimate the rate at which turbulent convection supplies energy to the solar p -modes. To do so, we generalize the calculations of mode excitation presented for simplified model atmospheres by Goldreich & Kumar (1990, hereafter GK). This earlier work successfully accounts for the excitation rates of low-frequency modes, but severely overestimates those of high-frequency modes. In the present paper we calculate excitation rates for the modes of solar models provided by Christensen-Dalsgaard (1982). These rates are then compared with those deduced from observations summarized by Libbrecht & Woodard (1991, hereafter LW). We refer the reader to independent attacks on the same problem by Osaki (1990) and Balmforth (1992b).

The plan of this paper is as follows. The observational determination of the rate at which energy is supplied to p -modes is discussed in § 2. Particular attention is given to the relation between a mode's line width and its energy damping rate. The line widths are also the entire subject of a companion paper Goldreich & Murray (1994, hereafter GM). In § 3 we derive an expression for the excitation rate due to turbulent convection. Numerical evaluations of excitation rates as functions of frequency ν and angular degree ℓ are carried out in § 4. A short discussion is provided in § 5. Throughout, we make frequent reference to the acoustic modes of plane-parallel atmospheres.

These atmospheres sit in a constant gravitational field and consist of two layers, the upper isothermal and the lower polytropic. The density and scale height ($p/\rho g$) are discontinuous across the boundary. The density ratio (isothermal over isentropic, ρ_i/ρ_d) at the boundary is denoted by ϵ . Figure 1 compares the run of scale height in a real solar model with that in several plane-parallel models having different values of ϵ . Details of these model atmospheres and their modes are relegated to the Appendix.

2. DIRECT DEDUCTIONS FROM OBSERVATIONAL DATA

2.1. Mode Energies

The first step in the determination of mode energies is the measurement of Doppler velocities on the solar disk. These velocities depend upon the height above the photosphere at which the relevant spectral line is formed. The surface velocities of individual modes are obtained from three-dimensional (two space dimensions plus time) power spectra of the data. Each mode's velocity thus determined is an rms value averaged over the length of the data string. These velocities are shown as a function of ν in Figure 2, taken from LW.

To deduce the time-averaged energy in a mode from its rms surface velocity requires knowledge of the mode's eigenfunction. Eigenfunctions are obtained by solving a linear wave equation in the background of a solar model. This step intro-

FIG. 1.—Pressure scale height, $H = p/\rho g$, vs. pressure. The dashed lines are from plane-parallel models with ϵ equal to (left to right) 1.0, 1.8, 2.6, and 3.4. The solid line is from the plane-parallel model with $\epsilon = 2.2$. The curve is obtained from Christensen-Dalsgaard's model.

duces uncertainty through the choice of solar model. It proves convenient to define a mode mass by

$$M_\alpha \equiv \frac{4\pi}{\omega_\alpha^2 \int d\Omega |\xi_\alpha(R_\odot, \theta, \phi)|^2} \quad (1)$$

FIG. 2.—Root mean square surface velocity per mode for low- ℓ p -modes. The data shown here represent an average for modes with $5 \leq \ell \leq 60$ scaled to give the amplitudes of $\ell = 0$ modes. The crosses at high frequency come from a separate analysis of $\ell = 60$ data (LW).

FIG. 3.—Crosses indicate mode masses as defined in eq. (1), computed using the solar model of Christensen-Dalsgaard (1982). The lines show mode masses for plane-parallel atmospheres with different values of the density ratio ϵ . The solid line is for $\epsilon = 2.2$.

for modes normalized such that

$$\omega_\alpha^2 \int d^3x \rho |\xi_\alpha(x)|^2 = 1. \quad (2)$$

The mode mass is the coefficient of proportionality between the energy and the mean square surface velocity.¹ Mode masses for $\ell = 0$ modes computed from the solar model of Christensen-Dalsgaard (1982) are shown in Figure 3.

Standard solar models fit the observationally determined mode frequencies to better than 1%, so one might assume that they yield accurate mode masses. However, the masses are sensitive to the poorly determined structure of the superadiabatic upper layers of the convection zone. This is particularly true for the high-frequency modes because they are propagating at the top of the convection zone. The steeper the entropy gradient is at the top of the convection zone, the more abrupt is the reflection of waves incident upon it from below.² The more abrupt the reflection is, the smaller is the surface mode amplitude, and thus the larger the mode mass becomes.

To illustrate the dependence of mode mass on wave reflection at the top of the convection zone, we compute mode masses (more precisely, masses per unit surface area multiplied by $4\pi R_\odot^2$) for the sequence of plane-parallel atmospheres referred to in § 1 and described in detail in the Appendix. The eigenfunctions are evaluated at the same pressure level in the isothermal layer for all models. Runs of mode mass as a function of ω for different values of ϵ are displayed in Figure 3. The masses for the plane-parallel atmosphere with $\epsilon = 2.2$ provide a good fit to those of the solar model.

The average mode energies, computed by multiplying the mean square surface velocities from Figure 2 by the mode

¹ The mode mass depends on the precise location of the “surface.”

² All modes suffer complete reflection below the temperature minimum.

FIG. 4.—Average mode energy given by $E = Mv^2$, with M from Fig. 3 and v from Fig. 2 (LW).

masses from Christensen-Dalsgaard's model (cf. Fig. 3), are depicted in Figure 4 (LW). The most highly excited modes have energies of order 10^{28} ergs.

Solar granules have lifetimes that are comparable to the periods of these modes, and their kinetic energies are similar to the modes' energies. This suggests that turbulent convection may be responsible for both the excitation and the damping of the modes (Goldreich & Keeley 1977; Christensen-Dalsgaard & Frandsen 1983b; Goldreich & Kumar 1988; Goldreich & Kumar 1990; Kumar & Goldreich 1989; Balmforth 1992b). Under this hypothesis, mode energies are set by the ratio of the

rates of stochastic excitation and turbulent damping. However, there are two problems with this hypothesis. First, the mode energies should decrease monotonically with increasing frequency; the higher a mode's frequency, the smaller and less energetic the eddies with which it interacts most strongly. That this expectation is not met by the observationally determined mode energies is obvious from Figure 4. Indeed, the mode energies actually increase with increasing frequency below the peak at $\nu \approx 3$ mHz. Second, at fixed ν the mode energies should be independent of n (or ℓ), whereas the energy per mode appears to decline with decreasing n as illustrated in Fig. 5 (Rhodes, Cacciani, & Korzennik 1991). Note that the f -modes ($n = 0$) have the lowest energy of all. (One must be cautious in interpreting Fig. 5; "seeing" reduces the amplitudes of high- ℓ modes relative to those of low- ℓ modes. The energies in Figure 5 have been corrected for seeing. Since the seeing correction is uncertain, it is not impossible that the n -dependence is due entirely to seeing.) The current paper goes some way toward resolving these problems. Another piece of the puzzle is presented in GM.

2.2. Mode Lifetimes

The frequency widths of the peaks in the power spectra, $\Delta\nu$, if greater than the inverse of the observation time, contain information on damping rates of the modes.³ The mode line widths are found to increase with both frequency ν and angular degree ℓ . A line width in excess of the inverse observation time could arise from variations in either the frequency or the amplitude of the mode's surface velocity. Although it is possible to invent scenarios in which the mode velocity is either purely frequency-modulated or purely amplitude-modulated, it seems most plausible that the two types of modulation make comparable contributions to the line width. That is the expectation for stochastically excited modes. The observational evidence bearing on this point is meager. However, a recent study,

³ $\Delta\nu$ is defined as the full width at half-maximum.

FIG. 5.—Energy per mode as a function of ν for different radial orders n . From Rhodes, Cacciani, & Korzennik (1991)

FIG. 6.—Line width as a function of ν for low- ℓ solar p -modes extrapolated to $\ell = 0$. The lowest frequency line widths, below about 1.5 mHz, are set by the length of the observations (LW).

based on 6 months of data obtained by the *Phobos* spacecraft, reports that the mode amplitudes vary as would be expected for stochastic excitation (Toutain & Frohlich 1992).

Line widths are shown as a function of ν in Figure 6 (LW). At fixed ℓ the line widths increase with increasing ν . At fixed ν they increase with increasing ℓ in rough proportion to the inverse mode mass. Presumably both trends arise because the non-adiabatic processes are concentrated close to the solar surface.

Nonadiabatic p -mode eigenfrequencies have been calculated by, among others, Ando & Osaki (1977), Goldreich & Keeley (1977a), Antia, Chitre, & Narasimha (1986), Christensen-Dalsgaard & Frandsen (1983a), Kidman & Cox (1984), Christensen-Dalsgaard, Gough, & Libbrecht (1989), and Balmforth (1992a). All of these calculations yield similar magnitudes for the imaginary parts of the eigenfrequencies, ω_i , even those carried out prior to the measurements of the line widths. However, there is less agreement as to the sign of ω_i , so the linear overstability of the modes remains an open theoretical issue.⁴ What is striking is that the calculated magnitudes of ω_i fall well below the measured values at low ω_r . This hints that there is an unmodeled process which dominates the line widths at low frequencies (Goldreich & Kumar 1991). Scattering is a plausible candidate (Balmforth 1992a; GM).

Turbulent velocity fluctuations scatter acoustic waves and thereby transfer energy among modes of similar ν and different ℓ .⁵ Scattering definitely contributes to the line widths, but its role as a source of damping is less apparent. Elastic scattering tends to equalize the energies of the modes it couples; it does not act as a source of damping if it couples modes of equal energy. However, because the mode energies decrease with decreasing n at fixed ν (Kaufman 1990; Rhodes et al. 1991;

Fernandes et al. 1992; Willette 1993), scattering damps p -modes by transferring their energy to modes of similar ν but smaller n .

2.3. $E\Gamma$

On the assumption that scattering represents true damping, the radian line width, $\Gamma \equiv 2\pi\Delta\nu$, is the energy decay rate (Kumar, Franklin, & Goldreich 1988). The power going into each mode is given by $E\Gamma$. By modeling this quantity, we test the hypothesis of stochastic excitation while avoiding the more difficult issue of mode damping. A plot of $E\Gamma(\nu)$ is displayed in Figure 7. It is formed by multiplying the values of E given in Figure 4 by $\Gamma = 2\pi\Delta\nu$ obtained from Figure 6. Note that $E\Gamma \propto \omega^7$ at low frequency and $E\Gamma \propto \omega^{-4.4}$ at high ω . Libbrecht (1988) was the first to draw attention to the remarkable power-law behavior at low ω .

3. STOCHASTIC EXCITATION

In this section we set up and solve the inhomogeneous wave equation that governs the stochastic excitation of p -modes by turbulent convection.

3.1. Wave Equation with Source Terms

Our limited ability to model the dynamics of turbulent convection precludes a rigorous derivation of the acoustic emissivity. The best we can provide is a heuristic derivation of an inhomogeneous wave equation whose source terms account for both the expansion and the contraction of fluid elements due to entropy fluctuations and the redistribution of momentum by the Reynolds stress. The derivation provided here generalizes that given in GK.⁶

⁶ We retain the nonessential simplification of neglecting the gravitational field perturbation (Cowling 1941).

FIG. 7.—Rate $E\Gamma$ at which energy is supplied to solar p -modes obtained by multiplying E from Fig. 4 by $\Gamma = 2\pi\Delta\nu$ from Fig. 6. A least-squares fit to the data gives $E\Gamma \propto \omega^7$ below 3.0 mHz and $E\Gamma \propto \omega^{-4.4}$ above 3.8 mHz.

⁴ Other arguments suggest that the modes are stable (Kumar & Goldreich 1989).

⁵ Scattering by subsonic turbulence is nearly elastic.

We start from linearized versions of the equations for mass and momentum conservation. The displacement vector is written as ξ , and Eulerian perturbations of the thermodynamic variables are denoted by a subscript 1. The momentum equation is augmented by the turbulent Reynolds stress. Thus

$$\rho_1 + \nabla \cdot (\rho \xi) = 0, \quad (3)$$

and

$$\rho \frac{\partial^2 \xi}{\partial t^2} + \nabla p_1 - \rho_1 g = -\nabla \cdot (\rho v v) \equiv F. \quad (4)$$

We write the equation of state, $p = p(\rho, s)$, in linearized form as

$$p_1 = \frac{\partial p}{\partial \rho} \rho_1 + \frac{\partial p}{\partial s} s_1, \quad (5)$$

where

$$s_1 = \tilde{s} - (\xi \cdot \nabla) s. \quad (6)$$

Here ∇s denotes the background entropy gradient, and $\tilde{s}$ is the entropy fluctuation associated with turbulent convection. Equation (6) is the Eulerian version of the statement that the Lagrangian entropy perturbation is due entirely to turbulent convection. In other words, we approximate the waves as adiabatic. For later use we note that

$$\frac{\partial p}{\partial s} \nabla s = \nabla p - \frac{\partial p}{\partial \rho} \nabla \rho. \quad (7)$$

The inhomogeneous wave equation,

$$\begin{aligned} \rho \frac{\partial^2 \xi}{\partial t^2} - \nabla [c^2 \nabla \cdot (\rho \xi) + \rho \xi \cdot g - c^2 \rho \xi \cdot \nabla \ln \rho] + g \nabla \cdot (\rho \xi) \\ = -\nabla \left(\frac{\partial p}{\partial s} \tilde{s} \right) - \nabla \cdot (\rho v v) = S, \end{aligned} \quad (8)$$

is obtained by combining equations (3)–(7). As advertized, the source terms on the right-hand side of equation (8) arise from the entropy fluctuations and the Reynolds stress. The normal modes, the eigenfunctions of the linear differential operator on the left-hand side of equation (8), satisfy

$$\begin{aligned} \int d^3x \rho [\omega_\alpha^2 \xi_\alpha \xi_\beta^* - c^2 \nabla \cdot \xi_\alpha \nabla \cdot \xi_\beta^* - (g \cdot \xi_\alpha \nabla \cdot \xi_\beta^* \\ + \nabla \cdot \xi_\alpha g \cdot \xi_\beta^*) - \chi(x) g \cdot \xi_\alpha g \cdot \xi_\beta^*] = 0. \end{aligned} \quad (9)$$

The derivation of equation (9) requires use of the relation $g \times \nabla \rho = 0$, obtained by taking the curl of the equation of hydrostatic equilibrium. This relation allows us to define a scalar function $\chi(v)$ such that $\chi(v)g = \nabla(\ln \rho)$. Setting $\beta = \alpha$ in equation (9) shows that ω_α^2 is real. It follows that

$$(\omega_\alpha^2 - \omega_\beta^2) \int d^3x \rho \xi_\alpha \cdot \xi_\beta^* = 0, \quad (10)$$

which establishes the orthogonality of the modes.

3.2. The Amplitude Equation

We expand the displacement, $\xi(x, t)$, in terms of the normal modes, $\xi_\alpha(x)$, as

$$\xi(x, t) = \frac{1}{\sqrt{2}} \sum_\alpha [A_\alpha \xi_\alpha \exp(-i\omega_\alpha t) + A_\alpha^* \xi_\alpha^* \exp(i\omega_\alpha t)]. \quad (11)$$

The mode energy, E_α , is related to the mode amplitude, A_α , by $E_\alpha = |A_\alpha|^2$. To evaluate the time evolution of the amplitudes, we substitute the eigenfunction expansion, equation (11), in the inhomogeneous wave equation (8). Making use of the orthonormality of the eigenfunctions and the approximation $|\partial A_\alpha / \partial t| \ll \omega_\alpha |A_\alpha|$, we arrive at

$$\begin{aligned} \frac{\partial A_\alpha}{\partial t} = \frac{i\omega_\alpha e^{i\omega_\alpha t}}{\sqrt{2}} \int d^3x S \cdot \xi_\alpha^* \\ = \frac{i\omega_\alpha e^{i\omega_\alpha t}}{\sqrt{2}} \int d^3x \left(\frac{\partial p}{\partial s} \tilde{s} \nabla \cdot \xi_\alpha^* + \rho v v : \nabla \xi_\alpha^* \right), \end{aligned} \quad (12)$$

where the second form follows after an integration by parts.

Christensen-Dalsgaard's solar models parameterize the convective flux, F_c , in terms of the mixing length, Λ , where Λ/H is a dimensionless constant of order unity, and $H = p/g\rho$ is the local pressure scale height (Vitense 1953). The convective flux is given by

$$F_c = \rho v T \delta s, \quad (13)$$

with the entropy perturbation

$$\delta s = -\frac{\Lambda}{2} \frac{ds}{dr} \quad (14)$$

and the convective velocity

$$v^2 = \frac{\Lambda g}{4} \left(\frac{\partial \ln p}{\partial s} \right)_p \delta s. \quad (15)$$

The quantity δs is the mixing-length estimate for the entropy perturbation, $\tilde{s}$, associated with an energy-bearing eddy. Combining equations (13), (14), and (15), we arrive at

$$F_c = \frac{4H}{\Lambda} \left(\frac{\partial \ln p}{\partial \ln T} \right)_s \rho v^3. \quad (16)$$

For later use we note that equations (14) and (15) yield

$$\mathcal{R} \equiv \frac{1}{\rho v^2} \left(\frac{\partial p}{\partial s} \right)_p \delta s = \frac{4H}{\Lambda} \left(\frac{\partial \ln p}{\partial \ln \rho} \right)_s. \quad (17)$$

We take the vertical dimensions of the energy-bearing eddies to be Λ . Inertial range eddies of vertical size $h \lesssim \Lambda$ are assumed to follow the Kolmogorov scaling, that is, their velocities, v_h , and entropy fluctuations, s_h , satisfy

$$v_h \sim \left(\frac{h}{\Lambda} \right)^{1/3} v \quad (18)$$

and

$$s_h \sim \left(\frac{h}{\Lambda} \right)^{1/3} \delta s. \quad (19)$$

The characteristic correlation time $\tau_h \sim h/v_h$. Acoustic emission at frequency ω arises from eddies with $\omega\tau_h \lesssim 1$. These issues have been discussed in more detail by Goldreich & Kumar (1988, 1990). We introduce a shape parameter, $\mathcal{S}$, to describe the ratio of the horizontal to vertical correlation lengths of turbulent eddies.

As the entropy of a fluid element fluctuates, so does its volume. The fluctuating volume is a monopole source for acoustic waves. In a stratified medium the fluctuating buoyancy force adds a dipole source. By transferring momentum

among neighboring fluid elements, the Reynolds stress acts as a quadrupole source.⁷ The anisotropy of a stratified medium blurs the distinction between monopole, dipole, and quadrupole sources. It allows for destructive interference between the monopole and dipole amplitudes. Although the monopole and dipole amplitudes are individually larger than the quadrupole amplitude, their sum is of comparable size to that of the quadrupole. That this applies to energy-bearing eddies follows directly from equations (12) and (17), but its justification for inertial range eddies requires a subtle argument (cf. GK).

A few simplifications aid further progress. We note that the stochastic excitation of acoustic modes is confined to the upper layers of the convection zone because this is where the correlation times of the energy-bearing eddies match the periods of the modes. In this region the displacement vectors of the acoustic modes are nearly radial and vary most rapidly in the radial direction.⁸ Writing

$$\xi_\alpha = \left(\tilde{\xi}_\alpha^r(r), \tilde{\xi}_\alpha^h(r) \frac{\partial}{\partial \theta}, \tilde{\xi}_\alpha^h(r) \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) Y_{lm}(\theta, \phi), \quad (20)$$

we have

$$\xi_\alpha \approx \tilde{\xi}_\alpha Y_{lm} e_r, \quad (21)$$

and

$$\nabla \xi_\alpha \approx \frac{\partial \tilde{\xi}_\alpha^r}{\partial r} Y_{lm} e_r e_r, \quad (22)$$

where e_r is the unit radial vector. For low-frequency modes, $\omega_\alpha \ll \omega_{ac}$, an additional simplification proves useful, namely,

$$\frac{\partial \tilde{\xi}_\alpha^r}{\partial r} \sim \frac{\omega_\alpha^2}{g} \tilde{\xi}_\alpha^r. \quad (23)$$

The above relation characterizes the solutions of the homogeneous wave equation that are regular at a boundary where $\rho = 0$.

3.3. The Excitation Rate

The incremental amplitude, ΔA_α , produced during the lifetime of a single turbulent eddy is estimated from equation (12):

$$\frac{\Delta A_\alpha}{\tau_h} \sim \frac{i\omega_\alpha}{\sqrt{2}} \mathcal{S}^2 h^3 \left(\frac{\partial p}{\partial s} s_h \mathcal{C}_\alpha(r) + \rho v_h^2 \right) \frac{\partial \tilde{\xi}_\alpha^{r*}}{\partial r}, \quad (24)$$

where we define the compressibility, $\mathcal{C}_\alpha$, by

$$\nabla \cdot \xi = \mathcal{C}_\alpha \frac{\partial \tilde{\xi}_\alpha^r}{\partial r}. \quad (25)$$

Equation (24) accounts for the entire entropy driving, but only includes excitation by the radial component of the Reynolds stress. The latter is valid for $n > 1$. Figure 8 displays $\mathcal{C}^2$ evaluated at the top of the convection zone as a function of ℓ for several different values of v . For each v , $\mathcal{C}^2$ declines monotonically with decreasing n , and nearly vanishes for the high- ℓ f -modes.

⁷ We classify acoustic sources as monopole, dipole, or quadrupole according to whether they produce a change in volume, add net momentum, or merely redistribute momentum.

⁸ These statements do not apply to f -modes.

FIG. 8.—Square of the compressibility, $\mathcal{C}^2$, evaluated at the top of the convection zone as a function of ℓ for several different values of v . The points on each curve correspond to modes with different values of n , starting at the bottom with $n = 0$.

We introduce a parameter η such that only eddies for which $\omega\tau_h \leq \eta$ are included in mode driving. Summing over eddies, we then obtain

$$\mathcal{P}_\alpha \sim 2\pi\omega_\alpha^2 \int dr r^2 \rho^2 \left| \frac{\partial \tilde{\xi}_\alpha^r}{\partial r} \right|^2 \int_0^{h_{\max}(r)} \frac{dh}{h} v_h^3 h^4 (\mathcal{C}_\alpha^2 \mathcal{R}^2 + 1) \mathcal{S}^2, \quad (26)$$

where $h_{\max} \leq \Lambda$ is the largest eddy at r for which $\omega_\alpha \tau_h \leq \eta$. Equation (26) generalizes a similar expression (eq. [56]) derived by GK. Note that $\mathcal{C}_\alpha^2 \mathcal{R}^2$ measures the ratio of the excitation by entropy fluctuations to that due to fluctuations of the Reynolds stress. Figure 9 displays $\mathcal{R}^2$ as a function of pressure in the upper part of the convection zone. The clear implication is that the excitation of acoustic modes is dominated by entropy fluctuations (Stein & Nordlund 1991).

4. EVALUATING $\mathcal{P}_\alpha$

4.1. $\mathcal{P}_\alpha$ as a Function of v at Fixed ℓ ⁹

The expression for the acoustic emissivity includes the factor $|\partial \tilde{\xi}_\alpha^r(r)/\partial r|^2$. We show below that in regions more than 10^8 cm below the photosphere the eddy correlation times are so long that the integrand in equation (26) is negligible. Between the photosphere and a depth of 10^8 cm the derivative of the eigenfunction is roughly independent of r , $|\partial \tilde{\xi}_\alpha^r(r)/\partial r|^2 \approx |\partial \tilde{\xi}_\alpha^r(R_\odot)/\partial r|^2$. We plot this factor against v in Figure 10. Its slope exhibits a break of 5.4 powers of v associated with photospheric reflection. For comparison, we also include graphs of $|\partial \tilde{\xi}_\alpha^r(z)/\partial z|^2$ as derived for a sequence of values of ϵ . Note that the larger ϵ , the more the curves flatten out as $f \equiv \omega/\omega_{ac} \rightarrow 1$. A good fit is obtained with the plane-parallel model having $\epsilon = 2.2$. Figure 1 compares plots of $\log H$ versus $\log p$ from

⁹ The numerical results in this subsection are for modes with $\ell = 0$, so we set the compressibility $\mathcal{C}_\alpha = 1$.

FIG. 9.—Ratio $\mathcal{R}^2$ of entropy to Reynolds stress excitation evaluated near the solar photosphere as a function of the logarithm of the pressure.

Christensen-Dalsgaard's solar model with those from a sequence of plane-parallel envelopes. We observe that the abrupt drop in H found for the plane-parallel model with $\epsilon = 2.2$ matches the magnitude of the more gradual decline seen in H from Christensen-Dalsgaard's solar model.

FIG. 10.—Square of the radial gradient of the radial component of the p -mode eigenfunction, $|\partial \tilde{\xi}_r / \partial r|^2$, evaluated at the top of the convection zone. The points are for the $\ell = 0$ modes of Christensen-Dalsgaard's solar model. The dashed lines show $|\partial \tilde{\xi}_r / \partial r|^2$ for the plane-parallel model with density ratio ϵ equal to (top to bottom) 1.0, 1.8, 2.6, and 3.4. The solid line is for $\epsilon = 2.2$. The least-squares fits of straight lines to the curve for the solar model have slopes equal to 7.6 between 1.7 and 2.8 mHz and 2.2 between 3.7 and 4.8 mHz.

Figure 7 shows that $E\Gamma$ scales as ω^7 at low frequency and as $\omega^{-4.4}$ at high frequency. There is little doubt that photospheric reflection is responsible for a significant fraction of the break in slope, although the precise amount is model-dependent. According to our parameterization, the remaining break in slope arises because only inertial range eddies have short enough correlation times to contribute to $E\Gamma$ at frequencies above about 3.5 mHz. However, one can easily contemplate other possibilities. For example, in the formulation of convection put forth by Canuto & Mazzitelli (1991), the correlation length of the energy-bearing eddies in the top scale height is proportional to their distance from the top of the convection zone.

We proceed to evaluate equation (26). Performing the integral over dh , we find

$$\mathcal{P}_\alpha \sim \frac{2\pi\omega_\alpha^2}{5} \int_0^{R_\odot} dr r^2 \rho^2 v_\Lambda^3 \left| \frac{\partial \tilde{\xi}_\alpha}{\partial r} \right|^2 \times \Lambda^4 (\mathcal{R}^2 + 1) \mathcal{S}^2 \frac{1 + (\omega_\alpha \tau_\Lambda / \eta)^p}{1 + (\omega_\alpha \tau_\Lambda / \eta)^{p+1/2}}, \quad (27)$$

where we take $\mathcal{S}$ to be the same for energy-bearing and inertial-range eddies. In what follows we assume that $\mathcal{S}$ is independent of depth. We employ the last term (the “rollover term”) to interpolate between the contribution of energy-bearing eddies (with $\omega\tau_\Lambda < \eta$) and inertial-range eddies (with $\omega\tau_\Lambda > \eta$). The parameter p determines the sharpness of the transition; in numerical work we set $p = 10$. This choice of rollover function is arbitrary; all we can specify is the asymptotic behavior and the sharpness of the break in slope. The break required to fit the observations is surprisingly abrupt.

In our most precise evaluations of $\mathcal{P}_\alpha$, we directly compute the integral in equation (27). However, it proves revealing to dissect the integrand into several pieces. We note that the correlation times of the energy-bearing eddies rise rapidly with depth; in Figure 11 we plot τ_Λ versus depth below the top of the convection zone, Δr . A least-squares fit for the top 10^8 cm yields

$$\tau \approx \tau \begin{cases} 1, & 0 \leq \Delta r \leq \Delta r_0, \\ \left(\frac{\Delta r}{\Delta r_0} \right)^{0.59}, & \Delta r_0 \leq \Delta r \leq 10^8 \text{ cm}, \end{cases} \quad (28)$$

increasing from $\tau_i \approx 60$ s at the surface to $\tau \approx 600$ s at $\Delta r = 10^8$ cm. The depth at which the power-law behavior starts is $\Delta r_0 = 2.5 \times 10^6$ cm. For depths greater than 10^8 cm, $\tau \propto (\Delta r)^{1.6}$. Below $\Delta r \approx 10^7$ cm, the eddy turnover time is longer than $1000/2\pi$ s, so that energy-bearing eddies at these depths do not contribute to the driving of observable modes. The greatly reduced emissivity of subscale eddies ensures that the contribution to $\mathcal{P}_\alpha$ from these depths is negligible. Since $|\partial \tilde{\xi}_\alpha / \partial r|$ is roughly constant over the interval $\Delta r \leq 10^8$ cm, it can be pulled outside the integral.

Next we consider the frequency-independent quantity

$$J \equiv \left(\frac{2\pi r^2 \mathcal{S}^2 \rho v_\Lambda^3}{5} \right) (\mathcal{R}^2 + 1) (\rho \Lambda^4). \quad (29)$$

Figure 12 shows it to be nearly constant for $p \lesssim 3 \times 10^5$, corresponding to $\Delta r \lesssim 2 \times 10^7$ cm, so that we may also remove J from under the integral for all but the longest period modes. The constancy of J results from a number of coincidences and is worthy of comment. Using equation (16), we transform the

FIG. 11.—Eddy correlation time, $\tau = \Lambda/v$, versus distance, Δr , from the top of the convection zone from Christensen-Dalsgaard's model. The solid line is a least-squares fit to the portion $1.6 \times 10^6 \text{ cm} < \Delta r < 3.2 \times 10^8 \text{ cm}$, showing that $\tau \propto \Delta r^{0.59}$.

leading factor to

$$J_1 \equiv \frac{2\pi r^2 \mathcal{S}^2 \rho v^3}{5} = \frac{\Lambda \mathcal{S}^2}{40H} \left(\frac{\partial \ln T}{\partial \ln p} \right)_s L_c, \quad (30)$$

where L_c is the convective luminosity. The middle factor is

$$J_2 \equiv \mathcal{R}^2 + 1 = \left[\frac{4H}{\Lambda} \left(\frac{\partial \ln p}{\partial \ln \rho} \right)_s \right]^2 + 1. \quad (31)$$

Both J_1 and J_2 would be constant in the absence of ionization zones. Because the final factor,

$$J_3 \equiv \rho \Lambda^4, \quad (32)$$

increases rapidly with depth, one might expect J to increase rapidly with depth as well. However, the ionization of hydrogen causes J_1 and J_2 to decline with depth, thereby delaying the rise of J . As a consequence, J is roughly constant in the upper scale heights of the convection zone, where most of the acoustic emission takes place. Figure 12 displays the depth dependence of J and its three factors J_1 , J_2 , and J_3 .

All that is left to evaluate is the integral of the rollover term,

$$I(\omega) = \int_0^{R_\odot} dr \frac{1 + (\omega\tau/\eta)^p}{1 + (\omega\tau/\eta)^{p+15/2}}. \quad (33)$$

This may be done analytically with the aid of equation (28) in the limit that $p \rightarrow \infty$. We break the integral into two parts, a contribution from the energy-bearing eddies for which the integrand equals unity and a contribution from the inertial-range eddies for which the integrand is equal to $(\omega\tau/\eta)^{-15/2}$. Figure 13 compares $I(\omega)$ evaluated analytically with the more accurate numerical computation for $p = 10$.

We have now assembled all the ingredients needed to determine $\mathcal{P}_\alpha$. We find $J_t \approx 2 \times 10^{56}$ in cgs units, where we take $L_c = L_\odot/2$, since about half of the solar luminosity is carried

FIG. 12.—Solid line shows $J(r)$ divided by its surface value as a function of $\log p$. Similarly normalized curves for the three separate factors J_1 , J_2 , and J_3 are shown by the dot-dash line, the short-dashed line, and the long-dashed line, respectively. All quantities are evaluated from the solar model of Christensen-Dalsgaard.

by radiation at the top of the convection zone. Thus

$$\mathcal{P}_\alpha \sim 8 \times 10^{62} \omega^2 \mathcal{S}^2 \left| \frac{\partial \xi}{\partial r} \right|^2 \begin{cases} \left(\frac{\omega\tau_t}{\eta} \right)^{-1.7}, & \left(\frac{\omega\tau_t}{\eta} \right) < 1, \\ \left(\frac{\omega\tau_t}{\eta} \right)^{-15/2}, & \left(\frac{\omega\tau_t}{\eta} \right) > 1 \end{cases}, \quad (34)$$

FIG. 13.—Integral of the rollover function, $I(\nu)$, is plotted as the solid line. The dashed line shows the analytic approximation. The low- and high-frequency slopes obtained from least-squares fits are $s = -1.7$ and $s = -7.5$.

in cgs units. The above expression peaks at $\omega \approx \eta/\tau_i$, whereas the observationally determined peak occurs at $\nu \approx 3.5$ mHz. This forces the choice $\eta \approx \pi/2$. At $\nu \approx 3.5$ mHz, $|\partial \xi / \partial r|_{R_0}^2 = 2 \times 10^{-38} \text{ g}^{-1} \text{ cm}^{-2} \text{ s}^2$ (cf. Fig. 10). Thus our simplified calculation yields a peak value of $\mathcal{P}_\alpha \sim 8 \times 10^{21} \mathcal{S}^2 \text{ ergs s}^{-1}$, as compared with the maximum value of $E\Gamma \approx 3 \times 10^{22} \text{ ergs s}^{-1}$.

Figure 14 displays $\mathcal{P}_\alpha$ both as evaluated from equation (34) (dashed line) and also as computed from the expression given in equation (27) (solid line). We have chosen $\mathcal{S} = 1.8$, so that the computed values fit those of $E\Gamma$, which are shown for comparison. The more detailed calculation yields a higher value for $\mathcal{P}_\alpha$ at low frequency as a consequence of the increase of J with depth (cf. Fig. 12). Figure 15 plots the integrand of equation (27) as a function of pressure for a number of different frequencies. These figures demonstrate that the excitation occurs very close to the top of the convection zone.

4.2. The Dependence of $\mathcal{P}_\alpha$ on n at Fixed ν

The relative strength of mode excitation by entropy fluctuations and by the Reynolds stress is proportional to the square of the compressibility. This quantity is very small for f -modes of high ℓ , and essentially unity for p -modes with $n \gg 1$. It would be nice to assess the effect of the reduced compressibility on $E\Gamma$ for the low- n , high- ℓ modes. Unfortunately, this cannot be done directly with the available data. Although several observational estimates of the energies of these modes have been made, the corresponding line widths have not been measured. This stymies the determination of $E\Gamma$.

Since a direct comparison between $\mathcal{P}_\alpha$ and $E\Gamma$ is impossible, we set a lower goal. We compute $\mathcal{P}_\alpha$ for low- n modes. Then we assume that the product of line width and mode mass, $M_\alpha \Gamma_\alpha$, is independent of n at fixed ν ; this relation is known to hold for $\ell \leq 160$ (LW). With this assumption we can predict values of E_α for low- n modes by using the observed $\ell = 0$ line widths. These energies are shown in Figure 16. The decline of E_α with

FIG. 14.—Comparison of $\mathcal{P}_\alpha$ and $E\Gamma$. The points are from Fig. 6. The dashed line is the prediction of eq. (34). The solid line is obtained by numerically integrating the expression in eq. (27) using the solar model of Christensen-Dalsgaard.

FIG. 15.—Integrand in the expression for $\mathcal{P}_\alpha$, eq. (27), as a function of pressure for the following frequencies: $\nu = 1.6$ mHz, $\nu = 2.6$ mHz, $\nu = 3.7$ mHz, and $\nu = 4.7$ mHz. The sharp cutoff on the high-pressure side is due to the fact that the correlation time of the energy-bearing eddies exceeds $1/\omega$, combined with the reduced emissivity $[\propto (\omega\tau/\eta)^{-1/2}]$ of subscale eddies.

decreasing n is entirely due to the reduction of $\mathcal{C}_\alpha^2$ with decreasing n . The observationally determined mode energies also decline with decreasing n at fixed ν . All investigations seem to agree on this point. It is our impression that this decline is steeper than that shown in Figure 16, at least for the p -modes.

FIG. 16.—Energy per mode, E_α , calculated as described in § 4.2, as a function of ν for several different orders n . The dotted line gives the energy of the f -mode and the dashed line gives the energy of the $n = 1$ p -mode. The solid line gives the energies of modes with n between 2 and 6, with energy generally increasing with n .

However, this is not a strong statement because large seeing corrections limit the precision with which E_α is known.

5. DISCUSSION

The principal limitations of our calculations are reviewed below. An inadequate understanding of turbulent convection is responsible for most of the uncertainties. This problem is exacerbated because mode excitation is concentrated at the top of the convection zone. Mixing-length models are particularly suspect where the distance from the upper boundary of the convection zone is less than the pressure scale height, and where the turbulent velocity is near sonic.

The entropy gradient in the superadiabatic layers of the upper convection zone affects the reflection of high-frequency acoustic waves, and thereby the break in the slope of $\log \mathcal{P}$ as a function of $\log \nu$. Although the total increase in specific entropy between the top and bottom of the convection zone is tightly constrained by helioseismology, the entropy gradient in the near surface layers is not. For example, the solar models calculated by Christensen-Dalsgaard (1982) and by Canuto & Mazzitelli (1991) satisfy all observational constraints. However, the entropy gradient at the top of the convection zone is much steeper in the latter than in the former.

The acoustic emissivity depends upon finer details of the entropy and velocity fluctuations, in particular their relative magnitudes and their temporal and spatial correlation functions. We set the correlation times of the energy-bearing eddies equal to Λ/ν , where ν is determined by the mixing-length *Ansatz*. Christensen-Dalsgaard (1982) adopts a crude version of the mixing-length prescription (Vitense 1953; Böhm-Vitense 1958) which yields $\tau_i \approx 1$ minute. The ω at which $E\Gamma$ peaks is given by η/τ_i , so we choose $\eta \approx \pi/2$ to fit the observations. However, τ_i/η is much shorter than the several-minute-long lifetimes of solar granules. The horizontal dimensions of granules are typically about 1500 km, much greater than the 150 km pressure scale height at the photosphere. We include a shape factor $\mathcal{S}$ to account for the ratio of the horizontal to the vertical correlation lengths. We use the same $\mathcal{S}$ at all depths, and for both energy-bearing and inertial-range eddies. The excitation rate is proportional to $\mathcal{S}^2$. A best fit of $\mathcal{P}_\alpha$ to $E\Gamma$ is obtained with $\mathcal{S} \approx 1.8$.

According to our calculations, entropy fluctuations are about an order of magnitude more effective in driving p -modes than are fluctuations of the Reynolds stress (Stein & Nordlund 1991). Mode excitation by entropy fluctuations is proportional to the square of the compressibility. The compressibility declines monotonically with increasing ℓ (decreasing n) at fixed ν , and is almost zero for the f -modes. It is tempting to conclude that the decline of mode energies with increasing ℓ at fixed ν (Kaufman 1990; Rhodes et al. 1991; Fernandes et al. 1992; Willette 1993) is due to the dominance of entropy driving. However, for reasons set out below, such a conclusion seems premature.

There is a strong possibility that entropy driving is associated with a corresponding damping mechanism. Entropy damping would be proportional to the square of the compressibility; it would appear as a form of bulk viscosity. If entropy

damping were significant, the line widths would rise less rapidly than the mode mass as $n \rightarrow 0$ at fixed ν .

A more complete understanding must also account for the role that scattering plays in the transfer of energy among modes of similar ν . Scattering may be the dominant source of damping for p -modes and the principal source of excitation for the f -modes.

The horizontal velocity components of the mode eigenfunctions approach the values of the radial components as $n \rightarrow 0$. Thus low- n modes are subject to sources of driving and damping in addition to those that affect modes of high n . It would be easy to include these extra sources in a formal manner, but there is little incentive for doing so. If nonradial components of $\nabla \xi_\alpha$ were retained, it would be necessary to resolve the turbulent velocity into radial and horizontal components for which there is no reliable observational or theoretical basis. Horizontal components of the turbulent velocity may also make a significant contribution to the line widths of the f -modes, thus further complicating the relation between Γ_α and M_α .

There are three differences between the present work and that of GK: (1) GK derived an expression for $\mathcal{P}_\alpha$ only for a plane-parallel atmosphere, which (2) had no discontinuity between the isothermal and the adiabatic layers to mimic the superadiabatic temperature gradient in real solar models, and (3) they lumped entropy driving together with Reynolds stress driving because their simple scaling arguments showed them to be comparable for low- ℓ modes. The more careful, but still uncertain, analysis in this paper suggests that entropy driving may exceed Reynolds stress driving by as much as an order of magnitude.

The generalization of the expression for $\mathcal{P}_\alpha$ given here allows us to use eigenfunctions from real solar models. In addition to the use of a proper solar model, we also employ a more realistic plane-parallel model. This allows us to see where 5 powers of ω in the bend of $\mathcal{P}_\alpha$ (out of a total of 11.4 powers) come from. Both in this paper and in GK, 6.4 powers of ω in the bend are presumed to come from the lack of high-frequency energy-bearing eddies.

Balmforth (1992a, b) investigates many of the issues addressed in our paper. He employs a sophisticated, nonlocal, version of mixing-length theory. Two of his conclusions are of particular relevance here. He finds entropy fluctuations to be less important than Reynolds stress fluctuations in exciting p -modes, but finds turbulent pressure (bulk viscosity) to be a stronger source of damping than turbulent shear viscosity. Clearly, these points merit further attention. Balmforth (1992b) stresses the lack of predictive power of stochastic excitation calculations, a point with which we agree.

We are indebted to K. Libbrecht and M. Woodard for sharing their mode amplitudes and line widths with us, and for many informative discussions. We thank Sylvain Korzennik for the use of Figure 5, and M. Rast and G. Willette for useful comments. This research was supported in part by NSF grant AST 89-13664 and NASA grant NAGW 3018.

APPENDIX

We study model atmospheres that mimic the temperature drop at the photosphere found in realistic solar models. Schmitz & Fleck (1992) and Worrall (1991) examine simpler two-layer models with continuous and discontinuous temperature steps, respectively. It might be noted that Christensen-Dalsgaard & Gough (1980) made an analysis very similar to that carried out here, although restricted to vertical oscillations and for $\epsilon = 1$.

A1. THE MODEL ATMOSPHERE

Our model atmospheres are plane-parallel, sit in a constant gravitational field, and have two layers, the upper isothermal and the lower polytropic and isentropic. They are in hydrostatic equilibrium,

$$\frac{dp}{dz} = \rho g . \quad (\text{A1})$$

The z coordinate measures depth from the level at which the polytropic layer would reach $p = 0$ if it were not overlain by the isothermal layer. The pressure is continuous at z_t , the boundary between the two layers. Its value there, p_t , is the parameter that varies along the sequence of models.¹⁰ The gravitational acceleration is set equal to the photospheric value, $g = 2.74 \times 10^4 \text{ cm s}^{-2}$.

The equation of state in the polytropic layer reads

$$p = \kappa \rho^\Gamma , \quad (\text{A2})$$

with the adiabatic index Γ (not to be confused with the radian line width, which we denote by the same symbol) related to the polytropic index m by

$$\Gamma = \frac{m+1}{m} . \quad (\text{A3})$$

We set $m = 4$ to match the run of p and ρ in the upper convection zone of the solar model of Christensen-Dalsgaard. The pressure and density increase with depth as

$$p = p_t \left(\frac{z}{z_t} \right)^{m+1} , \quad \rho = \rho_t \left(\frac{z}{z_t} \right)^m , \quad (\text{A4})$$

where

$$p_t = \frac{1}{\kappa^m} \left(\frac{gz_t}{m+1} \right)^{m+1} , \quad \rho_t = \frac{1}{\kappa^m} \left(\frac{gz_t}{m+1} \right)^m . \quad (\text{A5})$$

The temperature behaves as

$$T = T_t \left(\frac{z}{z_t} \right) . \quad (\text{A6})$$

The adiabatic sound speed $c^2 = gz/m$, and the pressure scale height $H = z/(m+1)$.

In the isothermal layer we specify the acoustic cutoff frequency $\omega_{ac} \approx 3.3 \times 10^{-2} \text{ s}^{-1}$, the adiabatic exponent $\gamma = 3/2$, and the mean molecular weight $\mu = 2.0 \times 10^{-24} \text{ g}$. The scale height H_i , the adiabatic sound speed c_i , and the temperature T_i follow from the relations

$$H_i = \frac{\gamma g}{4\omega_{ac}^2} , \quad c_i^2 = \gamma g H_i , \quad (\text{A7})$$

and

$$k_B T_i = \mu g H_i . \quad (\text{A8})$$

At the lower boundary of the isothermal atmosphere the density $\rho_i = p_i/gH_i$. The ratio $\epsilon \equiv \rho_i/\rho_t$ is related to z_t by

$$\epsilon = \frac{z_t}{(m+1)H_i} = \frac{H_t}{H_i} . \quad (\text{A9})$$

A2. THE WAVE EQUATIONS

We take the Eulerian enthalpy perturbation, $Q = p_1/\rho$, as the dependent variable in the wave equations. Since the unperturbed model atmospheres are homogeneous in horizontal planes, we write

$$Q(\mathbf{x}) = \frac{1}{\sqrt{2}} \sum_{\alpha} \{ A_{\alpha} \exp [i(\mathbf{k}_{\alpha} \cdot \mathbf{x} - \omega_{\alpha} t)] + A_{\alpha}^* \exp [-i(\mathbf{k}_{\alpha} \cdot \mathbf{x} - \omega_{\alpha} t)] \} Q_{\alpha}(z) , \quad (\text{A10})$$

where $\mathbf{k}_{\alpha}$ lies in the x - y plane.

The wave equation in the polytropic layer reads

$$\frac{d^2 Q_{\alpha}}{dz^2} + \frac{m}{z} \frac{dQ_{\alpha}}{dz} + \left(\frac{\omega_{\alpha}^2}{c^2} - k_{\alpha}^2 \right) Q_{\alpha} = 0 . \quad (\text{A11})$$

¹⁰ Quantities carrying the subscript t refer to the top of the polytropic layer.

In the isothermal layer, we have

$$\frac{d^2 Q_\alpha}{dz^2} + \frac{1}{H_i} \frac{dQ_\alpha}{dz} + \left(\frac{\omega_\alpha^2}{c_i^2} - k_\alpha^2 \right) Q_\alpha = 0. \quad (\text{A12})$$

A3. EIGENFUNCTIONS

Eigenfunctions are obtained by solving equations (A11) and (A12) subject to appropriate boundary conditions. At z_t this demands the continuity of the vertical displacement, $g \cdot \xi_\alpha$, and the Lagrangian pressure perturbation, $\Delta p = p_1 + \rho g \cdot \xi_\alpha$. For $z \gg (n_\alpha + 1)/k_\alpha$ the exponentially decaying solution to equation (A11) must be selected. For $z \rightarrow -\infty$ the less rapidly growing solution to equation (A12) is the proper choice. We are interested in high-order, $n \gg 1$ acoustic modes. Each mode propagates in an acoustic cavity defined by

$$\frac{g}{\omega_\alpha^2} \lesssim z \lesssim \frac{n_\alpha + 1}{k_\alpha}, \quad (\text{A13})$$

and is evanescent elsewhere.¹¹

In the polytropic layer equation (A11) admits exact analytic solutions that involve confluent hypergeometric functions. In the limit $z_t \rightarrow 0$ those solutions that satisfy the relevant boundary conditions may be expressed in terms of Laguerre polynomials (Christensen-Dalsgaard 1980; Christensen-Dalsgaard & Gough 1980). For finite z_t it proves convenient to work with approximate formulae that hold in the region $z \ll (n + 1)/k$, namely,

$$Q(z) = \mathcal{A} \frac{J_{m-1}(u)}{u^{m-1}} + \mathcal{B} \frac{N_{m-1}(u)}{u^{m-1}}, \quad (\text{A14})$$

where J_ν and N_ν denote Bessel functions of the first and second kinds. The variable u is defined by

$$u \equiv 2 \left(\frac{m\omega^2 z}{g} \right)^{1/2}. \quad (\text{A15})$$

For $z \gg (n + 1)/k$ we demand that $Q \propto \exp(-kz)$. The displacement vector is obtained from

$$\xi_\alpha = \frac{1}{\omega_\alpha^2} \nabla Q_\alpha. \quad (\text{A16})$$

In the isothermal atmosphere, the physically relevant solution of equation (A12) is

$$Q \approx \mathcal{C} \exp(-Kz), \quad (\text{A17})$$

where

$$K \approx \frac{1}{2H_i} [1 - (1 - f^2)^{1/2}] \quad (\text{A18})$$

for $n \gg 1$. The dimensionless ratio f is defined by

$$f \equiv \frac{\omega}{\omega_{ac}}. \quad (\text{A19})$$

Since we are interested in trapped modes, $f \leq 1$. The horizontal and vertical components of the displacement vector, ξ_h and ξ_z , are obtained from

$$\xi_h = i \frac{k}{\omega^2} Q \quad (\text{A20})$$

and

$$\xi_z = \frac{1}{\omega^2 - \omega_b^2} \left[\frac{\partial Q}{\partial z} + \frac{(\gamma - 1)}{\gamma H_i} Q \right]. \quad (\text{A21})$$

The buoyancy frequency, ω_b , is given by

$$\omega_b^2 \equiv \frac{(\gamma - 1)g}{\gamma H_i}. \quad (\text{A22})$$

We normalize the modes in such a way that

$$\omega^2 \int_{-\infty}^{\infty} dz \rho \xi^2 = 1. \quad (\text{A23})$$

¹¹ Henceforth we drop the subscript α .

For the high-order modes of interest here, the dominant contribution to the normalization integral comes from the acoustic cavity which lies entirely inside the polytropic layer. Thus, the integral written in terms of Q using equation (A16) reads

$$\int_{g/\omega^2}^{(n+1)/k} dz \frac{\rho}{c^2} Q^2 \approx 1. \quad (\text{A24})$$

In the acoustic cavity the integrand consists of a factor that oscillates between zero and one, multiplied by a slowly varying envelope. A WKB analysis shows that regions between consecutive nodes contribute equally to the integral; that is, equal amounts of energy are stored between consecutive nodes. Thus, to evaluate the integral, we multiply the contribution from the region between the top two nodes by n . Substituting the expression for Q from equation (A14) into the normalization integral, we obtain

$$\mathcal{A}^2 + \mathcal{B}^2 = \frac{g}{2m\rho_t} \left(\frac{4m\omega^2 z_t}{g} \right)^m. \quad (\text{A25})$$

From the dispersion relation

$$\omega^2 \approx \frac{2}{m} gk \left(n + \frac{m}{2} \right), \quad (\text{A26})$$

which is exact for $z_t \rightarrow 0$, and still a good approximation for $\omega^2 z_t/g \ll 1$, we find

$$\mathcal{A}^2 + \mathcal{B}^2 = \frac{4gkz_t}{m\rho_t} \left(\frac{4m\omega^2 z_t}{g} \right)^{m-1}, \quad (\text{A27})$$

for $n \gg 1$. In applications we always use the more accurate equation (A25) together with the dispersion relation determined by numerical integration.

Now we apply the boundary conditions at z_t .¹² The continuity of the Lagrangian pressure perturbation implies

$$Q(z_{t+}) - (\epsilon - 1) \frac{g}{\omega^2} \frac{dQ(z_{t+})}{dz} = \epsilon Q(z_{t-}), \quad (\text{A28})$$

and the continuity of the vertical displacement yields

$$\frac{dQ(z_{t+})}{dz} = - \frac{\omega^2 \Re}{g} Q(z_{t-}). \quad (\text{A29})$$

Here

$$\Re \equiv \frac{1 + [1 - 4(\gamma - 1)/\gamma^2]^{1/2}}{(1 - f^2)^{1/2} + [1 - 4(\gamma - 1)/\gamma^2]^{1/2}}. \quad (\text{A30})$$

Combining equations (A28) and (A29), we arrive at

$$\frac{dQ(z_{t+})}{dz} = - \frac{\omega^2 \Re}{\delta g} Q(z_{t+}), \quad (\text{A31})$$

where

$$\delta \equiv \epsilon - (\epsilon - 1)\Re. \quad (\text{A32})$$

The ratio $\mathcal{B}/\mathcal{A}$ is found by substituting equation (A14) in equation (A31). It reads

$$\frac{\mathcal{B}}{\mathcal{A}} = \frac{2m \delta J_m(u_t) - \Re u_t J_{m-1}(u_t)}{2m \delta N_m(u_t) - \Re u_t N_{m-1}(u_t)}. \quad (\text{A33})$$

Together, equations (A25) and (A33) suffice to determine both $\mathcal{A}$ and $\mathcal{B}$.

A4. EXCITATION RATE

We use equation (A16) to transform the expression for the mode excitation rate given by equation (26). Its new form is

$$\mathcal{P}_\alpha \sim \frac{1}{\omega_\alpha^2} \int dz \rho^2 \left| \frac{\partial^2 Q}{\partial z^2} \right|^2 \int_0^{h_{\max}(z)} \frac{dh}{h} v_h^3 h^4 \left[\frac{1}{(\rho v)^2} \left(\frac{\partial p}{\partial s} s_h \right)^2 + 1 \right]. \quad (\text{A34})$$

Aside from the entropy term in the square brackets, equation (A34) is formally identical to equation (56) of GK. The other difference from GK is in the eigenfunction (Q) used in that paper. The sharp temperature drop at the top of the convection zone, modeled here by the discontinuity in T between the two layers, causes a sharp decrease in the surface amplitude of Q at frequencies near the acoustic cutoff.

¹² We denote the top of the polytropic layer and the base of the isothermal layer by the notation $z_{t\pm}$.

REFERENCES

- Ando, H., & Osaki, Y. 1977, *PASJ*, 29, 221
 Antia, H. M., Chitre, S. M., & Narasimha, K. 1986, *Ap&SS*, 118, 169
 Balmforth, N. J. 1992a, *MNRAS*, 255, 603
 ———. 1992b, *MNRAS*, 255, 639
 Böhm-Vitense, E. 1958, *Z. Astrophys.*, 46, 108
 Canuto, V. J., & Mazzitelli, I. 1991, *ApJ*, 370, 295
 Christensen-Dalsgaard, J. 1980, *MNRAS*, 190, 765
 ———. 1982, *MNRAS*, 199, 735
 Christensen-Dalsgaard, J., & Frandsen, S. 1983a, *Sol. Phys.*, 82, 165
 ———. 1983b, *Sol. Phys.*, 82, 469
 Christensen-Dalsgaard, J., & Gough, D. O. 1980, *Nature*, 288, 544
 Christensen-Dalsgaard, J., Gough, D. O., & Libbrecht, K. G. 1989, *ApJ*, 341, L103
 Cowling, T. G. 1941, *MNRAS*, 101, 367
 Fernandes, D. N., Scherrer, P. H., Tarbell, R. D., & Title, A. M. 1992, *ApJ*, 392, 736
 Goldreich, P., & Keeley, D. A. 1977, *ApJ*, 211, 934
 Goldreich, P., & Kumar, P. 1988, *ApJ*, 326, 462
 ———. 1990, *ApJ*, 363, 694 (GK)
 ———. 1991, *ApJ*, 374, 366
 Goldreich, P., & Murray, N. 1994, *ApJ*, 423, 480 (GM)
 Kaufman, J. M. 1990, Ph.D. thesis, California Inst. Technology
 Kidman, R. B., & Cox, A. N. 1984, in *Solar Seismology from Space*, ed. R. K. Ulrich, J. Harvey, E. J. Rhodes, Jr., & J. Toomre (JPL Publ. 84-84), 335
 Kumar, P., Franklin, J., & Goldreich, P. 1988, *ApJ*, 328, 879
 Kumar, P., & Goldreich, P. 1989, *ApJ*, 342, 558
 Libbrecht, K. G. 1988, *ApJ*, 334, 510
 Libbrecht, K. G., & Woodard, M. F. 1991, *Science*, 253, 152 (LW)
 Osaki, Y. 1990, in *Progress of Seismology of the Sun and Stars*, ed. Y. Osaki & H. Shibahashi (Berlin: Springer-Verlag), 75
 Rhodes, E. J., Cacciani, A., & Korzenik, S. G. 1991 in *Adv. Space Res.*, 11, 17
 Schmitz, F., & Fleck, B. 1992, *A&A*, 260, 447
 Stein, F., & Nordlund, A. 1991, in *Challenges to Theories of the Structure of Moderate-Mass Stars*, ed. D. Gough & J. Toomre (Berlin: Springer-Verlag), 195
 Toutain, T., & Frohlich, C. 1992, *A&A*, 257, 287
 Vitense, E. 1953, *Z. Astrophys.*, 32, 135
 Willette, G. 1993, in preparation
 Worrall, G. 1991, *MNRAS*, 251, 427