THE ASTROPHYSICAL JOURNAL, 432:612–621, 1994 September 10

TOWARD A THEORY OF INTERSTELLAR TURBULENCE. I. WEAK ALFVÉNIC TURBULENCE

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Received 1993 September 29; accepted 1994 March 18

ABSTRACT

We study *weak* Alfvénic turbulence of an incompressible, magnetized fluid in some detail, with a view to developing a firm theoretical basis for the dynamics of small-scale turbulence in the interstellar medium. We prove that resonant 3-wave interactions are absent. We also show that the Iroshnikov-Kraichnan theory of incompressible, magnetohydrodynamic turbulence—which is widely accepted—describes *weak* 3-wave turbulence; consequently, it is incorrect. Physical arguments, as well as detailed calculations of the coupling coefficients are used to demonstrate that these interactions are empty. We then examine resonant 4-wave interactions, and show that the resonance relations forbid energy transport to small spatial scales along the direction of the mean magnetic field, for both the *shear* Alfvén wave and the *pseudo* Alfvén wave. The three-dimensional inertial-range energy spectrum of 4-wave shear Alfvén turbulence guessed from physical arguments reads $E(k_z, k_\perp) \sim V_A v_L L^{-1/3} k_\perp^{-10/3}$, where V_A is the Alfvén speed, and v_L is the velocity difference across the outer scale L. Given this spectrum, the velocity difference across $\lambda_\perp \sim k_\perp^{-1}$ is $v_{\lambda_\perp} \sim v_L(\lambda_\perp/L)^{2/3}$. We derive a kinetic equation, and prove that this energy spectrum is a stationary solution and that it implies a positive flux of energy in $\boldsymbol{k}$-space, along directions perpendicular to the mean magnetic field. Using this energy spectrum, we deduce that 4-wave interactions strengthen as the energy cascades to small, perpendicular spatial scales; beyond an upper bound in perpendicular wavenumber, $k_\perp L \sim (V_A/v_L)^{3/2}$, weak turbulence theory ceases to be valid. Energy excitation amplitudes must be very small for the 4-wave inertial-range to be substantial. When the excitation is strong, the width of the 4-wave inertial-range shrinks to zero. This seems likely to be the case in the interstellar medium. The physics of *strong* turbulence is explored in Paper II.

Subject headings: ISM: general — MHD — turbulence

1. INTRODUCTION

Electron density fluctuations in the ionized interstellar medium (ISM) scatter radio waves, giving rise to scintillation of radio pulsars and compact radio sources (see Rickett 1990 and Narayan 1992 for reviews). The phenomenon was first understood in its essence by Scheuer (1968), who explained pulsar variability as an effect of the inhomogeneous, ionized ISM. In his model, blobs of some characteristic scale (the scale lying somewhere in the range 10^9–10^{13} cm) scattered radio waves. Lee & Jokipii (1976) proposed a power-law spectrum of electron density fluctuations, extending from small scales ($<10^{11}$ cm) to parsec scales of interstellar clouds. They also postulated a turbulent origin for the fluctuations, and suggested that the power spectrum follows an isotropic "Kolmogorov law." The measurements of the decorrelation bandwidths of several pulsars as a function of the frequency of observation by Cordes, Weisberg, & Boriakoff (1985) showed that deviations from the predictions of Scheuer's single scale spectrum were consistent with the Lee & Jokipii model. Further support arrived with the explanation of long time variability in pulsars and compact extragalactic radio sources by Rickett, Coles, & Bourgois (1984) as refractive scintillation on scales $\sim 10^{13}$–10^{14} cm. Observations of the fluctuations of dispersion measures of pulsars (e.g., Phillips & Wolszczan 1991) provide more confirmation on similar scales. In summary, radio scattering observations seem to be consistent with a three-dimensional electron density (fluctuation) power spectrum of the form (wavenumber)$^{-\alpha}$, where α is close to 11/3, over a range of scales 10^8–10^{14} cm. Armstrong, Cordes, & Rickett (1981) note that an extrapolation of this power spectrum to scales ~ 100 pc is consistent with the known structure of the ISM on these scales, and suggest that the spectrum might well span 12 decades in wavenumber!

The long span (at least 6 decades in wavenumber) of the density spectrum, and the closeness of the spectral index ($\alpha \approx 11/3$) to the Kolmogorov value for neutral fluids have lent weight to the idea of a turbulent origin for the electron density fluctuations—although Kolmogorov's theory (and its more sophisticated descendents) of turbulence in neutral fluids might be of little relevance in the ionized ISM! The ubiquitous presence of an interstellar magnetic field (see, e.g., Heiles 1987) of $\sim 3\mu$ gauss in the ionized ISM endows the magnetic field with a non-negligible dynamical role. A theory of interstellar turbulence should, presumably, be based on a theory of turbulence in a magnetized plasma. Plasma turbulence being the notoriously complicated subject it is, one might do not too badly by studying magnetohydrodynamic (MHD) turbulence. Montgomery, Brown, & Matthaeus (1987) assume an equation of state for a magnetized plasma, and estimate the density perturbations from the pressure variations; the latter are estimated from fluctuations of the velocity and magnetic fields. They claim a Kolmogorov form for the density spectrum, although they need to impose this form on the velocity and magnetic field fluctuations. Higdon (1984) proposed that the density fluctuations play a passive role, behaving like a passive scalar contaminant that is advected (and mixed) by a turbulent velocity field. Drawing on work done by Montgomery (1982) on

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incompressible MHD turbulence, Higdon suggested that the velocity and magnetic field fluctuations would be highly anisotropic (in fact, nearly two dimensional), and concentrated in directions perpendicular to the mean magnetic field. The electron density fluctuations were assumed to be isobaric entropy fluctuations with oppositely directed gradients of density and temperature. A nice feature of this model is that in the presence of a strong mean magnetic field, transport coefficients transverse to the field are drastically decreased compared to the case without a strong mean field; small dissipation would allow the inertial-range of MHD turbulence to reach small spatial scales, thereby removing one difficulty in imagining a MHD turbulent cascade extending to the small scales that give rise to diffractive scintillation. However, Montgomery's theory is little more than a plausibility argument that incompressible MHD turbulence in the presence of a mean magnetic field might become two-dimensional if only Alfvén waves could be excluded from the dynamics! Since Alfvén waves form such a large part of the dynamics, it seems worthwhile to study the dynamics of the entire system, rather than modes that are nearly two-dimensional to begin with anyway. In § 2 we shall have occasion to comment on work of a similar nature done by Shebalin, Matthaeus, & Montgomery (1983).

A theory of incompressible MHD turbulence due to Iroshnikov (1963) and Kraichnan (1965)—*hereafter referred to as the IK theory*—is widely accepted. The physical picture of turbulence in this theory is very appealing, but it is incorrect. The reasons lie far below the surface. We discuss this in some detail in § 2 as well as § 3. It seems to us that there does not exist even a phenomenological picture of incompressible MHD turbulence in the presence of a mean magnetic field. We begin a study of this problem with a view to applying the results to interstellar turbulence.

In this paper we study the turbulence that develops when an incompressible, magnetized fluid is weakly perturbed. In § 2 we discuss linear as well as nonlinear Alfvén waves. We outline the elements of weak turbulence theory, and show that the IK theory is based on resonant 3-wave interactions. Physical arguments are used to demonstrate that these interactions are absent, thereby rendering the IK theory incorrect. The resonance relations for 4-wave interactions are analysed, and we conclude on general grounds that 4-wave interactions forbid transfer of energy to small spatial scales along directions parallel to the mean magnetic field. Heuristic arguments are used to guess the 4-wave inertial-range energy spectrum for *shear* Alfvén waves. Interactions between waves strengthen as the cascade proceeds to large perpendicular wavenumbers; the assumptions on which weak turbulence theory is based are violated to a greater degree at large perpendicular wavenumbers, thereby restricting the inertial-range of the 4-wave energy cascade.

Our motivation for studying shear Alfvén waves derives from a result of Barnes (1966) showing that, in a high β collisionless plasma (where β is the ratio of gas pressure to magnetic pressure—for an incompressible, magnetized fluid, β is formally infinite, while for the warm, ionized ISM $\beta \sim 1$), the fast and the slow magnetosonic modes are heavily damped by kinetic effects, while the shear Alfvén wave is not damped by such an effect.

Section 3 provides a rigorous basis for the physical approach taken in § 2. A variational principle is used to calculate the coupling coefficients that govern resonant 3-wave and 4-wave interactions. Resonant 3-wave interactions are shown to be empty. A kinetic equation for 4-wave interactions is derived, and we prove that the 4-wave energy spectrum that was guessed in § 2 on physical grounds is indeed a stationary solution to the kinetic equation. We show in the Appendix that this solution corresponds to a positive flux of energy traveling to large perpendicular wavenumbers.

In § 4, we discuss the implications of the limitation of the 4-wave inertial-range, and mention issues that need further investigation.

2. WEAK ALFVÉNIC TURBULENCE; THE PHYSICAL PICTURE

2.1. *Failure of the IK Theory, and Resonant 3-Wave Interactions*

The dynamics of an incompressible, conducting fluid with constant transport properties can be described by the following equations of magnetohydrodynamics (MHD):

$$\begin{gathered} \partial_t \boldsymbol{b} = \nabla \times (\boldsymbol{v} \times \boldsymbol{b}) + \kappa \nabla^2 \boldsymbol{b} , \\ \partial_t \boldsymbol{v} = -(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} - \nabla p + \gamma \nabla^2 \boldsymbol{v} , \\ \nabla \cdot \boldsymbol{v} = \nabla \cdot \boldsymbol{b} = 0 , \end{gathered} \tag{1}$$

where $\boldsymbol{v}$ is the velocity, $\boldsymbol{b} = \boldsymbol{B}/(4\pi\rho)^{1/2}$ is the magnetic field in velocity units and p is the ratio of total (mechanical plus magnetic) pressure to the density. In this paper we assume that the only role of the magnetic diffusivity (κ) and the viscosity (γ) is to provide a sink at small spatial scales. Equations (1) allow for a stable, static equilibrium in which $\boldsymbol{v}_0 = 0$ and $\boldsymbol{B}_0 = B_0 \hat{z}$. *Shear* Alfvén waves and *pseudo*-Alfvén waves are the two linear perturbations about this equilibrium; the latter is the incompressible limit of the slow magnetosonic wave. Both kinds of waves have the same dispersion relation, namely, $\omega = V_A |k_z|$, where $V_A = B_0/(4\pi\rho)^{1/2}$ is the Alfvén speed. The perturbed velocity and magnetic field are related by $\boldsymbol{v}_1 = \pm \boldsymbol{b}_1$, where the upper/lower signs correspond to waves traveling antiparallel/parallel to B_0 (with $k_z < 0$ and $k_z > 0$, respectively).

Weak turbulence theory (see, e.g., Zakharov, L'vov, & Falkovich 1992) deals with the effects of the nonlinear terms (in the equations of motion) in a systematic, perturbative manner. When the nonlinear terms are ignored, the Fourier amplitudes and phases of the waves are constant in time. However, the nonlinearity will make the amplitudes change slowly, over many wave periods. It is this secular change in the amplitudes that measures energy transfer among the linear modes. A "kinetic equation" for the rate of change of energy in a mode with wavevector $\boldsymbol{k}$ describes how other modes in the system affect the energy in this mode. To lowest order in the nonlinearity, the kinetic equation takes account of interactions among modes taken three at a time. If the wavevectors of the three modes are $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, and $\boldsymbol{k}$ (clearly, $\boldsymbol{k}$ must be one of the three!), then they must satisfy a "triangle equality," namely

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k} . \tag{2}$$

This can be understood, mathematically, as simply arising from performing a Fourier transform on quadratic quantities, or physically, as conservation of momentum in every "elementary interaction" among 3-waves. The slowness of the secular change of the amplitudes implies that certain resonance relations—between the frequencies of the modes that form an elementary interaction—must be satisfied;

$$\omega_1 + \omega_2 = \omega(\boldsymbol{k}) , \tag{3}$$

where $\omega_j = \omega(\boldsymbol{k}_j)$ (for Alfvén waves $\omega_j = V_A |\hat{z} \cdot \boldsymbol{k}_j|$). Physically, this can be interpreted as the creation or annihilation of "quanta" (or "quasi-particles") in the classical limit of large occupation numbers. The ω's are the energies of the quanta, and equations (2) and (3) are statements of conservation of momentum and energy respectively in every elementary interaction. The energy in a mode with wave vector $\boldsymbol{k}$ is equal to the product of ω_k and the occupation number of the mode. Although we find it convenient to phrase the physics in a quantum-mechanical language, it must be understood that all our considerations are entirely classical, and concepts such as "quanta," "quasi-particles," or "occupation numbers" are to be interpreted in the limit that Planck's constant becomes very small, while quantities like the energy per mode remain finite. When equations (2) and (3) can be satisfied, one says that a "3-wave" resonant interaction is allowed. When 3-wave resonant interactions are forbidden (either because the dispersion relation prevents eqs. [2] and [3] from being satisfied, or because the 3-wave resonant coupling coefficients vanish), one must consider the effects of 4-wave resonant interactions. The resonance relations (see eq. [4]) in this case can always be satisfied. The elementary interactions are waves scattering elastically off each other, and the number of quanta are conserved ("two in and two out"). So long as the resonant coupling coefficients do not vanish, 4-wave resonant interactions are always permitted, and one need not usually worry about the effects of higher order terms. *It must be noted that the entire formalism depends on the linear wave equations being valid as a zeroth approximation. This means that, for problems dealing with fluids, the displacement of a fluid particle must be much smaller than a wavelength. After one has worked out a weakly turbulent spectrum, one must determine the range of k over which the displacement is smaller than* k^{-1}.

Let $\delta\boldsymbol{v}$ and $\delta\boldsymbol{b}$ be the perturbations in velocity and magnetic field respectively, about a static equilibrium in which $\boldsymbol{v}_0 = 0$ and $\boldsymbol{B}_0 = B_0\,\hat{z}$. If $\delta\boldsymbol{v}(\boldsymbol{x}) = -\delta\boldsymbol{b}(\boldsymbol{x})$ at some instant of time, $t = 0$, it can be checked that $\delta\boldsymbol{v}(x, y, z - V_A t) = -\delta\boldsymbol{b}(x, y, z - V_A t)$ for all time, irrespective of the functional form of $\delta\boldsymbol{v}(\boldsymbol{x})$ (see Parker 1979). This nonlinear solution (of eq. [1], with $\gamma = \kappa = 0$) describes a wave packet of arbitrary form traveling nondispersively in the direction of $\boldsymbol{B}_0$. Similarly, we can also construct another class of nonlinear solutions, with $\delta\boldsymbol{v} = \delta\boldsymbol{b}$, that travels nondispersively in a direction opposite to $\boldsymbol{B}_0$. Both types of nonlinear solutions are stable, and the dynamics is simple so long as there is no spatial overlap ("collisions") between oppositely moving wave packets. In the IK theory of Alfvén turbulence, a collision between two oppositely moving wave packets creates small distortions in each of the wave packets, and successive collisions of one of these wave packets with other oppositely moving wave packets are assumed to add with random phases, until the distortions build up to an amplitude of order unity; at that point the wave packet has lost memory of its initial state and its energy has cascaded to a smaller spatial scale. IK assume that the inertial-range energy spectrum is *isotropic* in $\boldsymbol{k}$-space, and that the energy transfer is *local* in $\boldsymbol{k}$-space (this is equivalent to assuming that only collisions between oppositely moving wave packets of similar spatial extent are effective in transferring energy to smaller spatial scales). Then, the collision time for packets of size k^{-1} is of order $(kV_A)^{-1}$. If $E(k)$ is the three-dimensional energy spectrum, the perturbation in the velocity, v_λ, on a spatial scale $\lambda \sim k^{-1}$, is $v_\lambda \sim [E(k)k^3]^{1/2}$. IK assume (from the form of the nonlinear terms in eq. [1]) that the fractional change in the velocity (or the magnetic) perturbation is of order $v_\lambda/V_A \ll 1$ during one collision—this is equivalent to assuming 3-wave interactions. Since subsequent collisions are assumed to contribute with random phases, the number of collisions, N, needed for a typical wave packet to lose memory of its initial state is $N \sim (V_A/v_\lambda)^2 \gg 1$. The cascade time on scale k^{-1} is $t_{\rm cas} \sim N\omega_k^{-1}$. Assuming a k-independent rate of energy transfer per unit mass, $v_\lambda^2/t_{\rm cas} \sim \epsilon$, IK find that $E(k) \propto k^{-7/2}$ and $v_\lambda \propto \lambda^{1/4}$ (for comparison, the well-known Kolmogorov spectrum for neutral fluids has $E(k) \propto k^{-11/3}$, giving $v_\lambda \propto \lambda^{1/3}$). We noted earlier that the IK theory is based on 3-wave interactions, although neither Iroshnikov nor Kraichnan thought of their theory as describing *resonant* 3-wave interactions. Let us imagine that energy is injected into the fluid on some spatial scale L, and that the velocity perturbation on this scale is strong ($v_L \sim V_A$). Then, in just one (or at most a few) collisions, energy will be transferred to a smaller scale (say, of order $L/2$). As the transfer continues to smaller spatial scales (the inertial-range), the velocity perturbations decrease $[v_\lambda \sim V_A(\lambda/L)^{1/4}]$. Also $N \sim (V_A/v_\lambda)^2 \sim (L/\lambda)^{1/2} \gg 1$, implying that in the inertial-range, it takes very many wave periods for energy to be transferred through, say, one octave in wavenumber. On some scale k^{-1}, the equality between the ω's in equation (3) is allowed to be violated by $\Delta\omega \sim t_{\rm cas}^{-1} \sim (\omega_k/N)$, which can be understood as an "uncertainty relation." Since N increases with k, we expect that the resonance relations must be satisfied ever more accurately as we go deeper (larger k) into the inertial range. Even if the excitation is strong to begin with (i.e., on scale L), the interaction—and hence the turbulence—gets weaker as the cascade proceeds. Therefore, in the inertial-range, the IK theory is based on *resonant* 3-wave interactions. Moreover, N increases with k, formally granting the theory an infinite inertial-range, which is a desirable feature.

While the physical arguments given above for the IK picture seem plausible, we argue below that the IK theory is basically incorrect. The reason is that the resonant 3-wave coupling coefficients vanish! Neither Iroshnikov nor Kraichnan took account of the 3-wave resonance relations, since they did not view their theories as being based on resonant interactions. We have demonstrated that the IK theory is indeed based on resonant 3-wave interactions. Now, we examine the resonance relations (eqs. [2] and [3]) and show that the resonant coupling coefficients must vanish. We consider three cases:

(i) *k_{1z} and k_{2z} have the same sign.*—The coupling coefficients must vanish since there exist nonlinear, nondispersive packets of arbitrary form.

(ii) *k_{1z} and k_{2z} have opposite signs.*—From equations (2) and (3) we have

$$k_{1z} + k_{2z} = k_z\,, \quad |k_{1z}| + |k_{2z}| = |k_z|\,.$$

For definiteness, let $k_{1z} \geq 0$ and $k_{2z} \leq 0$. When $k_{2z} \neq 0$, these equations do not have a solution, implying that oppositely directed packets do not interact via the resonant 3-wave process. This conclusion was reached by Shebalin, Matthaeus, & Montgomery (1983). Moreover, they proposed that only interactions with $k_{2z} = 0$ waves would survive, leading to anisotropic turbulence; however, we shall see that this is incorrect since the $k_{2z} = 0$ waves possess no power to contribute to resonant interactions.

(iii) $k_{2z} = 0$.—This is an interaction of a $\boldsymbol{k}_1$-wave with a zero-mode to produce a $\boldsymbol{k}$ wave. If $\boldsymbol{\xi} = \boldsymbol{\xi}_0\, e^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp}$ is the displacement of a fluid particle corresponding to a zero-mode, then the associated velocity and magnetic field (in velocity units) are proportional to $\omega_k \xi$ for linear waves. Since linear waves are all that are used to construct resonant 3-wave couplings, a zero-mode (for which $\omega_k = 0$, by definition) makes no contribution at all to resonant 3-wave interactions.

The IK theory, which is based on resonant 3-wave interactions between oppositely directed wave packets, must be incorrect since the couplings are empty in this case. At this stage there does not seem to be a clear connection between the physical picture of colliding wave packets and the 3-wave resonance conditions. We defer the discussion of this point to Paper II. Here we wish to note that in a theory of Strong Alfvénic turbulence that we develop there, the assumption of isotropy—that seems innocuous enough in the IK theory—breaks down, leading to long correlation lengths along the mean magnetic field, and strong interactions which lead to a cascade so rapid that the allowed violation of the ω resonance conditions become of order unity. What does survive from the IK theory is Kraichnan's insight into the nature of Alfvénic turbulence; that the cascade results from collisions between oppositely directed packets.

2.2. *Resonant 4-Wave Interactions*

Since 3-wave interactions are absent, we must examine 4-wave interactions. There seems to be some misconception in the literature regarding this; for instance, Bondeson (1985) states that only resonant 3-wave interactions are possible since equation (1) has only quadratic nonlinearities. This is incorrect since, in general, the nonlinearities in the equations of motion do not in any manner limit the *order* to which the equations could be solved perturbatively! In § 3 we will analyze 4-wave interactions in detail. Here, we reach some general conclusions and explore the physics without using the specific form of the 4-wave couplings. The conservation laws that must be satisfied for a 4-wave interaction are

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4 \;, \quad \omega_1 + \omega_2 = \omega_3 + \omega_4 \;. \tag{4}$$

These describe an elastic scattering of two waves into two other waves. The existence of the nonlinear solutions implies that the coupling coefficients must vanish when $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ have z-components of the same sign. So we need only consider cases when their z-components are oppositely directed. For definiteness, let $k_{1z} > 0$ and $k_{2z} < 0$. Using $\omega_j = V_{\rm A} |\hat{z} \cdot \boldsymbol{k}_j|$, the conservation laws imply that the z-components of $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$ must also be oppositely directed. Let $k_{3z} > 0$ and $k_{4z} < 0$. From equation (4) we get

$$k_{1z} + k_{2z} = k_{3z} + k_{4z} \;, \quad k_{1z} - k_{2z} = k_{3z} - k_{4z} \;. \tag{5}$$

Therefore $k_{1z} = k_{3z}$ and $k_{2z} = k_{4z}$; the scattering process leaves the z-components unaltered. This implies that waves with values of k_z neither present initially nor subsequently injected cannot be created by 4-wave interactions. Furthermore, since scattering conserves quasi-particle numbers, *the net result of 4-wave interactions is to shuffle quasi-particles around in **k**-space without changing their k_z components. Energy cannot cascade along k_z.* The specific form of the couplings determines how quasi-particles at k_z' affect the shuffling of quasi-particles at k_z.

For an isotropic dispersion relation, $\omega(k)$, and $d\omega/dk > 0$, 4-wave interactions usually result in a direct cascade of energy and inverse cascade of quasi-particles. The inverse cascade occurs because the creation of one quasi-particle with large ω (and large k) requires input from many quasi-particles with middling values of ω. Conservation of total number of quasi-particles implies that many of these should be created at small ω (and small k)—hence the inverse cascade. For the problem being studied here, equation (5) implies that the ω's do not change during an elementary scattering process. Therefore the energy and quasi-particle budget are automatically satisfied without the need to create many low-energy quasi-particles. *Hence there is no inverse cascade of quasi-particles.*

A general incompressible perturbation is a linear combination of *shear* and *pseudo*-Alfvén waves. Shear Alfvén waves have $v_z = b_z = 0$, while pseudo-Alfvén waves in general have nonzero v_z and b_z. We limit our further considerations to only the shear Alfvén waves.[3] This is a consistent procedure since the coupling between shear and pseudo-Alfvén waves is weak; the fractional energy loss per wave period due to the generation of pseudo-Alfvén waves by shear Alfvénic turbulence is small. We now study the direct cascade of energy. Assuming *locality* of interactions in **k**-space, we consider the collision between an Alfvén wave packet of size $(k_z^{-1}, k_\perp^{-1})$ with another packet of comparable size that propagates in the opposite direction. During a collision time of order $(k_z V_{\rm A})^{-1}$ the change in the fluid velocity amplitude,[4] v_λ, of one of the packets is

$$|\delta v_\lambda| \sim \left| \frac{d^2 v_\lambda}{dt^2} (k_z V_{\rm A})^{-2} \right| .$$

Note that if the 3-wave process were in consideration, the right side would have been

$$|(dv_\lambda/dt)(k_z V_{\rm A})^{-1}| .$$

On dimensional grounds, from equation (1)

$$\frac{d^2 v_\lambda}{dt^2} \sim \frac{d}{dt} (k_\perp v_\lambda^2) \sim k_\perp v_\lambda \frac{dv_\lambda}{dt} \sim k_\perp^2 v_\lambda^3 .$$

The quantity $k_\perp v_\lambda$ arises because, for shear Alfvén waves, $(\boldsymbol{v} \cdot \nabla) = \pm(\boldsymbol{b} \cdot \nabla) \sim v_\lambda \nabla_\perp \sim k_\perp v_\lambda$. So, in one collision the fractional change in v_λ is

$$\left| \frac{\delta v_\lambda}{v_\lambda} \right| \sim \left(\frac{k_\perp v_\lambda}{k_z V_{\rm A}} \right)^2 . \tag{6}$$

When this is small, subsequent collisions contribute roughly equally with random phases. Therefore, the number of collisions needed for the packet to lose memory of its initial state is

$$N \sim \left(\frac{k_z V_{\rm A}}{k_\perp v_\lambda} \right)^4 . \tag{7}$$

Energy cascades only along $k_\perp$ and the cascade time is $t_c \sim N(k_z V_{\rm A})^{-1}$. Let ϵ be the energy pumped into the system per unit time, per unit mass, per unit logarithmic interval of $|k_z|$. Since there is equipartition between kinetic and magnetic energies, the energy per unit mass is $\sim v_\lambda^2$. Therefore, $\epsilon \sim v_\lambda^2/t_c$. The three-dimensional energy spectrum, E, is defined by

$$\sum v_\lambda^2 = \int E(k_z, k_\perp) \frac{d^3k}{8\pi^3} ,$$

[3] As noted earlier in the Introduction, in collisionless plasmas, the pseudo Alfvén waves undergo Barnes damping. This is a kinetic effect and is not described by the fluid equations (1). Since we are interested in applying our results to high β plasmas, we look only at resonant 4-wave interactions of shear Alfvén waves.

[4] Consistency with earlier notation would require v_λ to carry the subscripts $\lambda_\perp$ and $\lambda_\perp$ (where $\lambda_\perp \sim k_\perp^{-1}$ and $\lambda_z \sim \lambda_z^{-1}$), instead of the single subscript λ. We use the single subscript to avoid clutter.

where $\sum$ is a sum over wave packets of various scales. For a constant rate of cascade, we get

$$E(k_z, k_\perp) \sim \epsilon^{1/3} V_A k_\perp^{-10/3} , \tag{8}$$

where ϵ can depend only on k_z. This spectrum is valid only when the 4-wave process is the dominant interaction. A necessary condition is that the fractional change in the velocity per collision, equation (6), be small (equivalently, $N \gg 1$). Equation (8) implies that $v_\lambda \propto k_\perp^{-2/3}$. Using this in equation (7), we find that the number of collisions per cascade time is $N \propto k_\perp^{-4/3}$ at fixed k_z; *N decreases as the cascade proceeds to higher* $k_\perp$. When N becomes of order unity, the interactions become too strong to be described within the framework of the perturbative expansions of weak turbulence theory (not surprisingly, this coincides with fluid displacements becoming of order $k_\perp^{-1}$). Thus equation (8) is valid only for values of $k_\perp$ less than that for which $N \sim 1$. Writing $\epsilon \sim v_L^3/L$, where L is the spatial scale on which energy is injected, and v_L is the fluid velocity amplitude on that scale, we find that

$$(k_\perp L)_{\max} \sim \left(\frac{V_A}{v_L}\right)^{3/2} . \tag{9}$$

For this range in $k_\perp$ (i.e., the inertial range for the 4-wave energy cascade) to be substantial, the excitation amplitude v_L must be very small compared to V_A. When $v_L \sim V_A$, the inertial range shrinks to zero!

3. WEAK ALFVÉNIC TURBULENCE; THE THEORY IN SOME DETAIL

We use Newcomb's (1962) action principle for ideal MHD to compute the coupling coefficients for 4-wave interactions. Enroute, we will show that the third-order terms vanish, implying the absence of resonant 3-wave interactions. Following standard procedures, we set up a kinetic equation for 4-wave interactions; the energy spectrum (eq. [8]) is a stationary solution of the kinetic equation. The details of the proof are given in the Appendix.

3.1. *The Absence of 3-Wave Interactions*

We have modified Newcomb's approach to simplify calculations in the incompressible case. The basic variable is the displacement, $\boldsymbol{\xi}(\boldsymbol{x}_0, t)$, of a fluid element:

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{\xi}(\boldsymbol{x}_0, t) . \tag{10}$$

The equations of motion—equivalent to equations (1)—for the fluid are obtained by varying the action (S) with respect to $\boldsymbol{\xi}(\boldsymbol{x}_0, t)$, and requiring that the variation be stationary (i.e., $\delta S = 0$). The action is

$$S = \int dt\, \mathscr{L} , \tag{11}$$

and $\mathscr{L}$ is the Lagrangian, given by

$$\mathscr{L} = \int d^3x \left(\frac{\rho}{2}\left|\frac{\partial \boldsymbol{\xi}}{\partial t}\right|^2 - \frac{B^2}{8\pi}\right) . \tag{12}$$

Here $\boldsymbol{B}(\boldsymbol{x}, t)$ is the magnetic field at the displaced $(\boldsymbol{x}_0 + \boldsymbol{\xi})$ position of the fluid element. For an incompressible fluid, the Jacobian of the transformation (10) should be unity:

$$J \equiv |\delta_{ij} + \xi_{ij}| = 1 , \tag{13}$$

where $\xi_{ij} = \partial \xi_i / \partial x_j$. The constraint $J = 1$ must be used in the variation δS. We now use this to rewrite the magnetic energy term in the Lagrangian. If the magnetic field in the undisturbed fluid is B_{0i}, then $B_i = J^{-1}(\delta_{ij} + \xi_{ij})B_{0j}$. Using $J = 1$ and $B_{0j} = B_0 \delta_{j3}$, the magnetic term in the Lagrangian is

$$\int d^3x \frac{B^2}{8\pi} = \frac{B_0^2}{8\pi} \int d^3x_0 (1 + 2\xi_{33} + \xi_{i3}^2) . \tag{14}$$

The ξ_{33} term integrates to zero. Dropping the constant term and rewriting the Lagrangian in Fourier variables, we have

$$\mathscr{L} = \frac{\rho}{2} \int \frac{d^3k}{8\pi^3} [\dot{\boldsymbol{\xi}}(\boldsymbol{k}) \cdot \dot{\boldsymbol{\xi}}(-\boldsymbol{k}) - \omega^2(\boldsymbol{k}) \boldsymbol{\xi}(\boldsymbol{k}) \cdot \boldsymbol{\xi}(-\boldsymbol{k})] , \tag{15}$$

where $\omega(\boldsymbol{k}) = V_A |k_z|$. Let

$$\xi_i(\boldsymbol{k}) = \eta_i(\boldsymbol{k}) + \frac{k_i \phi(\boldsymbol{k})}{k} . \tag{16}$$

Here $\boldsymbol{\eta}$ and ϕ are the transverse and longitudinal parts of $\boldsymbol{\xi}$, respectively. We require $k_i \eta_i(\boldsymbol{k}) = 0$. The Lagrangian can be written as $\mathscr{L} = \mathscr{L}_\perp + \mathscr{L}_\parallel$. Then the Lagrangian is the sum of

$$\mathscr{L}_\perp = \frac{\rho}{2} \int \frac{d^3k}{8\pi^3} [\dot{\boldsymbol{\eta}}(\boldsymbol{k}) \cdot \dot{\boldsymbol{\eta}}(-\boldsymbol{k}) - \omega^2(\boldsymbol{k}) \boldsymbol{\eta}(\boldsymbol{k}) \cdot \boldsymbol{\eta}(-\boldsymbol{k})] , \tag{17}$$

and

$$\mathscr{L}_\parallel = -\frac{\rho}{2} \int \frac{d^3k}{8\pi^3} [\dot{\phi}(\boldsymbol{k})\dot{\phi}(-\boldsymbol{k}) - \omega^2(\boldsymbol{k})\phi(\boldsymbol{k})\phi(-\boldsymbol{k})] . \tag{18}$$

The constraint $J = 1$ has not yet been completely imposed on the Lagrangian. The Jacobian has terms of up to third-order in the spatial derivatives of ξ_i. For our purposes, an expansion of J up to second-order will suffice:

$$J \equiv 1 + \xi_{ii} - \tfrac{1}{2}\xi_{ij}\xi_{ji} + \tfrac{1}{2}\xi_{ii}^2 + O(\xi_{ij}^3) = 1 . \tag{19}$$

To first-order in ξ_{ij}, we have the familiar condition of incompressibility, namely $\nabla \cdot \boldsymbol{\xi} = 0$. This implies that $\phi(\boldsymbol{k}) = 0$, while there is no constraint at all on $\boldsymbol{\eta}$. Working to the next higher order, we find that $\phi(\boldsymbol{k})$ depends on $\boldsymbol{\eta}$:

$$\phi_k = i4\pi^3 \int \frac{d^3k_1}{8\pi^3} \frac{d^3k_2}{8\pi^3} \frac{(\boldsymbol{k}_1 \cdot \boldsymbol{\eta}_2)(\boldsymbol{k}_2 \cdot \boldsymbol{\eta}_1)}{|\boldsymbol{k}_1 + \boldsymbol{k}_2|} \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}) + O(\eta^3) , \tag{20}$$

where we have used a shortened notation, $\phi_k \equiv \phi(\boldsymbol{k})$, $\boldsymbol{\eta}_i \equiv \boldsymbol{\eta}(\boldsymbol{k}_i)$, and so on. Using equation (20) in equation (18), we see that $\mathscr{L}_{\text{long}}$ is, to lowest nonvanishing order, fourth order in $\boldsymbol{\eta}$. At this stage it is clear that the third-order terms are absent in the Lagrangian, implying that there are no resonant 3-wave interactions.[5] We note that the expression (20), for ϕ_k, has been obtained by solving equation (19) perturbatively. Such a solution converges only when ξ_{ij} is small, which is precisely the condition that the turbulence be weak (when ξ_{ij} becomes of

[5] The vanishing of the third-order terms means that even nonresonant 3-wave interactions are absent. If, instead of $\boldsymbol{\xi}$ we had used different variables—for instance, the Elsasser variables used in Paper II—we would, in general, find that the third order terms do not vanish. However, the resonant third-order terms will vanish, since these are invariant under near-identity change of variables. The situation is very similar to perturbation theory in Classical Mechanics, where a small perturbation of the Hamiltonian of an integrable system can be "absorbed" into the unperturbed Hamiltonian by a near-identity transformation of the action variables, so long as "small denominators" due to resonances don't arise.

order unity, we cannot conclude that 3-wave couplings are absent—this case will be treated in Paper II). Also, the constraint $J = 1$ has now been completely implemented.

3.2. *Derivation of the 4-Wave Kinetic Equation*

The true degrees of freedom are $\boldsymbol{\eta}_k$, and the Lagrangian to fourth-order is $\mathscr{L} = \mathscr{L}_2 + \mathscr{L}_4$, where $\mathscr{L}_2 = \mathscr{L}_\perp$, and $\mathscr{L}_4$ is

$$\begin{aligned}\mathscr{L}_4 = \pi^3\rho \int \frac{d^3k_1}{8\pi^3}\frac{d^3k_2}{8\pi^3}\frac{d^3k_3}{8\pi^3}\frac{d^3k_4}{8\pi^3}\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)\\ \times \frac{1}{|\boldsymbol{k}_1+\boldsymbol{k}_2||\boldsymbol{k}_3+\boldsymbol{k}_4|}\\ \times \{[(\boldsymbol{k}_1\cdot\boldsymbol{\eta}_2)(\boldsymbol{k}_2\cdot\partial_t\boldsymbol{\eta}_1)+(\boldsymbol{k}_2\cdot\boldsymbol{\eta}_1)(\boldsymbol{k}_1\cdot\partial_t\boldsymbol{\eta}_2)]\\ \times[(\boldsymbol{k}_3\cdot\boldsymbol{\eta}_4)(\boldsymbol{k}_4\cdot\partial_t\boldsymbol{\eta}_3)+(\boldsymbol{k}_4\cdot\boldsymbol{\eta}_3)(\boldsymbol{k}_3\cdot\partial_t\boldsymbol{\eta}_4)]\\ -\omega_k^2(\boldsymbol{k}_1\cdot\boldsymbol{\eta}_2)(\boldsymbol{k}_2\cdot\boldsymbol{\eta}_1)(\boldsymbol{k}_3\cdot\boldsymbol{\eta}_4)(\boldsymbol{k}_4\cdot\boldsymbol{\eta}_3)\}\,. \quad (21)\end{aligned}$$

In general, $\boldsymbol{\eta}_k$ is a linear combination of a shear Alfvén and a pseudo-Alfvén wave. As before we limit our considerations to the shear Alfvén wave. Then, we can write

$$\boldsymbol{\eta}_k = i\psi_k(\hat{\boldsymbol{k}}_\perp \times \hat{z})\,, \quad (22)$$

where ψ_k is the Fourier amplitude of a shear Alfvén wave whose wave vector is along $\boldsymbol{k}$. The "^" denotes a unit vector, and $\boldsymbol{k}_\perp = \boldsymbol{k} - \hat{z}(\boldsymbol{k}\cdot\hat{z})$. It should also be noted that $\boldsymbol{\eta}^*_{-k} = \boldsymbol{\eta}_k$ and $\psi^*_{-k} = \psi_k$. Substituting equation (22) in the Lagrangian, we arrive at

$$\begin{aligned}\mathscr{L} = \frac{\rho}{2}\int\frac{d^3k}{8\pi^3}(|\dot\psi_k|^2 - \omega_k^2|\psi_k|^2)\\ +\pi^3\rho\int\frac{d^3k_1}{8\pi^3}\frac{d^3k_2}{8\pi^3}\frac{d^3k_3}{8\pi^3}\frac{d^3k_4}{8\pi^3}\\ \times\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)R_{1234}\\ \times[(\psi_1\psi_2)\dot{}(\psi_3\psi_4)\dot{} - \omega_{(1+2)}^2\psi_1\psi_2\psi_3\psi_4]\,, \quad (23)\end{aligned}$$

where the coupling coefficient

$$R_{1234} = (q_1q_2q_3q_4)^{-1}\frac{|\boldsymbol{q}_1\times\boldsymbol{q}_2|^2}{|\boldsymbol{k}_1+\boldsymbol{k}_2|}\frac{|\boldsymbol{q}_3\times\boldsymbol{q}_4|^2}{|\boldsymbol{k}_3+\boldsymbol{k}_4|}\,. \quad (24)$$

We use $\boldsymbol{q}_i = (\boldsymbol{k}_\perp)_i$ and $p_i = (k_z)_i$ to avoid a multiplicity of subscripts. In equation (23), $\omega_{(1+2)}$ stands for $\omega(\boldsymbol{k}_1+\boldsymbol{k}_2)$. The resonant Lagrangian, $\mathscr{L}_R$, is obtained as follows: In equation (23), ψ_k is rewritten in terms of canonical variables, c_k and c_k^*,

$$\psi_k = \sqrt{\frac{2}{\omega_k}}\,(c_k e^{-i\omega_k t} + c^*_{-k}e^{i\omega_k t})\,.$$

Then, noting that the effect of the nonlinear terms is to force a slow change of the amplitude of c_k, we drop products of $\dot c_k$ and $\dot c_k^*$ with themselves in the second-order terms. Averaging the resulting expression over time, we arrive at the following expression for the resonant Lagrangian:

$$\begin{aligned}\mathscr{L}_R = 2i\rho\int\frac{d^3k}{8\pi^3}(c_k^*\dot c_k - c_k\dot c_k^*)\\ +8\pi^3\rho\int\frac{d^3k_1}{8\pi^3}\frac{d^3k_2}{8\pi^3}\frac{d^3k_3}{8\pi^3}\frac{d^3k_4}{8\pi^3}\\ \times\delta(\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)R_{1234}\\ \times\frac{[(\omega_3+\omega_4)^2-\omega_{(3+4)}^2]}{(\omega_1\omega_2\omega_3\omega_4)^{1/2}}c_1^*c_2^*c_3c_4e^{i(\omega_1+\omega_2-\omega_3-\omega_4)t}\,. \quad (25)\end{aligned}$$

The Euler-Lagrange equations of motion, $\partial_t(\delta\mathscr{L}/\delta\dot c_k^*) = \delta\mathscr{L}/\delta c_k^*$, can be written in the following standard form:

$$\begin{aligned}i\dot c_k = \tfrac{1}{2}\int d^3k_2\,d^3k_3\,d^3k_4\,\delta(\boldsymbol{k}+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\\ \times T_{k234}c_2^*c_3c_4e^{i(\omega_k+\omega_2-\omega_3-\omega_4)t}\,, \quad (26)\end{aligned}$$

where

$$T_{k234} = \frac{1}{64\pi^6}R_{k234}\frac{[\omega_{(3+4)}^2-(\omega_3+\omega_4)^2]}{(\omega_k\omega_2\omega_3\omega_4)^{1/2}}\,. \quad (27)$$

When k_{3z} and k_{4z} have the same sign, $T_{k234} = 0$. As anticipated in § 2.2, there are no interactions between waves traveling in the same direction (parallel or antiparallel to $\hat{z}$).

The slow time evolution of the amplitudes is described by a kinetic equation for the second order correlator, n_k, defined by

$$\langle c_k c_{k'}^*\rangle = n_k\,\delta(\boldsymbol{k}-\boldsymbol{k}')\,,$$

where n_k is the wave action (or "quanta," in quantum-mechanical language). The energy (per unit mass) per mode is $E(\boldsymbol{k}, t) = \omega_k n(\boldsymbol{k}, t)$, and the inertial-range energy spectrum of a 4-wave turbulent cascade therefore arises as a stationary solution of a kinetic equation for n_k. The derivation of the 4-wave kinetic equation is standard (see, e.g., Zakharov et al. 1992)—here we simply quote the result. The generic form is

$$\dot n_k = \mathscr{C}(\boldsymbol{k}, t)\,, \quad (28)$$

where the "collision" term $\mathscr{C}$ is

$$\begin{aligned}\mathscr{C}(\boldsymbol{k}, t) = \frac{\pi}{2}\int d^3k_2\,d^3k_3\,d^3k_4\,\delta(\boldsymbol{k}+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\\ \times\delta(\omega_k+\omega_2-\omega_3-\omega_4)\\ \times|T_{k234}|^2[n_3n_4(n_k+n_2)-n_kn_2(n_3+n_4)]\,. \quad (29)\end{aligned}$$

We find it convenient to write this explicitly as

$$\begin{aligned}\mathscr{C}(\boldsymbol{k}, t) = A\int d^2q_2\,d^2q_3\,d^2q_4\,\delta(\boldsymbol{q}+\boldsymbol{q}_2-\boldsymbol{q}_3-\boldsymbol{q}_4)\\ \times\frac{|\boldsymbol{q}\times\boldsymbol{q}_2|^4|\boldsymbol{q}_3\times\boldsymbol{q}_4|^4}{(qq_2q_3q_4)^2}\\ \times\int dp_2\,dp_3\,dp_4\,\delta(p+p_2-p_3-p_4)\\ \times\delta(|p|+|p_2|-|p_3|-|p_4|)\\ \times\frac{1}{|\boldsymbol{k}+\boldsymbol{k}_2|^2|\boldsymbol{k}_3+\boldsymbol{k}_4|^2}\frac{(p_3p_4-|p_3p_4|)^2}{|pp_2p_3p_4|}\\ \times[n_3n_4(n_k+n_2)-n_kn_2(n_3+n_4)]\,, \quad (30)\end{aligned}$$

where $A = 2\pi/V_A(64\pi^6)^2$, $n_k = n(\boldsymbol{q}, p)$, and so on.

In the integrals over p_3 and p_4, the presence of the delta-functions as well as the factor $(p_3p_4 - |p_3p_4|)^2$ ensure that the integrals are nonzero only when

(i) p and p_2 have opposite signs, and

(ii) either $p_3 = p$, $p_4 = p_2$, OR $p_4 = p$, $p_3 = p_2$.

These conditions are identical to those that we found in the discussion following equations (4) and (5). Since the entire expression on the right side of equation (30) is symmetric in the indices 3 and 4, we choose $p_3 = p$ and $p_4 = p_2$ and multiply the right side by a factor of 2.

3.3. *Stationary Solution of the 4-Wave Kinetic Equation*

Motivated by equation (8), we look for a stationary solution of the form

$$n(q, p) = \frac{f(p)}{q^{\nu}}, \quad (31)$$

where $f(p)$ is an arbitrary positive function of p that depends on the nature of the excitation. If the excitation occurs on an (outer) scale L, then we expect $f(p)$ to be small for $|p| > L^{-1}$. We assume that f is symmetric in p, so that waves traveling in opposite directions have equal strengths.[6]

$$\mathscr{C}_k = 8A \int d^2q_2\, d^2q_3\, d^2q_4\, \delta(\boldsymbol{q} + \boldsymbol{q}_2 - \boldsymbol{q}_3 - \boldsymbol{q}_4) \times \frac{|\boldsymbol{q} \times \boldsymbol{q}_2|^4 |\boldsymbol{q}_3 \times \boldsymbol{q}_4|^4}{(qq_2q_3q_4)^{2+\nu}} \int dp_2 \frac{S(p, p_2)[f(p)f(p_2)]^2}{|\boldsymbol{k} + \boldsymbol{k}_2|^4} \times \left\{\frac{q^{\nu}}{f(p)} + \frac{q_2^{\nu}}{f(p_2)} - \frac{q_3^{\nu}}{f(p)} - \frac{q_4^{\nu}}{f(p_2)}\right\}, \quad (32)$$

where the function $S(p, p_2)$ equals unity when p and p_2 have opposite signs, and equals zero otherwise. We simplify the right side by effectively doing the integral over p_2:

(i) Note that $|\boldsymbol{k} + \boldsymbol{k}_2|^2 = (p + p_2)^2 + |\boldsymbol{q} + \boldsymbol{q}_2|^2$. Since $f(p)$ and $f(p_2)$ are effectively cutoff when $|p| > L^{-1}$ or $|p_2| > L^{-1}$, retaining the $(p + p_2)$ term only introduces corrections due to a finite outer-scale; these are negligible in the inertial-range. Therefore, we let $|\boldsymbol{k} + \boldsymbol{k}_2|^4 \simeq |\boldsymbol{q} + \boldsymbol{q}_2|^4 = |\boldsymbol{q}_3 + \boldsymbol{q}_4|^4$, where the last equality is ensured by the delta-function.

(ii) We now integrate equation (32) over p. Defining a positive constant

$$g = 8A \int dp\, dp_2\, S(p, p_2) f(p) f^2(p_2) = 8A \int dp\, dp_2\, S(p, p_2) f^2(p) f(p_2),$$

that depends only on the specific form of the excitation, we can write the (p-integrated) kinetic equation as the difference of two positive terms:

$$\dot{N}_q \equiv \int dp\, \dot{n}_k = g(I^+ - I^-), \quad (33)$$

where

$$I^+ = \int d^2q_2\, d^2q_3\, d^2q_4\, \delta(\boldsymbol{q} + \boldsymbol{q}_2 - \boldsymbol{q}_3 - \boldsymbol{q}_4) \times \frac{|\boldsymbol{q} \times \boldsymbol{q}_2|^4 |\boldsymbol{q}_3 \times \boldsymbol{q}_4|^4}{(qq_2q_3q_4)^{2+\nu} |\boldsymbol{q} + \boldsymbol{q}_2|^4} (q^{\nu} + q_2^{\nu}), \quad (34)$$

$$I^- = \int d^2q_2\, d^2q_3\, d^2q_4\, \delta(\boldsymbol{q} + \boldsymbol{q}_2 - \boldsymbol{q}_3 - \boldsymbol{q}_4) \times \frac{|\boldsymbol{q} \times \boldsymbol{q}_2|^4 |\boldsymbol{q}_3 \times \boldsymbol{q}_4|^4}{(qq_2q_3q_4)^{2+\nu} |\boldsymbol{q}_3 + \boldsymbol{q}_4|^4} (q_3^{\nu} + q_4^{\nu}). \quad (35)$$

The energy per mode is $E(q, p) = \omega_k n(q, p)$. We find it convenient to integrate this over p; hence we define

$$\mathscr{E}(q) = \frac{1}{8\pi^3} \int dp E(q, p), \quad (36)$$

[6] When $f(p)$ is not symmetric, there are more general solutions for which the power-law indices differ for waves traveling in opposite directions.

which is the energy density in q-space. This satisfies a continuity equation,

$$\partial_t \mathscr{E} + \nabla_q \cdot [\hat{q}\mathscr{F}] = 0. \quad (37)$$

where the energy flux at $\boldsymbol{q}$ is $\hat{q}\mathscr{F}(q)$. Equation (30) describes the scattering of the $\boldsymbol{q}_3$ waves into $\boldsymbol{q}$ waves when the former run into waves traveling in the opposite direction. Then, if $\mathscr{E}(\boldsymbol{q}_3 \to \boldsymbol{q})$ is the rate of change of energy density at $\boldsymbol{q}$ due to scattering of waves from $\boldsymbol{q}_3$, we have

$$\dot{\mathscr{E}}(\boldsymbol{q}_3 \to \boldsymbol{q}) = \frac{AV_A}{\pi^3} \int d^2q_2\, d^2q_4\, \delta(\boldsymbol{q} + \boldsymbol{q}_2 - \boldsymbol{q}_3 - \boldsymbol{q}_4) \times \frac{|\boldsymbol{q} \times \boldsymbol{q}_2|^4 |\boldsymbol{q}_3 \times \boldsymbol{q}_4|^4}{(qq_2q_3q_4)^{2+\nu}} \times \int dp\, dp_2 |p| \frac{S(p, p_2)[f(p)f(p_2)]^2}{|\boldsymbol{k} + \boldsymbol{k}_2|^4} \times \left\{\frac{q^{\nu}}{f(p)} + \frac{q_2^{\nu}}{f(p_2)} - \frac{q_3^{\nu}}{f(p)} - \frac{q_4^{\nu}}{f(p_2)}\right\}. \quad (38)$$

The symmetry of this expression implies a detailed balance in the elementary scattering processes: i.e., $\dot{\mathscr{E}}(\boldsymbol{q}_3 \to \boldsymbol{q}) = -\dot{\mathscr{E}}(\boldsymbol{q} \to \boldsymbol{q}_3)$. It is now easy to write down an expression for the magnitude of the energy flux:

$$2\pi q \mathscr{F}(q) = \int_{q' > q} d^2q' \int_{q_3 < q} d^2q_3\, \dot{\mathscr{E}}(\boldsymbol{q}_3 \to \boldsymbol{q}'). \quad (39)$$

From equation (33), it is obvious that $\nu = 0$ gives a stationary solution. Upon inspection, it is also evident from equations (38) and (39), that this solution carries zero energy flux. This then describes thermodynamic equilibrium, with the expected ultraviolet divergence of energy in $\boldsymbol{q}$-space.

We give a simple argument to show why $\nu = 10/3$ yields a stationary solution of equation (33) corresponding to a constant flux of energy. From the continuity equation (37), we see that $\mathscr{F} \propto q^{-1}$ for the energy density to be time-independent. Noting that, in equation (38), replacing $|\boldsymbol{k} + \boldsymbol{k}_2|^4$ by $|\boldsymbol{q} + \boldsymbol{q}_2|^4$ introduces only a negligible error, it can be checked that equations (38) and (39) imply that $\mathscr{F} \propto q^{(9-3\nu)}$. Therefore the solution describing stationary, turbulent transport energy in $\boldsymbol{q}$-space has an index $\nu = 10/3$—in fact, this is the only solution of equation (33) that corresponds to a constant flux of energy. In the Appendix we show that this energy flux is positive.

4. DISCUSSION

The principal results of the present work are

1. The absence of resonant 3-wave interactions in Alfvénic turbulence, and the consequent failure of the IK theory,

2. A relatively complete analysis of resonant 4-wave interactions for shear Alfvén waves. The inertial-range energy spectrum is given in equation (8). Using this energy spectrum, we deduce that the fractional change, per collision, in either the velocity or the magnetic field perturbations *increases* with $k_{\perp}$. If the excitation amplitudes are much smaller than V_A, the cascade time is much longer than the wave period on the scale of the excitation, and the disturbance undergoes a weak, 4-wave cascade. We noted earlier that the resonance condition on the frequencies may be violated by an amount $\Delta\omega \sim t_{cas}^{-1}$. Weak turbulence theory deals with the limit in which $\Delta\omega$ is much smaller than a typical ω that is involved in wave interactions. For a weak excitation, this condition is satisfied (by definition) on the scale of the excitation. As interactions

strengthen at high $k_\perp$, the cascade time decreases relative to the wave period, and the allowed violation of the frequency resonance condition (or frequency "conservation law") increases with $k_\perp$, until $t_{\rm cas} \sim \omega^{-1}$, when the allowed violation is of order unity. At this stage in the energy cascade, the basic conditions of validity of weak turbulence are not satisfied. The $k_\perp$ at which this occurs sets the upper bound on $k_\perp$ for the validity of the inertial-range energy spectrum given in equation (8). For larger $k_\perp$, the turbulence will be too *strong* to be described by weak 4-wave interactions. On the other hand, if the perturbation amplitudes on the excitation scale are of order V_A, or perhaps even larger, the interactions are so strong that the width of the 4-wave inertial-range shrinks to zero. In this case, the turbulence is *strong* from the beginning; we explore this case in Paper II.

Following Higdon (1984), we imagine that the electron density fluctuations are isobaric, entropy fluctuations having little effect on the dynamics of the turbulent, ionized medium. The electron density fluctuations will be advected (and mixed) by the turbulent velocity field. In neutral fluids undergoing an isotropic Kolmogorov cascade, the power spectrum of a passive scalar contaminant acquires the same shape as the power spectrum of the turbulent velocity fluctuations (see, e.g., Lesieur 1990). The velocity power spectrum of the weak 4-wave cascade (eq. [8]) has a two-dimensional index equal to 10/3. Line of sight variations in the direction of the mean magnetic field would isotropize the spectrum so that it mimicked a three-dimensional spectrum with index 13/3. The observed interstellar electron density power spectrum appears to have a three-dimensional index close to 11/3. Clearly, interstellar turbulence cannot be weak 4-wave Alfvénic turbulence. Moreover, the inertial-range is of limited extent in $k_\perp$ (see eq. [9]). For this range to span at least 6 decades in wavenumber, the excitation amplitudes would have to be unrealistically small! The theory of *strong* Alfvén turbulence developed in Paper II describes an anisotropic inertial-range energy spectrum with two-dimensional index equal to 8/3. Since line of sight averaging would make this mimic a three-dimensional spectrum with index equal to 11/3, it is a plausible candidate for a theory of interstellar turbulence.

We end the paper on a cautionary note. Approximating the ionized interstellar medium as an incompressible medium is indeed an idealization we would like to relax! We must study turbulence in a $\beta \sim 1$ plasma. In a compressible plasma, the shear Alfvén wave traveling in one direction is no longer an exact nonlinear solution to the equations of motion; nonlinear interactions result in a steepening of the wave (Cohen & Kulsrud 1974; Kennel et al. 1988). This allows for the possibility of resonant 3-wave interactions. We invoked Barnes' damping as a motivation to avoid consideration of the fast and slow magnetosonic modes. However, it must be noted that, for propagation nearly perpendicular to the mean magnetic field, the fast mode suffers negligible damping. Even in those directions in which Barnes' damping is significant, we must check that the damping is efficient enough to kill fast and slow waves quicker than the time over which they can themselves undergo a cascade. All these caveats are but aspects of the complications encountered in dealing with a compressible medium that must be faced in the process of building a theory of interstellar turbulence.

We would like to express our gratitude to some of our colleagues. Discussions with Roger Blandford, Alice Quillen, and Yanqin Wu were useful in clarifying several points in this paper. Sharadini Rath helped with the numerical evaluation of the energy flux. Jeremy Goodman demonstrated to us how a new choice of variables (eq. [16]) facilitates the implementation of the "Jacobian constraint" (eq. [13]). An anonymous referee's comments on the manuscript were very useful, and a free exchange of ideas with Robert Kraichnan was especially influential in shaping our work. This research was supported in part by NSF grant 89-13664 and NASA grant NAGW 2372.

APPENDIX

We have already shown in § 3.3 *that a stationary solution of equation* (33) *that possesses a nonzero energy flux in* $\boldsymbol{q}$*-space must have* $\nu = 10/3$. *Here we prove that the* $\boldsymbol{q}$*-space energy flux is positive as assumed in the heuristic treatment given in* § 2.2.

For a stationary solution, we must have $I^+ = I^-$. We adopt the strategy of transforming equations (34) and (35) until they resemble each other as much as possible:

(i) In equation (34) defining I^+, eliminate $\boldsymbol{q}_2$ using the δ-function, and let $\boldsymbol{q} \to -\boldsymbol{q}$.

(ii) In equation (35) defining I^-, eliminate $\boldsymbol{q}_4$ using the δ-function, replace the dummy index $\boldsymbol{q}_2$ by another dummy index $-\boldsymbol{q}_4$, and let $\boldsymbol{q} \to -\boldsymbol{q}$.

(iii) Since both I^+ and I^- depend only on $|\boldsymbol{q}|$, we integrate them over θ_q, which is the (polar) angle giving the direction of $\boldsymbol{q}$. Then, we have

$$2\pi I^+ = q^{-(2+\nu)} \int_0^\infty \int_0^\infty \frac{dq_3\, dq_4}{(q_3 q_4)^{(1+\nu)}} \left[q^\nu F(q, q_3, q_4; 2+\nu) + F(q, q_3, q_4; 2)\right],$$

$$2\pi I^- = q^{-(2+\nu)} \int_0^\infty \int_0^\infty \frac{dq_3\, dq_4}{(q_3 q_4)^{(1+\nu)}} \left[q_3^\nu F(q_3, q, q_4; 2+\nu) + F(q_3, q, q_4; 2)\right],$$

where $F(a, b, c; \mu)$ is a function of three positive numbers a, b, c, and an index μ. It is constructed as follows. Given a, b, c, form three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ lying in a plane. Then

$$F(a, b, c; \mu) = \oint d\theta_a\, d\theta_b\, d\theta_c \frac{|\boldsymbol{b} \times \boldsymbol{c}|^4 |\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c})|^4}{|\boldsymbol{b} + \boldsymbol{c}|^4 |\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}|^\mu},$$

is positive, symmetric in b and c, and homogeneous in a, b, and c:

$$F(\lambda a, \lambda c, \lambda b; \mu) = \lambda^{(12-\mu)} F(a, b, c; \mu).$$

(iv) In I^+, let $q_3 = qr$, and $q_4 = qs$. In I^-, let $q_3 = q/r$, and $q_4 = q(s/r)$. Noting that F is homogeneous, we get

$$2\pi I^+ = q^{(8-3\nu)} \int_0^\infty \int_0^\infty \frac{dr\, ds}{(rs)^{(1+\nu)}} [F(1, r, s; 2+\nu) + F(1, r, s; 2)] ,$$

$$2\pi I^- = q^{(8-3\nu)} \int_0^\infty \int_0^\infty \frac{dr\, ds}{(rs)^{(1+\nu)}} [F(1, r, s; 2+\nu) + F(1, r, s; 2)] r^{(3\nu-10)} .$$

As found earlier in § 3.3, $\nu = 10/3$ gives a stationary solution.

We next compute the energy flux associated with this spectrum. Let us define two positive constants, W and m, that depend only on the form of the excitation (m is of order unity);

$$W = \frac{AV_A}{\pi^3} \int dp\, dp_2 |p| S(p, p_2) \frac{[f(p)f(p_2)]^2}{f(p_2)} ,$$

$$mW = \frac{AV_A}{\pi^3} \int dp\, dp_2 |p| S(p, p_2) \frac{[f(p)f(p_2)]^2}{f(p)} .$$

We use the detailed balance relation, $\dot{\mathscr{E}}(\boldsymbol{q}_3 \to \boldsymbol{q}) = -\dot{\mathscr{E}}(\boldsymbol{q} \to \boldsymbol{q}_3)$, to rewrite equation (39). Detailed balancing implies that

$$\int_{q'>q} \int_{q_3>q} d^2q'\, d^2q_3\, \dot{\mathscr{E}}(\boldsymbol{q}_3 \to \boldsymbol{q}') = 0 .$$

Adding this expression to the right side of equation (39), we get a more convenient expression for the energy flux:

$$2\pi q \mathscr{F}(q) = \int_{q'>q} d^2q' \int d^2q_3 \dot{\mathscr{E}}(\boldsymbol{q}_3 \to \boldsymbol{q}') ;$$

together with equation (38), this allows us to write the flux as the difference of two positive terms:

$$2\pi q \mathscr{F}(q) = W(\mathscr{I}^+ - \mathscr{I}^-) ,$$

where

$$\mathscr{I}^+ = \int_{q'>q} d^2q' \int d^2q_2\, d^2q_3\, d^2q_4\, \delta(\boldsymbol{q}' + \boldsymbol{q}_2 - \boldsymbol{q}_3 - \boldsymbol{q}_4) \frac{|\boldsymbol{q}' \times \boldsymbol{q}_2|^4 |\boldsymbol{q}_3 \times \boldsymbol{q}_4|^4}{|\boldsymbol{q}' + \boldsymbol{q}_2|^4 (q' q_2 q_3 q_4)^{(2+\nu)}} (mq'^{\nu} + q_2^{\nu}) ,$$

$$\mathscr{I}^- = \int_{q'>q} d^2q' \int d^2q_2\, d^2q_3\, d^2q_4\, \delta(\boldsymbol{q}' + \boldsymbol{q}_2 - \boldsymbol{q}_3 - \boldsymbol{q}_4) \frac{|\boldsymbol{q}' \times \boldsymbol{q}_2|^4 |\boldsymbol{q}_3 \times \boldsymbol{q}_4|^4}{|\boldsymbol{q}_3 + \boldsymbol{q}_4|^4 (q' q_2 q_3 q_4)^{(2+\nu)}} (mq_3^{\nu} + q_4^{\nu}) .$$

It is evident that $\mathscr{I}^+$ and $\mathscr{I}^-$ bear a strong resemblence to I^+ and I^-. We follow the same steps [(i)–(iv) given above] we used to simplify I^+ and I^-. Then

$$\mathscr{I}^+ = \int_q^\infty dq' q'^{(9-3\nu)} \int_0^\infty \int_0^\infty \frac{dr\, ds}{(rs)^{(1+\nu)}} [mF(1, r, s; 2+\nu) + F(1, r, s; 2)] ,$$

$$\mathscr{I}^- = \int_q^\infty dq' q'^{(9-3\nu)} \int_0^\infty \int_0^\infty \frac{dr\, ds}{(rs)^{(1+\nu)}} [mF(1, r, s; 2+\nu) + F(1, r, s; 2)] r^{(3\nu-10)} ,$$

and

$$2\pi q F(q) = W \frac{q^{(10-3\nu)}}{(3\nu-10)} \int_0^\infty \int_0^\infty \frac{dr\, ds}{(rs)^{(1+\nu)}} [mF(1, r, s; 2+\nu) + F(1, r, s; 2)](1 - r^{(3\nu-10)}) ,$$

is well defined for $\nu > 10/3$. As $\nu \to \nu_0 = 10/3$ from above, we have an indeterminate ratio of the form 0/0. Resolving this using L'Hospital's rule, we have

$$2\pi q \mathscr{F}\bigg|_{\nu_0} = \frac{W}{3} \frac{\partial}{\partial \nu} \int_0^\infty \int_0^\infty \frac{dr\, ds}{(rs)^{(1+\nu)}} [mF(1, r, s; 2+\nu) + F(1, r, s; 2)](1 - r^{(3\nu-10)})\bigg|_{\nu_0}$$

$$= \frac{-W}{2} \int_0^\infty \int_0^\infty \frac{dr\, ds}{(rs)^{(1+\nu)}} [mF(1, r, s; 2+\nu) + F(1, r, s; 2)] \ln(rs)\bigg|_{\nu_0} .$$

In the $r - s$ plane, the region where $rs < 1$ makes a positive contribution to the flux, while the region $rs > 1$ makes a negative contribution. We have evaluated the integrals numerically, and we find that the flux is positive.

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