Acausality of Massive Gravity

S. Deser*

Lauritsen Laboratory, Caltech, Pasadena, California 91125, USA and Physics Department, Brandeis University, Waltham, Massachusetts 02454, USA

A. Waldron†

Department of Mathematics, University of California, Davis, California 95616, USA

(Received 23 December 2012; published 12 March 2013)

We show, by analyzing its characteristics, that the ghost-free, 5 degree of freedom, Wess-Zumino massive gravity model admits superluminal shock wave solutions and thus is acausal. Ironically, this pathology arises from the very constraint that removes the (sixth) Boulware-Deser ghost mode.

DOI: 10.1103/PhysRevLett.110.111101 PACS numbers: 04.50.Kd, 04.20.Gz, 14.70.Kv

Introduction.—Over four decades ago, Isham, Salam and Strathdee proposed a two-tensor "f-g" theory [1] by adding to the Einstein action that of a second vierbein $f_\mu{}^m$, plus a nonderivative coupling term, leaving a single common coordinate invariance. Of particular interest is the limit of nondynamical (say, flat) f, giving gravitons a finite range due to the coupling "mass" term. It was rapidly shown [2], however, that, unlike their linearized massive spin 2 Fierz-Pauli (FP) limits, these models suffered from a ghost problem: Generic nonlinearities reinstate a 6th degree of freedom (DoF) beyond the linearized $2s+1=5$ DoF, one of which is necessarily ghostlike. A final twist, also from that time, was the Wess-Zumino [3] discovery of a distinguished set of f-g mass terms. At least one of these models turned out to be immune from this disease, keeping 5 DoF. Because Ref. [3] was only published without detail in lecture notes, it remained unknown. Separately, other analyses showed that the linearized theory's matter coupling seemed to suffer a zero-mass discontinuity [4], as well as a failure of the Birkhoff theorem [5]. Hence, the subject remained moribund until the recent (independent) rediscovery [6] of the results [3] plus two new f-g models. This exhumation has, unsurprisingly, generated an immense industry (see the recent survey [7]). Our purpose is to reinter f-g. We will show that the 5 DoF Wess-Zumino model is acausal [8]. Our methods also show that, of the two remaining 5 DoF models [6], one is definitely acausal and the other is likely so [9]. Paradoxically, acausalities arise precisely because of the very constraint that removes the ghost. Note that there is no conflict between acausality and ghostlessness, as witnessed by the old "charged" higher ($s>1$) spin interactions with Maxwell and gravity, say, those of $s=(3/2,2)$ [14–16], that are also invalidated only by acausality.

Our results will be obtained by using the method of characteristics, analyzing the constraints' shock wave discontinuities, in particular, that of the "fifth" scalar one that results from combining the trace and double divergence of the field equations, just as is done in the linear FP model, to find a derivative-free constraint.

The model and the fifth constraint.—Our concrete 5 DoF model is

$$\mathcal{G}_{\mu\nu} := G_{\mu\nu}(g) + m^2(f_{\mu\nu} - g_{\mu\nu}f) = 0, \tag{1}$$

where all indices are moved by the dynamical metric $g_{\mu\nu}$ and its associated vierbein $e_\mu{}^m$; in particular, $f_{\mu\nu}$ is the fixed background vierbein $f_\mu{}^m$ times $e_{\nu m}$ and is manifestly symmetric on shell. The vanishing of its antisymmetric part yields six conditions. Taking the reference $f_\mu{}^m$ field as the flat bein is a popular choice but is not physically required; in fact, our results, both for acausality and the absence of the sixth ghost mode, depend neither on f being flat nor on the dimensionality of spacetime. The parameter m^2 reduces to the FP mass in the weak e-field limit. Next, we proceed as in the FP development and seek five constraints to reduce the *a priori* 10 metric DoF (now that coordinate invariance is lost due to the preferred background). The single derivative four-vector constraint is obviously (by the Bianchi identity) the covariant g divergence of Eq. (1),

$$0 = \mathcal{C}_\nu := \nabla^\mu \mathcal{G}_{\mu\nu} = m^2(\nabla\cdot f_\nu - \nabla_\nu f).$$

The scalar constraint results from taking the (covariantized) FP combination

$$0 = \mathcal{C} := \nabla_\mu(\ell^{\mu\nu}\nabla\cdot\mathcal{G}_\nu) + \frac{m^2}{2}\mathcal{G}, \tag{2}$$

with $\ell^{\mu\nu} := \ell^\mu{}_m e^{\nu m}$, where $\ell^\mu{}_m$ is the inverse of the background vierbein $f_\mu{}^m$. The proof that $\mathcal{C}$ is indeed a constraint, i.e., devoid of second derivatives, is simple: In the spirit of Ref. [17], we observe that the (torsion-free) Levi-Cività spin connection $\omega(e)_\mu{}^m{}_n$ corresponding to the vierbein $e_\mu{}^m$ will in general become torsionful when employed as the spin connection for the nondynamical vierbein $f_\mu{}^m$. The difference between this connection and the Levi-Cività spin connection $\omega(f)_\mu{}^m{}_n$ of $f_\mu{}^m$ yields the contorsion tensor

$$K_\mu{}^m{}_n := \omega(e)_\mu{}^m{}_n - \omega(f)_\mu{}^m{}_n.$$

It measures the failure of parallelograms of the dynamical metric to close with respect to the background metric (and vice versa). As will become apparent, it is important to emphasize that flatness of the background metric does *not* ensure vanishing contorsion. In these terms, the vector constraint reads

$$0 = \mathcal{C}_\mu = m^2 K_\nu{}^{\nu\rho} f_{\mu\rho}.$$

In particular, this means that metric derivatives enter the vector constraint only through the trace of the spin connection $\omega(e)$. However, the leading (second) derivative terms of the scalar curvature R are proportional to $\partial_\mu \omega(e)_\nu{}^{\nu\mu}$. Hence, the linear combination of the divergence of the vector constraint and the trace of the equation of motion quoted in Eq. (2) yields the remaining scalar constraint $\mathcal{C} = 0$. This ensures that the model does not propagate spurious ghost degrees of freedom and thus evades the generic difficulties associated with massive gravity theories [2].

For our purposes, an explicit evaluation of the scalar constraint $\mathcal{C}$ is needed: we first express the scalar curvature in terms of the contorsion

$$\begin{aligned} R = {} & 2\nabla_\mu K_\nu{}^{\nu\mu} - K_{\mu\nu\rho}K^{\nu\rho\mu} - K_\mu{}^{\mu\rho}K_\nu{}^\nu{}_\rho \\ & + e^\nu{}_m e^\mu{}_n R(f)_{\mu\nu}{}^{mn}, \end{aligned}$$

where $R(f)$ is the Riemann tensor corresponding to the vierbein $f_\mu{}^m$. Observing that the second K^2 term is the square of the vector constraint $-\frac{1}{m^4}\mathcal{C}_\mu \ell^{\mu\nu}\ell_{\nu\rho}\mathcal{C}^\rho$, we have the modified constraint

$$\begin{aligned} 0 &= \mathcal{C} - \frac{1}{2m^2}(\mathcal{C}\cdot\ell_\nu)^2 \\ &= -\frac{3m^4}{2} f - \frac{m^2}{2} e^\nu{}_m e^\mu{}_n R(f)_{\mu\nu}{}^{mn} + \frac{m^2}{2} K_{\mu\nu\rho}K^{\nu\rho\mu}. \end{aligned}$$

The first term is the familiar FP trace, and the second one vanishes for flat $f_\mu{}^m$. We will see in the next section that it is the third term that has dire consequences for the causality of the model. While it does vanish for special solutions whose contorsion obeys $K_{\mu\nu\rho} - K_{\rho\nu\mu} = 0$, imposing this condition as an additional constraint would remove further field theoretical DoF, an obviously unacceptable tradeoff.

Acausality.—We study the causality of the model via its characteristics, using a method first introduced in a field theoretical context in Refs. [14,18]. This allows us to determine the maximum speed of propagation by studying a shock whose second derivatives are discontinuous across its wave front. Since the model is second order in derivatives, we assume that the dynamical metric $g_{\mu\nu}$ and its first derivatives are continuous across the hypersurface spanned by the shock's wave front Σ. The inert $f_\mu{}^m$ background is of course continuous. Note that we are studying causality with respect to the dynamical metric g, not the background, this being a putative theory of the metric field. (Actually, our conclusions are equally valid with respect to the background metric.) Then, g, being smooth across Σ, defines local light cones that allow us to decide whether the shock wave front corresponds to superluminal propagation.

To start, we denote the leading discontinuity in the metric across Σ by square brackets

$$[\partial_\alpha\partial_\beta g_{\mu\nu}]_\Sigma = \xi_\alpha\xi_\beta\gamma_{\mu\nu},$$

where ξ_μ is a vector normal to the characteristic and $\gamma_{\mu\nu}$ is some nonvanishing symmetric tensor defined on the characteristic surface. Propagation is acausal whenever the field equations admit characteristics with a timelike normal ξ_μ, i.e.,

$$\xi^\mu g_{\mu\nu}\xi^\nu < 0;$$

it can be analyzed by studying the field equations and any combinations of field equations and their derivatives that are of degree 2 or less in derivatives on $g_{\mu\nu}$ and so have a well-defined discontinuity across Σ. This, of course, amounts to studying the discontinuity of $\mathcal{G}_{\mu\nu}$ and the constraints $\mathcal{C}_\mu$ and $\mathcal{C}$ across Σ.

First, we consider the antisymmetric part of the equation of motion $\mathcal{G}_{\mu\nu}$ implying $f_{\mu\nu} = f_{\nu\mu}$. For this, we must compute the discontinuity of the vierbein. Since these depend algebraically on the metric, we have

$$[\partial_\alpha\partial_\beta e_\mu{}^m]_\Sigma = \xi_\alpha\xi_\beta\mathcal{E}_\mu{}^m,$$

where $\mathcal{E}_\mu{}^m$ is some tensor defined on the characteristic surface. Computing the discontinuity of the relation $e_\mu{}^m\eta_{mn}e_\nu{}^n = g_{\mu\nu}$ gives $\xi_\alpha\xi_\beta(\mathcal{E}_{\mu\nu} + \mathcal{E}_{\nu\mu}) = \xi_\alpha\xi_\beta\gamma_{\mu\nu}$. At this point, we proceed by contradiction by taking ξ_μ as timelike. Without loss of generality, we may therefore set

$$\xi^\mu g_{\mu\nu}\xi^\nu = -1$$

and thus learn that

$$\mathcal{E}_{\mu\nu} + \mathcal{E}_{\nu\mu} = \gamma_{\mu\nu}.$$

A similar computation based on the symmetry of $f_{\mu\nu}$ gives

$$f_\mu{}^\rho\mathcal{E}_{\nu\rho} = f_\nu{}^\rho\mathcal{E}_{\mu\rho}. \tag{3}$$

Next, we compute the leading discontinuity in the field equation $\mathcal{G}_{\mu\nu}$ and in turn its trace $\mathcal{G}$. Since this amounts to studying the second derivative terms in these equations, the result coincides with that of the FP theory computed long ago in Refs. [15,16] (save that the indices are raised and lowered with the metric $g_{\mu\nu}$):

$$\begin{aligned} &\xi^2\gamma_{\mu\nu} - \xi_\mu\xi\cdot\gamma_\nu - \xi_\nu\xi\cdot\gamma_\mu + \xi_\mu\xi_\nu\gamma = 0, \\ &\qquad\qquad\qquad\qquad \xi^2\gamma - \xi\cdot\xi\cdot\gamma = 0. \end{aligned} \tag{4}$$

It is clearly useful to decompose our variables with respect to the (unit) timelike vector ξ_μ. In particular, for a vector, symmetric tensor, and antisymmetric tensor we have, respectively,

$$\begin{aligned} &V_\mu := V^\perp_\mu - \xi_\mu\xi\cdot V, \\ &S_{\mu\nu} := S^\perp_{\mu\nu} - \xi_\mu S^\perp_\nu - \xi_\nu S^\perp_\mu + \xi_\mu\xi_\nu\xi\cdot\xi\cdot S \quad (S_\mu := \xi\cdot S_\mu), \\ &A_{\mu\nu} := A^\perp_{\mu\nu} + \xi_\mu A^\perp_\nu - \xi_\nu A^\perp_\mu \qquad\qquad (A^\perp_\mu := A_{\mu\nu}\xi^\nu). \end{aligned}$$

In this language, Eq. (4) implies that $\gamma^\perp_{\mu\nu} = 0$, so

$$\gamma_{\mu\nu} = -\xi_\mu\gamma^\perp_\nu - \xi_\nu\gamma^\perp_\mu + \xi_\mu\xi_\nu\xi\cdot\xi\cdot\gamma. \tag{5}$$

The next task is to compute the discontinuity in the vector constraint:

$$\begin{aligned} [\xi^\alpha\partial_\alpha\mathcal{C}_\mu]_\Sigma &= m^2\xi^\alpha[\partial_\alpha\omega(e)_\rho{}^{\rho\sigma}]_\Sigma f_{\mu\sigma} \\ &= -m^2(\mathcal{E}_\nu{}^\nu\xi^\sigma - \mathcal{E}^{\sigma\nu}\xi_\nu)f_{\mu\sigma}. \end{aligned}$$

Since $f_{\mu\nu}$ is assumed to be invertible, by decomposing

$$2\mathcal{E}_{\mu\nu} = \gamma_{\mu\nu} + a_{\mu\nu}$$

into its symmetric and antisymmetric parts, we learn that

$$0 = \gamma^\perp_\mu + a^\perp_\mu. \tag{6}$$

Together, Eqs. (5) and (6) give $2\mathcal{E}_{\mu\nu} = a^\perp_{\mu\nu} - 2\xi_\mu\gamma^\perp_\nu + \xi_\mu\xi_\nu\xi\cdot\xi\cdot\gamma$ so that Eq. (3) becomes

$$\begin{aligned} 0 = f^{\perp\rho}_\mu a^\perp_{\nu\rho} + \xi_\mu(2f^{\perp\rho}_\nu\gamma^\perp_\rho - f^\perp_\rho a^{\perp\rho}_\nu - \xi\cdot\xi\cdot\gamma f^\perp_\nu) \\ - (\mu \leftrightarrow \nu). \end{aligned} \tag{7}$$

The terms perpendicular and parallel to ξ_μ must vanish separately, so

$$f^{\perp\rho}_\mu a^\perp_{\nu\rho} - f^{\perp\rho}_\nu a^\perp_{\mu\rho} = 0 = 2f^{\perp\rho}_\nu\gamma^\perp_\rho - f^\perp_\rho a^{\perp\rho}_\nu - \xi\cdot\xi\cdot\gamma f^\perp_\nu. \tag{8}$$

The first set of these equations *generically* gives three independent linear conditions on as many unknowns ($a^\perp_{\mu\nu}$) and so enforces $a^\perp_{\mu\nu} = 0$. The second set then gives three conditions on the four remaining nonvanishing unknowns $\gamma^\perp_\mu$ and $\xi\cdot\xi\cdot\gamma$. Thus, *generically* three linear combinations of these vanish, leaving one nonzero linear combination. If this were to vanish, we would have established the absence of shock wave fronts Σ with timelike normal ξ_μ. (Of course, one still would have to verify the absence of special cases for the two italicized appearances of "generically" in the preceding argument; those are irrelevant in the face of the generic acausality we are about to exhibit.)

At this stage, then, the model is left requiring one more condition on $\mathcal{E}_\mu{}^m$ for its causal consistency. That condition can only be derived from the remaining scalar constraint $\mathcal{C}$, whose discontinuity across Σ we compute next. To begin, to better exhibit the problem we are about to find, let us make the assumption that the background is flat and that the contorsion vanishes so that the remaining constraint implies $f = 0$ whose discontinuity across Σ implies $f^{\mu\nu}\mathcal{E}_{\mu\nu} = 0$. This provides the remaining independent linear relation between $\xi\cdot\xi\cdot\gamma$ and $\gamma^\perp_\mu$ required to establish that $\mathcal{E}_\mu{}^m = 0$ and in turn the absence of superluminal shocks—*so long as the contorsion vanishes*.

However, the contorsion does not vanish as a consequence of the field equations (in fact, as discussed above, this would imply too many conditions on the field theoretic DoF). Thus, a proper computation of the discontinuity of $\mathcal{C}$ reads

$$\begin{aligned} \left[\xi^\alpha\partial_\alpha\left(\mathcal{C} - \frac{1}{2m^2}(\mathcal{C}\cdot\ell_\nu)^2\right)\right]_\Sigma &= \frac{m^2}{2}\xi^\alpha[\partial_\alpha(K_{\mu\nu\rho}K^{\nu\rho\mu})]_\Sigma \\ &= -\frac{m^2}{2}\xi_\nu K^{\mu\nu\rho}\mathcal{E}_{\rho\mu} \\ &= \frac{m^2}{4}\xi_\nu K^{\mu\nu\rho}a^\perp_{\mu\rho}. \end{aligned}$$

Thus, instead of a relation involving $\xi\cdot\xi\cdot\gamma$ and $\gamma^\perp_\mu$, we find the seemingly additional, but in fact redundant, requirement $\xi_\nu K^{\mu\nu\rho}a^\perp_{\mu\rho} = 0$ on $a^\perp_{\mu\nu}$. Therefore, since some linear combination of $\xi\cdot\xi\cdot\gamma$ and $\gamma^\perp_\mu$ does not vanish, timelike shock normals are allowed. This establishes the promised presence of acausal characteristics for any choice of background.

Discussion.—We have just shown that one otherwise ghost-free, acceptable finite range gravity model is excluded. How far does this no-go result extend to all three possible such combinations, quite apart from other previously mentioned obstacles to these models? Very recently, causality for models with mass terms quadratic in f has been ruled out [9] using methods similar to the present ones. This leaves only the third candidate mass term, cubic in f: Any model of the form $G_{\mu\nu}(g) = T_{\mu\nu}(f, e)$ with algebraic T universally yields Eq. (5) for the shock; the structure of the fifth constraint is at the root of the acausality [19]. Its covariant version for the third mass term is as yet unknown, but, if it takes the generic form $f^3 + f^2K^2$ where K is the contorsion, the argument of Ref. [9] already establishes its acausality. Even if it does not, there is a potentially new source of discontinuity, closer to that of the charged massive spin 3/2 and 2 systems [14–16,18]. Namely, zeros in the characteristic matrix can allow superluminal characteristics, just as critical values of the background E/M field permit superluminal signal propagation in the charged $s = (3/2, 2)$ models. In fact, for those models, acausality can be traced to nonpositivity of equal time commutators, a fatal physical flaw [21]. Furthermore, tachyonic excitations would interact with, and thereby also affect, matter sources.

We conclude, therefore, that the acausality we have exhibited is an unavoidable pathology of f-g massive gravity, barring some miracle of the cubic model or some (as yet unknown) underlying "rescue" modification [22] that also yields a smooth massless limit [24]. Indeed, the fact that neither GR nor Yang-Mills have massive "neighbors" is a self-sharpening Occam's razor that further ornaments these fundamental pillars!

We thank M. Sandora for a very useful discussion; A. W. acknowledges the hospitality of the Lauritsen Lab, Caltech. S. D. was supported in part by NSF Grant No. PHY-1064302 and DOE Grant No. DE-FG02-164 92ER40701.

*deser@brandeis.edu
†wally@math.ucdavis.edu

[1] A. Salam and J. Strathdee, Phys. Rev. **184**, 1750 (1969); **184**, 1760 (1969); C. J. Isham, A. Salam, and J. Strathdee, Phys. Lett. **31B**, 300 (1970).
[2] S. Deser and D. G. Boulware, Phys. Rev. D **6**, 3368 (1972).
[3] B. Zumino, in *Brandeis University Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT, Cambridge, MA, 1970), Vol. 2, p. 437.
[4] H. van Dam and M. J. G. Veltman, Nucl. Phys. **B22**, 397 (1970); V. I. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. **12**, 447 (1970) [JETP Lett. **12**, 312 (1970)].
[5] P. van Nieuwenhuizen, Phys. Rev. D **7**, 2300 (1973).
[6] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011); Phys. Lett. B **711**, 190 (2012); C. de Rham and G. Gabadadze, Phys. Rev. D **82**, 044020 (2010); S. F. Hassan and R. A. Rosen, Phys. Rev. Lett. **108**, 041101 (2012).
[7] K. Hinterbichler, Rev. Mod. Phys. **84**, 671 (2012).
[8] An earlier massive gravity causality study [10] found superluminal behavior in the auxiliary fields of the model's Stückelberg formulation. However, these superluminal modes amount to unphysical background metrics [11]. Nonetheless, this effect could well be related to known horizon and null energy difficulties of one of either one of the metrics of a bimetric theory [12]. We emphasize that our classical results are valid for all (nonzero) m; hence, even though these—like any classical—models have a limited range of validity (as has been argued from a quantum viewpoint in Ref. [13]), their acausality is surely present in that region.
[9] S. Deser, M. Sandora, and A. Waldron, arXiv:1301.5621.
[10] A. Gruzinov, arXiv:1106.3972.
[11] C. de Rham, G. Gabadadze, and A. J. Tolley, arXiv:1107.0710.
[12] C. Deffayet and T. Jacobson, Classical Quantum Gravity **29**, 065009 (2012); V. Baccetti, P. Martin-Moruno, and M. Visser, J. High Energy Phys. 08 (2012) 148.
[13] C. Burrage, N. Kaloper, and A. Padilla, arXiv:1211.6001.
[14] G. Velo and D. Zwanziger, Phys. Rev. **186**, 1337 (1969).
[15] M. Kobayashi and A. Shamaly, Phys. Rev. D **17**, 2179 (1978); Prog. Theor. Phys. **61**, 656 (1979).
[16] S. Deser and A. Waldron, Nucl. Phys. **B631**, 369 (2002).
[17] C. Deffayet, J. Mourad, and G. Zahariade, arXiv:1207.6338.
[18] J. Madore and W. Tait, Commun. Math. Phys. **30**, 201 (1973); J. Madore, Phys. Lett. **55B**, 217 (1975).
[19] Given the scalar constraint's nefarious role, one might try to turn it into a harmless Bianchi identity by taking a de Sitter background and a partially massless limit where the scalar helicity does not propagate [20]. However, very recently it has been shown that no partially massless limit of ghost-free massive f-g theories exists [9] (this result has since been confirmed in Ref. [27]).
[20] S. Deser and R. I. Nepomechie, Ann. Phys. (N.Y.) **154**, 396 (1984); Phys. Lett. **132B**, 321 (1983); S. Deser and A. Waldron, Phys. Rev. Lett. **87**, 031601 (2001); Nucl. Phys. **B607**, 577 (2001); Phys. Lett. B **508**, 347 (2001); **513**, 137 (2001); **603**, 30 (2004); Phys. Rev. D **74**, 084036 (2006); C. de Rham and S. Renaux-Petel, arXiv:1206.3482; S. Deser, E. Joung, and A. Waldron, Phys. Rev. D **86**, 104004 (2012).
[21] K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N.Y.) **13**, 126 (1961).
[22] One example, in a different context, is the use of string theoretically-inspired nonminimal couplings for charged higher spins [23].
[23] M. Porrati, R. Rahman, and A. Sagnotti, Nucl. Phys. **B846**, 250 (2011); M. Porrati and R. Rahman, Phys. Rev. D **80**, 025009 (2009); **84**, 045013 (2011).
[24] The massless van Dam-Veltman-Zhakarov discontinuity [4] can actually be averted by introducing a cosmological constant and setting the mass to zero before limiting to flat space [25]. Also, it was suggested long ago that a similar mechanism applies to the interchange of massless and free limits in a putative nonlinear massive theory [26].
[25] I. I. Kogan, S. Mouslopoulos, and A. Papazoglou, Phys. Lett. B **503**, 173 (2001); M. Porrati, Phys. Lett. B **498**, 92 (2001); S. Deser and A. Waldron, Phys. Lett. B **501**, 134 (2001).
[26] A. I. Vainshtein, Phys. Lett. **39B**, 393 (1972).
[27] C. de Rham, K. Hinterbichler, and R. A. Rosen, arXiv:1302.0025.