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Identification of Nonlinear Vibrating Structures: Part I—Formulation

A self-starting multistage, time-domain procedure is presented for the identification of nonlinear, multi-degree-of-freedom systems undergoing free oscillations or subjected to arbitrary direct force excitations and/or nonuniform support motions. Recursive least-squares parameter estimation methods combined with nonparametric identification techniques are used to represent, with sufficient accuracy, the identified system in a form that allows the convenient prediction of its transient response under excitations that differ from the test signals. The utility of this procedure is demonstrated in a companion paper.

1 Introduction

1.1 Background. The identification and modeling of nonlinear multidegree-of-freedom (MDOF) dynamic systems through the use of experimental data is a problem of considerable importance in the applied mechanics area. Since the model structure in many practical dynamics problems is by no means clear, an increasing amount of attention has recently been devoted to nonparametric identification methods.

One rather general nonparametric nonlinear identification approach is based on the expansion of the nonlinear restoring force functions in a power series or generalized Fourier series involving orthogonal polynomial functions. In applications, it is generally assumed that such series are rapidly convergent so that only a few terms need be retained for identification purposes. In such an approach, the coefficients of the retained terms from the series become parameters of the system which may be identified by many well-known techniques, such as least-squares fit in the time domain.

The origins of this basic approach are very classical and diverse, with roots in the theory of analytic functions and in the theory of Fourier series, and with applications in many engineering disciplines as well as operations research, economics, and the physical sciences. With regard to the engineering literature, the basic approach is outlined in the book by Graupe (1976). Applications of the method in the mechanical sciences appear to have originated in the early 1950's in several NACA technical notes (Greenberg, 1951; Shinbrot, 1951; Shinbrot, 1952; Briggs and Jones, 1953; and Shinbrot, 1954) and in the papers by Klotter (1953) and Shinbrot (1957). In the following years, interest in similar time series methods for nonlinear system identification of structures expanded, as attested to by the representative publications of Kohr (1963), Hoberock and Kohr (1967), Sprague and Kohr (1969), Sehitoglu and Klein (1975), Masri et al. (1982), Natke (1982), Masri et al. (1984), Tomlinson (1985), and Hac and Spanos (1987).

Most of the research in this area has been concerned with SDOF systems with nonlinearities of varying complexity. The basic identification method becomes generally impractical for complex MDOF systems due to excessive computation and computer memory requirements caused by slow convergence of the series expansions. However, Masri et al. (1982) demonstrated by example that rapid series convergence (and hence practical identification results) may be obtained in at least some MDOF structural applications by basing the identification procedure on a set of generalized coordinates corresponding to the mode shapes of a comparison linear structural system.

In the paper by Masri et al. (1982), certain restrictions were made on the class of nonlinear structural systems to be identified. In particular, it was assumed that (1) the system mass matrix M is diagonal and known; (2) the equivalent linear system stiffness matrix K is symmetric and known; and (3) the excitation to the system is furnished through forces directly applied to the discrete mass locations. The requirement of knowing the linearized system parameters pertaining to M and K, as well as the exclusion of the class of problems involving support motion (such as in the case of earthquake ground motion), limited the utility of the approach in practical cases.

The present paper further extends the above-referenced work by generalizing the approach to handle, approximately, the case of arbitrary nonlinear MDOF dynamic systems with multiple inputs and outputs under the action of force excitations and/or nonuniform support motion. The method is based on the use of time-domain estimation techniques to identify the parameters of an equivalent linear model whose eigenvectors are then used to estimate the "modes" of the nonlinear system. Regression techniques involving the use of two-dimensional orthogonal functions are then employed to develop an approximate expression for the system generalized

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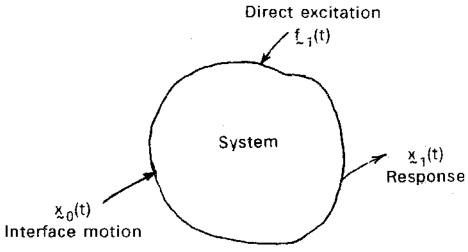

Fig. 1 Model of system

restoring forces in terms of the corresponding generalized system state variables.

Section 2 of this paper extends the work of previous investigators by presenting a unified approach for handling the time-domain identification of the system matrices associated with a variety of classes of linear problems arising in the field of structural dynamics. The formulation under discussion includes the cases of free vibrations as well as direct force and/or independent support motion.

Section 3 incorporates the results of Section 2 in the identification of nonlinear vibrating structures. The "calibration" of this approach is accomplished in the companion paper (Masri et al., 1987) by applying the method under discussion to a representative multi-input/multi-output nonlinear system incorporating polynomial as well as hysteretic nonlinearities.

1.2 Formulation of Time Domain Identification Procedure. Consider a discrete multi-degree-of-freedom (MDOF) system of the type shown in Fig. 1, which is subjected to directly applied excitation forces $\mathbf{f}_1(t)$ as well as prescribed support motions $\mathbf{x}_0(t)$. The motion of this multi-input/multi-output nonlinear system is governed by the set of equations

$$\mathbf{f}_T(\ddot{\mathbf{x}},\dot{\mathbf{x}},\mathbf{x})=\mathbf{f}_1(t) \tag{1}$$

where:

$\mathbf{f}_T$ = an n_1 column vector representing the total sum of all the inertia and restoring forces acting on the system,
$\mathbf{f}_1(t)$ = an n_1 column vector of directly applied forces,
$\mathbf{x}(t)$ = $(\mathbf{x}_1(t),\ \mathbf{x}_0(t))^T$ = system displacement vector of order $(n_1 + n_0)$,
$\mathbf{x}_1(t)$ = active degree-of-freedom (DOF) displacement vector of order n_1,
$\mathbf{x}_0(t)$ = prescribed support displacement vector of order n_0.

Let $\mathbf{f}_T(t)$ be expressed as

$$\mathbf{f}_T(t)={}^L\mathbf{f}_1(t)+{}^S\mathbf{f}_1(t)+{}^L\mathbf{f}_0(t)+{}^S\mathbf{f}_0(t)+\mathbf{f}_N(t), \tag{2}$$

where

$${}^L\mathbf{f}_1(t)=M_{11}\ddot{\mathbf{x}}_1(t)+C_{11}\dot{\mathbf{x}}_1(t)+K_{11}\mathbf{x}_1(t), \tag{3}$$

$${}^S\mathbf{f}_1(t)=M_{11}^S\ddot{\mathbf{x}}_1(t)+C_{11}^S\dot{\mathbf{x}}_1(t)+K_{11}^S\mathbf{x}_1(t), \tag{4}$$

$${}^L\mathbf{f}_0(t)=M_{10}\ddot{\mathbf{x}}_0(t)+C_{10}\dot{\mathbf{x}}_0(t)+K_{10}\mathbf{x}_0(t), \tag{5}$$

$${}^S\mathbf{f}_0(t)=M_{10}^S\ddot{\mathbf{x}}_0(t)+C_{10}^S\dot{\mathbf{x}}_0(t)+K_{10}^S\mathbf{x}_0(t), \tag{6}$$

M_{11}, C_{11}, K_{11} = constant matrices that characterize the inertia, damping, and stiffness forces associated with the unconstrained DOF of the system, each of order $n_1 \times n_1$,
$M_{11}^S, C_{11}^S, K_{11}^S$ = response-dependent matrices that characterize the inertia, damping, and stiffness forces associated with the unconstrained DOF of the system, each of order $n_1 \times n_1$,
M_{10}, C_{10}, K_{10} = constant matrices that characterize the inertia, damping, and stiffness forces associated with the support motions, each of order $n_1 \times n_0$,
$M_{10}^S, C_{10}^S, K_{10}^S$ = response-dependent matrices that characterize the inertia, damping, and stiffness forces associated with the support motions, each of order $n_1 \times n_0$,
${}^L\mathbf{f}_1(t)$ = an n_1 column vector of linear forces involving $\mathbf{x}_1(t)$,
${}^S\mathbf{f}_1(t)$ = an n_1 column vector of response-dependent forces involving $\mathbf{x}_1(t)$,
${}^L\mathbf{f}_0(t)$ = an n_1 column vector of linear forces involving $\mathbf{x}_0(t)$,
${}^S\mathbf{f}_0(t)$ = an n_1 column vector of response-dependent forces involving $\mathbf{x}_1(t)$ as well as $\mathbf{x}_0(t)$,
$\mathbf{f}_N(t)$ = an n_1 column vector of nonlinear nonconservative forces involving $\mathbf{x}_1(t)$ as well as $\mathbf{x}_0(t)$.

Making use of equation (2), the system equation of motion (1) can be expressed as

$$M_{11}^e\ddot{\mathbf{x}}_1(t)+C_{11}^e\dot{\mathbf{x}}_1(t)+K_{11}^e\mathbf{x}_1(t)+ M_{10}^e\ddot{\mathbf{x}}_0(t)+C_{10}^e\dot{\mathbf{x}}_0(t)+K_{10}^e\mathbf{x}_0(t)+\mathbf{f}_N(t)=\mathbf{f}_1(t), \tag{7}$$

where:

$$\begin{aligned} M_{11}^e&=M_{11}+M_{11}^S, & M_{10}^e&=M_{10}+M_{10}^S,\\ C_{11}^e&=C_{11}+C_{11}^S, & C_{10}^e&=C_{10}+C_{10}^S,\\ K_{11}^e&=K_{11}+K_{11}^S, & K_{10}^e&=K_{10}+K_{10}^S, \end{aligned} \tag{8}$$

This study is concerned with a time-domain method for the identification of the system matrices appearing in equation (7) as well as the nonlinear forces acting on the system. The representation of the identifed system will be in a form that allows the prediction of its transient response under arbitrary excitations, by using conventional numerical techniques for initial-value problems in ordinary differential equations.

Note that equation (7) can be expressed as

$$\ddot{\mathbf{x}}_1(t)=[M_{11}^e]^{-1}(\mathbf{f}_1(t)-\mathbf{f}_L^e(t)-\mathbf{f}_N(t)), \tag{9}$$

where:

$$\mathbf{f}_L^e(t)=\mathbf{b}_1(t)+\mathbf{b}_0(t), \tag{10}$$

$$\mathbf{b}_1(t)=C_{11}^e\dot{\mathbf{x}}_1(t)+K_{11}^e\mathbf{x}_1(t), \tag{11}$$

$$\mathbf{b}_0(t)=M_{10}^e\ddot{\mathbf{x}}_0(t)+C_{10}^e\dot{\mathbf{x}}_0(t)+K_{10}^e\mathbf{x}_0(t). \tag{12}$$

Thus, by introducing the state vector $\mathbf{y}$ of order $2n_1$ where

$$y_{2i-1}=x_{1_i} \tag{13}$$

$$y_{2i}=\dot{x}_{1_i} \qquad i=1, 2, \ldots, n_1 \tag{14}$$

standard time-marching techniques can be used to solve

$$\dot{\mathbf{y}}=\mathbf{g}(\mathbf{y}, \mathbf{f}_1, \mathbf{x}_0). \tag{15}$$

2 Time-Domain Identification of Linear System Matrices

The use of least-squares methods to estimate unknown parameters is a well known and developed approach which occupies significant portions of numerous books devoted to the subject of parameter estimation, particularly in the field of electrical engineering control and system theory (Mendel, 1973; Graupe, 1976; Hsia, 1977; Sorenson, 1980). While this approach has also been frequently applied in the field of structural dynamics (Caravani and Thomson, 1974, 1977; Ibrahim and Mikulcik, 1973, 1976, 1977; Ibrahim and Pappa, 1982; Ibrahim, 1977, 1978, 1983; Junkins, 1978; Beck and Jennings, 1980; Yao, 1985; Torkamani and Hart, 1975; Shinozuka et al., 1982; Rajaram and Junkins, 1985; Hac and Spanos, 1987), there is a paucity of studies that are concerned with the problems encountered by this approach when applied to realistic problems arising in the vibration field. Consequently, the present section of this paper is devoted to presenting an in-depth, unified, and efficient approach for using least-squares parameter estimation methods to identify the needed system matrices associated with a wide variety of realistic situations commonly encountered when dealing with experimental measurements of vibrating structures.

2.1 Formulation. Consider a linearized version of the system shown in Fig. 1, and assume it is governed by

$$M_{11}\ddot{\mathbf{x}}_1(t)+C_{11}\dot{\mathbf{x}}_1(t)+K_{11}\mathbf{x}_1(t)+ M_{10}\ddot{\mathbf{x}}_0(t)+C_{10}\dot{\mathbf{x}}_0(t)+K_{10}\mathbf{x}_0(t)=\mathbf{f}_1(t). \tag{16}$$

Let the response vector $\mathbf{r}(t)$ of order $3(n_1+n_0)$ be defined as

$$\mathbf{r}(t)=(\ddot{\mathbf{x}}_1^T(t), \dot{\mathbf{x}}_1^T(t), \mathbf{x}_1^T(t), \ddot{\mathbf{x}}_0^T(t), \dot{\mathbf{x}}_0^T(t), \mathbf{x}_0^T(t))^T. \tag{17}$$

For clarity of presentation, let the six matrices appearing in equation (16) be denoted by ${}^1A, {}^2A, \ldots, {}^6A$, respectively.

Let $\langle {}^jA_i\rangle$ = ith row of a generic matrix ${}^jA$, and introduce the parameter vector $\boldsymbol{\alpha}_i$.

$$\boldsymbol{\alpha}_i=(\langle {}^1A_i\rangle, \langle {}^2A_i\rangle, \langle {}^3A_i\rangle, \langle {}^4A_i\rangle, \langle {}^5A_i\rangle, \langle {}^6A_i\rangle)^T. \tag{18}$$

Suppose that the excitation and the response of the system governed by equation (16) is measured at times $t_1, t_2, \ldots, t_N$. Then at every t_k,

$${}^1A\ddot{\mathbf{x}}_1(t_k)+{}^2A\dot{\mathbf{x}}_1(t_k)+{}^3A\mathbf{x}_1(t_k)+{}^4A\ddot{\mathbf{x}}_0(t_k)+ {}^5A\dot{\mathbf{x}}_0(t_k)+{}^6A\mathbf{x}_0(t_k)=\mathbf{f}_1(t_k);\quad k=1,2,\ldots,N. \tag{19}$$

Introducing matrix R

$$R=\begin{bmatrix}\mathbf{r}^T(t_1)\\ \mathbf{r}^T(t_2)\\ \cdot\\ \cdot\\ \cdot\\ \mathbf{r}^T(t_N)\end{bmatrix} \tag{20}$$

and using the notation above, the grouping of the measurements can be expressed concisely as

$$\hat{R}\hat{\boldsymbol{\alpha}}=\hat{\mathbf{b}} \tag{21}$$

where $\hat{R}$ is a block-diagonal matrix whose diagonal elements are equal to R, $\hat{\boldsymbol{\alpha}}=(\boldsymbol{\alpha}_1^T, \boldsymbol{\alpha}_2^T, \ldots, \boldsymbol{\alpha}_{n_1}^T)^T$, and $\hat{\mathbf{b}}$ is the corresponding vector of excitation measurements.

Keeping in mind that $\hat{R}$ is of order $m\times n$ where $m=Nn_1$, and $n=3n_1(n_1+n_0)$, then if a sufficient number of measurements is taken, this will result in $m>n$. Under these conditions, least-squares procedures can be used to solve for all the system parameters that constitute the entries in $\hat{\boldsymbol{\alpha}}$:

$$\hat{\boldsymbol{\alpha}}=\hat{R}^{\dagger}\hat{\mathbf{b}} \tag{22}$$

where $\hat{R}^{\dagger}$ is the pseudoinverse of $\hat{R}$ (Golub and Van Loan, 1983).

Using the weighted least-squares approximation to minimize the cost function, J, results in the approximate solution

$$\hat{\boldsymbol{\alpha}}=(\hat{R}^TW\hat{R})^{-1}\hat{R}^TW\hat{\mathbf{b}}, \tag{23}$$

where W is the error weighting matrix.

2.2 Computational Efficiency.

2.2.1 Decoupling. One way to reduce the order of equation (21) to a manageable level is by making use of the diagonal nature of partitioned matrix $\hat{R}$, thus resulting in a set of n_1 decoupled matrix equations each of the form

$$R\boldsymbol{\alpha}_i=\mathbf{b}_i,\quad i=1,2,\ldots,n_1. \tag{24}$$

Comparing the order of R in equation (24) with that of $\hat{R}$ in equation (21), shows that the order of R is smaller by a factor of n_1^2 compared to $\hat{R}$. Least-squares techniques can again be used to obtain the components of the n_1 parameter vectors $\boldsymbol{\alpha}_i$:

$$\hat{\boldsymbol{\alpha}}_i=R^{\dagger}\hat{\mathbf{b}}_i;\quad i=1,2,\ldots,n_1. \tag{25}$$

Note that $R^{\dagger}$ needs to be computed only *once*.

While the formulation in equation (24) is obviously superior to the corresponding formulation in equation (21), the former suffers from a significant (practical) limitation pertaining to the number of system DOFs simultaneously excited.

2.2.2 Recursive Solutions. Suppose that a set of m equations

$$\hat{R}_k\hat{\boldsymbol{\alpha}}=\hat{\mathbf{b}}^{(k)} \tag{26}$$

has been used to obtain a weighted least-squares estimate for $\hat{\boldsymbol{\alpha}}$, denoted by $\hat{\boldsymbol{\alpha}}^{(k)}$:

$$\hat{\boldsymbol{\alpha}}^{(k)}=(\hat{R}_k^TW_k\hat{R}_k)^{-1}\hat{R}_k^TW_k\hat{\mathbf{b}}^{(k)}. \tag{27}$$

Using an additional set of relations

$$\hat{R}_{(k+1)}\hat{\boldsymbol{\alpha}}=\hat{\mathbf{b}}^{(k+1)} \tag{28}$$

a new estimate of $\hat{\boldsymbol{\alpha}}$, denoted by $\hat{\boldsymbol{\alpha}}^{(k+1)}$, can be obtained without reprocessing the whole set of equations involving $(\hat{\mathbf{b}}^{(k)}, \hat{\mathbf{b}}^{(k+1)})$ (Brogan, 1985).

2.3 Special Cases. In the work of Masri et al. (1987a), special cases that influence the application of the method in practical situations are discussed and steps are provided for alleviating some of the problems appearing in realistic cases. Among these topics are the uniqueness issues, partial knowledge of system parameters, conditions under which the approach fails to yield desired results, symmetry assumptions, and recursive approaches to enhance computational efficiency.

3 Identification of Nonlinear Systems

Consider the nonlinear system governed by equation (1) and assume that the identification procedure discussed in Section 2 has yielded the system matrices needed to determine the equivalent linear internal force vector $\mathbf{f}_L^e(t)$ appearing in equation (9) and defined by equation (10).

3.1 Restoring Force Estimation. Solving equation (7) for the nonlinear force vector $\mathbf{f}_N(t)$ results in

$$\mathbf{f}_N(t)=\mathbf{f}_1(t)-(M_{11}^e\ddot{\mathbf{x}}_1(t)+\mathbf{f}_L^e(t)). \tag{29}$$

Since all the terms appearing on the right-hand side of equation (29) are available from measurements or have been previously identified, the time history of $\mathbf{f}_N$ can be determined. Note from equation (29) that $\mathbf{f}_N(t)$ can be interpreted as the residual force vector corresponding to the difference between the excitation vector $\mathbf{f}_1(t)$ and the equivalent linear force vector composed of the inertia, damping, and the stiffness terms.

An alternative form of equation (29) is

$$\mathbf{f}_R(t)\equiv\mathbf{f}_N(t)+\mathbf{f}_L^e(t)=\mathbf{f}_1(t)-M_{11}^e\ddot{\mathbf{x}}_1(t), \tag{30}$$

where $\mathbf{f}_R(t)$ represents the difference between the excitation and equivalent linear inertia forces associated with the active degrees of freedom. The force $\mathbf{f}_R$ can be thought of as the "restoring force" of the system.

Let $h_i(t)$ represent either the ith component of the nonlinear residual force vector $\mathbf{f}_N(t)$ as defined by equation (29) or the restoring force vector $\mathbf{f}_R(t)$ as defined by equation (30). In general, vector $\mathbf{h}$ depends simultaneously on all the components of the system acceleration, velocity, and displacement vectors associated with the n_1 active DOF as well as the n_0 support components:

$$\mathbf{h}(t)=\mathbf{h}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}). \tag{31}$$

The central idea of the present method is that, in the case of nonlinear dynamic systems commonly encountered in the applied mechanics field, a judicious assumption is that each component of $\mathbf{h}$ can be expressed in terms of a series of the form:

$$h_i(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})\approx\sum_{j=1}^{J_{\max_i}}\hat{h}_i^{(j)}(v_{1_i}^{(j)}, v_{2_i}^{(j)}) \tag{32}$$

where the v_1's and v_2's are suitable generalized coordinates which, in turn, are linear combinations of the physical displacements, velocities, and accelerations. The approximation indicated in equation (32) is that each component h_i of the nonlinear force vector $\mathbf{h}$ can be adequately estimated by a collection of terms $\hat{h}_i^{(j)}$, each one of which involves a pair of generalized coordinates. The particular choice of combinations and permutations of the generalized coordinates and the number of terms $J_{\max_i}$ needed for a given h_i depends on the nature and extent of the nonlinearity of the system and its effect on the specific DOF i.

3.2 Eigenvector Expansion. If $h_i(t)$ is chosen as the ith component of $\mathbf{f}_N(t)$, then the procedure expressed by equation (32) will directly estimate the corresponding component of the unknown nonlinear force. For certain structural configurations (e.g., localized nonlinearities) and/or relatively low-order systems, the choice of suitable generalized coordinates for the series in equation (32) is a relatively straightforward task. However, in many practical cases involving distributed nonlinearities coupled with a relatively high-order system, an improved rate of convergence of the series in equation (32) can be achieved by performing the least-squares fit of the nonlinear forces in the "modal" domain as outlined below.

Using the identification results for the linear system discussed in Section 2, the eigenvalue problem associated with $M_{11}^{-1}K_{11}$ is solved resulting in the eigenvector or modal matrix Φ and the corresponding vector of generalized coordinates $\mathbf{u}$:

$$\mathbf{h}_1(\mathbf{u}, \dot{\mathbf{u}}) = \Phi^T \mathbf{f}_N(t) \tag{33}$$

with

$$\mathbf{u}(t) = \Phi^{-1}\mathbf{x}(t) \tag{34}$$

With this formulation in mind, equation (32) can be viewed as allowing for "modal" interaction between all generalized coordinates, taken two at a time. Note that the formulation in equation (32) allows for "modal" interaction between all "modal" displacements, velocities, and accelerations.

3.3 Series Expansion. The individual terms appearing in the series expansion of equation (32) may be evaluated by using the least-squares approach to determine the optimum fit for the time history of each h_i. Thus $\hat{h}_i^{(1)}$ may be expressed as a double series involving a suitable choice of generalized coordinates:

$$\hat{h}_i^{(1)}(v_{1_i}^{(1)}, v_{2_i}^{(1)}) \equiv \sum_k \sum_\ell {}^{(1)}C_{k\ell}^{(i)} T_k(v_{1_i}^{(1)}) T_\ell(v_{2_i}^{(1)}) \tag{35}$$

where the C_{kl}'s are a set of undetermined constants and $T_k(.)$ are suitable basis functions, such as orthogonal polynomials. Let $h_i^{(2)}$, the deviation (residual) error between h_i and its first estimate $\hat{h}_i^{(1)}$, be given by

$$h_i^{(2)}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = h_i(\mathbf{x}_1, \dot{\mathbf{x}}_1, \ddot{\mathbf{x}}_1) - \hat{h}_i^{(1)}(v_{1_i}^{(1)}, v_{2_i}^{(1)}). \tag{36}$$

Equation (32) accounts for the contribution to the nonlinear force h_i of generalized coordinates $v_{1_i}^{(1)}$ and $v_{2_i}^{(1)}$ appearing in the form $(v_{1_i}^{(1)})^k (v_{2_i}^{(1)})^\ell$. Consequently, the residual error as defined by equation (32) can be further reduced by fitting $h_i^{(2)}$ by a similar double series involving variables $v_{1_i}^{(2)}$ and $v_{2_i}^{(2)}$:

$$\mathbf{h}_i^{(2)}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) \approx \hat{h}_i^{(2)}(v_{1_i}^{(2)}, v_{2_i}^{(2)}) \tag{37}$$

where

$$\hat{h}_i^{(2)}(v_{1_i}^{(2)}, v_{2_i}^{(2)}) \equiv \sum_k \sum_\ell {}^{(2)}C_{k\ell}^{(i)} T_k(v_{1_i}^{(2)}) T_\ell(v_{2_i}^{(2)}). \tag{38}$$

By extending this procedure to account for all DOFs that have significant interaction with DOF i, equation (32) is obtained with

$$h_i^{(j+1)}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = h_i^{(j)}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) - \hat{h}_i^{(j)}(v_{1_i}^{(j)}, v_{2_i}^{(j)}); \quad j = 1, 2, \ldots, J_{\max_i} \tag{39}$$

where

$$h_i^{(1)}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) \equiv h_i(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}), \tag{40}$$

and

$$\hat{h}_i^{(j)}(v_{1_i}^{(j)}, v_{2_i}^{(j)}) \equiv \sum_k \sum_\ell {}^{(j)}C_{k\ell}^{(i)} T_k(v_{1_i}^{(j)}) T_\ell(v_{2_i}^{(j)}). \tag{41}$$

Note that, in general, the range of the summation indices k and ℓ appearing in equation (41) may vary with the series index j and DOF index i. Similarly, $J_{\max_i}$, the total number of series terms needed to achieve a given level of accuracy in fitting the nonlinear force time history, depends on the DOF index i.

3.4 Least Squares Fit for Nonlinear Forces. Using two-dimensional orthogonal polynomials $T_k(.)$ to estimate each h_i $(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})$ by a series of approximating functions $\hat{h}_i^{(j)}$ of the form indicated in equation (41), then the numerical value of the $C_{k\ell}$ coefficients can be determined by invoking the applicable orthogonality conditions for the chosen polynomials. While there is a wide choice of suitable basis functions for least-squares application, the orthogonal nature of the Chebyshev polynomials and their "equal-ripple" characteristics make them convenient to use in the present work.

Let each generalized coordinate v appearing in equation (32) be normalized to lie in the range -1 to 1:

$$v' = [v - (v_{\max} + v_{\min})/2]/[(v_{\max} - v_{\min})/2]. \tag{42}$$

If the Chebyshev polynomials, given by

$$T_n(\xi) = \cos(n \cos^{-1}\xi), \quad -1 < \xi < 1 \tag{43}$$

and satisfying the weighted orthogonality property

$$\int_{-1}^{1} w(x) T_n(\xi) T_m(\xi) d\xi = \begin{cases} 0, & n \neq m \\ \pi/2, & n = m \neq 0 \\ \pi, & n = m = 0 \end{cases} \tag{44}$$

where $w(x) = (1 - x^2)^{-1/2}$ is the weighting function, are used, then the $C_{k\ell}$ coefficients would be given by

$$C_{k\ell} = \begin{cases} (2/\pi)^2 S_{k\ell}, & k \text{ and } \ell \neq 0 \\ (2/\pi^2) S_{k\ell}, & k \text{ or } \ell = 0 \\ (1/\pi^2) S_{k\ell}, & k \text{ and } \ell = 0 \end{cases} \tag{45}$$

where

$$S_{k\ell} = \int_0^\pi \int_0^\pi h(\cos^{-1}v_1, \cos^{-1}v_2) T_k(\theta_1) T_\ell(\theta_2) d\theta_1 d\theta_2 \tag{46}$$

and

$$v_i = \cos\theta_i, \quad i = 1, 2. \tag{47}$$

Note that in the special case when no cross-product terms are involved in any of the series terms, functions h can be expressed as the sum of two one-dimensional orthogonal polynomial series instead of a single two-dimensional series of the type under discussion.

4 Summary and Conclusions

An approximate time-domain method is presented for the identification of nonlinear multi-degree-of-freedom systems subjected to arbitrary direct force excitations and/or not-necessarily-identical support motions. This self-starting method uses recursive least-squares parameter estimation methods, combined wtih nonparametric identification techniques, to generate a reduced-order nonlinear mathematical model suitable for use in subsequent studies to predict, with good fidelity, the response of the test article under arbitrary dynamic excitations. The utility of this procedure is demonstrated in a companion paper.

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