FRAÏSSÉ LIMITS, RAMSEY THEORY, AND TOPOLOGICAL DYNAMICS OF AUTOMORPHISM GROUPS

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0. Introduction

(A) We study in this paper some connections between the Fraïssé theory of amalgamation classes and ultrahomogeneous structures, Ramsey theory, and topological dynamics of automorphism groups of countable structures.

A prime concern of topological dynamics is the study of continuous actions of (Hausdorff) topological groups G on (Hausdorff) compact spaces X. These are usually referred to as (compact) G-*flows*. Of particular interest is the study of *minimal* G-*flows*, those for which every orbit is dense. Every G-flow contains a minimal subflow. A general result of topological dynamics asserts that every topological group G has a *universal minimal flow* $M(G)$, a minimal G-flow which can be homomorphically mapped onto any other minimal G-flow. Moreover, this is uniquely determined, by this property, up to isomorphism. (As usual a *homomorphism* $\pi : X \to Y$ between G-flows is a continuous G-map and an *isomorphism* is a bijective homomorphism.) For separable, metrizable groups G, which are the ones that we are interested in here, the universal minimal flow of G is an inverse limit of manageable, i.e., metrizable G-flows, but itself may be very complicated, for example non-metrizable. In fact, for the "simplest" infinite G, i.e., the countable discrete ones, $M(G)$ is a very complicated compact G-invariant subset of the space βG of ultrafilters on G and is always non-metrizable.

Rather remarkably, it turned out that there are non-trivial topological groups G for which $M(G)$ is actually trivial, i.e., a singleton. This is equivalent to saying that G has a very strong fixed point property, namely every G-flow has a fixed point (i.e., a point x such that $g \cdot x = x$, $\forall g \in G$). (For separable, metrizable groups this is also equivalent to the fixed point property restricted to metrizable G-flows.) Such groups are said to have the *fixed point on compacta property* or be *extremely amenable.* The latter name comes from one of the standard characterizations of second countable locally compact amenable groups. A second countable locally compact group G is *amenable* iff every metrizable G-flow has an invariant (Borel probability) measure. However, no non-trivial locally compact group can be extremely amenable, because, by a theorem of Veech [83], every such group admits a *free* G-flow (i.e., a flow for which $g \cdot x = x \Rightarrow g = 1_G$). Nontriviality of the universal minimal flow for locally compact groups also follows from the earlier results of Granirer-Lau [36]. This probably explains the rather late emergence of extreme amenability. Note that the corresponding property for semigroups is much more common and easier to come by, and in fact the study of the fixed point on compacta property was initiated in the context of topological semigroups by Mitchell [49], followed by Granirer [35]. In 1966, Mitchell [49] asked the question of existence of non-trivial extremely amenable topological groups. The first examples of such groups were constructed by Herer-Christensen [39]. They found Polish abelian so-called *pathological groups*, i.e., topological groups with no non-trivial unitary representations. Then they showed (see Theorem 4 in their paper) that every amenable pathological group is extremely amenable. Remarkably though it turned out that a lot of important (non-locally compact) Polish groups are indeed extremely amenable. Gromov-Milman [38] showed that the unitary group of infinite dimensional separable Hilbert space is extremely amenable, Furstenberg-Weiss (unpublished) and independently Glasner [28] showed that the group of measurable maps from $I = [0, 1]$ to the unit circle $\mathbb{T}$ is extremely amenable, Pestov [64] (see also [63]) showed that the groups $H_+(I)$, $H_+(\mathbb{R})$ of increasing homeomorphisms of $I, \mathbb{R}$, resp., are extremely amenable, and Pestov [64] showed that the group $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ of automorphisms of the rationals is extremely amenable. More recently, Pestov [66] proved that the universal Polish group $\mathrm{Iso}(\boldsymbol{U})$, of all isometries of the Urysohn space $\boldsymbol{U}$, is extremely amenable, and Giordano-Pestov [26, 27] showed that the group $\mathrm{Aut}(I, \lambda)$ (resp., $\mathrm{Aut}^*(I, \lambda)$) of measure preserving automorphisms of I with Lebesgue measure λ (resp., measure-class preserving automorphisms of I, λ) is extremely amenable.

In most known examples of extremely amenable groups, beginning with the result by Gromov and Milman [38] on the unitary group, extreme amenability was established by using the phenomenon of concentration of measure on high-dimensional structures, see, e.g., Milman and Schechtman [48] or Ledoux [44]. However, we will not touch upon this subject here, referring the reader instead to the introductory article [68] or the most recent work in this direction [27] and references therein.

Beyond the extremely amenable groups there were very few cases of metrizable universal minimal flows that had been computed. The first such example is in Pestov [64], where the author shows that the universal minimal flow of $H_+(\mathbb{T})$, the group of orientation preserving homeomorphisms of the circle, has as a universal minimal flow its natural (evaluation) action on $\mathbb{T}$. Then Glasner-Weiss [30] showed that the universal minimal flow of S_∞, the infinite symmetric group of all permutations of $\mathbb{N}$, is its canonical action on the space of all linear orderings on $\mathbb{N}$. Finally, Glasner-Weiss [31] showed that the universal minimal flow of $H(2^{\mathbb{N}})$, the group of homeomorphisms of the Cantor space, is its canonical action on the space of maximal chains of compact subsets of $2^{\mathbb{N}}$, a space introduced in [81].

(B) Motivated by Pestov's result that $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ is extremely amenable and the Glasner-Weiss computation of the universal minimal flow of S_∞, we develop in this paper a general framework, in which such results can be viewed as special instances. In particular, this gives many new examples of automorphism groups that are extremely amenable and calculations of universal minimal flows. There are two main ingredients that come into play here. The first is the Fraïssé theory of amalgamation classes and ultrahomogeneous structures, and the second is the structural Ramsey theory that arises in the works of Graham, Leeb, Rothschild, Nešetřil and Rödl. As repeatedly stressed by one of the present authors (see, e.g., Pestov [68]), extreme amenability is related to Ramsey-type phenomena. For instance, Pestov's proof that $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ is extremely amenable depends on the classical finite Ramsey theorem and in fact it is equivalent to it. Generalizing this, we will see that, once things are put in the proper context, extreme amenability of automorphism groups and calculation of universal minimal flows turn out to have equivalent formulations in terms of concepts that have arisen in structural Ramsey theory.

(C) Let us first review some basic facts of the Fraïssé theory. A (countable) *signature* consists of a set of symbols $L = \{R_i\}_{i\in I} \cup \{f_j\}_{j\in J}$ (I, J countable), to each of which there is an associated *arity* $n(i) \in \{1, 2, \dots\}$ ($i \in I$) and $m(j) \in \mathbb{N}$ ($j \in J$). We call R_i the *relation symbols* and f_j the *function symbols* of L. A *structure* for L is of the form $\boldsymbol{A} = \langle A, \{R_i^{\boldsymbol{A}}\}_{i\in I},\ \{f_j^{\boldsymbol{A}}\}_{j\in J}\rangle$, where $A \neq \emptyset$, $R_i^{\boldsymbol{A}} \subseteq A^{n(i)}$, $f_j^{\boldsymbol{A}} : A^{m(j)} \to A$. The set A is called the *universe* of the structure. An *embedding* between structures $\boldsymbol{A}, \boldsymbol{B}$ for L is an injection $\pi : A \to B$ such that $R_i^{\boldsymbol{A}}(a_1, \dots, a_{n(i)}) \Leftrightarrow R_i^{\boldsymbol{B}}(\pi(a_1), \dots, \pi(a_{n(i)}))$ and $\pi(f_j^{\boldsymbol{A}}(a_1, \dots, a_{m(j)})) = f_j^{\boldsymbol{B}}(\pi(a_1), \dots, \pi(a_{m(j)}))$. If π is the identity, we say that $\mathbf{A}$ is a *substructure* of $\mathbf{B}$. An *isomorphism* is an onto embedding. We write $\boldsymbol{A} \leq \boldsymbol{B}$ if $\boldsymbol{A}$ can be embedded in $\boldsymbol{B}$ and $\boldsymbol{A} \cong \boldsymbol{B}$ if $\boldsymbol{A}$ is isomorphic to $\boldsymbol{B}$.

A class $\mathcal{K}$ of finite structures for L is *hereditary* if $\boldsymbol{A} \leq \boldsymbol{B} \in \mathcal{K}$ implies $\boldsymbol{A} \in \mathcal{K}$. It satisfies the *joint embedding property* if for any $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$ there is $\boldsymbol{C} \in \mathcal{K}$ with $\boldsymbol{A} \leq \boldsymbol{C},\ \boldsymbol{B} \leq \boldsymbol{C}$. Finally, it satisfies the *amalgamation property* if for any embeddings $f : \boldsymbol{A} \to \boldsymbol{B},\ g : \boldsymbol{A} \to \boldsymbol{C}$, with $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$, there is $\boldsymbol{D} \in \mathcal{K}$ and embeddings $r : \boldsymbol{B} \to \boldsymbol{D}$ and $s : \boldsymbol{C} \to \boldsymbol{D}$, such that $r \circ f = s \circ g$. We call $\mathcal{K}$ a *Fraïssé class* if it is hereditary, satisfies joint embedding and amalgamation, contains only countably many structures, up to isomorphism, and contains structures of arbitrarily large (finite) cardinality.

If now $\boldsymbol{A}$ is a countable structure, which is locally finite (i.e., finitely generated substructures are finite), its *age*, $\mathrm{Age}(\boldsymbol{A})$, is the class of all finite structures which can be embedded in $\boldsymbol{A}$. We call $\boldsymbol{A}$ *ultrahomogeneous* if every isomorphism between finite substructures of $\boldsymbol{A}$ can be extended to an automorphism of $\boldsymbol{A}$. We call a locally finite, countably infinite, ultrahomogeneous structure a *Fraïssé structure*.

There is a canonical 1-1 correspondence between Fraïssé classes and structures, discovered by Fraïssé. If $\boldsymbol{A}$ is a Fraïssé structure, then $\mathrm{Age}(\boldsymbol{A})$ is a Fraïssé class. Conversely, if $\mathcal{K}$ is a Fraïssé class, then there is a unique Fraïssé structure, the *Fraïssé limit of* $\mathcal{K}$, denoted by $\mathrm{Flim}(\mathcal{K})$, whose age is exactly $\mathcal{K}$. Here are a couple of examples: the Fraïssé limit of the class of finite linear orderings is $\langle \mathbb{Q}, < \rangle$, and the Fraïssé limit of the class of finite graphs is the *random graph*.

(D) We now come to structural Ramsey theory. Let $\mathcal{K}$ be a hereditary class of finite structures in a signature L. For $\boldsymbol{A} \in \mathcal{K},\ \boldsymbol{B} \in \mathcal{K}$ with $\boldsymbol{A} \leq \boldsymbol{B}$, we denote by $\binom{\boldsymbol{B}}{\boldsymbol{A}}$ the set of all substructures of $\boldsymbol{B}$ isomorphic to $\boldsymbol{A}$. If $\boldsymbol{A} \leq \boldsymbol{B} \leq \boldsymbol{C}$ are in $\mathcal{K}$ and $k = 2, 3, \dots$, we write

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_k,$$

if for every coloring $c : \binom{\boldsymbol{C}}{\boldsymbol{A}} \to \{1, \dots, k\}$, there is $\boldsymbol{B}' \in \binom{\boldsymbol{C}}{\boldsymbol{B}}$ which is homogeneous, i.e., $\binom{\boldsymbol{B}'}{\boldsymbol{A}}$ is monochromatic. We say that $\mathcal{K}$ satisfies the *Ramsey property* if for every $\boldsymbol{A} \leq \boldsymbol{B}$ in $\mathcal{K}$ and $k \geq 2$, there is $\boldsymbol{C} \in \mathcal{K}$ with $\boldsymbol{B} \leq \boldsymbol{C}$ such that $\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_k$. For example, the classical finite Ramsey theorem is equivalent to the statement that the class of finite linear orderings has the Ramsey property. Also Nešetřil and Rödl showed that the class of finite ordered graphs has the Ramsey property, and Graham-Leeb-Rothschild showed that the class of finite-dimensional vector spaces over a finite field has the Ramsey property.

(E) Consider now automorphism groups $\mathrm{Aut}(\boldsymbol{A})$ of countably infinite structures $\boldsymbol{A}$, for which we may as well assume that $A = \mathbb{N}$. Thus, with the pointwise convergence topology, $\mathrm{Aut}(\boldsymbol{A})$ is a closed subgroup of S_∞, the infinite symmetric group. Conversely, given a closed subgroup $G \leq S_\infty$, G is the automorphism group of some structure on $A = \mathbb{N}$ (in some signature).

Assume now L is a signature containing a distinguished binary relation symbol $<$. An *order structure* $\boldsymbol{A}$ for L is a structure $\boldsymbol{A}$ for which $<^{\boldsymbol{A}}$ is a linear ordering. An *order class* $\mathcal{K}$ for L is one for which all $\boldsymbol{A} \in \mathcal{K}$ are order structures.

We obtain the following result (Theorem 4.7):

♦ *The extremely amenable closed subgroups of S_∞ are exactly the groups of the form* $\mathrm{Aut}(\boldsymbol{A})$*, where $\boldsymbol{A}$ is the Fraïssé limit of a Fraïssé order class with the Ramsey property.*

Another way to formulate this result is by saying that *the group* $\mathrm{Aut}(\boldsymbol{A})$ *of automorphisms of the Fraïssé limit of a Fraïssé order class $\mathcal{K}$ is extremely amenable if and only if $\mathcal{K}$ has the Ramsey property.*

(F) We can now use this, and known results of structural Ramsey theory, to find many new examples of extremely amenable automorphism groups (see Section 6). Notice that, by the preceding result, the extreme amenability of these groups is in fact equivalent to the corresponding Ramsey theorem.

Consider the class of finite ordered graphs. Its Fraïssé limit is the random graph with an appropriate linear ordering. We call it the *random ordered graph*. Let K_n be the complete graph with n elements, $n = 3, 4, \dots$. Consider the class of K_n-free finite ordered graphs, whose Fraïssé limit we call the *random K_n-free ordered graph*. Finally consider the class of finite linear orderings. Its Fraïssé limit is $\langle \mathbb{Q}, < \rangle$.

All of the above classes satisfy the Ramsey property. This is due to Nešetřil-Rödl [56],[58] (see also Nešetřil [50] and [51]) for the graph cases, and it is of course the classical Finite Ramsey Theorem for the last case. So all the corresponding automorphism groups of their Fraïssé limits are extremely amenable.

This can be generalized to hypergraphs. Let $L_0 = \{R_i\}_{i\in I}$ be a finite relational signature with the arity of each R_i at least 2. A *hypergraph of type* L_0 is a structure $\boldsymbol{A}_0 = \langle A_0, \{R_i^{\boldsymbol{A}_0}\}_{i\in I}\rangle$ in which $(a_1,\dots a_{n(i)}) \in R_i^{\boldsymbol{A}} \Rightarrow a_1,\dots,a_{n(i)}$ are distinct, and $R_i^{\boldsymbol{A}_0}$ is closed under permutations. Thus, essentially, $R_i^{\boldsymbol{A}_0} \subseteq [A_0]^{n(i)} =$ the set of subsets of A_0 of cardinality $n(i)$. Consider the class of all finite ordered hypergraphs of type L_0, whose Fraïssé limit we call the *random ordered hypergraph of type* L_0. More generally, for every class $\mathcal{A}$ of finite irreducible hypergraphs of type L_0 (where $\boldsymbol{A}_0$ is *irreducible* if it has at least two elements and for every $x \neq y$ in A_0 there is $i \in I$ with $\{x, y\} \subseteq R_i^{\boldsymbol{A}_0}$), let $\mathcal{O}Forb(\mathcal{A})$ be the class of all finite ordered hypergraphs of type L_0 which omit $\mathcal{A}$ (i.e., no element of $\mathcal{A}$ can be embedded in them). We call the Fraïssé limit of $\mathcal{O}Forb(\mathcal{A})$ the *random $\mathcal{A}$-free ordered hypergraph of type* L_0. Again Nešetřil-Rödl [56], [58], and independently Abramson–Harrington [1] for $\mathcal{A} = \emptyset$, showed that these classes have the Ramsey property, so the corresponding automorphism groups are extremely amenable.

There are similar results for metric spaces. Consider the class of finite ordered metric spaces with rational distances. Its Fraïssé limit is the so-called rational Urysohn space with an appropriate ordering. We call it the *ordered rational Urysohn space*. In response to an inquiry by the authors, Nešetřil [52] verified that the class of finite ordered metric spaces has the Ramsey property. Thus the automorphism group of the ordered rational Urysohn space is extremely amenable. We also show how this result can be used to give a new proof of the result of Pestov [66] that the isometry group of the Urysohn space is extremely amenable.

We next consider some other kinds of examples. We first look at the class of all finite *convexly ordered equivalence relations*, where convexly ordered means that each equivalence class is convex (whenever two elements are in it every element between them is also in it). Their Fraïssé limit is the rationals with the usual ordering and an equivalence relation whose classes are convex, order isomorphic to the rationals, and moreover the set of classes itself is ordered like the rationals. We show that the automorphism group of this structure is extremely amenable. This implies that the corresponding class has the Ramsey property, a fact that can also be proved directly.

Further we consider finite-dimensional vector spaces over a fixed finite field F. A *natural ordering* on such a vector space is one induced antilexicographically by an ordering of a basis. These were considered in Thomas [76], who showed that the class of naturally ordered finite-dimensional spaces over F is a Fraïssé class. We call its limit the $\aleph_0$-*dimensional vector space over* F *with the canonical ordering*. The Ramsey property for the class of naturally ordered finite-dimensional vector spaces over F is easily seen to be equivalent to the Ramsey property for the class of finite-dimensional vector spaces over F, which was established in [32]. So the corresponding automorphism group of the Fraïssé limit is extremely amenable.

Finally, we consider the class of naturally ordered finite Boolean algebras, where a *natural ordering* on a finite Boolean algebra is one antilexicographically induced by an ordering of its atoms. By analogy with Thomas' result, we show that this is also a Fraïssé class, and we call its limit the *countable atomless Boolean algebra with the canonical ordering*. The Ramsey property for the class of naturally ordered finite Boolean algebras is again easily seen to be equivalent to the Ramsey property for the class of finite Boolean algebras and this is trivially equivalent to the Dual Ramsey Theorem of Graham-Rothschild [33]. Thus the corresponding automorphism group is extremely amenable.

(G) Finally we use the results in **(E)**, and some additional considerations, to compute universal minimal flows. In **(E)** we have seen a host of examples of Fraïssé order classes $\mathcal{K}$ in a signature $L \supseteq \{<\}$. Let $L_0 = L \setminus \{<\}$, the signature without the distinguished symbol for the ordering. For any structure $\boldsymbol{A}$ for L, we denote by $\boldsymbol{A}_0 = \boldsymbol{A}|L_0$ its *reduct* to L_0, i.e., $\boldsymbol{A}_0$ is the structure $\boldsymbol{A}$ with $<^{\boldsymbol{A}}$ dropped. Denote also by $\mathcal{K}_0 = \mathcal{K}|L$ the class of all reducts $\boldsymbol{A}_0 = \boldsymbol{A}|L_0$ for $\boldsymbol{A} \in \mathcal{K}$. When $\mathcal{K}$ satisfies a mild (and easily verified in every case we are interested in) condition, in which case we call $\mathcal{K}$ *reasonable* (see 5.1 below for the precise definition), then $\mathcal{K}_0$ is a Fraïssé class, whose limit is the reduct of the Fraïssé limit of $\mathcal{K}$. Put $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0)$, $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K})$, so that $\boldsymbol{F}_0 = \boldsymbol{F}|L_0$, i.e., $\boldsymbol{F} = \langle \boldsymbol{F}_0, <^{\boldsymbol{F}} \rangle$. In particular, $F_0 = F$. Put $<^{\boldsymbol{F}} = \prec_0$. It is natural now to look at the action of $\mathrm{Aut}(\boldsymbol{F}_0)$ on the space of all linear orderings on F_0. Denote then by $X_{\mathcal{K}}$ the orbit closure $\overline{G \cdot \prec_0}$ of $\prec_0$ in this action. It is easy to see that $X_{\mathcal{K}}$ is the space of all linear orderings $\prec$ on F_0 which have the property that for any finite substructure $\boldsymbol{B}_0$ of $\boldsymbol{F}_0$, $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec |B_0 \rangle \in \mathcal{K}$. We call these $\mathcal{K}$-*admissible* orderings. This is clearly a compact $\mathrm{Aut}(\boldsymbol{F}_0)$-invariant subset of $2^{F_0 \times F_0}$ in the natural action of $\mathrm{Aut}(\boldsymbol{F}_0)$ on $2^{F_0 \times F_0}$, so $X_{\mathcal{K}}$ is an $\mathrm{Aut}(\boldsymbol{F}_0)$-flow. If $\mathcal{K}$ has the Ramsey property, it turns out that any minimal subflow of $X_{\mathcal{K}}$ is the universal minimal flow of $\mathrm{Aut}(\boldsymbol{F}_0)$, and $X_{\mathcal{K}}$ is itself minimal precisely when $\mathcal{K}$ additionally satisfies a natural property called the ordering property, which also plays an important role in structural Ramsey theory (see Nešetřil-Rödl [57] and Nešetřil [51]). We say that $\mathcal{K}$ satisfies the *ordering property* if for every $\boldsymbol{A}_0 \in \mathcal{K}_0$, there is $\boldsymbol{B}_0 \in \mathcal{K}_0$ such that for every linear ordering $\prec$ on A_0 and every linear ordering $\prec'$ on B_0, if $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec \rangle \in \mathcal{K}$ and $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec' \rangle \in \mathcal{K}$, then $\boldsymbol{A} \leq \boldsymbol{B}$. Now Theorems 7.4, 7.5, and 10.8 provide a toolbox for computing universal minimal flows of automorphism groups, which can be summarized as follows.

♦ *In the above assumptions, let* $X_{\mathcal{K}}$ *be the* $\mathrm{Aut}(\boldsymbol{F}_0)$*-flow of* $\mathcal{K}$*-admissible orderings on* F_0 $(= F)$*. Then the following are equivalent:*

(i) $X_{\mathcal{K}}$ *is a minimal* $\mathrm{Aut}(\boldsymbol{F}_0)$*-flow,*

(ii) $\mathcal{K}$ *satisfies the ordering property.*

Moreover the following are equivalent:

(iii) $X_{\mathcal{K}}$ *is the universal minimal* $\mathrm{Aut}(\boldsymbol{F}_0)$*-flow.*

(iv) $X_{\mathcal{K}}$ *satisfies the Ramsey and ordering properties.*

Now all the classes $\mathcal{K}$, considered in **(F)** above, satisfy the ordering property. This is due to Nešetřil-Rödl [57] for the case of graphs and hypergraphs, Nešetřil [53] for metric spaces, and is easily verified in all the other cases. Therefore, we have the following computations of universal minimal flows (see Section 8):

⧫ *If* $\boldsymbol{F}_0$ *is one of the following structures, then the universal minimal flow* $M(\mathrm{Aut}(\boldsymbol{F}_0))$ *of the group* $\mathrm{Aut}(\boldsymbol{F}_0)$ *of automorphisms of* $\boldsymbol{F}_0$ *is its action on the space of linear orderings on the universe* F_0 *of* $\boldsymbol{F}_0$*:*

(a) The random graph.

(b) The random $\mathcal{K}_n$*-free graph,* $n = 2, 3, \ldots$

(c) The structure $\langle \mathbb{N} \rangle$ *(where* $\mathrm{Aut}(\boldsymbol{F}_0) = S_\infty$*).*

(d) The random hypergraph of type L_0*.*

(e) The random $\mathcal{A}$*-free hypergraph of type* L_0*, where* $\mathcal{A}$ *is a class of irreducible finite hypergraphs of type* L_0*.*

(f) The rational Urysohn space.

⧫ *If* $\boldsymbol{F}_0$ *is the equivalence relation on a countable set with infinitely many classes, each of which is infinite, then* $M(\mathrm{Aut}(\boldsymbol{F}_0))$ *is the space of all linear orderings on that set for which each equivalence class is convex.*

⧫ *If* $\boldsymbol{F}_0 = \boldsymbol{V}_F$ *is the* $\aleph_0$*-dimensional vector space* $\boldsymbol{V}_F$ *over a finite field* F*, then* $M(\mathrm{Aut}(\boldsymbol{F}_0))$ *is the space of all orderings on* V_F*, whose restrictions to finite-dimensional subspaces are natural.*

⧫ *If* $\boldsymbol{F}_0 = \boldsymbol{B}_\infty$ *is the countable atomless Boolean algebra* $\boldsymbol{B}_\infty$*, then* $M(\mathrm{Aut}(\boldsymbol{F}_0))$ *the space of all linear orderings on* B_∞*, whose restrictions to finite subalgebras are natural.*

In particular, in all these cases, the universal minimal flow is metrizable.

Of course (i), (c) is the result of Glasner-Weiss [30]. Very recently, Glasner-Weiss [31] computed the universal minimal flow of $H(2^{\mathbb{N}})$, the homeomorphism group of the Cantor space $2^{\mathbb{N}}$, as the space of maximal chains of compact subsets of $2^{\mathbb{N}}$, which is metrizable. Since the group in (iv) above is, by Stone duality, isomorphic to $H(2^{\mathbb{N}})$, we have another proof that the universal minimal flow is metrizable and a different description of this flow. Of course these two flows are isomorphic and in fact an explicit isomorphism can be found.

There is actually quite a bit more that we can say in this context and this has some further interesting connections with Ramsey theory. Let $\mathcal{K}_0$ be a Fraïssé class in a signature L_0, let $L = L_0 \cup \{<\}$ and call any class $\mathcal{K}$ in L with $\mathcal{K}|L_0 = \mathcal{K}_0$ an *expansion* of $\mathcal{K}_0$. We define in Section 9 a canonical notion of equivalence of expansions of $\mathcal{K}_0$, called *simple bi-definability*. Intuitively, if $\mathcal{K}', \mathcal{K}''$ are simply bi-definable expansions, we view $\mathcal{K}', \mathcal{K}''$ as "trivial" perturbations of each other. Using dynamical ideas, e.g., the uniqueness of the universal minimal flow, we prove the following (see 9.2, 10.7):

♦ *If $\mathcal{K}$ is a reasonable Fraïssé order class in L which is an expansion of $\mathcal{K}_0$ and has the Ramsey property, then there is $\mathcal{K}' \subseteq \mathcal{K}$, a reasonable Fraïssé order class in L which is an expansion of $\mathcal{K}_0$ and has both the Ramsey and ordering properties.*

♦ *If $\mathcal{K}', \mathcal{K}''$ are reasonable Fraïssé order classes in L that are expansions of $\mathcal{K}_0$ and satisfy both the Ramsey and ordering properties, then $\mathcal{K}', \mathcal{K}''$ are simply bi-definable.*

Thus among all the expansions of $\mathcal{K}_0$ that satisfy the Ramsey property there are canonical ones, those that also satisfy the ordering property. These are unique, up to simple bi-definability, and are "extremal", i.e., least under inclusion, again up to simple bi-definability. In several cases, like, e.g., Boolean algebras and vector spaces, we can also list explicitly all such classes.

If $\mathcal{K}_0$ is as above and $\boldsymbol{A}_0 \in \mathcal{K}_0$, let $t(\boldsymbol{A}_0, \mathcal{K}_0)$ be the smallest number $t \in \mathbb{N}$, if it exists, that satisfies: For every $\boldsymbol{A}_0 \leq \boldsymbol{B}_0$ in $\mathcal{K}_0, k \geq 2$, there is $\boldsymbol{C}_0 \in \mathcal{K}_0$ with $\boldsymbol{C}_0 \geq \boldsymbol{B}_0$ and

$$\boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_{k,t},$$

where this notation means that for every coloring $c : \binom{\boldsymbol{C}_0}{\boldsymbol{A}_0} \to \{1, \dots, k\}$, there is $\boldsymbol{B}'_0 \in \binom{\boldsymbol{C}_0}{\boldsymbol{B}_0}$ such that c on $\binom{\boldsymbol{B}'_0}{\boldsymbol{A}_0}$ takes at most t many values. (Thus $\boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_k \Leftrightarrow \boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_{k,1}$.) This variation of the original arrow-notation of Erdös-Rado [19] is due to Erdös-Hajnal-Rado [18] and has already reappeared in several other areas of Ramsey theory. Following Fouché [21], we call $t(\boldsymbol{A}_0, \mathcal{K}_0)$ the *Ramsey degree* of $\boldsymbol{A}_0$ in $\mathcal{K}_0$. For every expansion $\mathcal{K}$ of $\mathcal{K}_0$ let, for $\boldsymbol{A}_0 \in \mathcal{K}_0$

$$X^{\boldsymbol{A}_0}_{\mathcal{K}} = \{\prec : \prec \text{ is a linear ordering on } A_0 \text{ and } \langle \boldsymbol{A}_0, \prec \rangle \in \mathcal{K}\}.$$

Then $\mathrm{Aut}(\boldsymbol{A}_0)$ acts in the obvious way on $X^{\boldsymbol{A}_0}_{\mathcal{K}}$ and we let $t_{\mathcal{K}}(\boldsymbol{A}_0)$ be the number of orbits. Clearly,

$$t_{\mathcal{K}}(\boldsymbol{A}_0) = \frac{\mathrm{card}(X^{\boldsymbol{A}_0}_{\mathcal{K}})}{\mathrm{card}(\mathrm{Aut}(\boldsymbol{A}_0))}.$$

For example, if $\mathcal{K}_0 =$ the class of finite graphs, $\mathcal{K} =$ the class of finite ordered graphs, then $t_{\mathcal{K}}(\boldsymbol{A}_0) = \frac{\mathrm{card}(\boldsymbol{A}_0)!}{\mathrm{card}(\mathrm{Aut}(\boldsymbol{A}_0))}$. The following can be proved by standard methods in Ramsey theory (see Section 10).

♦ *Let $\mathcal{K}_0$ be a Fraïssé class in a signature $L_0, L = L_0 \cup \{<\}$. If $\mathcal{K}_0$ admits a reasonable Fraïssé order class in L which is an expansion of $\mathcal{K}_0$ and has the Ramsey property, then $t(\boldsymbol{A}_0, \mathcal{K}_0)$ exists for all $\boldsymbol{A}_0 \in \mathcal{K}_0$. Moreover, $t(\boldsymbol{A}_0, \mathcal{K}_0) \leq t_{\mathcal{K}}(\boldsymbol{A}_0)$, for any reasonable Fraïssé order class $\mathcal{K}$ in L which is an expansion of $\mathcal{K}_0$ and has the Ramsey property, and $t(\boldsymbol{A}_0, \mathcal{K}_0) = t_{\mathcal{K}'}(\boldsymbol{A}_0)$ for any such class $\mathcal{K}'$ that also has the ordering property.*

For example, if $\mathcal{K}_0 =$ the class of finite graphs, then the Ramsey-theoretic results of Nešetřil-Rödl [56], [57], [58] and Abramson-Harrington [78] can be interpreted as saying that $t(\boldsymbol{A}_0, \mathcal{K}_0) = \frac{\mathrm{card}(A_0)!}{\mathrm{card}(\mathrm{Aut}(\boldsymbol{A}_0))}$. This seems to have been first pointed out in print by Fouché [22], who in a series of other papers [20], [22], [23] has computed Ramsey degrees of several classes of finite structures.

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1. Topological dynamics

(A) We will survey here some basic concepts and results of topological dynamics, which we will need in this paper. More detailed treatments can be found in Ellis [15], Auslander [3], de Vries [12], Pestov [63], [65], Glasner [29], Uspenskij [82].

Recall that an action $(g, x) \in G \times X \mapsto g \cdot x \in X$ of a topological group G on a topological space X is *continuous* if it is continuous as a map from $G \times X$ into X. We will consider continuous actions of (Hausdorff) topological groups G on (non-$\emptyset$) compact, Hausdorff spaces X. Actually we are primarily interested in metrizable topological groups G and in fact only separable metrizable ones. So although we will state in this survey several standard results for general topological groups, we will often give a sketch of an argument for the metrizable case only. It should be kept in mind that if G is metrizable (equivalently is Hausdorff and has a countable nbhd basis at the identity), then G admits a right-invariant compatible metric d_r, which of course can always be taken to be bounded by 1, by replacing it, if necessary, by $\frac{d_r}{1+d_r}$. See, e.g., [6], p. 28.

Let G be a topological group and X a compact, Hausdorff space. If we equip $H(X)$, the group of homeomorphisms of X, with the compact-open topology, i.e., the topology with subbasis $\{f \in H(X) : f(K) \subseteq V\}$, with $K \subseteq X$ compact, $V \subseteq X$ open, then $H(X)$ is a topological group, and a continuous action of G on X is simply a continuous homomorphism of G into $H(X)$. We will also refer to a continuous action of G on X as a *G-flow* on X. If the action is understood, we will often simply use X to refer to the flow.

Given a G-flow on X and a point $x \in X$, the *orbit* of x is the set

$$G \cdot x = \{g \cdot x : g \in G\}$$

and the *orbit closure* of x, the set

$$\overline{G \cdot x}.$$

This is a G-invariant, compact subset of X. In general, a (non-$\emptyset$) compact, G-invariant subset $Y \subseteq X$ defines a *subflow* by restricting the G-action to Y. A G-flow on X is *minimal* if it contains no proper subflows, i.e., there is no (non-$\emptyset$) compact G-invariant set other than X. Thus X is minimal iff every orbit is dense. A simple application of Zorn's Lemma shows that every G-flow X contains a minimal subflow $Y \subseteq X$.

Among minimal flows of a given group G, there is a largest (universal) one, called the *universal minimal flow*. To define this, we first need the concept of homomorphism of G-flows. Let X, Y be two G-flows. A *homomorphism* of the G-flow X to the G-flow Y is a continuous map $\pi : X \to Y$, which is also a *G-map*, i.e.,

$$\pi(g \cdot x) = g \cdot \pi(x), \quad x \in X, g \in G.$$

Notice that if Y is minimal, then any homomorphism of X into Y is surjective. An *isomorphism* of X to Y is a bijective homomorphism $\pi : X \to Y$ (notice then that π^{-1} is also a homomorphism). We now have the following basic fact in topological dynamics. (For a proof see Auslander [3], Ch. 8, or Uspenskij [82], §3.)

Theorem 1.1. *Given a topological group G, there is a minimal G-flow $M(G)$ with the following property: For any minimal G-flow X there is a homomorphism $\pi : M(G) \to X$. Moreover, $M(G)$ is uniquely determined up to isomorphism by this property.* $\dashv$

The space $M(G)$ is called the *universal minimal flow* of G. In order to get an intuition about this space, we will discuss a standard way of looking at it. For that we first need to discuss the concept of a *pointed G-flow* or *ambit*.

Let G be a topological group. A *G-ambit* is a G-flow X with a distinguished point $x_0 \in X$, whose orbit is dense in X. We often abbreviate this by (X, x_0). A *homomorphism of G-ambits* $(X, x_0), (Y, y_0)$ is a homomorphism $\pi : X \to Y$ of the G-flows such that $\pi(x_0) = y_0$. If such a homomorphism exists, it is clearly unique. Similarly we define the concept of isomorphism of G-ambits.

It is another basic fact of topological dynamics that there is again a largest (universal) G-ambit.

Theorem 1.2. *Given a topological group G, there is a G-ambit $(S(G), s_0)$ with the following property: For any G-ambit (X, x_0) there is a homomorphism of $(S(G), s_0)$ to (X, x_0). Moreover, $(S(G), s_0)$ is uniquely determined up to isomorphism by this property.* $\dashv$

The space $(S(G), s_0)$, often simply written as $S(G)$, is called the *greatest G-ambit*.

The uniqueness part of the preceding result is obvious from the definitions. To establish existence, we will describe a particularly useful way of constructing the greatest ambit.

Consider the space $RUC^b(G)$ of bounded right-uniformly continuous functions $x : G \to \mathbb{C}$. Recall that $x : G \to \mathbb{C}$ is *right-uniformly continuous* if for each $\epsilon > 0$ there is a nbhd V of the identity 1_G of G so that

$$gh^{-1} \in V \Rightarrow |x(g) - x(h)| < \epsilon.$$

If G is metrizable, with right-invariant compatible metric d_r, then right-uniformly continuous means uniformly continuous with respect to d_r:

$$\forall\epsilon\exists\delta(d_r(g,h) < \delta \Rightarrow |x(g) - x(h)| < \epsilon).$$

Under pointwise addition, multiplication and conjugation, and with the sup norm $\|x\|_\infty = \sup\{|x(g)| : g \in G\}$, $RUC^b(G)$ is an abelian C^*-algebra which is unital (with multiplicative identity the constant 1 function). Denote by $S(G)$ the maximal ideal space of $RUC^b(G)$, i.e., the space of all continuous homomorphisms $\varphi : RUC^b(G) \to \mathbb{C}$. Equipped with the topology generated by the maps $\hat{x} : S(G) \to \mathbb{C}, \hat{x}(\varphi) = \varphi(x)$, for $x \in RUC^b(G)$ (i.e., the smallest topology in which all $\hat{x}, x \in RUC^b(G)$, are continuous), this is a compact, Hausdorff space. Moreover, by the Gelfand-Naimark theorem, $RUC^b(G)$ can be canonically identified with $C(S(G))$, the C^*-algebra of all continuous complex-valued functions on $S(G)$, identifying $x \in RUC^b(G)$ with $\hat{x}$ (see, e.g., [74], 11.18).

Now G acts continously by C^*-automorphisms on $RUC^b(G)$ by left-shift

$$g \cdot x(h) = x(g^{-1}h)$$

and thus acts canonically on $S(G)$ via

$$g \cdot \varphi(x) = \varphi(g^{-1} \cdot x).$$

It is also easy to check that this action is continuous, so $S(G)$ is a G-flow. We will now identify a canonical element of $S(G)$, that will turn it into an ambit.

For each $g \in G$, let $\varphi_g \in S(G)$ be defined by $\varphi_g(x) = x(g), x \in RUC^b(G)$. Then one can see that $g \mapsto \varphi_g$ is a homeomorphism of G with a dense subset of $S(G)$. For example, when G is metrizable with bounded compatible right-invariant metric d_r, and $g_0 \neq h_0$, then for $x(g) = d_r(g,h_0), \varphi_{g_0}(x) \neq \varphi_{h_0}(x)$ so $\varphi_{g_0} \neq \varphi_{h_0}$, i.e., this map is 1-1. The verification that it is homeomorphism is straightforward. Finally $\{\varphi_g : g \in G\}$ is dense in $S(G)$, since, otherwise, there is $f \in C(S(G))$, so that $f = \hat{x}$ for some $x \in RUC^b(G)$, with $f \neq 0$ but $f(\varphi_g) = \hat{x}(\varphi_g) = x(g) = 0, \forall g \in G$, which implies that $x = 0$, thus $f = 0$, a contradiction.

So from now on we will identify g with φ_g and think of G as a dense subset of $S(G)$. Moreover G is an invariant subset of $S(G)$ and the restriction of the action to G is simply left-translation: $(g,h) \mapsto gh$. We now have the following standard fact.

Theorem 1.3. *The G-ambit $(S(G), 1_G)$ is the greatest G-ambit.*

Proof. Since the orbit of 1_G in $S(G)$ is G, which is dense, clearly $(S(G), 1_G)$ is a G-ambit. Consider now an arbitrary G-ambit (X, x_0). Suppose $f \in C(X)$. Define then $f^* : G \to \mathbb{C}$ by

$$f^*(g) = f(g \cdot x_0).$$

We verify that $f^* \in RUC^b(G)$. Since the action of G on X is continuous, an easy compactness argument shows that given $\epsilon > 0$, there is a nbhd V of the identity of G such that $g \in V$ implies $|f(g \cdot x) - f(x)| < \epsilon, \forall x \in X$. So if $gh^{-1} \in V$, then

$|f^*(g) - f^*(h)| = |f(g \cdot x_0) - f(h \cdot x_0)| = |f(gh^{-1} \cdot (h \cdot x_0)) - f(h \cdot x_0)| < \epsilon$, and thus $f^* \in RUC^b(G)$ (f^* is clearly bounded).

Identifying, as usual, $RUC^b(G)$ with $C(S(G))$, the map $f \mapsto f^*$ is a unital C^*-algebra monomorphism of $C(X)$ into $C(S(G))$. Now it is a well known fact that a unital C^*-algebra monomorphism $\pi : C(K) \to C(L)$, where K, L are (non-$\emptyset$) compact spaces, is of the form $\pi(f) = f \circ \Pi$ for a *uniquely* determined continuous surjection $\Pi : L \to K$ (see, e.g., [10], 2.4.3.6). From this it also follows that if K, L are actually G-flows, and we let G act on $C(K), C(L)$ by shift, $g \cdot f(x) = f(g^{-1} \cdot x)$, then if π is a G-map, Π is also a G-map. Applying this to $f \mapsto f^*$, we see that there is a homomorphism of G-flows $\Phi : S(G) \to X$ with $f^* = f \circ \Pi$. It only remains to check that $\Pi(1_G) = x_0$. But for any $f \in C(X), f^*(1_G) = f(x_0) = f(\Pi(1_G))$, so we must have $\Pi(1_G) = x_0$, and the proof is complete. ⊣ ⊣

Using this it is now immediate to obtain the following description of the universal minimal flow of G.

Corollary 1.4. *Let $M(G)$ be a minimal subflow of $S(G)$ (i.e., $M(G)$ is a minimal G-invariant compact subset of $S(G)$). Then $M(G)$ is the universal minimal flow (up to isomorphism).*

Proof. Let X by any minimal G-flow. Fix $x_0 \in X$. Then (X, x_0) is a G-ambit, so let $\pi : (S(G), 1_G) \to (X, x_0)$ be a homomorphism. Then clearly the restriction of π to $M(G)$ is also a homomorphism, and we are done. ⊣

In particular, it follows from the uniqueness part of Theorem 1.1, that all minimal subflows of $S(G)$ are isomorphic. (This uniqueness part, which is not proved here, is based on techniques from semigroup theory.) As we will soon see, the space $M(G)$ can be extremely complicated, e.g., non-metrizable, even when the group G is very "small", e.g., an infinite countable discrete G. However, we will verify that when G is separable, metrizable, $M(G)$ is at least an inverse limit of metrizable G-flows.

Fix a topological group G. An *inverse system* of G-flows consists of a directed set $\langle I, \preceq \rangle$, a family $\{X_i\}_{i \in I}$ of G-flows and a family of homomorphisms $\pi_{ij} : X_j \to X_i$, for each $i \preceq j$, such that $\pi_{ii} =$ the identity of X_i, and $i \preceq j \preceq k \Rightarrow \pi_{ik} = \pi_{ij} \circ \pi_{jk}$. The *inverse limit* $\lim_{\leftarrow} X_i$ is the G-flow defined as follows: Consider the product topological space $\prod_{i \in I} X_i$ and let

$$\lim_{\leftarrow} X_i = \{\{x_i\}_i \in \prod_i X_i : \forall i \preceq j (\pi_{ij}(x_j) = x_i)\}.$$

By a simple application of compactness, $\lim_{\leftarrow} X_i \neq \emptyset$ and is clearly a compact subset of $\prod_{i \in I} X_i$. The group G acts on $\lim_{\leftarrow} X_i$ coordinatewise: $g \cdot \{x_i\} = \{g \cdot x_i\}$ and this is clearly a continuous action. Define

$$\pi_i : \lim_{\leftarrow} X_i \to X_i$$

by $\pi_i(\{x_i\}_{i\in I}) = x_i$. Then π_i is a homomorphism and if $i \preceq j$, then $\pi_i = \pi_{ij} \circ \pi_j$. Finally, if X is a G-flow and there are homomorphisms $\varphi_i : X \to X_i$ with $i \preceq j \Rightarrow \varphi_i = \varphi_{ij} \circ \varphi_j$, then there is a unique homomorphism $\varphi : X \to \lim_{\leftarrow} X_i$ such that $\varphi_i = \pi_i \circ \varphi$.

Similarly we define inverse systems of G-ambits. We now have the following folklore fact.

Theorem 1.5. *Let G be a separable, metrizable group. Then the greatest ambit $(S(G), 1_G)$ is the inverse limit of a system of metrizable G-ambits. Similarly, the universal minimal flow is the inverse limit of a system of metrizable minimal G-flows.*

Proof. We will first derive the second assertion from the first. Suppose $\{(X_i, x_i^0)\}$ is an inverse system of metrizable G-ambits, so that $(X, x_0) = \lim_{\leftarrow}(X_i, x_i^0)$ is the greatest G-ambit. Let $\pi_i : X \to X_i$ be the corresponding homomorphism. In particular, $X = \lim_{\leftarrow} X_i$ as a G-flow. Now fix a minimal subflow $M \subseteq X$, so that, as we have seen earlier, M is the universal minimal flow. Put $\pi_i(M) = M_i \subseteq X_i$. Then M_i is a subflow of X_i and it is clearly minimal and metrizable. So it is enough to check that $M = \lim_{\leftarrow} M_i$, which is an easy compactness argument.

For the first assertion, fix a countable dense set $D \subseteq G$. Let $(I, \preceq)$ be the following directed set: I consists of all separable, closed, unital G-invariant C^*-subalgebras of $RUC^b(G)$. Since a closed, unital C^*-subalgebra of $RUC^b(G)$ is G-invariant iff it is D-invariant, clearly $\bigcup I = RUC^b(G)$. Let for $A, B \in I$

$$A \preceq B \Leftrightarrow A \subseteq B.$$

For $A \in I$ denote by X_A the maximal ideal space of A, which is compact metrizable, since A is separable. Since G acts continuously by C^*-automorphisms on A, it acts continuously on X_A. Moreover, as before, we can identify each $g \in G$ with an element of X_A, so that G can be thought as a dense invariant subset of X_A and the G action on it is by left-translation. Thus again $(X_A, 1_G)$ is a metrizable G-ambit. We view as usual A as identified with $C(X_A)$ via $x \mapsto \hat{x}_A$. When $A \preceq B$, the identity is an injective unital C^*-homomorphism from A to B, so there is a unique surjection $\pi_{AB} : X_B \to X_A$ such that for $x \in A$, $\hat{x}_B = \hat{x}_A \circ \pi_{AB}$, therefore for $\varphi \in X_B, \hat{x}_A(\pi_{AB}(\varphi)) = \hat{x}_B(\varphi)$ or $\pi_{AB}(\varphi)(x) = \varphi(x)$, i.e., $\pi_{AB}(\varphi) = \varphi|A$. Similarly, there is a surjection $\pi_A : S(G) \to X_A$ given by $\pi_A(\varphi) = \varphi|A$ for each $A \in I$, so that $\pi_A = \pi_{AB} \circ \pi_B$ for $A \preceq B$. Moreover, $\pi_A(1_G) = \pi_{AB}(1_G) = 1_G$. Thus $\varphi \mapsto (\pi_A(\varphi))_{A\in I}$ is a homomorphism from $(S(G), 1_G)$ to $\lim_{\leftarrow}(X_A, 1_G)$. Also given $(\varphi_A) \in \lim_{\leftarrow} X_A$, we have, for $A \preceq B$, that $\varphi_A = \varphi_B|A$, so that there is a unique $\varphi \in S(G)$ with $\pi_A(\varphi) = \varphi_A$. Thus $\varphi \mapsto (\pi_A(\varphi))_{A\in I}$ is an isomorphism of the G-ambit $(S(G), 1_G)$ and $\lim_{\leftarrow}(X_A, 1_G)$. ⊣

(B) We will now discuss the case of infinite countable discrete groups G and see that in this case $M(G)$ is an extremely large space, in particular it is not metrizable.

For an infinite countable discrete G, it is clear that $RUC^b(G)$ is identical with $\ell^\infty(G)$, the C^*-algebra of bounded complex functions on G with the supremum norm. It is then easy to see that $S(G)$ is identical with βG, the space of ultrafilters on G with the topology whose basis consists of the sets $\hat{A} = \{U \in \beta G : A \in U\}$, for $\emptyset \neq A \subseteq G$. This is a non-metrizable, compact, Hausdorff space. The action of G on $S(G)$ is given by

$$A \in g \cdot U \Leftrightarrow g^{-1}A \in U$$

for $A \subseteq G$.

The copy of G in $S(G)$ consists simply of the principal ultrafilters. So the distinguished point is the principal ultrafilter on 1_G. Finally, $M(G)$ is any minimal subflow of βG.

First we point out that G acts *freely* on $S(G)$, i.e., $g \cdot x \neq x,\ \forall g \neq 1_G,\ x \in S(G)$ (this is due to Ellis [14]). For that it is of course enough to find a free G-flow X (since $S(G)$ can be homomorphically mapped to X). Again it is enough to find for each $g \in G,\ g \neq 1_G$, a G-flow X_g, such that $g \cdot x \neq x,\ \forall x \in X_g$. Because then $X = \prod_{g \in G \setminus \{1_G\}} X_g$ with the action $h \cdot (x_g) = (h \cdot x_g)$ works. Consider the shift action of G on $3^G,\ g \cdot p(h) = p(g^{-1}h)$. Suppose we can find $p_g \in 3^G$ such that $(*) : \forall h \in G(g \cdot (h \cdot p_g)(1_G) \neq (h \cdot p_g)(1_G))$. Then we can take X_g to be the orbit closure of p_g. Now $(*)$ is equivalent to: $\forall h \in G(p_g(h^{-1}g^{-1}) \neq p_g(h^{-1}))$ or $\forall h(p_g(h) \neq p_g(hg))$. Consider a coset $h\langle g \rangle$ of the cyclic subgroup $\langle g \rangle$. Define p_g on $h\langle g \rangle$, so that $p_g(hg^k) \neq p_g(hg^{k+1})$, for any $k \in \mathbb{Z}$. (We used 3^G instead of 2^G to take care of the case when g has finite order.) This clearly works.

The space $M(G)$ is quite big, see e.g., the references in [12], p. 391, **11**. Let us verify for instance that it is not metrizable. The space $M(G)$ is a closed subset of βG. If it was metrizable and infinite, it would have a non-trivial convergent sequence, which is impossible in the extremally disconnected space βG (see [16], Ex. 6.2.G(a) on p. 456). So if $M(G)$ is metrizable, it has to be finite, contradicting the fact that G acts freely on $M(G)$.

(C) More generally, Veech [83] has shown that when G is locally compact, then G acts freely on $S(G)$ and thus on $M(G)$. For a simpler version of the proof see Pym [71]. See also [2] for the second countable case. In an Appendix to this paper, we also give a new proof of Veech's Theorem. Note that Veech's Theorem implies that if G is second countable, then G admits a free metrizable G-flow. To see this, notice that it is enough to find for each $1_G \neq g \in G$ a metrizable G-flow X_g with $g \cdot x \neq x$, if $x \in X_g$. Indeed, if we can do that, then, by compactness, there is an open nbhd V_g of g with $1_G \notin V_g$ and $h \in V_g \Rightarrow (h \cdot x \neq x,\ \forall x \in X_g)$. Find now $g_0, g_1, \cdots \in G \setminus \{1_G\}$ so that $\{V_{g_n}\}_{n \in \mathbb{N}}$ is an open cover of $G \setminus \{1_G\}$. Then $X = \prod_n X_{g_n}$ with the coordinatewise action is a free metrizable G-flow. (This argument comes from [2].) So fix $g \in G,\ g \in 1_G$, in order to find X_g. Write $S(G) = \lim_{\leftarrow} X_i,\ X_i$ a metrizable G-flow. If none of the X_i can be $X_g,\ \{x_i \in X_i : g \cdot x_i = x_i\} = Y_i$ is non-$\emptyset$, and if $\pi_{ij} : X_j \to X_i$, for $i \preceq j$, are the corresponding homomorphisms, then $\pi_{ij}(Y_j) \subseteq Y_i$, so the inverse limit of $(Y_i, \pi_{ij}|Y_j)$ is non-$\emptyset$ and thus there is $\{y_i\} \in S(G)$ with $g \cdot \{y_i\} = \{y_i\}$, a contradiction, as G acts freely on $S(G)$.

We also show in an Appendix that when G is non-compact, locally compact, $M(G)$ is non-metrizable. Stronger results in special cases were obtained in Turek [78], Lau-Milner-Pym [43]. Of course when G is compact, $M(G)$ is G itself with the left-translation action.

(D) Rather remarkably, there are nontrivial groups G for which $M(G)$ trivializes, i.e., consists of a single point. Such groups are called extremely amenable. Thus a topological group is *extremely amenable* if any G-flow X has a fixed point, i.e., there is $x \in X$ with $g \cdot x = x$, $\forall g \in G$. (For this reason, sometimes extremely amenable groups are described as groups having the *fixed point on compacta property.*) By Veech's Theorem nontrivial such groups cannot be locally compact. As it turned out, a number of important, non-locally compact Polish groups are extremely amenable. Among them are: the unitary group $U(H)$ of the infinite dimensional separable Hilbert space H (Gromov-Milman [38]); $L(I, \mathbb{T})$, the group of measurable maps from $I = [0, 1]$ to $\mathbb{T}$, with pointwise multiplication, and the topology of convergence in measure (Furstenberg-Weiss, Glasner [28]); $H_+(I)$ and $H_+(\mathbb{R})$, the groups of orientation preserving homeomorphisms of I and $\mathbb{R}$, with the compact-open topology (Pestov [64]); $\mathrm{Aut}(I, \lambda)$ (resp., $\mathrm{Aut}^*(I, \lambda)$), the groups of measure preserving (resp., measure-class preserving) automorphisms of Lebesgue measure λ on I, with the weak topology (Giordano-Pestov [26]); $\mathrm{Iso}(\mathbf{U})$, the group of isometries of the Urysohn space, with the pointwise convergence topology (Pestov [66])), and $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$, the group of automorphisms of the rationals with the usual ordering, with the pointwise convergence topology (Pestov [64]). For more about extreme amenability, see also Pestov [65], [67], [68] and Uspenskij [82].

In case the group G is separable metrizable, we can restate the definition of extreme amenability in terms of metrizable flows only. In other words, a separable metrizable group G is extremely amenable iff every metrizable G-flow has a fixed point. Indeed, if every metrizable G-flow has a fixed point, every minimal metrizable G-flow is a singleton, and thus so is $M(G)$, being an inverse limit of such G-flows.

(E) Except for the case of compact metrizable or extremely amenable groups, there were very few cases where the universal minimal flow $M(G)$ was known to be metrizable. Pestov [64] first computed that for the group $H_+(\mathbb{T})$ of orientation-preserving homeomorphisms of the circle $\mathbb{T}$, the canonical evaluation action on $\mathbb{T}$ is the universal minimal flow. Then Glasner-Weiss [30] computed the universal minimal flow of S_∞, the group of permutations of $\mathbb{N}$ with the pointwise convergence topology. It turns out to be the canonical action on the compact, metrizable space of linear orderings on $\mathbb{N}$. Finally Glasner-Weiss [31] computed the universal minimal flow of the group $H(2^{\mathbb{N}})$ of the homeomorphisms of the Cantor space $2^{\mathbb{N}}$. It is the action of $H(2^{\mathbb{N}})$ on the space of maximal chains of compact subsets of $2^{\mathbb{N}}$, invented by Uspenskij [81].

Let us point out here that only one of the following is possible:

(i) $M(G)$ is finite.

(ii) $M(G)$ is perfect (i.e., has no isolated points), and thus $\mathrm{card}(M(G)) = 2^{\aleph_0}$, if $M(G)$ is metrizable.

To see this, note that if $M(G)$ has an isolated point x_0, then, as every orbit of $M(G)$ is dense, x_0 belongs to every orbit, thus there is only one orbit in $M(G)$, and so every point of $M(G)$ is isolated. Since $M(G)$ is compact, $M(G)$ is finite.

We have already discussed examples of metrizable $M(G)$ which consist of exactly one point or are perfect. It is easy to see that also any finite cardinality for $M(G)$ is possible. Because if H is extremely amenable, and $G = H \times \mathbb{Z}_n$, then the obvious action of G on $\mathbb{Z}_n$ is the universal minimal flow of G, thus $\mathrm{card}(M(G)) = n$.

Our goal in this paper is to study extreme amenability and universal minimal flows of closed subgroups of S_∞, i.e., automorphism groups of countable structures. In particular, we find new examples of extremely amenable groups and also new cases where the universal minimal flow is metrizable and can be computed.

2. Fraïssé Theory

We will review here some basic ideas of model theory concerning the Fraïssé construction and ultrahomogeneous countable structures. Our main reference here is Hodges [40], Ch. 7. See also [9] and [8].

A (countable) *signature* is a countable collection $L = \{R_i\}_{i\in I} \cup \{f_j\}_{j\in J}$ of (distinct) *relation* and *function symbols* each of which has an associated number, called its *arity*. The arity $n(i)$ of each relation symbol R_i is a positive integer and the arity $m(j)$ of each function symbol f_j is a non-negative integer. A structure for L is an object of the form

$$\boldsymbol{A} = \langle A, \{R_i^{\boldsymbol{A}}\}_{i\in I}, \{f_j^{\boldsymbol{A}}\}_{j\in J}\rangle,$$

where A is a *non-empty* set, called the *universe* of $\boldsymbol{A}$, $R_i^{\boldsymbol{A}} \subseteq A^{n(i)}$, i.e., $R_i^{\boldsymbol{A}}$ is a $n(i)$-ary relation on A, and $f_j : A^{m(j)} \to A$, i.e., $f_j^{\boldsymbol{A}}$ is an m_j-ary function on A. (When $m(j) = 0$, $f_j^{\boldsymbol{A}}$ is a distinguished element of A.)

Given two structures $\boldsymbol{A}, \boldsymbol{B}$ of the same signature L, a *homomorphism* of $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $\pi : A \to B$ such that

$$R_i^{\boldsymbol{A}}(a_1, \ldots, a_{n(i)}) \Leftrightarrow R_i^{\boldsymbol{B}}(\pi(a_1), \ldots, \pi(a_{n(i)}))$$

and

$$\pi(f_j^{\boldsymbol{A}}(a_1, \ldots, a_{m(j)})) = f_j^{\boldsymbol{B}}(\pi(a_1), \ldots, \pi(a_{m(j)})).$$

We write also in this case $\pi : \boldsymbol{A} \to \boldsymbol{B}$. (*Caution.* Sometimes in the definition of homomorphism, one only requires the left-to-right implication concerning $R_i^{\boldsymbol{A}}, R_i^{\boldsymbol{B}}$. We will use here only the stronger version above.) If π is also 1-1, it is called a *monomorphism* or *embedding*. Finally, if π is 1-1 and onto it is called an *isomorphism*. If there is an isomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$, we say that $\boldsymbol{A}, \boldsymbol{B}$ are *isomorphic*, in symbols $\boldsymbol{A} \cong \boldsymbol{B}$. An *automorphism* of $\boldsymbol{A}$ is an isomorphism of $\boldsymbol{A}$ to itself. We denote by $\mathrm{Aut}(\boldsymbol{A})$ the group of automorphisms of $\boldsymbol{A}$.

We will be primarily interested in countable structures $\boldsymbol{A}$ (i.e., the universe A is countable). For countable $\boldsymbol{A}$, the group $\mathrm{Aut}(\boldsymbol{A})$, with the pointwise convergence topology, is Polish, in fact it is a closed subgroup of S_A, the Polish group of permutations of A with the pointwise convergence topology. Conversely, given a closed subgroup $G \subseteq S_A$, there is a signature L and a structure $\boldsymbol{A}_G$ with universe A, so that $\mathrm{Aut}(\boldsymbol{A}_G) = G$. To see this, let for each $n \geq 1, \mathcal{O}_1^n, \mathcal{O}_2^n, \dots$ be the orbits of G on A^n, the action of G on A^n being defined by $g \cdot (a_1, \dots, a_n) = (g(a_1), \dots, g(a_n))$. Let $L = \{R_{n,i}\}_{n \geq 1}$, where each $R_{n,i}$ is an n-ary relation symbol. Define $\boldsymbol{A}_G$ by letting $R_{n,i}^{\boldsymbol{A}_G} = \mathcal{O}_i^n \subseteq A^n$. Then it is easy to check that $\mathrm{Aut}(\boldsymbol{A}_G) = G$. We call $\boldsymbol{A}_G$ the *induced structure* associated to G (see Hodges [40], 4.1.4, where this is called the canonical structure for G - we use however the term canonical for other purposes in this paper).

A substructure $\boldsymbol{B}$ of $\boldsymbol{A}$ has as universe a (non-empty) subset $B \subseteq A$ closed under each $f_j^{\boldsymbol{A}}$, and $R_i^{\boldsymbol{B}} = R_i^{\boldsymbol{A}} \cap B^{n(i)}, f_j^{\boldsymbol{B}} = f_j^{\boldsymbol{A}} | B^{m(j)}$. We write $\boldsymbol{B} \subseteq \boldsymbol{A}$ in this case. For each $X \subseteq A$, there is a smallest substructure containing X, called the *substructure generated* by X. A substructure is *finitely generated* it is generated by a finite set. A structure is *locally finite* if all its finitely generated substructures are finite. For example, if L is *relational*, i.e., $J = \emptyset$, the substructure generated by X has universe X and so every finitely generated substructure is finite. This is also true if J is finite and each f_j has arity 0.

A structure $\boldsymbol{A}$ is called *ultrahomogeneous* if every isomorphism between finitely generated substructures $\boldsymbol{B}, \boldsymbol{C}$ of $\boldsymbol{A}$ can be extended to an automorphism of $\boldsymbol{A}$. For example, $\langle \mathbb{Q}, < \rangle$, the rationals with the usual order, is an ultrahomogeneous structure. Fraissé's theory provides a general analysis of ultrahomogeneous countable structures.

Let $\boldsymbol{A}$ be a structure for L. The *age* of A, $\mathrm{Age}(\boldsymbol{A})$ is the collection of all finitely generated structures in L that can be embedded in $\boldsymbol{A}$, i.e., the closure under isomorphism of the collection of finitely generated substructures of $\boldsymbol{A}$. Clearly the class $\mathcal{K} = \mathrm{Age}(\mathcal{A})$ is non-empty, and satisfies the following two properties:

(i) *Hereditary property (HP)*: If $\boldsymbol{B} \in \mathcal{K}$ and $\boldsymbol{C}$ is a finitely generated structure that can be embedded in $\boldsymbol{B}$, then $\boldsymbol{C} \in \mathcal{K}$.

(ii) *Joint embedding property (JEP)*: If $\boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$, there is $\boldsymbol{D} \in \mathcal{K}$ such that $\boldsymbol{B}, \boldsymbol{C}$ can be embedded in $\boldsymbol{D}$.

When $\boldsymbol{A}$ is moreover ultrahomogeneous, it is easy to see that $\mathcal{K} = \mathrm{Age}(\boldsymbol{A})$ satisfies also the following crucial property:

(iii) *Amalgamation property (AP)*: If $\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D} \in \mathcal{K}$ and $f : \boldsymbol{B} \to \boldsymbol{C}, g : \boldsymbol{B} \to \boldsymbol{D}$ are embeddings, then there is $\boldsymbol{E} \in \mathcal{K}$ and embeddings $r : \boldsymbol{C} \to \boldsymbol{E}, s : \boldsymbol{D} \to \boldsymbol{E}$ such that $r \circ f = s \circ g$.

We summarize:

Proposition 2.1. *Let $\boldsymbol{A}$ be an ultrahomogeneous structure. Then $\mathcal{K} = \mathrm{Age}(\boldsymbol{A})$ is non-empty, and satisfies HP, JEP and AP.* $\dashv$

If $\boldsymbol{A}$ is countable, then clearly $\text{Age}(\boldsymbol{A})$ contains only countably many isomorphism types. Abusing language, we say that a class $\mathcal{K}$ of structures is *countable* if it contains only countably many isomorphic types.

We now have the following main result of Fraïssé [24].

Theorem 2.2 (Fraïssé)**.** *Let L be a signature and $\mathcal{K}$ a class of finitely generated structures for L, which is non-empty, countable, and satisfies HP, JEP and AP. Then there is a unique, up to isomorphism, countable structure $\boldsymbol{A}$ such that $\boldsymbol{A}$ is ultrahomogeneous and $\mathcal{K} = \text{Age}(\boldsymbol{A})$.* ⊣

We call this structure the *Fraïssé limit* of $\mathcal{K}$,

$$\boldsymbol{A} = \text{Flim}(\mathcal{K}).$$

Thus a countable ultrahomogeneous structure is the Fraïssé limit of its age. For example, $\text{Age}(\langle \mathbb{Q}, <\rangle) =$ the class of finite linear orderings $= \mathcal{L}O$ and so $\langle \mathbb{Q}, <\rangle = \text{Flim}(\mathcal{L}O)$.

We note here the following alternative characterization of ultrahomogeneity.

Proposition 2.3. *Let $\boldsymbol{A}$ be a countable structure. Then $\boldsymbol{A}$ is ultrahomogeneous iff it satisfies the following extension property:*

If $\boldsymbol{B}, \boldsymbol{C}$ are finitely generated and can be embedded in $\boldsymbol{A}$, $f : \boldsymbol{B} \to \boldsymbol{A}, g : \boldsymbol{B} \to \boldsymbol{C}$ are embeddings, then there is an embedding $h : \boldsymbol{C} \to \boldsymbol{A}$ such that $h \circ g = f$. ⊣

It follows that if $\boldsymbol{A}$ is countable, $\boldsymbol{B}$ is countable ultrahomogeneous with $\text{Age}(\boldsymbol{A}) \subseteq \text{Age}(\boldsymbol{B})$ and $\boldsymbol{C} \subseteq \boldsymbol{A}$ is finitely generated, then every embedding $f : \boldsymbol{C} \to \boldsymbol{B}$ can be extended to an embedding $g : \boldsymbol{A} \to \boldsymbol{B}$.

In particular, $\boldsymbol{B}$ is *universal* for the class of all countable structures $\boldsymbol{A}$ whose age is contained in that of $\boldsymbol{B}$, i.e., every such $\boldsymbol{A}$ can be embedded in $\boldsymbol{B}$. For example, any countable linear ordering can be embedded in $\langle \mathbb{Q}, <\rangle$.

We will be primarily interested in the case when the classes $\mathcal{K}$ in 2.2 actually consist of *finite* structures and the Fraïssé limit of $\mathcal{K}$ is countably infinite. It will be convenient then to introduce, for later use, the following terminology, where the *cardinality* of a structure $\boldsymbol{A} = \langle A, \dots\rangle$ is the cardinality of its universe A.

Definition 2.4. Given a signature L, a Fraïssé class in L is a class of *finite* structures in L, which contains structures of arbitrary large (finite) cardinality, is countable, and satisfies HP, JEP and AP. A Fraïssé structure in L is a countably *infinite* structure which is locally finite and ultrahomogeneous.

Thus the map $\mathcal{K} \mapsto \text{Flim}(\mathcal{K})$ is a bijection between Fraïssé classes and Fraïssé structures (up to isomorphism) with inverse the map $\boldsymbol{A} \mapsto \text{Age}(\boldsymbol{A})$.

We would like to point out here that for G a closed subgroup of S_A, the induced structure $\boldsymbol{A}_G$ is ultrahomogeneous and, since the associated signature is relational, it is locally finite, so it is a Fraïssé structure, provided A is infinite.

Finally, for further reference, we recall the following definition. A class $\mathcal{K}$ of structures for L satisfies the *strong amalgamation property* (SAP) if for any $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$

and embeddings $f : \boldsymbol{A} \to \boldsymbol{B}, g : \boldsymbol{A} \to \boldsymbol{C}$, there is $\boldsymbol{D} \in \mathcal{K}$ and embeddings $r : \boldsymbol{B} \to \boldsymbol{D}, s : \boldsymbol{C} \to \boldsymbol{D}$ with $r \circ f = s \circ g$, such that moreover $r(B) \cap s(C) = r(f(A))$ $(= s(g(A)))$.

Similarly, we say that $\mathcal{K}$ satisfies the *strong joint embedding property* (SJEP) if for any $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$ there is $\boldsymbol{C} \in \mathcal{K}$ and embeddings $f : \boldsymbol{A} \to \boldsymbol{C}, g : \boldsymbol{B} \to \boldsymbol{C}$ such that $f(A) \cap g(B) = \emptyset$.

Remark 2.5. In retrospect, one can say that Fraïssé's construction was anticipated by Urysohn [79], who considered the special case of the class of finite metric spaces with rational distances. He constructed a countable metric space $\boldsymbol{U}_0$ (see Section 6, **(E)** below) whose completion $\boldsymbol{U}$, known as the *Urysohn space*, is the unique universal ultrahomogeneous (with respect to isometries) complete separable metric space. Note that we can view metric spaces (X, d) as structures in a countable signature $L = \{R_q\}_{q \in \mathbb{Q}},\ R_q$ binary, identifying (X, d) with $\boldsymbol{X} = (X, R_q^{\boldsymbol{X}})$, where $(x, y) \in R_q^{\boldsymbol{X}} \Leftrightarrow d(x, y) < q$.

3. Structural Ramsey Theory

We will now recall some concepts and results from Ramsey theory, for which we refer the reader to Nešetřil [51], Nešetřil-Rödl [60] and Graham-Rothschild-Spencer [34].

Let $\boldsymbol{A}, \boldsymbol{B}$ be structures in a signature L. We write

$$\boldsymbol{A} \leq \boldsymbol{B}$$

if $\boldsymbol{A}$ can be embedded into $\boldsymbol{B}$. If $\boldsymbol{A} \leq \boldsymbol{B}$, we let

$$\binom{\boldsymbol{B}}{\boldsymbol{A}} = \{\boldsymbol{A}_0 : \boldsymbol{A}_0 \text{ is a substructure of } \boldsymbol{B} \text{ isomorphic to } \boldsymbol{A}\}.$$

For $\boldsymbol{A} \leq \boldsymbol{B} \leq \boldsymbol{C},\ k = 2, 3, \dots$, we write, using the Erdös-Rado [19] arrow-notation,

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_k$$

if for any coloring $c : \binom{\boldsymbol{C}}{\boldsymbol{A}} \to \{1, \dots, k\}$ with k colors, there is $\boldsymbol{B}_0 \in \binom{\boldsymbol{C}}{\boldsymbol{B}}$ which is *homogeneous*, in the sense that for some $1 \leq i \leq k$, and all $\boldsymbol{A}_0 \in \binom{\boldsymbol{B}_0}{\boldsymbol{A}},\ c(\boldsymbol{A}_0) = i$, i.e., $\binom{\boldsymbol{B}_0}{\boldsymbol{A}}$ is monochromatic.

Let now $\mathcal{K}$ be a class of finite structures in a signature L. We say that $\mathcal{K}$ satisfies the *Ramsey property* if $\mathcal{K}$ is hereditary (i.e., satisfies HP) and for any $\boldsymbol{A} \leq \boldsymbol{B}$ in $\mathcal{K},\ k = 2, 3, \dots$, there is $\boldsymbol{C} \in \mathcal{K}$ with $\boldsymbol{B} \leq \boldsymbol{C}$ such that

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_k.$$

Note that by a simple induction, we can restrict this condition to $k = 2$.

Let us now mention some examples of classes with the Ramsey property:

(i) Let $L = \{<\}$ and let $\mathcal{LO}$ be the class of finite linear orderings. Then $\mathcal{LO}$ has the Ramsey property, by the classical Ramsey theorem.

(ii) Let $L = \{<, E\}$, $<, E$ binary relation symbols and let $\mathcal{OG}$ be the class of all finite ordered graphs $\boldsymbol{A} = \langle A, <^{\boldsymbol{A}}, E^{\boldsymbol{A}}\rangle$ (i.e., $<^{\boldsymbol{A}}$ is a linear ordering of A and $E^{\boldsymbol{A}}$ is a symmetric, irreflexive relation). Then Nešetřil-Rödl [56], [58] showed that $\mathcal{OG}$ has the Ramsey property.

(iii) Let F be a finite field and let $L = \{+\} \cup \{f_\alpha\}_{\alpha \in F}$ (all function symbols), where $+$ has arity 2 and each f_α is unary. Any vector space over F can be viewed as a structure in this language with $+$ representing addition and f_α scalar multiplication by $\alpha \in F$. A substructure of a vector space is clearly a subspace. Let $\mathcal{V}_F$ be the class of all finite-dimensional vector spaces over F. Clearly $\boldsymbol{A} \leq \boldsymbol{B} \Leftrightarrow \dim \boldsymbol{A} \leq \dim \boldsymbol{B}$. It was shown in Graham-Leeb-Rothschild [32], that $\mathcal{V}_F$ has the Ramsey property.

(iv) Let now $L = \{0, 1, -, \wedge, \vee\}$ (all function symbols), where $0, 1$ have arity 0 and $-, \wedge, \vee$ have arities 1, 2, 2 resp. Any Boolean algebra is a structure in this language (with $-$ representing Boolean complementation). Substructures are again subalgebras. Let $\mathcal{BA}$ be the class of all finite Boolean algebras. Then the so-called Dual Ramsey Theorem of Graham-Rothschild [33] can be equivalently reformulated by saying that $\mathcal{BA}$ has the Ramsey property (see Nešetřil [51], 4.13).

Finally, let us point out the following connection between the Ramsey property and the amalgamation property, discussed in the previous section (see Nešetřil-Rödl [56], p. 294, Lemma 1).

Let $\mathcal{K}$ be a class of finite *rigid* (i.e., having no non-trivial automorphisms) structures in a signature L, which is hereditary. If $\mathcal{K}$ has the JEP and the Ramsey property, then $\mathcal{K}$ has the AP. To see this, fix $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$ and embeddings $f : \boldsymbol{A} \to \boldsymbol{B}$, $g : \boldsymbol{A} \to \boldsymbol{C}$. By the JEP, find $\boldsymbol{E} \in \mathcal{K}$ in which both $\boldsymbol{B}, \boldsymbol{C}$ can be embedded. Then find $\boldsymbol{D} \in \mathcal{K}$ such that $\boldsymbol{D} \to (\boldsymbol{E})^{\boldsymbol{A}}_4$ and consider the coloring $c : \binom{\boldsymbol{D}}{\boldsymbol{A}} \to \{x : x \subseteq \{\boldsymbol{B}, \boldsymbol{C}\}\}$ defined as follows: Given $\boldsymbol{A}_0 \in \binom{\boldsymbol{D}}{\boldsymbol{A}}$, $\boldsymbol{B} \in c(\boldsymbol{A}_0) \Leftrightarrow$ there is an embedding $r : \boldsymbol{B} \to \boldsymbol{D}$ with $r \circ f(\boldsymbol{A}) = \boldsymbol{A}_0$, and similarly for $\boldsymbol{C}$. Let $\boldsymbol{E}_0 \in \binom{\boldsymbol{D}}{\boldsymbol{E}}$ be a homogeneous set. Then $c(\boldsymbol{A}_0) = \{\boldsymbol{B}, \boldsymbol{C}\}$, for all $\boldsymbol{A}_0 \in \binom{\boldsymbol{E}_0}{\boldsymbol{A}}$. For such $\boldsymbol{A}_0$, there is $r : \boldsymbol{B} \to \boldsymbol{D}$ with $f \circ r(\boldsymbol{A}) = \boldsymbol{A}_0$ and $s : \boldsymbol{C} \to \boldsymbol{D}$ with $g \circ s(\boldsymbol{A}) = \boldsymbol{A}_0$. So $r \circ f$, $g \circ s$ are isomorphisms of $\boldsymbol{A}$ with $\boldsymbol{A}_0$. Since $\boldsymbol{A}, \boldsymbol{A}_0$ are rigid, it follows that $r \circ f = s \circ g$, so $\boldsymbol{D}, r, s$ verify the amalgamation property for $f : \boldsymbol{A} \to \boldsymbol{B}, g : \boldsymbol{A} \to \boldsymbol{C}$.

In particular, if $\mathcal{K}$ is a non-empty class of rigid finite structures which is countable, contains structures of arbitrarily large cardinality, and satisfies HP and JEP and the Ramsey property, then $\mathcal{K}$ is a Fraïssé class, i.e., the age of a countably infinite ultrahomogeneous structure.

4. Characterizing extremely amenable automorphism groups by a Ramsey property

We will first reformulate the condition that a G-flow has a fixed point in the following manner.

Lemma 4.1. *Let G be a topological group and X a G-flow. Then the following are equivalent:*

(i) The G-flow X has a fixed point.

(ii) For every $n = 1, 2, \dots,$ and continuous $f : X \to \mathbb{R}^n,\ \epsilon > 0,\ F \subseteq G$ finite, there is $x \in X$, such that $|f(x) - f(g \cdot x)| \leq \epsilon,\ \forall g \in F$, where $|\ |$ refers to Euclidean norm.

Proof. (i) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (i). We use a compactness argument. For $f : X \to \mathbb{R}^n$ continuous, $\epsilon > 0,\ F \subseteq G$ finite, put

$$A_{f,\epsilon,F} = \{x \in X : \forall g \in F(|f(x) - f(g \cdot x)| \leq \epsilon)\}.$$

Claim. $\bigcap_{f,\epsilon,F} A_{f,\epsilon,F} \neq \emptyset$.

Granting this, fix $x \in \bigcap_{f,\epsilon,F} A_{f,\epsilon,F}$. Then x is a fixed point, since otherwise there is $g \in G$ with $g \cdot x \neq x$, so there is a continuous $f : X \to \mathbb{R}$ with $f(x) = 0, f(g \cdot x) = 1$, thus $x \notin A_{f,1,\{g\}}$.

Proof of the claim. Notice that $A_{f,\epsilon,F}$ is closed, so, by compactness, it is enough to show that for any finite collection $(f_j, \epsilon_j, F_j),\ j = 1, \dots, m$, we have $\bigcap_{j=1}^m A_{f_j,\epsilon_j,F_j} \neq \emptyset$. Put

$$\bar{F} = F_1 \cup \cdots \cup F_m,\ \bar{\epsilon} = \min\{\epsilon_1, \dots, \epsilon_m\}$$
$$\bar{f} = (f_1, \dots, f_m) : X \to \mathbb{R}^{n_1+\cdots+n_m},$$

where $f_i : X \to \mathbb{R}^{n_i}$. Then $A_{\bar{f},\bar{\epsilon},\bar{F}} \subseteq \bigcap_{j=1}^m A_{f_j,\epsilon_j,F_j}$. But $A_{\bar{f},\bar{\epsilon},\bar{F}}$ is non-empty by (ii). $\dashv$

We use this to prove the following preliminary characterization.

Proposition 4.2. *Let S_∞ be the Polish group of permutations of $\mathbb{N}$ with the pointwise convergence topology. If $G \leq S_\infty$ is a closed subgroup, then the following are equivalent:*

(i) G is extremely amenable.

(ii) For any open subgroup V of G, every coloring $c : G/V \to \{1, \dots, k\}$, of the set of left-cosets hV of V, and every finite $A \subseteq G/V$, there is $g \in G$ and $1 \leq i \leq k$, such that $c(g \cdot a) = i,\ \forall a \in A$, where G acts on G/V in the usual way $g \cdot hV = ghV$.

Proof. (i) $\Rightarrow$ (ii): Fix V, k, c as in (ii), and consider the shift action of G on $Y = \{1, \dots, k\}^{G/V},\ g \cdot p(x) = p(g^{-1} \cdot x)$, for $p \in Y,\ x \in G/V$. This is a G-flow and $c \in Y$. Let $X = \overline{G \cdot c}$. By (i) find a fixed point $\gamma \in X$. Since G acts transitively on G/V, clearly $\gamma : G/V \to \{1, \dots, k\}$ is a constant function, say $\gamma(a) = i,\ \forall a \in G/V$. Fix now finite $A \subseteq G/V$. Since $\gamma \in \overline{G \cdot c}$, there is $g \in G$ such that $g^{-1} \cdot c|A = \gamma|A$, so $c(g \cdot a) = \gamma(a) = i,\ \forall a \in A$.

(ii) $\Rightarrow$ (i): Clearly (ii) is equivalent to the corresponding statement about the space V/G of right-cosets Vh of V on which G acts as usual by $g \cdot Vh = Vhg^{-1}$, and this is what we will use below. Using 4.1, it suffices to show that if X is a G-flow and $f : X \to \mathbb{R}^n$ is continuous, $\epsilon > 0$, $F \subseteq G$ is finite, then there is $x \in X$, with $|f(x) - f(h \cdot x)| \leq \epsilon$, $\forall h \in F$.

As in the proof of 1.3, there is an open nbhd of 1_G, say V, such that $\forall h \in V \forall x \in X |f(x) - f(h \cdot x)| \leq \epsilon/3$. But, since G is a closed subgroup of S_∞, we can assume that V is an open subgroup of G (see [4], 1.5). Partition now the compact set $f(X) \subseteq \mathbb{R}^n$ into sets $A_1, \dots, A_k$ of diameter $\leq \epsilon/3$.

Fix $x_0 \in X$ and let

$$U_i = \{g \in G : f(g \cdot x_0) \in A_i\}.$$

Put $VU_i = V_i$, so that V_i is a union of right-cosets of V and thus can be viewed as a subset of V/G. Since $\bigcup_{i=1}^k V_i = G/V$, we can find $c : G/V \to \{1, \dots, k\}$ such that $c^{-1}(\{i\}) \subseteq V_i$. So by (ii) there is $1 \leq i \leq k$ and $g \in G$ with $(F \cup \{1_G\})g \subseteq V_i = VU_i$. We will now show that $x = g \cdot x_0$ works.

Indeed, fix $h \in F$. Let $v \in V$ be such that $vhg \in U_i$, so that $f(vhg \cdot x_0) = f(vh \cdot x) \in A_i$. Since $|f(vh \cdot x) - f(h \cdot x)| \leq \frac{\epsilon}{3}$, it follows that $f(h \cdot x)$ is in the $\frac{\epsilon}{3}$-nbhd of A_i. Since $1_G \in F$, it follows that $|f(x) - f(h \cdot x)| \leq \epsilon$. ⊣

Clearly in 4.2 we can restrict V to any local basis at 1_G consisting of open subgroups. In particular, if for each non-empty finite $F \subseteq \mathbb{N}$ we let

$$G_{(F)} = \{g \in G : \forall i \in F(g(i) = i)\}$$

be the *pointwise stabilizer* of F, then as $\{G_{(F)} : \emptyset \neq F \subseteq \mathbb{N},\ F \text{ finite}\}$ is a local basis of 1_G, we can restrict V in 4.2 to be of the form $G_{(F)}$, and moreover it is enough to consider only F in any cofinal under inclusion collection of finite subsets of $\mathbb{N}$.

Remark. By the proof of 4.2, to test extreme amenability of a closed subgroup $G \leq S_\infty$ it is enough to find fixed points in compact invariant subsets of the G-flow $\{1, \dots, k\}^{G/V}$, $V \leq G$ open.

For the next result we will need some further notation and terminology. Each $G \leq S_\infty$ acts on the finite subsets of $\mathbb{N}$ in the obvious way

$$g \cdot F = \{g(i) : i \in F\}.$$

For each finite $\emptyset \neq F \subseteq \mathbb{N}$, we let then

$$G_F = \{g \in G : g \cdot F = F\}$$

be the stabilizer of F in this action, i.e., the *setwise stabilizer* of F. Clearly $G_{(F)} \leq G_F$ and $[G_F : G_{(F)}] < \infty$.

The G-*type* of $\emptyset \neq F \subseteq \mathbb{N},\ F$ finite, is the orbit $G \cdot F$ of F. A G-*type* σ is the G-type of some finite nonempty $F,\ \sigma = G \cdot F$. If ρ, σ are G-types, we write

$$\begin{aligned} \rho \leq \sigma \quad & \Leftrightarrow \exists F \in \sigma \exists F' \in \rho (F' \subseteq F). \\ & \Leftrightarrow \forall F \in \sigma \exists F' \in \rho (F' \subseteq F) \\ & \Leftrightarrow \forall F' \in \rho \exists F \in \sigma (F' \subseteq F). \end{aligned}$$

Finally, given a signature $L = \{R_i\}_{i \in I} \cup \{f_j\}_{j \in J}$, we denote by X_L the space of all structures for L with universe $\mathbb{N}$. Thus

$$X_L = \prod_i 2^{(\mathbb{N}^{n_i})} \times \prod_j \mathbb{N}^{(\mathbb{N}^{m_j})}$$

If L is relational, i.e., $J = \emptyset,\ X_L$ is compact (homeomorphic to $2^{\mathbb{N}}$). The group S_∞ acts canonically on X_L as follows: Given $\boldsymbol{A} = \langle \mathbb{N}, \{R_i^{\boldsymbol{A}}\},\ \{f_j^{\boldsymbol{A}}\} \rangle$ we let $g \cdot \boldsymbol{A} = \boldsymbol{B} = \langle \mathbb{N}, \{R_i^{\boldsymbol{B}}\},\ \{f_j^{\boldsymbol{B}}\} \rangle$, where

$$R_i^{\boldsymbol{B}}(a_1, \dots, a_{n(i)}) \Leftrightarrow R_i^{\boldsymbol{A}}(g^{-1}(a_1), \dots, g^{-1}(a_{n(i)}))$$
$$f_j^{\boldsymbol{B}}(a_1, \dots, a_{m(j)}) = f_j^{\boldsymbol{A}}(g^{-1}(a_1), \dots, g^{-1}(a_{m(i)})),$$

so that g is an isomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. This action, called the *logic action*, is clearly continuous. In particular, if L is relational, the action of any $G \leq S_\infty$ on X_L is a G-flow. Consider the language $L = \{<\},\ <$ a binary relation symbol, and denote by LO the compact S_∞-invariant subset of X_L consisting of all linear orderings $\boldsymbol{A} = \langle \mathbb{N}, <^{\boldsymbol{A}} \rangle$ on $\mathbb{N}$. Clearly for any $G \leq S_\infty$, LO is a subflow of the G-flow X_L. We say that G *preserves an ordering* if this subflow has a fixed point, i.e., there is an ordering $\prec$ on $\mathbb{N}$ such that for every $g \in G,\ a \prec b \Leftrightarrow g(a) \prec g(b)$.

We now have:

Proposition 4.3. *Let $G \leq S_\infty$ be a closed subgroup. Then the following are equivalent:*

(i) G is extremely amenable.

(ii) (a) For any finite $\emptyset \neq F \subseteq \mathbb{N},\ G_{(F)} = G_F$ and (b) For any two G-types ρ, σ with $\rho \leq \sigma$, and every finite coloring $c : \rho \to \{1, \dots, k\}$, there is $1 \leq i \leq k$ and $F \in \sigma$ such that $c(F') = i,\ \forall F' \subseteq F,\ F' \in \rho$.

(iii) (a)' G preserves an ordering and (b) as in (ii) above.

Proof. (i) $\Rightarrow$ (iii): Consider (a)' first. Since G is extremely amenable and LO is a G-flow, there is a fixed point, i.e., G preserves an ordering.

We next prove (b). Fix $\rho \leq \sigma,\ c : \rho \to \{1, \dots, k\}$. Say $G \cdot F' = \rho$. Then if $V = G_{(F')}$, we note that $V = G_{(F')} = G_{F'}$ by (a)' and we can identify G/V with $G \cdot F' = \rho$. Applying then (ii) of 4.2, to $V, c, A = \{F'_0 \subseteq F_0 : F'_0 \in \rho\}$, where $F_0 \in \sigma$, we find $1 \leq i \leq k$ and $g \in G$ with $c(g \cdot F'_0) = i,\ \forall F'_0 \in A$. Let $F = g \cdot F_0 \in \sigma$. If $F' \subseteq F,\ F' \in \rho$, then $g^{-1} \cdot F' = F'_0 \subseteq F_0$, and $F'_0 \in \rho$, so $c(g \cdot F'_0) = c(F') = i$.

(iii) $\Rightarrow$ (ii): Clearly, (a)' $\Rightarrow$ (a).

(ii) $\Rightarrow$ (i): We verify (ii) of 4.2 for V of the form $G_{(F)} = G_F,\ \emptyset \neq F \subseteq \mathbb{N}$ finite. If $V = G_F$, then G/V can be identified with $\rho = G \cdot F$. So fix $c : \rho \to \{1, \dots, k\}$

and $A \subseteq \rho$ finite. Put $\bigcup A = F_0,\ \sigma = G \cdot F_0$. Clearly $\rho \leq \sigma$, so there is $1 \leq i \leq k$ and $g \in G$ such that for all $F' \subseteq g \cdot F_0$ with $F' \in \rho$, we have $c(F') = i$. Thus $c(g \cdot F) = i,\ \forall F \in A$. ⊣

We will now use a compactness argument to put this characterization in a final form. It will be convenient first to introduce the following notation.

Definition 4.4. Let $G \leq S_\infty$. Let $\rho \leq \sigma$ be G-types. If $F \in \sigma$, we put

$$\binom{F}{\rho} = \{F' \subseteq F : F' \in \rho\}.$$

If $\rho \leq \sigma \leq \tau$ are G-types, we put

$$\tau \to (\sigma)^\rho_k,$$

where $k = 2, 3, \ldots,$ if for every $F \in \tau$ and coloring $c : \binom{F}{\rho} \to \{1, \ldots, k\}$, there is $F_0 \in \binom{F}{\sigma}$, which is homogeneous, i.e., c is monochromatic on $\binom{F_0}{\rho}$: for some $1 \leq i \leq k$, and every $F' \in \binom{F_0}{\rho}$, $c(F') = i$. (Note that this is equivalent to asserting that this is true for *some* $F \in \tau$.)

We say that G has the Ramsey property if for every G-types $\rho \leq \sigma$ and every $k = 2, 3, \ldots,$ there is a G-type $\tau \geq \sigma$ with $\tau \to (\sigma)^\rho_k$.

We now have

Theorem 4.5. *Let $G \leq S_\infty$ be a closed subgroup. Then the following are equivalent:*
(i) G is extremely amenable.
(ii) (a) G preserves an ordering and (b) G has the Ramsey property.

Proof. (i) ⇒ (ii). We have already seen (a). To prove (b), first note that, by a simple induction, it is enough to restrict ourselves in 4.4. to the case $k = 2$. So assume, towards a contradiction, that for some G-types $\rho \leq \sigma$, there is no $\tau \geq \sigma$ with $\tau \to (\sigma)^\rho_2$. Fix $F_0 \in \sigma$. Then for every finite set $E \supseteq F_0$ there is a coloring $c_E : \binom{E}{\rho} \to \{1, 2\}$ which does not have a homogeneous set $F \in \binom{E}{\sigma}$. Pick an ultrafilter U on the set I of finite non-empty subsets of $\mathbb{N}$ such that for every finite $F \subseteq \mathbb{N},\ \{E : F \subseteq E\} \in U$. Then for each $D \in \rho,\ \{E \supseteq D \cup F_0 : c_E(D) = 1\} \in U$ or $\{E \supseteq D \cup F_0 : c_E(D) = 2\} \in U$, so put $c(D) = i$ iff $\{E \supseteq D \cup F_0 : c_E(D) = i\} \in U$. This gives a coloring $c : \rho \to \{1, 2\}$. Then by 4.3, (ii) (b), there is $F \in \sigma$ such that c is monochromatic on $\binom{F}{\rho}$, say with value i. If $D \in \binom{F}{\rho}$, $A_D = \{E \supseteq F \cup F_0 : c_E(D) = c(D) = i\} \in U$, so pick $E \in \bigcap_{D \in \binom{F}{\rho}} A_D$.

Then $E \supseteq F_0$ and for each $D \in \binom{F}{\rho}$, $c_E(D) = i$, so $F \in \binom{E}{\sigma}$ is homogeneous for c_E, a contradiction.

(ii) ⇒ (i): It is of course enough to verify (b) of 4.3 (ii), which follows trivially from the assumption that G has the Ramsey property. ⊣

Remark 4.6. Let $G \leq S_\infty$. We call a set T of G-types *cofinal* if for every G-type ρ there is $\sigma \in T$ with $\rho \leq \sigma$. Then it is not hard to see that Theorem 4.5 still holds if in the definition 4.4 of G having the Ramsey property, we restrict the G-types to be in any given cofinal set of G-types.

We will finally tie-up extreme amenability of automorphism groups with the structural Ramsey theory of §3.

Let L be a signature with a distinguished binary relation symbol $<$ (and perhaps other symbols). An *order structure* for L is a structure $\boldsymbol{A}$ of L in which $<^{\boldsymbol{A}}$ is a linear ordering. If $\mathcal{K}$ is a class of structures of L, we say that $\mathcal{K}$ is an *order class* if every $\boldsymbol{A} \in \mathcal{K}$ is an order structure.

We also recall that up to (topological group) isomorphism the closed subgroups of S_∞ are exactly the same as the automorphism groups of countable structures and also the same as the Polish groups which admit a countable nbhd basis at the identity consisting of open subgroups (see [4], 1.5). So the next result provides a characterization of the groups in this last class that are extremely amenable.

Theorem 4.7. *Let $G \leq S_\infty$ be a closed subgroup. Then the following are equivalent:*

(i) G is extremely amenable.

(ii) $G = \mathrm{Aut}(\boldsymbol{A})$, where $\boldsymbol{A}$ is the Fraïssé limit of a Fraïssé order class with the Ramsey property.

Proof. (i) ⇒ (ii): Let $\boldsymbol{A}_G = \langle \mathbb{N}, \dots \rangle$ be the induced structure for G. As we pointed out in §2 (third paragraph before the last remark) $\boldsymbol{A}_G$ is ultrahomogeneous. Also, since G is extremely amenable, G preserves a linear order $\prec$ on $\mathbb{N}$. Let L be the signature obtained from the signature of $\boldsymbol{A}_G$ by adding a new binary relation symbol $<$. Let $\boldsymbol{A}$ be the expansion of the structure $\boldsymbol{A}_G$ in which $<^{\boldsymbol{A}} = \prec$. Clearly we have $\mathrm{Aut}(\boldsymbol{A}) = G$, in particular $\boldsymbol{A}$ is still ultrahomogeneous. Note also that the signature of $\boldsymbol{A}_G$ and thus of $\boldsymbol{A}$ is relational, so $\boldsymbol{A}$ is locally finite. Thus $\mathcal{K} = \mathrm{Age}(\boldsymbol{A})$ is a Fraïssé order class. Noting now that, by ultrahomogeneity, a G-type is exactly the collection of all substructures of $\boldsymbol{A}$ isomorphic to a given $\boldsymbol{A}_0 \in \mathrm{Age}(\boldsymbol{A})$, we see that the G having the Ramsey property is equivalent to $\mathrm{Age}(\boldsymbol{A})$ having the Ramsey property, so we are done by 4.5.

(ii) ⇒ (i): Since $\boldsymbol{A}$ is the Fraïssé limit of a Fraïssé order class it is a locally finite order structure. This implies that G preserves an ordering and also that the G-types of finite substructures are cofinal in all the G-types. As noted earlier, the G-type of a finite substructure $\boldsymbol{A}_0$ is the collection of all substructures of $\boldsymbol{A}$ isomorphic to $\boldsymbol{A}_0$, so, by 4.5 and 4.6, it is clear that G has the Ramsey property, so it is extremely amenable. $\dashv$

We make explicit the following fact observed in the preceding proof.

Theorem 4.8. *Let $\mathcal{K}$ be a Fraïssé order class and $\boldsymbol{A} = \mathrm{Flim}(\mathcal{K})$. Then the following are equivalent:*

(i) $\mathrm{Aut}(\boldsymbol{A})$ *is extremely amenable.*

(ii) $\mathcal{K}$ *has the Ramsey property.* $\dashv$

5. Reducts

Let $L = \{R_i\}_{i\in I} \cup \{f_i\}_{j\in J}$ be a signature and $\boldsymbol{A} = \langle A, \{R_i^{\boldsymbol{A}}\}, \{f_j^{\boldsymbol{A}}\}\rangle$ a structure for L. If $L_0 = \{R_i\}_{i\in I_0} \cup \{f_j\}_{j\in J_0}$, with $I_0 \subseteq I$, $J_0 \subseteq J$, so that $L_0 \subseteq L$, we let $\boldsymbol{A}_0 = \boldsymbol{A}|L_0 = \langle A, \{R_i^{\boldsymbol{A}}\}_{i\in I_0},\ \{f_j^{\boldsymbol{A}}\}_{j\in J_0}\rangle$ be the *reduct* of $\boldsymbol{A}$ to L_0. We also call $\boldsymbol{A}$ an *expansion* of $\boldsymbol{A}_0$. If $\mathcal{K}$ is a class of structures in L, we denote by

$$\mathcal{K}|L_0 = \{\boldsymbol{A}|L_0 : A \in \mathcal{K}\},$$

the class of reducts of elements of $\mathcal{K}$, called also the *reduct* of $\mathcal{K}$ to L_0.

We have seen that order classes of structures play a crucial role in extreme amenability of automorphism groups. We will now examine what happens to reducts of such classes, when the ordering is dropped.

Let L be a signature with a distinguished binary relation symbol $<$ and let $L_0 = L \setminus \{<\}$. For $\boldsymbol{A}$ a structure for L, we denote by $\boldsymbol{A}_0$ the reduct of $\boldsymbol{A}$ to L_0 and for any class $\mathcal{K}$ of structures for L, we denote by $\mathcal{K}_0$ the reduct of $\mathcal{K}$ to L_0. Conversely if $\boldsymbol{A}_0 = \langle A_0, \dots\rangle$ is a structure for L_0 and $\prec$ a binary relation on A_0, we denote by $\langle \boldsymbol{A}_0, \prec\rangle = \boldsymbol{A}$ the structure for L whose reduct to L_0 is $\boldsymbol{A}_0$ and $\prec\, = \,<^{\boldsymbol{A}}$ (thus also $A = A_0$).

Consider a Fraïssé order class $\mathcal{K}$ in a signature $L \supseteq \{<\}$ with Fraïssé limit $\boldsymbol{F}$. We characterize when $\mathcal{K}_0$ is also a Fraïssé class with limit $\boldsymbol{F}_0 = \boldsymbol{F}|L_0$.

Definition 5.1. Let L be a signature with $L \supseteq \{<\}$, and put $L_0 = L \setminus \{<\}$. Let $\mathcal{K}$ be a class of structures in L and put $\mathcal{K}_0 = \mathcal{K}|L_0$. We say that $\mathcal{K}$ is reasonable if for every $\boldsymbol{A}_0 \in \mathcal{K}_0$, $\boldsymbol{B}_0 \in \mathcal{K}_0$, embedding $\pi : \boldsymbol{A}_0 \to \boldsymbol{B}_0$, and linear ordering $\prec$ on A_0 with $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec\rangle \in \mathcal{K}$, there is a linear ordering $\prec'$ on B_0, so that $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec'\rangle \in \mathcal{K}$ and $\pi : \boldsymbol{A} \to \boldsymbol{B}$ is also an embedding (i.e., $a \prec b \Leftrightarrow \pi(a) \prec' \pi(b)$).

We now have

Proposition 5.2. *Let $L \supseteq \{<\}$ be a signature and $\mathcal{K}$ a Fraïssé order class in L. Let $L_0 = L \setminus \{<\}$, $\mathcal{K}_0 = \mathcal{K}|L_0$, $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K})$, $\boldsymbol{F}_0 = \boldsymbol{F}|L_0$. Then the following are equivalent:*

(i) $\mathcal{K}_0$ is a Fraïssé class and $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0)$.

(ii) $\mathcal{K}$ is reasonable.

Proof. (i) $\Rightarrow$ (ii). Let $\boldsymbol{A}_0 \in \mathcal{K}_0,\ \boldsymbol{B}_0 \in \mathcal{K}_0,\ \pi : \boldsymbol{A}_0 \to \boldsymbol{B}_0$ an embedding, and fix a linear ordering $\prec$ of A_0 with $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec\rangle \in \mathcal{K}$. Then there is an embedding $\varphi : \boldsymbol{A} \to \boldsymbol{F}$, and φ is of course also an embedding $\varphi : \boldsymbol{A}_0 \to \boldsymbol{F}_0$. By the extension property 2.3 (since $\boldsymbol{F}_0$ is ultrahomogeneous), there is an embedding $\psi : \boldsymbol{B}_0 \to \boldsymbol{F}_0$ with $\psi \circ \pi = \varphi$. Let $\prec' = \psi^{-1}(<^{\boldsymbol{F}} |\psi(B_0))$. Then $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec'\rangle \in \mathcal{K}$, as $\boldsymbol{B}$ is isomorphic to a substructure of $\boldsymbol{F}$, and moreover $\pi : \boldsymbol{A} \to \boldsymbol{B}$ is also an embedding.

(ii) $\Rightarrow$ (i): Clearly $\mathcal{K}_0 = \mathrm{Age}(\boldsymbol{F}_0)$, so to verify that $\mathcal{K}_0$ is a Fraïssé class, we only need to check that $\mathcal{K}_0$ satisfies the AP. Fix $\boldsymbol{A}_0, \boldsymbol{B}_0, \boldsymbol{C}_0 \in \mathcal{K}_0$ and embeddings $f : \boldsymbol{A}_0 \to \boldsymbol{B}_0,\ g : \boldsymbol{A}_0 \to \boldsymbol{C}_0$. Let then $\prec$ be a linear order on A_0 with $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec\rangle \in \mathcal{K}$. Since $\mathcal{K}$ is reasonable, we can find linear orders $\prec', \prec''$ on B_0, C_0 resp., so that the structure $\boldsymbol{B} = \langle B_0, \prec'\rangle \in \mathcal{K},\ \boldsymbol{C} = \langle \boldsymbol{C}_0, \prec''\rangle \in \mathcal{K}$ and $f : \boldsymbol{A} \to \boldsymbol{B}, g : \boldsymbol{A} \to \boldsymbol{C}$ are still embeddings. By AP for $\mathcal{K}$ find $\boldsymbol{D} \in \mathcal{K}$ and embeddings $r : \boldsymbol{B} \to \boldsymbol{D},\ s : \boldsymbol{C} \to \boldsymbol{D}$ with $r \circ f = s \circ g$. Let $\boldsymbol{D}_0 = \boldsymbol{D}|L_0$. Clearly $r : \boldsymbol{B}_0 \to \boldsymbol{D}_0, s : \boldsymbol{C}_0 \to \boldsymbol{D}_0$ and we are done.

Finally, we check that $\mathrm{Flim}(\mathcal{K}_0) = \boldsymbol{F}_0$, for which it is enough to verify that $\boldsymbol{F}_0$ has the extension property 2.3. So fix $\boldsymbol{A}_0, \boldsymbol{B}_0 \in \mathcal{K}_0, \pi : \boldsymbol{A}_0 \to \boldsymbol{B}_0$ an embedding, and $\varphi : \boldsymbol{A}_0 \to \boldsymbol{F}_0$ an embedding. Then let

$$\prec\ =\ \varphi^{-1}(<^{\boldsymbol{F}} |\varphi(\boldsymbol{A}_0)),$$

so that $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec\rangle \in \mathcal{K}$ and $\varphi : \boldsymbol{A} \to \boldsymbol{F}$ is an embedding. Since $\mathcal{K}$ is reasonable, let $\prec'$ be a linear ordering on B_0 with $\langle \boldsymbol{B}_0, \prec'\rangle \in \mathcal{K}$ and $\pi : \boldsymbol{A} \to \boldsymbol{B}$ still an embedding. Since $\boldsymbol{F}$ satisfies the extension property, there is an embedding $\psi : \boldsymbol{B} \to \boldsymbol{F}$ with $\psi \circ \pi = \varphi$ and, since clearly also $\psi : \boldsymbol{B}_0 \to \boldsymbol{F}_0$, we are done. $\dashv$

A common way to construct an order class $\mathcal{K}$ in $L \supseteq \{<\}$ is to start with a class $\mathcal{K}_0$ in $L_0 = L \setminus \{<\}$ and take

$$\mathcal{K} = \mathcal{K}_0 * \mathcal{LO} = \{\langle \boldsymbol{A}_0, \prec\rangle : \boldsymbol{A}_0 \in \mathcal{K}_0 \text{ and } \prec \text{ is a linear ordering on } A_0\}.$$

For example, if $\mathcal{K}_0$ is the class of finite graphs, $\mathcal{K}$ is the class of all finite ordered graphs. We now have

Proposition 5.3. *Let $L \supseteq \{<\}$ be a signature and let $L_0 = L \setminus \{<\}$. Let $\mathcal{K}_0$ be a class of structures in L_0 and put $\mathcal{K} = \mathcal{K}_0 * \mathcal{LO}$. Then the following are equivalent:*

(i) $\mathcal{K}$ satisfies the amalgamation property.

(ii) $\mathcal{K}$ satisfies the strong amalgamation property.

(iii) $\mathcal{K}_0$ satisfies the strong amalgamation property.

Proof. Suppose that $\mathcal{K}$ satisfies AP in order to show that $\mathcal{K}_0$ satisfies the strong amalgamation property. Fix $\boldsymbol{A}_0, \boldsymbol{B}_0, \boldsymbol{C}_0 \in \mathcal{K}_0$ and embeddings $f : \boldsymbol{A}_0 \to \boldsymbol{B}_0,\ g : \boldsymbol{A}_0 \to \boldsymbol{C}_0$. There are clearly linear orderings $\prec$ on $A_0, \prec'$ on B_0, and $\prec''$ on C_0 such that if $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec\rangle,\ \boldsymbol{B} = \langle \boldsymbol{B}_0, \prec'\rangle,\ \boldsymbol{C} = \langle \boldsymbol{C}_0, \prec''\rangle$ (which are all in $\mathcal{K}$), then $f : \boldsymbol{A} \to \boldsymbol{B},\ g : \boldsymbol{A} \to \boldsymbol{C}$ are still embeddings, and $f(A_0) \prec' (B_0 \setminus f(A_0))$ and $(C_0 \setminus g(A_0)) \prec'' f(A_0)$ (where if $\prec\prec$ is a linear order on a set X and $Y, Z \subseteq X$, then $Y \prec\prec Z \Leftrightarrow \forall y \in Y \forall z \in Z (y \prec\prec z)$). By the AP for $\mathcal{K}$, let $r : \boldsymbol{B} \to \boldsymbol{D},\ s : \boldsymbol{C} \to \boldsymbol{D}, \boldsymbol{D} \in \mathcal{K}$, be such that $r \circ f = s \circ g$. If, towards a contradiction, there is

$d \in r(B) \cap s(C),\ d \notin r(f(A))$ (where $A = A_0,\ B = B_0,\ C = C_0$), then $r(f(A)) <^{\boldsymbol{D}} d$, since $d \in r(B \setminus f(A))$ and $d <^{\boldsymbol{D}} r(f(A)) = s(g(A))$, since $d \in s(C \setminus g(A))$, which is absurd. So $r(B) \cap s(C) = r(f(A))$.

Now assume that $\mathcal{K}_0$ satisfies strong amalgamation. We verify that $\mathcal{K}$ satisfies strong amalgamation. Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{K}$ and let $f : \boldsymbol{A} \to \boldsymbol{B},\ g : \boldsymbol{A} \to \boldsymbol{C}$ be embeddings. Then also $f : \boldsymbol{A}_0 \to \boldsymbol{B}_0,\ g : \boldsymbol{B}_0 \to \boldsymbol{C}_0$ are embeddings, so, by strong amalgamation for $\mathcal{K}_0$, find $r : \boldsymbol{B}_0 \to \boldsymbol{D}_0,\ s : \boldsymbol{C}_0 \to \boldsymbol{D}_0,\ \boldsymbol{D}_0 \in \mathcal{K}_0$, so that $r \circ f = s \circ g$ and $r(B) \cap s(C) = r(f(A))$. Then $r(B) \setminus r(f(A)),\ s(C) \setminus s(g(A)),\ r(f(A))$ $(= s(g(A))$ are pairwise disjoint, so clearly there is an order $\prec$ on D_0 such that if $\boldsymbol{D} = \langle \boldsymbol{D}_0, \prec \rangle$, which is in $\mathcal{K}$, then $r : \boldsymbol{B} \to \boldsymbol{D},\ s : \boldsymbol{C} \to \boldsymbol{D}$. ⊣

The following is quite obvious:

Proposition 5.4. *Let $L, L_0, \mathcal{K}, \mathcal{K}_0$ be as in 5.3. Then $\mathcal{K}_0$ satisfies the strong joint embedding property iff $\mathcal{K}$ satisfies the strong joint embedding property.* ⊣

Clearly it is not true that if $\mathcal{K}$ satisfies the joint embedding property, then $\mathcal{K}_0$ satisfies the strong joint embedding property. Consider, e.g., $L = \{c\} \cup \{<\}$, where c is a 0-ary function symbol, $\mathcal{K}$ = all finite structures in L.

Finally, we note a condition that implies a connection between the Ramsey property for $\mathcal{K}$ and $\mathcal{K}_0$.

Definition 5.5. Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}$, $\mathcal{K}$ a class of structures in L and $\mathcal{K}_0 = \mathcal{K}|L_0$. We say that $\mathcal{K}$ is order forgetful if for every $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$, letting $\boldsymbol{A}_0 = \boldsymbol{A}|L_0,\ \boldsymbol{B}_0 = \boldsymbol{B}|L_0$, we have

$$\boldsymbol{A} \cong \boldsymbol{B} \Leftrightarrow \boldsymbol{A}_0 \cong \boldsymbol{B}_0.$$

(Notice that this does not say that any isomorphism of $\boldsymbol{A}_0$ with $\boldsymbol{B}_0$ is also an isomorphism of $\boldsymbol{A}$ with $\boldsymbol{B}$.)

An example of an order forgetful class is the class of finite-dimensional vector spaces $\boldsymbol{V}$ over a finite field F with antilexicographical ordering induced by an ordering of a basis of $\boldsymbol{V}$. This was considered in Thomas [76].

We now have the following fact, which is easy to prove:

Proposition 5.6. *Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}$ and let $\mathcal{K}$ be a class of finite structures in L which is hereditary. Put $\mathcal{K}_0 = \mathcal{K}|L_0$. If $\mathcal{K}$ is order forgetful, then the following are equivalent:*

(i) $\mathcal{K}$ satisfies the Ramsey property.

(ii) $\mathcal{K}_0$ satisfies the Ramsey property. ⊣

6. Extremely amenable automorphism groups

We will now apply the preceding general results to find many new examples of extremely amenable automorphism groups. We will use the following immediate consequence of earlier results.

Theorem 6.1. *Let L be a signature with $L \supseteq \{<\}$ and let $\mathcal{K}$ be a Fraïssé order class in L. Let $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K})$ be the Fraïssé limit of $\mathcal{K}$, so that $\boldsymbol{F}$ is an order structure. Then the following are equivalent:*
(i) $G = \mathrm{Aut}(\boldsymbol{F})$ is extremely amenable.
(ii) $\mathcal{K}$ has the Ramsey property.

Proof. (ii) $\Rightarrow$ (i) follows from 4.7, (ii) $\Rightarrow$ (i). (i) $\Rightarrow$ (ii) is as in the proof of (i) $\Rightarrow$ (ii) of 4.7. ⊣

Ramsey theory provides now many examples of $\mathcal{K}$ satisfying the Ramsey property and we use them to produce new examples of extremely amenable groups.

(A) Graphs

Let $L_0 = \{E\}$ be the signature with one binary relation symbol E. Let also $L = \{E, <\}$. A structure $\boldsymbol{A}_0 = \langle A_0, E^{\boldsymbol{A}_0}\rangle$ is a *graph* if $E^{\boldsymbol{A}_0}$ irreflexive and symmetric. An *ordered graph* is a structure $\boldsymbol{A} = \langle \boldsymbol{A}_0, <^{\boldsymbol{A}}\rangle$ for L in which $\boldsymbol{A}_0$ is a graph and $<^{\boldsymbol{A}}$ a linear ordering.

Lachlan-Woodrow [42] classified all Fraïssé classes $\mathcal{K}_0$ of finite graphs. They are exactly the following:

(i) $\mathcal{GR}$ = all finite graphs.

(ii) For $n = 3, 4, \dots$, $\mathcal{F}orb(K_n)$ = the class of all finite graphs omitting K_n, the complete graph on n vertices (i.e., the class of finite graphs that do not contain K_n as a substructure).

(iii) $\mathcal{EQ}$ = the class of finite equivalence relations.

(iv) For $n = 1, 2, \dots$, $\mathcal{EQ}_n$ = the class of finite equivalence relations with at most n classes.

(v) For $n = 1, 2, \dots$, $\mathcal{EQ}^*_n$ = the class of finite equivalence relations, all of whose classes have at most n elements.

(vi) The complement $\overline{\mathcal{K}_0}$ of one of the classes $\mathcal{K}_0$ listed in (ii)-(iv) above, where for any graph $\boldsymbol{A}_0 = \langle A_0, E^{\boldsymbol{A}_0}\rangle$ its *complement* is $\overline{\boldsymbol{A}_0} = \langle A_0, \overline{E^{\boldsymbol{A}_0}}\rangle$, where $(x, y) \in \overline{E^{\boldsymbol{A}_0}} \Leftrightarrow x \neq y$ and $(x, y) \notin E^{\boldsymbol{A}_0}$, and $\overline{\mathcal{K}_0} = \{\overline{\boldsymbol{A}_0} : \boldsymbol{A}_0 \in \mathcal{K}_0\}$.

Remark. Strictly speaking an equivalence relation is not a graph, because it is reflexive. So when we think of an equivalence relation $\boldsymbol{X} = \langle X, R\rangle$ as a graph, we identify it with $\langle X, R \setminus \{(x, x) : x \in X\}\rangle$.

Since the automorphism group of the complement of a given graph $\boldsymbol{A}_0$ is the same as the automorphism group of $\boldsymbol{A}_0$, we do not need to consider the classes of type (vi). For any one of the classes $\mathcal{K}_0$ of type (i)-(iv), let $\mathcal{OK}_0 = \mathcal{K}_0 * \mathcal{LO}$ = the class of finite ordered graphs $\boldsymbol{A} = \langle \boldsymbol{A}_0, <^{\boldsymbol{A}}\rangle$ with $\boldsymbol{A}_0 \in \mathcal{K}_0$. Now for $\mathcal{K}_0$ of type (v) with $n \geq 2$, it is easy to check that $\mathcal{K}_0$ does not have the strong amalgamation property, so $\mathcal{OK}_0$ is not a Fraïssé order class, by 5.3. For $\mathcal{K}_0$ of type (iv) with $n \geq 2$, $\mathcal{OK}_0$ is a Fraïssé order class, whose Fraïssé limit $\boldsymbol{F} = \langle F, E^{\boldsymbol{F}}, <^{\boldsymbol{F}}\rangle$, consists of an equivalence relation $E^{\boldsymbol{F}}$ on F with exactly n classes, which are all infinite, and in which $<^{\boldsymbol{F}}$ is an ordering isomorphic to the rationals such that every equivalence class is dense. But then it is easy to check that $\mathrm{Aut}(\boldsymbol{F})$ is not extremely amenable, since it acts (continously) on the finite (discrete) space $X =$ the set of $E^{\boldsymbol{F}}$ classes, without fixed point. Finally, notice that $\overline{\mathcal{EQ}_1} = \mathcal{EQ}_1^*$, so we only need to consider $\mathcal{K}_0 = \mathcal{GR}, \mathcal{F}orb(K_n), n = 3, 4, \dots,\ \mathcal{EQ}$ and $\mathcal{EQ}_1$.

Each one of the classes $\mathcal{K}_0 = \mathcal{GR}, \mathcal{F}orb(K_n), n = 3, 4, \dots,\ \mathcal{EQ}$ and $\mathcal{EQ}_1$, clearly satisfies the strong amalgamation and strong joint embedding properties, so $\mathcal{K} = \mathcal{OK}_0$ is a Fraïssé order class. $\mathcal{K}$ is reasonable in each case. Finally, Nešetřil and Rödl [56], [58], Nešetřil [50], have shown that each one of these classes $\mathcal{K}$, satisfies the Ramsey property (the case $\mathcal{K}_0 = \mathcal{EQ}_1$ is of course the classical Ramsey Theorem), except for $\mathcal{OEQ}$, for which, despite the claim in [50], it fails. Thus if $\boldsymbol{F}$ is the Fraïssé limit of $\mathcal{K}$, then $\mathrm{Aut}(\boldsymbol{F})$ is extremely amenable. We discuss now each case in some more detail:

(i) $\mathcal{K}_0 = \mathcal{GR}$: Then $\boldsymbol{R} = \mathrm{Flim}(\mathcal{GR})$ is called the *random graph.* It is natural to call $\boldsymbol{OR} = \mathrm{Flim}(\mathcal{OGR})$ the *random ordered graph.* It is of the form $\boldsymbol{OR} = \langle \boldsymbol{R}, <^{\boldsymbol{OR}} \rangle$, where $<^{\boldsymbol{OR}}$ is an appropriate linear order of the random graph, isomorphic to the rationals. Thus we have:

Theorem 6.2. *The automorphism group of the random ordered graph is extremely amenable.* ⊣

Of course the automorphism group of the random graph itself is not extremely amenable, since it does not preserve an ordering.

(ii) $\mathcal{K}_0 = \mathcal{F}orb(K_n)$ is called the *K_n-free random graph* and so we call $\boldsymbol{OR}^n = \mathrm{Flim}(\mathcal{OF}orb(K_n))$ the *random K_n-free ordered graph.* It is of the form $\boldsymbol{OR}^n = \langle \boldsymbol{R}_0^n, <^{\boldsymbol{OR}^n} \rangle$, with $<^{\boldsymbol{OR}^n}$ a linear ordering of the K_n-free random graph, isomorphic to the rationals. Thus we have:

Theorem 6.3. *The automorphism group of the random K_n-free ordered graph is extremely amenable.* ⊣

(iii) $\mathcal{K}_0 = \mathcal{EQ}$: Then $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{EQ})$ is the equivalence relation with infinitely many classes each of which is infinite. So $\boldsymbol{F} = \mathrm{Flim}(\mathcal{OEQ}) \cong \langle \mathbb{Q}, E, < \rangle$, where $<$ is the usual ordering on $\mathbb{Q}$ and E is an equivalence relation on $\mathbb{Q}$ with infinitely many classes each of which is dense. Here we have:

Theorem 6.4. *The class $\mathcal{OEQ}$ does not have the Ramsey property and so the automorphism group of the rationals with the usual order and an equivalence relation with infinitely many classes, all of which are dense, is not extremely amenable.*

Proof. Let $\boldsymbol{A} = \langle \{a, b\}, =, \prec \rangle$, where $a \prec b$, $\boldsymbol{B} = \langle \{a, b, c\}, E, \prec' \rangle$, where $a \prec' b \prec' c$,and the E-equivalence classes are $\{a, c\}$ and $\{b\}$. Then there is no $\boldsymbol{C}$ verifying the Ramsey property for $\boldsymbol{A}, \boldsymbol{B}$. To see this, order the equivalence classes of any potential such $\boldsymbol{C}$ acccording to the order of their least elements. Then color a copy $\{a', b'\}$ of $\boldsymbol{A}$ in $\boldsymbol{C}$ red if a' is in a lower class than b' and green otherwise. Then any copy of $\boldsymbol{B}$ in $\boldsymbol{C}$ realizes both colors.

One can also directly see that the automorphism group is not extremely amenable, as it acts without fixed points on the space of linear orderings of the set of equivalence classes. ⊣

(iv) $\mathcal{K}_0 = \mathcal{EQ}_1$: Then $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{EQ}_1)$ is clearly the complete graph on a countable infinite set and so, up to isomorphism, $\mathrm{Flim}(\mathcal{OEQ}_1) \cong \langle \mathbb{Q}, E, < \rangle$, where E is the complete graph on $\mathbb{Q}$. But the automorphism group of this structure is exactly that of $\langle \mathbb{Q}, < \rangle$, so we have

Theorem 6.5 (Pestov [64])**.** *The automorphism group of the rationals with the usual order is extremely amenable.* ⊣

In the preceding example the automorphism group of $\boldsymbol{F}_0$ is of course exactly S_∞, which is not extremely amenable as proved in Pestov [64]. This is clear, as S_∞ cannot preserve an ordering.

Finally, we discuss another order class which can be obtained from $\mathcal{EQ}$. This will also play a role in §8.

Let $L_0 = \{E\},\ L = \{E, <\}$, where $E, <$ are binary relation symbols. Let $\mathcal{K}$ be the class of structures $\boldsymbol{A} = \langle A, E^{\boldsymbol{A}}, <^{\boldsymbol{A}} \rangle$ in L such that $E^{\boldsymbol{A}}$ is an equivalence relation on A, $<^{\boldsymbol{A}}$ a linear order on A, and for every $a <^{\boldsymbol{A}} b <^{\boldsymbol{A}} c$, if $(a, c) \in E^{\boldsymbol{A}}$, then $(a, b) \in E^{\boldsymbol{A}}$, i.e., every $E^{\boldsymbol{A}}$ class is convex in $<^{\boldsymbol{A}}$. We call a structure in $\mathcal{K}$ a *convexly ordered finite equivalence relation*. Clearly $\mathcal{K}|L_0 = \mathcal{EQ}$. Now it is easy to check that $\mathcal{K}$ is a reasonable, Fraïssé order class. The Fraïssé limit of $\mathcal{K}$ is of the form $\boldsymbol{F} = \langle \boldsymbol{F}_0, <^{\boldsymbol{F}} \rangle$, where $\boldsymbol{F}_0$ is the equivalence relation with infinitely many classes each of which is infinite and $<^{\boldsymbol{F}}$ is an ordering such that each equivalence class is convex and isomorphic to the rationals, and the equivalence classes are also ordered like the rationals. So $\boldsymbol{F}$ up to isomorphism, is the same as $\langle \mathbb{Q}^2, E, <_\ell \rangle$, where $<_\ell$ is the lexicographical ordering on $\mathbb{Q}^2$ and E is the equivalence relation on $\mathbb{Q}^2$ given by $(r, s,)E(r', s') \Leftrightarrow r = r'$.

Theorem 6.6. *The automorphism group of $\mathbb{Q}^2$ with the lexicographical ordering and the equivalence relation $(r, s)E(r', s') \Leftrightarrow r = r'$ is extremely amenable.* ⊣

Proof. Put $\boldsymbol{F} = \langle \mathbb{Q}^2, E, <_\ell \rangle$. We will show that $\mathrm{Aut}(\boldsymbol{F})$ is extremely amenable.

Let $I_r = \{r\} \times \mathbb{Q}$, so that $r < s \Leftrightarrow I_r <_\ell I_s$. Let now $\pi \in \mathrm{Aut}(\boldsymbol{F})$. Let $f_\pi : \mathbb{Q} \to \mathbb{Q}$ be defined by $f_\pi(r) = s \Leftrightarrow \pi(I_r) = I_s$. Then clearly $f_\pi \in \mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$. Also let for each $r \in \mathbb{Q},\ (g_\pi)_r : \mathbb{Q} \to \mathbb{Q}$ be defined by $(g_\pi)_r(s) = t \Leftrightarrow \pi(r, s)) = (f_\pi(r), t)$. Thus again $(g_\pi)_r \in \mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$. Put $\Theta(\pi) = (f_\pi, g_\pi)$, where $g_\pi \in \mathrm{Aut}(\mathbb{Q})^{\mathbb{Q}},\ g_\pi = \{(g_\pi)_r\}$. Consider the semi-direct product $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle) \ltimes \mathrm{Aut}(\langle \mathbb{Q}, < \rangle)^{\mathbb{Q}}$, where $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ acts on $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)^{\mathbb{Q}}$ by shift: $g \cdot x(r) = x(g^{-1}(r))$. Then it is easy to check that Θ is a (topological group) isomorphism of $\mathrm{Aut}(\boldsymbol{F})$ with the group which is the semidirect product $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle) \ltimes \mathrm{Aut}(\langle \mathbb{Q}, < \rangle)^{\mathbb{Q}}$, so it is enough to check that the latter group is extremely amenable. This follows from the following standard closure properties of extreme amenability and 6.5. ⊣

Lemma 6.7. *i) Let G be a topological group and $\mathcal{H}$ an upward-directed, under inclusion, family of extremely amenable subgroups of G such that $\bigcup\mathcal{H}$ is dense in G. Then G is extremely amenable.*

ii) Let G be a topological group, $N \trianglelefteq G$ a closed normal subgroup. If $N, G/N$ are extremely amenable, so is G.

iii) The product of extremely amenable groups is extremely amenable.

Proof. i) Let X be a G-flow, so also an H-flow, for any $H \in \mathcal{H}$. Put $X_H = \{x \in X : \forall h \in H(h \cdot x = x)\}$. Then X_H is compact, non-empty and $\{X_H : H \in \mathcal{H}\}$ has the finite intersection property. So $\bigcap_{H\in\mathcal{H}} X_H \neq \emptyset$, and any $x \in \bigcap_{H\in\mathcal{H}} X_H$ is fixed by an $g \in \bigcup\mathcal{H}$, so by G.

ii) Let X be a G-flow, so also a N-flow. Then $X_N = \{x \in X : \forall g \in N(g \cdot x = x)\}$ is a compact, non-empty subset of X. It follows easily by the normality of N that X_N is G-invariant: Let $x \in X_N, g \in G$. If $h \in N$, then $h\cdot(g\cdot x) = hg\cdot x = g\cdot(g^{-1}hg)\cdot x = g\cdot x$ (as $g^{-1}hg \in N$), so $g \cdot x \in X_N$.

Define now an action of G/N on X_N as follows: $gN \cdot x = g \cdot x$ (this is clearly well-defined). It is easy to check that this is a continuous action, so there is a fixed point $x \in X_N$ which is clearly a fixed point of the G-flow on X.

iii) Suppose that each $G_i, i \in I$, is extremely amenable. Then, by ii), the products $\prod_{i\in I_0} G_i$, $I_0 \subseteq I$ finite, are extremely amenable, and identifying $\prod_{i\in I_0} G_i$ with the subgroup $\prod_{i\in I} G'_i$ of $\prod_{i\in I} G_i$, where $G'_i = G_i$, if $i \in I_0, G'_i = \{1_{G_i}\}$, if $i \notin I_0$, the family $\{\prod_{i\in I_0} G_i : I_0 \subseteq I \text{ finite}\}$ is upwards-directed under inclusion and its union is dense in $\prod_{i\in I} G_i$, so $\prod_{i\in I} G_i$ is extremely amenable. ⊣

Corollary 6.8. *The class of convexly ordered finite equivalence relations satisfies the Ramsey property.*

Proof. By 6.7 and 6.1. ⊣

This Ramsey result can be also proved directly (and in fact its unordered version has already been considered in the literature; see Rado [72], Graham-Rothschild-Spencer [34], §5, Theorem 5), but it seems interesting to reverse the roles here and derive it from an extreme amenability result.

(B) Hypergraphs

Let $L_0 = \{R_i\}_{i\in I}$ be a finite relational signature with R_i of arity $n(i) \geq 2$. A *hypergraph of type* L_0 is a structure $\boldsymbol{A}_0 = \langle A_0, \{R_i^{\boldsymbol{A}_0}\}_{i\in I}\rangle$ for L_0 for which each $R_i^{\boldsymbol{A}_0}$ is irreflexive and symmetric, i.e., $(a_1, \dots, a_{n(i)}) \in R_i^{\boldsymbol{A}_0} \Rightarrow a_1, \dots, a_{n(i)}$ are distinct, and for any permutation π of $\{1, \dots, n(i)\}, (a_1, \dots, a_{n(i)}) \in R_i^{\boldsymbol{A}_0} \Rightarrow (a_{\pi(i)}, \dots, a_{\pi(n(i))}) \in R_i^{\boldsymbol{A}_0}$. Thus, in essence, $R_i^{\boldsymbol{A}_0} \subseteq [A_0]^{n(i)} =$ the set of subsets of A_0 of cardinality $n(i)$.

Let $\mathcal{H}_{L_0}$ be the class of finite hypergraphs of type L_0, let $L = L_0 \cup \{<\}$, and let $\mathcal{OH}_{L_0} = \mathcal{H}_{L_0} * \mathcal{LO}$ be the class of structures in L which are finite ordered hypergraphs, i.e., of the form $\boldsymbol{A} = \langle \boldsymbol{A}_0, <^{\boldsymbol{A}}\rangle$, with $\boldsymbol{A}_0$ a finite hypergraph, and $<^{\boldsymbol{A}}$ a linear ordering. It is easy to check that $\mathcal{OH}_{L_0}$ is a reasonable Fraïssé order class (note that $\mathcal{H}_{L_0}$ satisfies strong amalgamation and joint embedding). The Fraïssé limit $\mathrm{Flim}(\mathcal{H}_{L_0}) = \boldsymbol{H}_{L_0}$ is called the *random hypergraph of type* L_0, so we call $\mathrm{Flim}(\mathcal{OH}_{L_0}) = \boldsymbol{OH}_{L_0}$ the *random ordered hypergraph of type* L_0. We have $\boldsymbol{OH}_{L_0} = \langle \boldsymbol{H}_{L_0}, <^{\boldsymbol{OH}_{L_0}} \rangle$, where $<^{\boldsymbol{OH}_{L_0}}$ is an appropriate ordering on H_{L_0} isomorphic to the rationals. Now Nešetřil-Rödl [56], [58] and Abramson-Harrington [1] (see also Nešetřil [51]) have shown that $\mathcal{OH}_{L_0}$ satisfies the Ramsey property, so we have

Theorem 6.9. *The automorphism group of the random ordered hypergraph of type L_0 is extremely amenable.* ⊣

In case $L_0 = \{E\}$, E a binary relation, $\boldsymbol{OH}_{L_0} = \boldsymbol{OR}$, the random ordered graph, so this generalizes 6.2. As another special case, consider $L_0 = \{R_i\}_{i \in I}$, where each $n(i) = 1$. Then $\boldsymbol{OH}_{L_0} \cong \langle \mathbb{Q}, \{A_i\}_{i \in I}, < \rangle$, where $<$ is the usual ordering on $\mathbb{Q}$ and each $A_i \subseteq \mathbb{Q}$ is dense and co-dense in $\mathbb{Q}$. Thus the automorphism group of the rationals with the usual ordering and a finite family of subsets each of which is dense and co-dense is extremely amenable.

The preceding example can be further generalized.

Definition 6.10. Let $L_0 = \{R_i\}_{i \in I}$ be a finite relational signature with $n(i) \geq 2$. A hypergraph $\boldsymbol{A}_0 = \langle A_0, \{R_i^{\boldsymbol{A}_0}\}_{i \in I} \rangle$ is called irreducible if A_0 has at least two elements and for every $a \neq b$ in A_0 there is $i \in I$ and $c_1, \dots, c_{n(i)-2} \in A_0$ such that $(a, b, c_1, \dots, c_{n(i)-2}) \in R_i^{\boldsymbol{A}_0}$.

Let $\mathcal{A}$ be a class of finite irreducible hypergraphs of type L_0. Then $\mathcal{F}orb(\mathcal{A})$ is the class of all finite hypergraphs of type L_0 which omit $\mathcal{A}$, i.e., do not contain a substructure isomorphic to a member of $\mathcal{A}$.

For example, for $L_0 = \{E\}$, E binary, $\mathcal{A} = \{K_n\}$, $n = 3, 4, \dots, \mathcal{F}orb(\mathcal{A}) = \mathcal{F}orb(K_n)$, the class of finite graphs that do not contain the complete graph of n elements as a substructure.

For $\mathcal{A}$ a class of finite irreducible hypergraphs of type L_0, we denote by $\mathcal{OF}orb(\mathcal{A})$ the class of finite ordered hypergraphs of type L_0 that omit $\mathcal{A}$. It is again easy to see that $\mathcal{OF}orb(\mathcal{A})$ is a reasonable Fraïssé class. We can call again $\mathrm{Flim}(\mathcal{F}orb(\mathcal{A}))$ the *random $\mathcal{A}$-free ordered hypergraph of type* L_0. Nešetřil-Rödl [56], [58] (see also Nešetřil [51]) proved that $\mathcal{OF}orb(\mathcal{A})$ satisfies the Ramsey property. So we have

Theorem 6.11. *For each class $\mathcal{A}$ of finite irreducible hypergraphs of type L_0, the automorphism group of the random $\mathcal{A}$-free ordered hypergraph of type L_0 is extremely amenable.* ⊣

(C) Vector spaces

We will now consider an example of a different type. Fix a finite field F and consider the signature $L_0 = \{+\} \cup \{f_\alpha\}_{\alpha \in F}$ with $+$ a binary function symbol and f_α a unary function symbol. Vector spaces over F can be viewed as structures in this signature. Let $\mathcal{V}_F$ be the class of finite vector spaces over F. This is a Fraïssé class. Let $L = L_0 \cup \{<\}$, and consider the following order class defined in Thomas [76]. Fix an ordering on F

such that the 0 of the field F is the least element in that ordering. If $\boldsymbol{V}_0$ is a finite-dimensional vector space over F of dimension n and B is a basis for $\boldsymbol{V}_0$, then every ordering $b_1 < \cdots < b_n$ of B gives an ordering on V_0 by

$$\alpha_1 b_1 + \cdots + \alpha_n b_n <_{a\ell} \beta_1 b_1 + \cdots + \beta_n b_n \Leftrightarrow$$

$$(\alpha_n < \beta_n) \text{ or } (\alpha_n = \beta_n \text{ and } \alpha_{n-1} < \beta_{n-1}) \text{ or } \ldots$$

i.e., $<_{a\ell}$ is the antilexicographical ordering induced by the ordering of B. A *natural ordering* of V_0 is one induced this way by an ordering of a basis. Let $\mathcal{OV}_F$ be the order class of all $\boldsymbol{V} = \langle \boldsymbol{V}_0, <^{\boldsymbol{V}} \rangle$, such that $\boldsymbol{V}_0$ is finite-dimensional vector space and $<^{\boldsymbol{V}}$ a natural ordering on V_0. Thomas [76] shows that this is a Fraïssé class. Next it is easy to check that $\mathcal{OV}_F$ is reasonable. Now the Fraïssé limit $\boldsymbol{V}_F$ of $\mathcal{V}_F$ is easily seen to be the vector space over F of countably infinite dimension, so if $\boldsymbol{OV}_F$ is the Fraïssé limit of $\mathcal{OV}_F$, then $\boldsymbol{OV}_F = \langle \boldsymbol{V}_F, <^{\boldsymbol{OV}_F} \rangle$, where $<^{\boldsymbol{OV}_F}$ is an appropriate linear order on V_F. Let us call $\boldsymbol{OV}_F$ the $\aleph_0$-*dimensional vector space over F with the canonical ordering.*

Finally $\mathcal{V}_F$ has the Ramsey property as shown in Graham-Leeb-Rothschild [32]. It is easy to see though that $\mathcal{OV}_F$ is order forgetful, according to Definition 5.5. Thus, by 5.6, $\boldsymbol{OV}_F$ has the Ramsey property too.

Thus we have:

Theorem 6.12. *The automorphism group of the $\aleph_0$-dimensional vector space over a finite field with the canonical ordering is extremely amenable.* ⊣

Of course the automorphism group of this vector space is not extremely amenable, as it cannot preserve an ordering.

(D) Boolean algebras

Let now $L_0 = \{0, 1, -, \wedge, \vee\}$, where $0, 1$ have arity 0, $-$ has arity 1 and $\wedge, \vee$ have arity 2. Boolean algebras are structures in L_0. Let $\mathcal{BA}$ be the class of finite Boolean algebras. Then it is not hard to check that $\mathcal{BA}$ is a Fraïssé class and its Fraïssé limit is $\boldsymbol{B}_\infty$, the countable atomless Boolean algebra.

We will next define natural orderings on finite Boolean algebras similar to example **(C)**. Let $\boldsymbol{B}_0$ be a finite Boolean algebra and A its set of atoms. Then every ordering $a_1 < \cdots < a_n$ of A gives an ordering of B_0 as follows: Given $x, y \in B_0$, we can write them uniquely as $x = \delta_1 a_1 \vee \cdots \vee \delta_n a_n,\ y = \epsilon_1 a_1 \vee \cdots \vee \epsilon_n a_n$, where $\delta_i, \epsilon_i \in \{0, 1\}$, and for $\epsilon \in \{0, 1\},\ b \in B_0$,

$$\epsilon b = \begin{cases} b, \text{ if } \epsilon = 1, \\ 0^{\boldsymbol{B}}, \text{ if } \epsilon = 0. \end{cases}$$

(Here and below, we simply write $\vee$ instead of $\vee^{\boldsymbol{B}}$, when the Boolean algebra $\boldsymbol{B}$ is understood.) Then put

$$x <_{a\ell} y \Leftrightarrow (\delta_n < \epsilon_n) \ \text{ or } \ (\delta_n = \epsilon_n \text{ and } \delta_{n-1} < \epsilon_{n-1}) \ \text{ or } \ \ldots$$

i.e., $<_{a\ell}$ is the antilexicographical ordering induced by the ordering $<$ of the atoms. Again a *natural ordering* of $\boldsymbol{B}$ is one induced this way from an ordering of the set of atoms.

Let $\mathcal{OBA}$ be the order class of all $\boldsymbol{B} = \langle \boldsymbol{B}_0, <^{\boldsymbol{B}} \rangle$, such that $\boldsymbol{B}_0$ is a finite Boolean algebra and $<^{\boldsymbol{B}}$ is a natural ordering of B_0. We now have:

Proposition 6.13. *$\mathcal{OBA}$ is a reasonable Fraïssé order class.*

Proof. First we check that $\mathcal{OBA}$ is reasonable (see Definition 5.1). Let $\boldsymbol{B}_1, \boldsymbol{B}_2$ be two finite Boolean algebras and let $\pi : \boldsymbol{B}_1 \to \boldsymbol{B}_2$ be an embedding. Let $<_{a\ell}$ be a natural ordering on $\boldsymbol{B}_1$ induced by an ordering $a_1 < a_2 < \cdots < a_n$ of the atoms of B_1. Let $\{c_1, \dots, c_k\}$ be the atoms of $\boldsymbol{B}_2$. Then $\pi(a_1) = \bigvee_{i=1}^{k_1} c_{1i}, \dots, \pi(a_n) = \bigvee_{i=1}^{k_n} c_{ni}$, where $\{c_{1i}\}_{i=1}^{k_1}, \dots, \{c_{ni}\}_{i=1}^{k_n}$ is a partition of $\{c_1, \dots, c_k\}$. Order then the atoms of $\boldsymbol{B}_2$ as follows:

$$c_{11} \prec c_{12} \prec \cdots \prec c_{1k_1} \prec c_{21} \prec \cdots \prec c_{2k_2} \prec \cdots \prec c_{n1} \prec \cdots \prec c_{nk_n},$$

and let $\prec_{a\ell}$ be the induced antilexicographical ordering on $\boldsymbol{B}_2$. Then clearly $\pi : \langle \boldsymbol{B}_1, <_{a\ell} \rangle \to \langle \boldsymbol{B}_2, \prec_{a\ell} \rangle$ is still an embedding.

We next check that $\mathcal{OBA}$ is hereditary. To see this, let $\boldsymbol{B}_2$ be a finite Boolean algebra and $\boldsymbol{B}_1$ a subalgebra. Let $<_{a\ell}$ be a natural ordering on $\boldsymbol{B}_2$ induced by an ordering $a_1 < \cdots < a_n$ of the atoms of $\boldsymbol{B}_2$. Let now $b_1 <_{a\ell} \cdots <_{a\ell} b_k$ be the atoms of $\boldsymbol{B}_1$. Write $b_i = c_{i1} \vee \cdots \vee c_{ik_i}$, where $c_{i1}, \dots, c_{ik_1}$ are atoms of $\boldsymbol{B}_2$ and $c_{i1} < \cdots < c_{ik_i}$. Then $c_{ik_i} < c_{jk_j}$ if $i < j$. From this it easily follows that $<_{a\ell} |B_1 =$ the antilexicographical ordering induced by $<_{a\ell} |\{b_1, \dots, b_k\}$, so a substructure of an element of $\mathcal{OBA}$ is also an element of $\mathcal{OBA}$.

Finally, we check that $\mathcal{OBA}$ satisfies the amalgamation property (from which JEP also follows, since the two element Boolean algebra embeds in any Boolean algebra).

Suppose $\boldsymbol{B}$ is a finite Boolean algebra and $b_1 <^{\boldsymbol{B}} \cdots <^{\boldsymbol{B}} b_k$ is an ordering of the atoms of $\boldsymbol{B}$ with induced antilexicographical ordering $<^{\boldsymbol{B}}_{a\ell}$. Let also $\boldsymbol{C}, \boldsymbol{D}$ be finite Boolean algebras with orderings $c_1 <^{\boldsymbol{C}} \cdots <^{\boldsymbol{C}} c_l,\ d_1 <^{\boldsymbol{D}} \cdots <^{\boldsymbol{D}} d_m$ of their atoms, and corresponding induced antilexicographical orderings $<^{\boldsymbol{C}}_{a\ell}, <^{\boldsymbol{D}}_{a\ell}$. Suppose we have embeddings

$$f : \langle \boldsymbol{B}, <^{\boldsymbol{B}}_{a\ell} \rangle \to \langle \boldsymbol{C}, <^{\boldsymbol{C}}_{a\ell} \rangle,\ g : \langle \boldsymbol{B}, <^{\boldsymbol{B}}_{a\ell} \rangle \to \langle \boldsymbol{D}, <^{\boldsymbol{D}}_{a\ell} \rangle.$$

We will find a Boolean algebra $\boldsymbol{E}$ with $m + \ell - k$ atoms and an ordering $<^{\boldsymbol{E}}$ on these atoms, so that, if $<^{\boldsymbol{E}}_{a\ell}$ is the induced antilexicographical ordering, then there are embeddings $r : \langle \boldsymbol{C}, <^{\boldsymbol{D}}_{a\ell} \rangle \to \langle \boldsymbol{E},\ <^{\boldsymbol{E}}_{a\ell} \rangle$, $s : \langle \boldsymbol{D}, <^{\boldsymbol{C}}_{a\ell} \rangle \to< \boldsymbol{E},\ <^{\boldsymbol{E}}_{a\ell} \rangle$, such that $r \circ f = s \circ g$. To specify r, s, it is of course enough to define where the atoms of $\boldsymbol{C}, \boldsymbol{D}$ go.

Let $f(b_i) = c_{i1} \vee \cdots \vee c_{ik_i}$, with $c_{i1} <^{\boldsymbol{C}} \cdots <^{\boldsymbol{C}} c_{ik_i}$ atoms in $\boldsymbol{C}$. Then $c_{ik_i} < c_{jk_j}$, if $i < j$. Similarly, $g(b_i) = d_{i1} \vee \cdots \vee d_{i\ell_i}$, where $d_{i1} <^{\boldsymbol{D}} \cdots <^{\boldsymbol{D}} d_{i\ell_i}$ are atoms in $\boldsymbol{D}$ with $d_{i\ell_i} < d_{j\ell_j}$, if $i < j$.

The Boolean algebra $\boldsymbol{E}$ will have atoms $\{\bar{c}_{ij}\}_{1 \le i \le k,\ 1 \le j \le k_i},\ \{\bar{d}_{ij}\}_{1 \le i \le k,\ 1 \le j \le \ell_i}$ all distinct, except that

$$\bar{c}_{ik_i} = \bar{d}_{i\ell_i},\ 1 \le i \le k.$$

We will now define the ordering $<^{\boldsymbol{E}}$ on these atoms and decide where the atoms of $\boldsymbol{C}, \boldsymbol{D}$ go.

We first order $\{\bar{c}_{11}, \dots, \bar{c}_{1k_1}\} \cup \{\bar{d}_{11}, \dots, \bar{d}_{1\ell_1}\}$ as follows:

$$\bar{c}_{11} <^{\boldsymbol{E}} \cdots <^{\boldsymbol{E}} \bar{c}_{1k_1}, \bar{d}_{11} <^{\boldsymbol{E}} \cdots <^{\boldsymbol{E}} \bar{d}_{1\ell_1} (= \bar{c}_{1k_1})$$

and extend $<^{\boldsymbol{E}}$ on the rest in an arbitrary way. Using the notation $(a, b] = \{x : a < x \le b\}$, $(-\infty, a] = \{x : x \le a\}$ in an arbitrary ordering, we now define

$$r(c_{11}) = \bigvee(-\infty, \bar{c}_{11}],\ r(c_{12}) = \bigvee(\bar{c}_{11}, \bar{c}_{12}], \dots, r(c_{1k_1}) = \bigvee(\bar{c}_{1,k_1-1}, \bar{c}_{1k_1}],$$

$$s(d_{11}) = \bigvee(-\infty, \bar{d}_{11}], \dots, r(d_{1\ell_1}) = \bigvee(\bar{d}_{1,\ell_1-1}, \bar{d}_{1,\ell_1}],$$

where, for example $\bigvee(\bar{c}_{11}, \bar{c}_{12}]$ means $a_1 \vee \cdots \vee a_p$, where $a_1, \dots, a_p$ are the elements of $\{\bar{c}_{11}, \dots, \bar{c}_{1k_1}\} \cup \{\bar{d}_{11}, \dots, \bar{d}_{1\ell_1}\}$ in the interval $(\bar{c}_{11}, \bar{c}_{12}]$ according to $<^{\boldsymbol{E}}$. Clearly r, s are order preserving from $\{c_{11}, \dots, c_{1k_1}\}, \{d_{11}, \cdots d_{1\ell_1}\}$, resp., to $<^{\boldsymbol{E}}_{a\ell}$ and $r \circ f(b_1) = s \circ g(b_1) = \bar{c}_{11} \vee \cdots \vee \bar{c}_{1k_1} \vee \bar{d}_{11} \vee \cdots \vee \bar{d}_{1\ell_1}$.

Next we extend $<^{\boldsymbol{E}}$ to $\{\bar{c}_{11}, \dots, \bar{c}_{1k_1}\} \cup \{\bar{c}_{21}, \dots, \bar{c}_{2k_2}\} \cup \{\bar{d}_{11}, \dots, \bar{d}_{1\ell_1}\} \cup \{\bar{d}_{21}, \dots, \bar{d}_{2\ell_2}\}$ and define $r(c_{2i}), s(d_{2i})$. We simply do that by requiring that $c_{ij} \mapsto \bar{c}_{ij}$ $(i = 1, 2,\ 1 \le j \le k_i)$ is order preserving, and also $d_{ij} \mapsto \bar{d}_{ij}$ $(i = 1, 2,\ 1 \le j \le \ell_i)$ is order preserving, and define it arbitrarily otherwise. Notice that this guarantees that $\bar{c}_{2k_2} = \bar{d}_{2\ell_2}$ is the largest element, in particular $\bar{c}_{2k_2} > \bar{c}_{1k_1}, \bar{d}_{2\ell_2} > \bar{d}_{1\ell_1}$. We then extend r, s by defining

$$r(c_{21}) = \bigvee(-\infty, \bar{c}_{21}], \dots, r(c_{2k_2}) = \bigvee(\bar{c}_{2,k_2-1}, \bar{c}_{2k_2}],$$

$$s(d_{21}) = \bigvee(-\infty, \bar{d}_{21}], \dots, s(d_{2\ell_2}) = \bigvee(\bar{d}_{2,\ell_2-1}, \bar{d}_{2\ell_2}],$$

where these intervals now refer to the ordering $<^{\boldsymbol{E}}$ restricted to

$$\{c_{21}, \dots, c_{2k_2}\} \cup \{d_{21}, \dots, d_{2\ell_2}\}.$$

Then r, s are still order preserving and $r \circ f(b_2) = s \circ g(b_2) = \bar{c}_{21} \vee \cdots \vee \bar{c}_{2k_2} \vee \bar{d}_{21} \vee \cdots \vee \bar{d}_{2\ell_2}$. Proceeding this way, we define $<^{\boldsymbol{E}}$ on all the atoms of $\boldsymbol{E}$ and r, s on all the atoms of $\boldsymbol{C}, \boldsymbol{D}$, resp., so that r, s are order preserving on the atoms and $r \circ f(b) = s \circ g(b)$, for any atom b of $\boldsymbol{B}$. Then r, s extend uniquely to embeddings from $\langle \boldsymbol{C}, <^{\boldsymbol{C}}_{a\ell} \rangle$ to $\langle \boldsymbol{E}, <^{\boldsymbol{E}}_{a\ell} \rangle$ and $\langle \boldsymbol{D}, <^{\boldsymbol{D}}_{a\ell} \rangle$ to $\langle \boldsymbol{E}, <^{\boldsymbol{E}}_{a\ell} \rangle$, resp., and $r \circ f = s \circ g$. ⊣

Finally, it is clear that $\mathcal{OBA}$ is order forgetful and, since $\mathcal{BA}$ satisfies the Ramsey property by Graham-Rothschild [33] (the Dual Ramsey Theorem), it follows that so does $\mathcal{OBA}$. Let $\boldsymbol{OB}_\infty = \langle \boldsymbol{B}_\infty, <^{\boldsymbol{OB}_\infty} \rangle$ be the Fraïssé limit of $\mathcal{OBA}$, which we call the *countable atomless Boolean algebra with the canonical ordering*. Then we have:

Theorem 6.14. *The automorphism group of the countable atomless Boolean algebra with the canonical ordering is extremely amenable.* ⊣

And we conclude this section by providing the following characterization of the group $\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ in terms of extreme amenability.

Proposition 6.15. *Let $G \leq S_\infty$ be a closed subgroup of S_∞ which acts transitively on $[\mathbb{N}]^n = \{A \subseteq \mathbb{N} : \mathrm{card}(A) = n\}, n = 1, 2, 3 \ldots.$ If G is extremely amenable, then there is an ordering $\prec$ on $\mathbb{N}$ with $\langle \mathbb{N}, \prec \rangle \cong \langle \mathbb{Q}, < \rangle$ and $G = \mathrm{Aut}(\langle \mathbb{N}, \prec \rangle)$.*

Proof. Since G preserves an ordering, it follows from 3.11 of [8] that there is an ordering $\prec$ on $\mathbb{N}$ with $\langle \mathbb{N}, \prec \rangle \cong \langle \mathbb{Q}, < \rangle$, such that $G \leq \mathrm{Aut}(\langle \mathbb{N}, \prec \rangle)$. Since, for each n, G acts transitively on increasing n-tuples in $\prec$, G is dense in $\mathrm{Aut}(\langle \mathbb{N}, \prec \rangle)$, so $G = \mathrm{Aut}(\langle \mathbb{N}, \prec \rangle)$. ⊣

(E) Metric spaces

We can view metric spaces (X, d) as structures for the language $L_0 = \{R_q\}_{q \in \mathbb{Q}}$, R_q binary, identifying (X, d) with $\boldsymbol{X} = \langle X, \{R_q^{\boldsymbol{X}}\}_{q \in \mathbb{Q}} \rangle$, where $(x, y) \in R_q^{\boldsymbol{X}} \Leftrightarrow d(x, y) < q$. Let $\mathcal{M}_\mathbb{Q}$ be the class of finite metric spaces with rational distances. Then it is not hard to check that $\mathcal{M}_\mathbb{Q}$ is a Fraïssé class (see, e.g., [7]). Its Fraïssé limit is $\boldsymbol{U}_0$, originally constructed in Urysohn [79], and which we will call the *rational Urysohn space*. Let also $\mathcal{OM}_\mathbb{Q} = \mathcal{M}_\mathbb{Q} * \mathcal{LO}$ be the class of finite ordered metric spaces with rational distances. Since actually $\mathcal{M}_\mathbb{Q}$ satisfies strong amalgamation and joint embedding, it is easy to verify that $\mathcal{OM}_\mathbb{Q}$ is a reasonable Fraïssé order class. Its Fraïssé limit $\mathrm{Flim}(\mathcal{OM}_\mathbb{Q}) = \boldsymbol{OU}_0$ will be called the *ordered rational Urysohn space*. In response to an inquiry of the authors, Nešetřil [52] has announced that $\mathcal{OM}_\mathbb{Q}$ satisfies the Ramsey property. So we have:

Theorem 6.16. *The automorphism group of the ordered rational Urysohn space is extremely amenable.* ⊣

This result has an interesting application, which actually was our motivation for looking at the case of metric spaces.

Let $\boldsymbol{U}$ be the so-called *Urysohn space*, see Urysohn [79]. This is the unique, up to isometry, complete separable metric space which contains (up to isometry) all finite metric spaces and is ultrahomogeneous, for isometries. Uspenskij [80] showed that $\mathrm{Iso}(\boldsymbol{U})$, with the pointwise convergence topology, is a universal Polish group, i.e., contains up to isomorphism any Polish group. Note that the topology of the group $\mathrm{Iso}(\boldsymbol{U})$ is that of pointwise convergence on the space $\boldsymbol{U}$ equipped with the metric topology, not the discrete one, unlike the case of $\mathrm{Aut}(\boldsymbol{OU}_0)$. Pestov [66], using quite different techniques than the ones used in our paper, showed that $\mathrm{Iso}(\boldsymbol{U})$ is extremely amenable. This result has several applications. We now use 6.16 to provide a quite different proof of this theorem.

Theorem 6.17 (Pestov [66]). *The group of isometries Iso($\boldsymbol{U}$) of the Urysohn space $\boldsymbol{U}$, with the pointwise convergence topology, is extremely amenable.*

Proof. We start with the following standard fact.

Lemma 6.18. *Let G, H be topological groups and $\pi : G \to H$ a continuous homomorphism with $\pi(G)$ dense in H. If G is extremely amenable, so is H.*

Proof. Let X be an H-flow. Denote by $\alpha : H \times X \to X$ the action. Define now $\tilde{\alpha} : G \times X \to X$ by $\tilde{\alpha}(g,x) = \alpha(\pi(g),x)$. This turns X into a G-flow, so there is a fixed point $x_0 \in X$. Clearly x_0 is a fixed point for the H-flow, since $\pi(G)$ is dense in H. ⊣

Now denote by $\langle \boldsymbol{U}_0, \prec\rangle$ the ordered rational Urysohn space (so that $\boldsymbol{U}_0$ is the rational Urysohn space). Already Urysohn [79] showed that the completion of $\boldsymbol{U}_0$ is $\boldsymbol{U}$, so we view $\boldsymbol{U}_0$ as a dense subspace of $\boldsymbol{U}$. Thus if $g \in \mathrm{Iso}(\boldsymbol{U}_0)$, there is a unique extension $\bar{g} \in \mathrm{Iso}(\boldsymbol{U})$. Since every $g \in \mathrm{Aut}(\langle \boldsymbol{U}_0, \prec\rangle)$ is in particular an isometry of $\boldsymbol{U}_0$, the map $g \mapsto \bar{g}$ is 1–1 from $\mathrm{Aut}(\langle \boldsymbol{U}_0, \prec\rangle)$ into $\mathrm{Iso}(\boldsymbol{U})$ and it is easy to check that it is continuous. It only remains to show that its range is dense in $\mathrm{Iso}(\boldsymbol{U})$ and then use 6.18 and 6.16.

Lemma 6.19. *Let $D \subseteq \mathrm{Iso}(\boldsymbol{U})$. Let d be the metric on $\boldsymbol{U}$. Then D is dense, if the following holds:*

$$\begin{gathered}(*)\ \forall \epsilon > 0 \forall x_1, \dots, x_n \in U \forall h \in \mathrm{Iso}(\boldsymbol{U}) \\ \exists x'_1, \dots, x'_n, y'_1, \dots, y'_n \in U \exists g \in D \\ (d(x_i, x'_i) < \epsilon,\ d(h(x_i), y'_i) < \epsilon,\ g(x'_i) = y'_i,\ i = 1, \dots, n).\end{gathered}$$

Proof. To check that D is dense, fix $\epsilon > 0,\ h \in \mathrm{Iso}(\boldsymbol{U}), x_1, \dots, x_n \in U$, in order to find $g \in D$ with $d(g(x_i), h(x_i)) < \epsilon$.

By $(*)$ find $x'_1, \dots, x'_n, y'_1, \dots y'_n$ and $g \in D$ for $\epsilon/2$. Then

$$\begin{aligned} d(g(x_i), h(x_i)) &\leq d(g(x_i), g(x'_i)) + d(g(x'_i), h(x_i)) \\ &= d(x_i, x'_i) + d(y'_i, h(x_i)) \\ &< \epsilon \end{aligned}$$

⊣

So to check that $\{\bar{g} : g \in \mathrm{Aut}(\langle \boldsymbol{U}_0, \prec\rangle)\}$ is dense in $\mathrm{Iso}(\boldsymbol{U})$, it is enough to show the following.

Lemma 6.20. *Given $x_1, \dots, x_n, y_1, \dots, y_n \in U$ such that $x_i \mapsto y_i,\ i = 1, \dots n$, is an isometry, and given $\epsilon > 0$, there are $x'_1, \dots, x'_n, y'_1, \dots y'_n \in U_0$ so that $x'_i \mapsto y'_i$ is an order preserving (with respect to $\prec$) isometry and*

$$d(x'_i, x_i) < \epsilon,\ d(y'_i, y_i) < \epsilon,\ i = 1, \dots, n.$$

Proof. By induction on n.

$\boldsymbol{n = 1}$: Simply choose $x'_1, y'_1 \in U_0$ with $d(x'_1, x_1) < \epsilon,\ d(y'_1, y_1) < \epsilon$.

$\boldsymbol{n \to n+1}$: Suppose $x_1, \dots, x_n, x_{n+1}, y_1, \dots, y_n, y_{n+1} \in U$ are given so that $x_i \mapsto y_i$ is an isometry. By induction hypothesis, find $x'_1, \dots x'_n, y'_1, \dots, y'_n \in U_0$, so that $x'_i \mapsto y'_i$ is an order preserving isometry and $d(x_i, x'_i) < \epsilon/2,\ d(y_i, y'_i) < \epsilon/2,\ i = 1, \dots, n$. Let $x^0_{n+1}, y^0_{n+1} \in U_0$ be such that

$$d(x^0_{n+1}, x_{n+1}) < \epsilon/2,\ d(y^0_{n+1}, y_{n+1}) < \epsilon/2.$$

Put $d(x^0_{n+1}, x'_i) = d_i,\ d(y^0_{n+1}, y'_i) = d'_i,\ 1 \leq i \leq n$. We can of course assume that ϵ is small enough so that $d_i, d'_i > \epsilon$.

Therefore,

$$\begin{aligned} |d_i - d(x_{n+1}, x_i)| = & \ |d(x^0_{n+1}, x'_i) - d(x_{n+1}, x_i)| \\ & \leq d(x^0_{n+1}, x_{n+1}) + d(x_i, x'_i) < \epsilon \end{aligned}$$

and

$$|d'_i - d(y_{n+1}, y_i)| < \epsilon,$$

so

$$\begin{aligned} |d_i - d'_i| & = |d_i - d(x_{n+1}, x_i) + d(x_{n+1}, x_i) - d(y_{n+1}, y_i) + d(y_{n+1}, y_i) - d'_i| \\ & < 2\epsilon. \end{aligned}$$

Put $e_i = \frac{d_i + d'_i}{2}$, and consider the ordered metric space

$$\langle \{x'_1, \ldots, x'_n, x^0_{n+1}, u\}, d', \prec' \rangle,$$

where $d'(x'_i, x'_j) = d(x'_i, x'_j),\ d'(x'_i, x^0_{n+1}) = d(x'_i, x^0_{n+1}),\ d'(u, x'_i) = e_i$, and $d'(u, x^0_{n+1})$ is any rational number satisfying the inequalities

$$d_i + e_i > 2\epsilon > d'(u, x^0_{n+1}) \geq |d_i - e_i|,\ i = 1, \ldots, n.$$

Notice here that

$$d_i + e_i = \frac{3d_i + d'_i}{2} > 2\epsilon$$

and

$$|d_i - e_i| = \tfrac{|d_i - d'_i|}{2} < \epsilon,$$

so such a number exists. We let $\prec'$ agree with the ordering $\prec$ (of U_0) for $x'_1, \ldots, x'_n,\ x^0_{n+1}$ and $x'_i \prec' u, x^0_{n+1} \prec' u$. We need of course to verify that d' is indeed a metric:

(i) Since $d'(x^0_{n+1}, x'_i) = d_i,\ d'(u, x'_i) = e_i$, we need to check that

$$|d_i - e_i| \leq d'(u, x^0_{n+1}) \leq d_i + e_i,$$

which is given by the definition of $d'(u, x^0_{n+1})$.

(ii) Let $\alpha_{ij} = d(x'_i, x'_j)$. We need to verify that

$$|e_i - e_j| \leq \alpha_{ij} \leq e_i + e_j.$$

We have

$$|d_i - d_j| \leq \alpha_{ij} \leq d_i + d_j,$$

since $d_i = d(x^0_{n+1}, x'_i)$. But also $\alpha_{ij} = d(y'_i, y'_j)$, so we also have

$$|d'_i - d'_j| \leq \alpha_{ij} \leq d'_i + d'_j.$$

Adding and dividing by 2, we get

$$|e_i - e_j| \leq \alpha_{ij} \leq e_i + e_j.$$

So by the properties of $\langle \boldsymbol{U}_0, \prec \rangle$, we can find a point $x'_{n+1} \in U_0$ with $x'_i \prec x'_{n+1},\ i = 1, \ldots n,\ x^0_{n+1} \prec x'_{n+1}$, and $d(x'_{n+1}, x'_i) = e_i,\ d(x'_{n+1}, x^0_{n+1}) = d'(u, x^0_{n+1}) < 2\epsilon$. Similarly we can find $y'_{n+1} \in U_0$ with $y'_i \prec y'_{n+1},\ i = 1, \ldots, n,\ y^0_{n+1} \prec y'_{n+1}$ and $d(y'_{n+1}, y'_i) = e_i,\ d(y'_{n+1}, y^0_{n+1}) < 2\epsilon$. Then $x'_i \mapsto y'_i,\ 1 \leq i \leq n+1$, is an order preserving isometry, and $d(x'_{n+1}, x_{n+1}) \leq d(x'_{n+1}, x^0_{n+1}) + d(x^0_{n+1}, x_{n+1}) < 3\epsilon$ and $d(y'_{n+1}, y_{n+1}) < 3\epsilon$, so the proof is complete. ⊣

⊣

A result similar to 6.16 can be proved for the ordered integer Urysohn space (where we consider the class of ordered finite metric spaces with integer distances), since Nešetřil [52] has also verified the corresponding Ramsey property. It is also conceivable that one can push those ideas to find a new proof of the result of Gromov and Milman [38] that the unitary group of the infinite-dimensional separable Hilbert space is extremely amenable, as well as a recent strengthening of this result by Pestov [66], who established extreme amenability of the group of (affine) isometries of the same Hilbert space.

7. Universal minimal flows and the ordering property

Consider now a signature $L \supseteq \{<\}$ and put $L_0 = L \setminus \{<\}$. Let $\mathcal{K}$ be a reasonable Fraïssé order class in L and put $\mathcal{K}_0 = \mathcal{K}|L_0$. Then by 5.2 we know that $\mathcal{K}_0$ is a Fraïssé class and if $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K})$, $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0)$, we have $\boldsymbol{F}_0 = F|L_0$. Let $<^{\boldsymbol{F}} = \prec_0$. Put $G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$ and consider the logic action of G_0 on LO, the compact space of linear orderings on $F_0 = F$ (which of course we can identify, if we want, with $\mathbb{N}$). Let $X_{\mathcal{K}}$ be the orbit closure of $\prec_0$, $\overline{G_0 \cdot \prec_0} \subseteq \mathrm{LO}$. We first note the following.

Proposition 7.1. *A linear ordering $\prec$ is in $X_{\mathcal{K}}$ iff for every finite substructure $\boldsymbol{B}_0$ of $\boldsymbol{F}_0$, $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec |B_0\rangle \in \mathcal{K}$.*

Proof. Assume $\prec \in X_{\mathcal{K}}$ and fix a finite substructure $\boldsymbol{B}_0$ of $\boldsymbol{F}_0$. Then as $\prec \in \overline{G_0 \cdot \prec_0}$, there is $g \in G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$ such that $\prec |B_0 = (g \cdot \prec_0)|B_0$. So if $g^{-1}(\boldsymbol{B}_0) = \boldsymbol{A}_0$, a substructure of $\boldsymbol{F}_0$, and $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec_0 |A_0\rangle$, which is in $\mathcal{K}$, we have that $g|A_0 : A_0 \to B_0$ is an isomorphism of $\boldsymbol{A}$ with $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec |B_0\rangle$, so $\boldsymbol{B} \in \mathcal{K}$.

Conversely, assume that for every finite substructure $\boldsymbol{B}_0$ of $\boldsymbol{F}_0$, $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec |B_0\rangle \in \mathcal{K}$. Then there is an embedding $\pi : \boldsymbol{B} \to \boldsymbol{F}$. If $\pi(\boldsymbol{B}) = \boldsymbol{A}$, then $\boldsymbol{A}$ is a substructure of $\boldsymbol{F}$ and π is an isomorphism of $\boldsymbol{B}, \boldsymbol{A}$, and thus in particular an isomorphism of $\boldsymbol{B}_0, \boldsymbol{A}_0 = \boldsymbol{A}|L_0$. But $\boldsymbol{B}_0, \boldsymbol{A}_0$ are finite substructures of $\boldsymbol{F}_0$, so, by ultrahomogeneity of $\boldsymbol{F}_0$, there is $g \in \mathrm{Aut}(\boldsymbol{F}_0) = G_0$ extending π^{-1}, so in particular, $\prec |B_0 = (g \cdot \prec_0)|B_0$. Since $\boldsymbol{B}_0$ was arbitrary, this shows that $\prec \in X_{\mathcal{K}}$. ⊣

Definition 7.2. We call any linear ordering in $X_{\mathcal{K}}$ a $\mathcal{K}$-admissible ordering.

Clearly, $X_{\mathcal{K}}$ is a G_0-flow. We will now derive necessary and sufficient conditions for $X_{\mathcal{K}}$ to be a minimal G_0-flow.

The following concept plays an important role in the Ramsey theory of graphs and hypergraphs, see Nešetřil-Rödl [57], Nešetřil [51], 5.2. We formulate it here in a general context.

Definition 7.3. Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}$, $\mathcal{K}$ a class of structures in L and let $\mathcal{K}_0 = \mathcal{K}|L_0$. We say that $\mathcal{K}$ satisfies the ordering property if for every $\boldsymbol{A}_0 \in \mathcal{K}_0$, there is $\boldsymbol{B}_0 \in \mathcal{K}_0$ such that for every linear ordering $\prec$ on A_0 and linear ordering $\prec'$ on B_0, if $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec\rangle \in \mathcal{K}$ and $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec'\rangle \in \mathcal{K}$, we have $\boldsymbol{A} \leq \boldsymbol{B}$.

For example, if $\mathcal{K}$ is the class of finite ordered graphs, so that $\mathcal{K}_0$ = the class of finite graphs, then $\mathcal{K}$ satisfies the ordering property by results of Nešetřil-Rödl (see, e.g., Nešetřil [51], 5.2 or Nešetřil-Rödl [57]). On the other hand, it is easy to see that the class $\mathcal{K} = \mathcal{OEQ}$ does not have the ordering property: Let $\boldsymbol{A}_0 = \langle A_0, E\rangle$ be an equivalence relation with two classes $\{a, b\}, \{c\}$ and consider the ordering $\prec$ on A_0 given by $a \prec c \prec b$. Then for any $\boldsymbol{B}_0 = \langle B_0, F\rangle$, if $\prec'$ is an ordering on B_0, so that each F-class is convex in $\prec'$, clearly $\langle \boldsymbol{A}_0, \prec\rangle \not\leq \langle \boldsymbol{B}_0, \prec'\rangle$.

Now we have

Theorem 7.4. *Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}$, $\mathcal{K}$ a reasonable Fraïssé order class in L. Let $\mathcal{K}_0 = \mathcal{K}|L_0$, and $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K})$, $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0) = \boldsymbol{F}|L_0$. Let $X_{\mathcal{K}}$ be the set of linear orderings $\prec$ on F $(= F_0)$ which are $\mathcal{K}$-admissible. Let also $G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$. Then the following are equivalent:*

(i) $X_{\mathcal{K}}$ is a minimal G_0-flow.

(ii) $\mathcal{K}$ satisfies the ordering property.

Proof. First we will reformulate (i) in a more explicit form. Below let $\boldsymbol{F} = \langle \boldsymbol{F}_0, \prec_0\rangle$.

Claim. Let $\prec$ be a linear ordering on F_0. Then $\prec_0 \in \overline{G_0 \cdot \prec}$ iff for every $\boldsymbol{A} \in \mathcal{K}$ there is a finite substructure $\boldsymbol{C}_0$ of $\boldsymbol{F}_0$ such that $\boldsymbol{C} = \langle \boldsymbol{C}_0, \prec |C_0\rangle \cong \boldsymbol{A}$.

Proof of claim. Suppose first that $\prec_0 \in \overline{G_0 \cdot \prec}$, and fix $\boldsymbol{A} \in \mathcal{K}$. Find $\boldsymbol{D} = \langle \boldsymbol{D}_0, \prec_0 |D_0\rangle$ a finite substructure of $\boldsymbol{F}$ with $\boldsymbol{D} \cong \boldsymbol{A}$. Then there is $g \in G_0$ such that $(g\cdot \prec)|D_0 = \prec_0 |D_0$. Clearly $\boldsymbol{C}_0 = g^{-1}(\boldsymbol{D}_0)$ is a substructure of $\boldsymbol{F}_0$ and $\boldsymbol{C} = \langle \boldsymbol{C}_0, \prec |C_0\rangle \cong \boldsymbol{D} \cong \boldsymbol{A}$.

To prove the converse, it is enough to show that given a finite substructure $\boldsymbol{A}_0$ of $\boldsymbol{F}_0$ we can find $g \in G_0$ such that $(g\cdot \prec)|A_0 = \prec_0 |A_0$. Since

$$\langle \boldsymbol{A}_0, \prec_0 |A_0\rangle = \boldsymbol{A} \in \mathcal{K}$$

there is a finite substructure $\boldsymbol{C}_0$ of $\boldsymbol{F}_0$ such that $\langle \boldsymbol{C}_0, \prec |\boldsymbol{C}_0\rangle \cong \boldsymbol{A}$, say via the isomorphism $\pi : C_0 \to A_0$. In particular, π is an isomorphism of $\boldsymbol{C}_0$ with $\boldsymbol{A}_0$ so, by the ultrahomogeneity of $\boldsymbol{F}_0$, there is $g \in G_0$ extending it. Then clearly $(g\cdot \prec)|A_0 = \prec_0 |A_0$ and the proof is complete. ⊣

Thus we see that (i) is equivalent to the following statement:

For every $\prec \in X_{\mathcal{K}}$ and every $\boldsymbol{A} \in \mathcal{K}$ there is a finite substructure $\boldsymbol{C}_0$ of $\boldsymbol{F}_0$ such that $\boldsymbol{C} = \langle \boldsymbol{C}_0, \prec |C_0\rangle \cong \boldsymbol{A}$.

We now proceed to prove the equivalence of (i) and (ii).

(ii) $\Rightarrow$ (i): Fix $\prec \in X_{\mathcal{K}}$, $\boldsymbol{A} \in \mathcal{K}$ and let $\boldsymbol{A}_0 = \boldsymbol{A}|L_0$. By (ii), find $\boldsymbol{B}_0 \in \mathcal{K}_0$ as in 7.3. We can of course assume that $\boldsymbol{B}_0$ is a substructure of $\boldsymbol{F}_0$. Then we have, since $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec |B_0\rangle \in \mathcal{K}$, $\boldsymbol{A} \leq \boldsymbol{B}$. Thus there is a substructure $\boldsymbol{C}$ of $\boldsymbol{B}$ isomorphic to $\boldsymbol{A}$. Clearly, if $\boldsymbol{C}_0 = \boldsymbol{C}|L_0$, $\boldsymbol{C} = \langle \boldsymbol{C}_0, \prec |C_0\rangle \cong \boldsymbol{A}$ and we are done.

(i) $\Rightarrow$ (ii): Notice first that, in order to verify the ordering property, it is enough to show that for every $\boldsymbol{A} \in \mathcal{K}$ there is $\boldsymbol{B}_0 \in \mathcal{K}_0$ such that for every linear ordering $\prec_0'$ on B_0, if $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec_0' \rangle \in \mathcal{K}$, then $\boldsymbol{A} \leq \boldsymbol{B}$. This follows from the JEP for $\mathcal{K}_0$.

So fix $\boldsymbol{A} \in \mathcal{K}$, and for every finite substructure $\boldsymbol{C}_0$ of $\boldsymbol{F}_0$, let

$$X_{\boldsymbol{C}_0} = \{\prec \in X_{\mathcal{K}} : \boldsymbol{A} \cong \langle \boldsymbol{C}_0, \prec |C_0\rangle\}.$$

Then (i) implies that

$$X_{\mathcal{K}} = \bigcup_{\boldsymbol{C}_0} X_{\boldsymbol{C}_0},$$

so, since each $X_{\boldsymbol{C}_0}$ is open, by compactness we have $\boldsymbol{C}_0^1, \dots, \boldsymbol{C}_0^n$ with $X_{\mathcal{K}} = \bigcup_{i=1}^n X_{\boldsymbol{C}_0^i}$, so that $\forall \prec \in X_{\mathcal{K}} \exists 1 \leq i \leq n (\boldsymbol{A} \cong \langle \boldsymbol{C}_0^i, \prec |C_0^i\rangle)$. Let $\boldsymbol{B}_0$ be the (finite) substructure of $\boldsymbol{F}_0$ generated by $\bigcup_{i=1}^n C_0^i$, so that

$$\forall \prec \in X_{\mathcal{K}} (\boldsymbol{A} \leq \langle \boldsymbol{B}_0, \prec |B_0\rangle).$$

Fix now $\prec_0'$, a linear ordering on B_0, such that $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec_0' \rangle \in \mathcal{K}$. If we can show that we can extend $\prec_0'$ to a linear ordering $\prec' \in X_{\mathcal{K}}$, then $\boldsymbol{A} \leq \langle \boldsymbol{B}_0, \prec' |B_0\rangle = \langle \boldsymbol{B}_0, \prec_0' \rangle = \boldsymbol{B}$, and this verifies the ordering property. To find such an extension, note that there is a finite substructure $\boldsymbol{D}_0$ of $\boldsymbol{F}_0$ and an isomorphism φ from $\boldsymbol{B}$ to $\boldsymbol{D} = \langle \boldsymbol{D}_0, \prec_0 |D_0\rangle$. In particular, φ is an isomorphism of $\boldsymbol{B}_0$ with $\boldsymbol{D}_0$, so, since $\boldsymbol{F}_0$ is ultrahomogeneous, there is $g \in \mathrm{Aut}(\boldsymbol{F}_0) = G_0$ extending φ. Then $\prec' = g^{-1} \cdot \prec_0 \in X_{\mathcal{K}}$ and clearly $\prec'$ extends $\prec_0'$. ⊣

We can finally show that $X_{\mathcal{K}}$ is the universal minimal flow of $G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$, when $\mathcal{K}$ has both the Ramsey and ordering properties.

Theorem 7.5. *Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}$, $\mathcal{K}$ a reasonable Fraïssé order class in L, and let $\mathcal{K}_0 = \mathcal{K}|L_0$ and $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K}), \boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0) = \boldsymbol{F}|L_0$. Let $G_0 = \mathrm{Aut}(\boldsymbol{F}_0), G = \mathrm{Aut}(\boldsymbol{F})$, and let $X_{\mathcal{K}}$ be the set of linear orderings of F $(= F_0)$ which are $\mathcal{K}$-admissible.*

(i) If $\mathcal{K}$ has the Ramsey property, the G_0-ambit $(X_{\mathcal{K}}, \prec_0)$ is the universal G_0-ambit with the property that G stabilizes the distinguished point, i.e., it can be mapped homomorphically to any G_0-ambit (X, x_0) with $G \cdot x_0 = \{x_0\}$. Thus any minimal subflow of $X_{\mathcal{K}}$ is the universal minimal flow of G_0. In particular, the universal minimal flow of G_0 is metrizable.

(ii) If $\mathcal{K}$ has the Ramsey and ordering properties, $X_{\mathcal{K}}$ is the universal minimal flow of G_0.

Proof. By 4.7, G is extremely amenable. Let also $<^{\boldsymbol{F}} = \prec_0$, so that $\boldsymbol{F} = \langle \boldsymbol{F}_0, \prec_0 \rangle$. By definition, $g \cdot \prec_0 = \prec_0$, for all $g \in G$.

Let X be a G_0-flow and let $x_0 \in X$ be such that $g \cdot x_0 = x_0$, $\forall g \in G$. We will find a homomorphism φ of the G_0-flow $X_{\mathcal{K}}$ to the G_0-flow X with $\varphi(\prec_0) = x_0$.

Let Φ be the closure of the set

$$\{(g \cdot \prec_0, g \cdot x_0) : g \in G_0\} \subseteq X_{\mathcal{K}} \times X$$

in the compact Hausdorff space $X_{\mathcal{K}} \times X$. We will show that Φ is the graph of a function $\varphi : X_{\mathcal{K}} \to X$. Granting this, we can easily verify that this φ works. First, since $(\prec_0, x_0) \in \Phi$, we have that $\varphi(\prec_0) = x_0$. Next, since the graph of φ is closed, φ is continuous. Finally, it is a G_0-map, since if $\varphi(\prec) = x$, then there is a net $\{g_i\}$ in G_0 such that $g_i\cdot \prec_0 \to \prec, g_i \cdot x_0 \to x$, so that $gg_i\cdot \prec_0 \to g\cdot \prec, gg_i \cdot x_0 \to g \cdot x$ and since $(gg_i\cdot \prec_0, gg_i \cdot x_0) \in \Phi$, we also have that $(g\cdot \prec, g\cdot x) \in \Phi$, i.e., $\varphi(g\cdot \prec) = g\cdot x = g\cdot\varphi(\prec)$.

So it is enough to prove that Φ is the graph of a function $\varphi : X_{\mathcal{K}} \to X$. First we notice that for any $\prec \in X_{\mathcal{K}}$, there is some $x \in X$ with $(\prec, x) \in \Phi$. Indeed, let $\{g_i\}$ be a net (actually a sequence) such that $g_i\cdot \prec_0 \to \prec$. Then $\{g_i \cdot x_0\}$ is a net in X, so there is a subnet $\{g_{i_j} \cdot x_0\}$ converging to some $x \in X$. Then $g_{i_j}\cdot \prec_0 \to \prec, g_{i_j} \cdot x_0 \to x$, so $(\prec, x) \in \Phi$. Finally, we show that if $(\prec, x) \in \Phi$ and $(\prec, x') \in \Phi$, then $x = x'$. This amounts to proving the following property:

$(*)$ If $\{g_i\}, \{h_j\}$ are nets in G_0 and $g_i\cdot \prec_0 \to \prec, h_j\cdot \prec_0 \to \prec$, and $g_i\cdot \prec_0 \to x, h_j\cdot \prec_0 \to x'$, then $x = x'$.

Recall that any compact Hausdorff space is regular, so as $(x, x') \notin \Delta = \{(y, y) : y \in X\}$, there are open nbhds U, U' of x, x', resp., and W of Δ with $(U \times U') \cap W = \emptyset$. For each $y \in X$, there is an open nbhd U_y of y with $U_y \times U_y \subseteq W$ and thus, by regularity again, there is an open nbhd V_y of y with $y \in V_y \subseteq \overline{V_y} \subseteq U_y$. So, by compactness, we can find compact sets $K_1, \dots, K_n$ and open sets $U_1, \dots, U_n$ such that $X = \bigcup_{i=1}^n K_i$, and $K_i \subseteq U_i, U_i^2 \subseteq W, i = 1, \dots, n$.

Since the action of G_0 on X is continuous, the map $g \mapsto \varphi_g$ from G_0 to $H(X)$, where $\varphi_g(y) = g \cdot y$, is continuous, where $H(X)$ has the compact-open topology. Since $K_i \subseteq U_i$, the set

$$\bigcap_{i=1}^n \{f \in H(X) : f(K_i) \subseteq U_i\}$$

is an open nbhd of the identity of $H(X)$, so if d_r is a right-invariant compatible metric for G_0, there is $\delta > 0$ such that $d_r(1_{G_0}, g) < \delta \Rightarrow g \cdot K_i \subseteq U_i, \forall 1 \le i \le n \Rightarrow \forall y \in X (y, g \cdot y) \in W$. So $d_r(g, h) < \delta \Rightarrow \forall y (y, gh^{-1} \cdot y) \in W \Rightarrow (g \cdot x_0, h \cdot x_0) \in W$.

We will now choose a standard right-invariant metric for G_0. Without loss of generality, we can assume that $F = F_0 = \mathbb{N}$, so that G_0 is a closed subgroup of S_∞. The following is then a left-invariant compatible metric on S_∞ and thus on G_0 : For $f \neq g$ in S_∞,

$$d_\ell(f, g) = 2^{-k-1}, \text{ where } k \text{ is least with } f(k) \neq g(k).$$

Let

$$d_r(f, g) = d_\ell(f^{-1}, g^{-1})$$

be the corresponding right-invariant compatible metric on S_∞ and G_0.

Next choose a finite substructure $\boldsymbol{A}_0$ of $\boldsymbol{F}_0$ such that for any $f, g \in \mathrm{Aut}(\boldsymbol{F}_0) = G_0$,

$$f|A_0 = g|A_0 \Rightarrow d_\ell(f, g) < \delta.$$

Since $g_i \cdot x_0 \to x, h_j \cdot x_0 \to x', g_i \cdot \prec_0 \to \prec, h_j \cdot \prec_0 \to \prec$, find N, M large enough so that $g_N \cdot x_0 \in U, h_M \cdot x_0 \in U'$ and

$$g_N \cdot \prec_0 |A_0 = h_M \cdot \prec_0 |A_0 = \prec |A_0.$$

Thus for $a, b \in A_0$,

$$g_N^{-1}(a) \prec_0 g_N^{-1}(b) \Leftrightarrow h_M^{-1}(a) \prec_0 h_M^{-1}(b).$$

Let $g_N^{-1} \cdot \boldsymbol{A}_0 = \boldsymbol{B}_0,\ h_M^{-1} \cdot \boldsymbol{A}_0 = \boldsymbol{C}_0$, so that $\boldsymbol{B}_0, \boldsymbol{C}_0$ are finite substructures of $\boldsymbol{F}_0$. Let $\boldsymbol{B} = \langle \boldsymbol{B}_0, \prec_0 |B_0\rangle,\ \boldsymbol{C} = \langle \boldsymbol{C}_0, \prec_0 |C_0\rangle$, so that $\boldsymbol{B}, \boldsymbol{C}$ are finite substructures of $\boldsymbol{F}$. Thus $\boldsymbol{B} \cong \boldsymbol{C}$ via the isomorphism $\rho : B \to C$, given by

$$\rho(g_N^{-1}(a)) = h_M^{-1}(a), \quad a \in A_0.$$

Since $\boldsymbol{F}$ is ultrahomogeneous, there is $r \in \mathrm{Aut}(\boldsymbol{F}) = G$ extending ρ, i.e., for $a \in A_0,\ r(g_N^{-1}(a)) = h_M^{-1}(a)$, so $r \circ g_N^{-1}|A_0 = h_M^{-1}|A_0$, thus $d_\ell(r \circ g_N^{-1}, h_M^{-1}) < \delta$, $d_r(g_N \circ r^{-1}, h_M) < \delta$, and therefore $(g_N \cdot (r^{-1} \cdot x_0), h_M \cdot x_0) < W$, and since $r^{-1} \cdot x_0 = x_0$ (because $r^{-1} \in G$), we have $(g_N \cdot x_0, h_M \cdot x_0) < W$, a contradiction. ⊣

The converse of 7.5 (ii) will be proved in 10.8 below. It follows from the preceding result that if $\mathcal{K}_0$ is a Fraïssé class in a signature L_0 with $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0), G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$, then it is important to understand when there is a reasonable Fraïssé order class $\mathcal{K}$ in $L = L_0 \cup \{<\}$ with $\mathcal{K}_0 = \mathcal{K}|L_0$, which also has the Ramsey property (because then any minimal subflow of the G_0-flow $X_{\mathcal{K}}$ is the universal minimal flow of G_0). Not every $\mathcal{K}_0$ admits such a $\mathcal{K}$. For example, let G_0 be an infinite countable discrete group. Via its left-regular representation, we can view G_0 as a closed subgroup of S_∞, so let $\boldsymbol{F}_0$ be the induced structure for G_0 which is ultrahomogeneous (see Section 2 third paragraph before the last Remark), and let $\mathcal{K}_0 = \mathrm{Age}(\boldsymbol{F}_0)$. Then $\mathcal{K}_0$ does not admit a $\mathcal{K}$ as above with the Ramsey property as then the universal minimal flow of G_0 would be metrizable. We will say more about this question in Section 10.

Assuming such a $\mathcal{K}$ with the Ramsey property exists, one can ask further whether another such $\mathcal{K}'$ exists with both the Ramsey and ordering properties (in that case $X_{\mathcal{K}'}$ would be the universal minimal flow of G_0). We will see in Section 10 that this is always the case. Note that by 7.5 any $\mathcal{K}'$ that has both the Ramsey and ordering properties has an important minimality property among all $\mathcal{K}$ that have the Ramsey property: The G_0-flow $X_{\mathcal{K}'}$ is (up to isomorphism) a subflow of $X_{\mathcal{K}}$. Moreover, any two such $X_{\mathcal{K}'}$ are isomorphic. We will exploit further these minimality and uniqueness properties of classes that have both the Ramsey and ordering properties in Section 9.

8. Calculating universal minimal flows

We now apply the results in §6, §7 to compute the universal minimal flows of several automorphism groups.

Consider first the classes $\mathcal{OGR}$, of finite ordered graphs, $\mathcal{OF}orb(K_n),\ n = 3, 4, \ldots$ of K_n-free finite ordered graphs, $\mathcal{OEQ}_1$ of complete finite ordered graphs, $\mathcal{OH}_{L_0}$ of finite ordered hypergraphs of type L_0, $\mathcal{OF}orb(\mathcal{A})$ of finite ordered hypergraphs of type L_0

that omit $\mathcal{A}$, where $\mathcal{A}$ is a class of finite irreducible hypergraphs of type L_0, and $\mathcal{OM}_{\mathbb{Q}}$ of finite ordered metric spaces with rational distances.

Each of these classes satisfies the ordering property and this follows easily from the fact (already used above in §6) that each of these classes satisfies the Ramsey property. (The case of $\mathcal{OEQ}_1$ is of course trivial.) This is done via a standard Sierpinski-style of coloring obtained by comparing two orderings (see, e.g., Nešetřil [51], p. 1376). A similar argument will deduce the ordering property for finite ordered metric spaces with rational distances from the corresponding Ramsey property. It should be mentioned, however, that typically the ordering property for a given class of structures is a result considerably easier to prove than the corresponding Ramsey property and can frequently be proved directly (see Nešetřil-Rödl [57] and Nešetřil [51],[53]).

In each one of the above cases, the space of admissible orderings is of course the space LO of all linear orderings on $\mathbb{N}$ (which we identify with the universe of the Fraïssé limit of each class). Thus, by 7.5, we have

Theorem 8.1. *For each one of the groups below, its universal minimal flow is the space* LO *of linear orderings on* $\mathbb{N}$*, so in particular it is metrizable:*

(i) The automorphism group of the random graph.

(ii) The automorphism group of the random K_n-free graph.

(iii) (Glasner-Weiss [30]) S_∞*, the permutation group of* $\mathbb{N}$.

(iv) The automorphism group of the random hypergraph of type L_0.

(v) The automorphism group of the random $\mathcal{A}$-free hypergraph of type L_0, where $\mathcal{A}$ is a class of irreducible finite hypergraphs of type L_0.

(vi) The isometry group of the rational Urysohn space $\boldsymbol{U}_0$. ⊣

Consider now the classes of convexly ordered finite equivalence relations, naturally ordered finite-dimensional spaces over a finite field F, and naturally ordered finite Boolean algebras. Each is easily seen to satisfy the ordering property. So we have:

Theorem 8.2. *(i) The automorphism group of the structure $\langle \mathbb{N}, E\rangle$, where E is an equivalence relation on $\mathbb{N}$ with infinitely many classes, each of which is infinite, has as universal minimal flow the space of all convex orderings on $\mathbb{N}$, i.e., all orderings on $\mathbb{N}$ for which each E-class is convex.*

(ii) Let $\boldsymbol{V}_F$ be the $\aleph_0$-dimensional vector space over a finite field F. The universal minimal flow of its automorphism group (i.e., $\mathrm{GL}(\boldsymbol{V}_F)$*) is the space of all orderings on V_F, whose restrictions to finite-dimensional subspaces are natural.*

(iii) Let $\boldsymbol{B}_\infty$ be the countable atomless Boolean algebra. The universal minimal flow of its automorphism group is the space of all orderings on B_∞, whose restrictions to finite subalgebras are natural. ⊣

In particular all these universal minimal flows are metrizable.

The question of whether the universal minimal flow of $\mathrm{GL}(\boldsymbol{V}_F)$ is nontrivial was brought to one of the authors' attention by Pierre de la Harpe.

Note that, by Stone duality, $\boldsymbol{B}_\infty$ can be identified with the algebra of clopen subsets of $2^{\mathbb{N}}$ and that every $g \in \mathrm{Aut}(\boldsymbol{B}_\infty)$ determines and is uniquely determined by a homeomorphism $\sigma(g) \in H(2^{\mathbb{N}})$. In Glasner-Weiss [31] there is another representation of the universal minimal flow of $H(2^{\mathbb{N}})$. They showed that the space $\Phi(2^{\mathbb{N}})$ of all maximal chains of closed subsets of $2^{\mathbb{N}}$, defined by Uspenskij [81], can serve as the universal minimal flow of the group $H(2^{\mathbb{N}})$. The existence of an isomorphism between the space $\mathcal{N}(\boldsymbol{B}_\infty)$, of all orderings of $\boldsymbol{B}_\infty$ whose restrictions to finite subalgebras are natural, and $\Phi(2^{\mathbb{N}})$ is of course a consequence of the uniqueness of the universal minimal flow but we exhibit below an explicit one.

Theorem 8.3. *There exists an (explicit) homeomorphism $\varphi : \Phi(2^{\mathbb{N}}) \to \mathcal{N}(\boldsymbol{B}_\infty)$ such that: $\varphi(\sigma(g) \cdot x) = g \cdot \varphi(x)$, for $x \in \Phi(2^{\mathbb{N}})$, $g \in \mathrm{Aut}(\boldsymbol{B}_\infty)$.*

Proof. Given a maximal chain $\mathcal{F}$ of closed subsets of $2^{\mathbb{N}}$, for every clopen subset A of $2^{\mathbb{N}}$, let

$$F_A = \bigcap \{F \in \mathcal{F} : F \cap A \neq \emptyset\}$$

By the maximality of $\mathcal{F}$, $A \cap F_A$ is a singleton. Note that if A is included in B, then F_A is included in F_B, though they can also be equal. Note however that if A and B are disjoint, then F_A and F_B are different, so for each finite Boolean algebra $\boldsymbol{B}$ contained in $\boldsymbol{B}_\infty$, we have a total ordering of the atoms of $\boldsymbol{B}$ and this induces the antilexicographical ordering on the Boolean algebra $\boldsymbol{B}$. These orderings cohere and produce an ordering $<_{\mathcal{F}}$ of $\boldsymbol{B}_\infty$ which is in $\mathcal{N}(B_\infty)$. Let $\varphi(F) =<_{\mathcal{F}}$. This defines a homeomorphism $\varphi : \Phi(2^{\mathbb{N}}) \to \mathcal{N}(\boldsymbol{B}_\infty)$ having the required property. ⊣

We conclude with another example of a calculation of a universal minimal flow, which turns out to be finite in this case.

Let $\langle \mathbb{Q}, <, E_n \rangle$ be the structure of the rationals with the usual ordering and an equivalence relation E_n with exactly n classes, each of which is dense in $\mathbb{Q}$. If, as in Section 6, **(A)**, we denote by $\mathcal{EQ}_n$ the class of finite equivalence relations with at most n classes and let $\mathcal{OEQ}_n = \mathcal{EQ}_n * \mathcal{LO}$, then $\langle \mathbb{Q}, <, E_n \rangle$ is the Fraïssé limit of $\mathcal{OEQ}_n$. As we pointed out in Section 6, **(A)**, the automorphism group G_n of this structure is not extremely amenable, when $n \geq 2$, as it acts on the finite space $\mathbb{Q}/E_n$ without fixed points. We will calculate below the universal minimal flow of G_n.

Theorem 8.4. *For each $n \geq 1$, let G_n be the automorphism group of the structure $\langle \mathbb{Q}, <, E_n \rangle$, where $<$ is the usual ordering on $\mathbb{Q}$ and E_n is an equivalence relation with exactly n classes each dense in $\mathbb{Q}$. Let $A_1, \dots, A_n$ be an enumeration of $\mathbb{Q}/E_n$ and let $H_n =\mathrm{Aut}(\langle \mathbb{Q}, <, A_1, \dots, A_n \rangle)$, where we view each A_i as a unary relation on $\mathbb{Q}$. Then H_n is a finite index clopen normal subgroup of G_n and (the natural action of G_n on) G_n/H_n is the universal minimal flow of G_n.*

Proof. Put $\boldsymbol{F}_n = \langle \mathbb{Q}, <, A_1, \dots, A_n \rangle$, so that $H_n = \mathrm{Aut}(\boldsymbol{F}_n)$. We will show that H_n is extremely amenable, from which it is straightforward to see that G_n/H_n is the universal minimal flow of G_n.

To prove that H_n is extremely amenable, let first $\mathcal{OP}_n$ denote the class of all finite structures of the form $\boldsymbol{A} = \langle A, <^{\boldsymbol{A}}, P_1^{\boldsymbol{A}}, \dots, P_n^{\boldsymbol{A}}\rangle$, with $<^{\boldsymbol{A}}$ a linear ordering of A and $\{P_i^{\boldsymbol{A}}\}_{i=1}^n$ a partition of A into disjoint sets. Then it is easy to verify that $\mathcal{OP}_n$ is a Fraïssé order class and $\boldsymbol{F}_n$ is its Fraïssé limit. So, by 6.1, it is enough to verify that $\mathcal{OP}_n$ has the Ramsey property.

Fix $\boldsymbol{A} \subseteq \boldsymbol{B}$ in $\mathcal{OP}_n$. We need to find $\boldsymbol{C} \in \mathcal{OP}_n$ such that $\boldsymbol{B} \leq \boldsymbol{C}$ and $\boldsymbol{C} \to (\boldsymbol{B})_2^{\boldsymbol{A}}$. Choose a partition $\mathbb{N} = N_1 \cup \dots \cup N_n$ into n pairwise disjoint infinite subsets. Let $\boldsymbol{C}_{\mathbb{N}} = \langle \mathbb{N}, <, N_1, \cdots, N_n\rangle$, where $<$ is the usual ordering of $\mathbb{N}$. As in 4.5, (i)⇒(ii), it suffices to show that $\boldsymbol{C}_{\mathbb{N}} \to (\boldsymbol{B})_2^{\boldsymbol{A}}$. For each $1 \leq i \leq n$, let $\mathcal{U}_i$ be a fixed nonprincipal ultrafilter on N_i. We use this to define another sequence $\vec{\mathcal{V}} = (\mathcal{V}_l : l = 0, 1, \cdots, k-1)$ of ultrafilters, as follows: let $\mathcal{V}_l = \mathcal{U}_i$, where $i \in \{1, \cdots, n\}$ is determined by $a_l \in P_i^A$ with $a_0, a_1, \cdots, a_{k-1}$ the $<^{\boldsymbol{A}}$ −increasing enumeration of the universe of $\boldsymbol{A}$. A $\vec{\mathcal{V}}$*−tree* is a nonempty subset of $T \subseteq \mathbb{N}^{[\leq k]}$, the set of increasing sequences from $\mathbb{N}$ of length $\leq k$, closed under the restrictions $t \to t|l$, such that, if $|t|$ = length of t,

$$A_t = \{m \in \mathbb{N} : t^\wedge m \in T\} \in \mathcal{V}_{|t|}, \text{ for all } t \in T \text{ of length } < k.$$

Note that a maximal node (i.e., a node of length k) of any $\vec{\mathcal{V}}$−tree naturally determines a copy of $\boldsymbol{A}$ inside $\boldsymbol{C}_{\mathbb{N}}$. Note also that the family of all $\vec{\mathcal{V}}$−trees forms a base for an ultrafilter of subsets of $\binom{\boldsymbol{C}_{\mathbb{N}}}{\boldsymbol{A}}$. So, given a coloring $c : \binom{\boldsymbol{C}_{\mathbb{N}}}{\boldsymbol{A}} \to \{1, 2\}$, one can find a $\vec{\mathcal{V}}$−tree T such that c is constant on the set $T^{\max} = T \cap \mathbb{N}^k$ of maximal nodes of T. Assume $\boldsymbol{B} = \langle\{0, 1, \cdots, m-1\}, <, P_1^{\boldsymbol{B}}, \cdots, P_n^{\boldsymbol{B}}\rangle$, where $<$ is the natural ordering of $\{0, 1, \cdots, m-1\}$. Recursively on $0 \leq j < m$ we construct a strictly increasing sequence l_j $(0 \leq j < m)$ of non-negative integers such that for all $0 \leq j < m$:

$$(a)\forall i \in \{1, \cdots, n\}(j \in P_i^{\boldsymbol{B}} \Leftrightarrow l_j \in N_i),$$

$$(b)\forall t \in T(\text{range}(t) \subseteq \{l_p : p < j\} \text{ and } j \in P_i^{\boldsymbol{B}} \text{ and } a_{|t|} \in P_i^{\boldsymbol{A}} \Rightarrow l_j \in A_t).$$

Then $\{l_j : 0 \leq j < m\}$ forms a substructure $\boldsymbol{B}'$ of $\boldsymbol{C}_{\mathbb{N}}$ isomorphic to $\boldsymbol{B}$ (via $j \mapsto l_j$) such that every $\boldsymbol{A}' \in \binom{\boldsymbol{B}'}{\boldsymbol{A}}$ is determined by a maximal node of T. So c is constant on $\binom{\boldsymbol{B}'}{\boldsymbol{A}}$, as required. ⊣

One can similarly see that the universal minimal flow of the automorphism group of the rationals with the usual ordering and an equivalene relation with infinitely many classes each of which is dense (see Theorem 6.4) is its action on the space of linear orderings on the set of equivalence classes.

9. A uniqueness result about the ordering property

We now return to the context of Theorem 7.5, in order to exploit another basic fact of topological dynamics, namely the uniqueness of the universal minimal flow (see 1.1).

Recall from Section 5 that if $L_0 \subseteq L$ are signatures and $\boldsymbol{A}$ is a structure for L, then $\boldsymbol{A}_0 = \boldsymbol{A}|L_0$ is its reduct to L_0. In this case we also call $\boldsymbol{A}$ an *expansion* of $\boldsymbol{A}_0$ to L. Similarly, if $\mathcal{K}$ is a class of structures in L and $\mathcal{K}_0 = \mathcal{K}|L_0$, then we call $\mathcal{K}$ an *expansion* of $\mathcal{K}_0$ to L.

Let $\mathcal{K}_0$ be a Fraïssé class of structures in a signature L_0. Theorem 7.5 shows that if $\mathcal{K}_0$ admits a reasonable Fraïssé order expansion $\mathcal{K}$ in $L = L_0 \cup \{<\}$, then $X_{\mathcal{K}}$ is the univeral minimal flow of G_0 =Aut(Flim($\mathcal{K}_0$)), provided $\mathcal{K}$ satisfies the Ramsey and ordering properties. Now, by the uniqueness of the universal minimal flow, if $\mathcal{K}'$ is another class with the same properties, then $X_{\mathcal{K}}$ and $X_{\mathcal{K}'}$ are isomorphic (as G_0-flows), which might suggest that $\mathcal{K}, \mathcal{K}'$ are the "same" in some sense. In other words, one concludes that among reasonable Fraïssé order classes that expand $\mathcal{K}_0$, there is "at most one" that satisfies both the Ramsey and ordering properties. As we will see below, there may be quite distinct expansions that satisfy just the Ramsey property, so this illustrates an interesting feature of the ordering property.

We will first formulate a quite general uniqueness result and then consider special cases in which it can be strengthened. We will need to introduce some concepts first.

Given a signature L, a *simple formula* in L is a quantifier-free formula in the infinitary language $L_{\omega_1\omega}$. Explicitly, this means that a simple formula is obtained from the atomic formulas of L by using only negations, countable conjunctions and disjunctions. In case we allow only negation and *finite* conjunctions and disjunctions, we call this a *first-order simple formula.* Consider now a Fraïssé class $\mathcal{K}_0$ in a signature L_0 and let $L = L_0 \cup \{<\}$. If $\mathcal{K}, \mathcal{K}'$ are reasonable Fraïssé order classes in L which are expansions of $\mathcal{K}$, (i.e., $\mathcal{K}_0 = \mathcal{K}|L_0 = \mathcal{K}'|L_0$), then we call $\mathcal{K}, \mathcal{K}'$ *simply bi-definable* if there are simple formulas $\varphi(x, y), \varphi'(x, y)$ in L, each with two variables, such that for any given $\boldsymbol{A}_0 \in \mathcal{K}_0$, φ and φ' define (uniformly) a bijection between the expansions of $\boldsymbol{A}_0$ in the signature L that are in $\mathcal{K}$ with those that are in $\mathcal{K}'$. More precisely, this means the following: Consider any $\boldsymbol{A}_0 \in \mathcal{K}_0$. If $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec\rangle$ is an expansion of $\boldsymbol{A}_0$ which is in $\mathcal{K}$, let $\prec'$ be the relation on A_0 defined by φ over $\boldsymbol{A}$, i.e.,

$$a \prec' b \Leftrightarrow \boldsymbol{A} \models \varphi[a, b].$$

Put

$$\Phi(\boldsymbol{A}) = \langle \boldsymbol{A}_0, \prec'\rangle.$$

Similarly for any $\boldsymbol{A}' \in \mathcal{K}'$ define $\Phi'(\boldsymbol{A}')$ using φ'. Then the above condition means that, for each $\boldsymbol{A}_0, \Phi$ is a bijection between the expansions of $\boldsymbol{A}_0$ in $\mathcal{K}$ with the expansions of $\boldsymbol{A}_0$ in $\mathcal{K}'$, with inverse Φ'.

The following is easy to verify:

Proposition 9.1. *Let $\mathcal{K}_0$ be a Fraïssé class in a signature L_0, let $L = L_0 \cup \{<\}$ and let $\mathcal{K}, \mathcal{K}'$ be reasonable Fraïssé order classes in L that are expansions of $\mathcal{K}_0$. Then, if $\mathcal{K}, \mathcal{K}'$ are simply bi-definable:*

(i) $\mathcal{K}$ satisfies the Ramsey property iff $\mathcal{K}'$ satisfies the Ramsey property.

(ii) $\mathcal{K}$ satisfies the ordering property iff $\mathcal{K}'$ satisfies the ordering property. ⊣

We can now state the main uniqueness result.

Theorem 9.2. *Let $\mathcal{K}_0$ be a Fraïssé class in a signature L_0, let $L = L_0 \cup \{<\}$, and let $\mathcal{K}, \mathcal{K}'$ be reasonable Fraïssé order classes in L that are expansions of $\mathcal{K}_0$. If $\mathcal{K}, \mathcal{K}'$ satisfy the Ramsey and ordering properties, then $\mathcal{K}, \mathcal{K}'$ are simply bi-definable.*

Proof. Let $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0)$, and $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K}) = \langle \boldsymbol{F}_0, \prec_0 \rangle$, $\boldsymbol{F}' = \mathrm{Flim}(\mathcal{K}') = \langle \boldsymbol{F}_0, \prec_0' \rangle$. Then, by 7.5, both $X_{\mathcal{K}}$ and $X_{\mathcal{K}'}$ are universal minimal flows of $G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$, so there are isomorphic (as G_0-flows), i.e., there is a homeomorphism $\pi : X_{\mathcal{K}} \to X_{\mathcal{K}'}$ such that $\pi(g\cdot \prec) = g \cdot \pi(\prec)$, for $\prec \in X_{\mathcal{K}}$.

Claim. $G_0 \cdot \prec_0$ is a dense G_δ set in $X_{\mathcal{K}}$ (and similarly for $G_0 \cdot \prec_0'$ in $X_{\mathcal{K}'}$).

Proof. By definition, $G_0 \cdot \prec_0$ is dense in $X_{\mathcal{K}}$.

Assuming, without loss of generality, that $F_0 = \mathbb{N}$, we will show that $G_0 \cdot \prec_0$ is G_δ in $X_{\mathcal{K}}$. Note that $\prec \in G_0 \cdot \prec_0$ iff $\langle \boldsymbol{F}_0, \prec \rangle \cong \boldsymbol{F}$ iff $\langle \boldsymbol{F}_0, \prec \rangle$ has age $\mathcal{K}$ and satisfies the extension property 2.3. It is now easy to verify that these properties can be expressed in a G_δ way. ⊣

Thus $G_0 \cdot \prec_0$ is the unique dense G_δ orbit of $X_{\mathcal{K}}$ and similarly $G_0 \cdot \prec_0'$ is the unique dense G_δ orbit of $X_{\mathcal{K}'}$. It follows that $\pi(G_0 \cdot \prec_0) = G_0 \cdot \prec_0'$. Put $\pi(\prec_0) = \prec_0^*$. Then $\boldsymbol{F}^* = \langle \boldsymbol{F}_0, \prec_0^* \rangle$ is also (isomorphic to) the Fraïssé limit of $\mathcal{K}'$ and since $\mathrm{Aut}(\boldsymbol{F})$ is the stabilizer of $\prec_0$ in the action of G_0 on $X_{\mathcal{K}}$, and similarly for $\mathrm{Aut}(\boldsymbol{F}^*)$ and $\prec_0^*$, it follows that $\mathrm{Aut}(\boldsymbol{F}) = \mathrm{Aut}(\boldsymbol{F}^*)$.

Thus (at the cost of replacing $\boldsymbol{F}'$ by its isomorphic copy $\boldsymbol{F}^*$) we may as well assume that

$$\mathrm{Aut}(\boldsymbol{F}) = \mathrm{Aut}(\boldsymbol{F}').$$

Consider now the action of S_∞ (and its subgroups) on $\mathbb{N}^n, n = 1, 2, \ldots$:

$$g \cdot (a_1, \ldots, a_n) = (g(a_1), \ldots, g(a_n)).$$

Claim. If $R \subseteq \mathbb{N}^n$ is $\mathrm{Aut}(\boldsymbol{F})$-invariant, then R is definable in $\boldsymbol{F}$ by a simple formula φ_R in L.

Proof. Write $R = \bigcup_{i \in I} R_i$, where R_i are the $\mathrm{Aut}(\boldsymbol{F})$-orbits on $\mathbb{N}^n$ contained in R. (Here I is a countable index set.) For each $i \in I$, fix $(a_1, \ldots, a_n) \in R_i$ and let

$$\varphi_i(x_1, \ldots, x_n)$$

be the simple formula in L which is the conjunction of all atomic or negations of atomic formulas $\psi(x_1, \ldots, x_n)$ in L such that

$$\boldsymbol{F} \models \psi[a_1, \ldots, a_n].$$

Then it is easy to see (using that $\boldsymbol{F}$ is ultrahomogeneous) that

$$(b_1, \dots, b_n) \in R_i \Leftrightarrow \boldsymbol{F} \models \varphi_i[b_1, \dots, b_n].$$

(Note here that if L is a finite relational language, then actually φ_i is first-order simple.)

Take then φ_R to be the disjunction of $\varphi_i, i \in I$. ⊣

Now the relation $\prec_0' \subseteq \mathbb{N}^2$ is invariant under $\mathrm{Aut}(\boldsymbol{F}') = \mathrm{Aut}(\boldsymbol{F})$, so there is a simple formula in L, φ, such that

$$a \prec_0' b \Leftrightarrow \boldsymbol{F} \models \varphi[a, b].$$

Similarly, there is a simple formula φ' that defines $\prec_0$ in $\boldsymbol{F}'$. It is easy now to see that φ, φ' witness that $\mathcal{K}, \mathcal{K}'$ are simply bi-definable. ⊣

Remark. Note that the preceding proof shows that if L_0 is finite and relational, then $\mathcal{K}, \mathcal{K}'$ are first-order simply bi-definable.

In certain special instances, we can actually assert in 9.2 that $\mathcal{K} = \mathcal{K}'$.

Proposition 9.3. *In the context of 9.2, if $\mathcal{K}, \mathcal{K}'$ are comparable under inclusion, i.e., $\mathcal{K} \subseteq \mathcal{K}'$ or $\mathcal{K}' \subseteq \mathcal{K}$, then $\mathcal{K} = \mathcal{K}'$. In particular, if $\mathcal{K} = \mathcal{K}_0 * \mathcal{LO}$, then $\mathcal{K} = \mathcal{K}'$.*

Proof. Say $\mathcal{K} \subseteq \mathcal{K}'$, but $\mathcal{K}' \setminus \mathcal{K} \neq \emptyset$, so that there is $\langle \boldsymbol{A}_0, \prec \rangle \in \mathcal{K}' \setminus \mathcal{K}$. Now apply the ordering property to $\boldsymbol{A}_0 \in \mathcal{K}'|L_0$ to find $\boldsymbol{B}_0 \in \mathcal{K}'|L_0 = \mathcal{K}|L_0$ and consider $\prec$ on A_0 and $\prec'$ on B_0, so that $\langle \boldsymbol{B}_0, \prec' \rangle \in \mathcal{K}$. Then $\langle \boldsymbol{A}_0, \prec \rangle \leq \langle \boldsymbol{B}_0, \prec' \rangle$, so, as $\mathcal{K}$ is hereditary, $\langle \boldsymbol{A}_0, \prec \rangle \in \mathcal{K}$, a contradiction. ⊣

We see from 9.3 that in all the classes considered in 8.1, i.e., $\mathcal{K}_0 =$ (finite) graphs, K_n-free graphs, complete graphs, hypergraphs of type L_0 that omit $\mathcal{A}$, and metric spaces with rational distances, $\mathcal{K} = \mathcal{K}_0 * \mathcal{LO}$ is the unique (reasonable order Fraïssé) expansion that satisfies both the Ramsey and ordering properties.

Note that 9.3 is also a trivial corollary of 9.2, since simple bi-definability implies, in particular, that the cardinality of the expansions of any $\boldsymbol{A}_0 \in \mathcal{K}|L_0 = \mathcal{K}'|L_0$ which are in $\mathcal{K}$ is the same as that of the expansions which are in $\mathcal{K}'$.

By these simple cardinality considerations, we can also see that 9.2 is not true if one of $\mathcal{K}, \mathcal{K}'$ fails to satisfy the ordering property. Consider, for example, the case when $\mathcal{K}_0$ is the class of all $\boldsymbol{A}_0 = \langle A_0, P, Q \rangle$, where $\{P, Q\}$ is a partition of A_0, $\mathcal{K}$ is the class of all $\boldsymbol{A} = \langle \boldsymbol{A}_0, < \rangle$, where $<$ is an ordering of A_0 with $P < Q$, and $\mathcal{K}' = \mathcal{K}_0 * \mathcal{LO}$. Then $\mathcal{K}$ satisfies the Ramsey and ordering properties, $\mathcal{K}'$ satisfies the Ramsey but not the ordering property, and $\mathcal{K}, \mathcal{K}'$ are not simply bi-definable.

In case one of $\mathcal{K}, \mathcal{K}'$ might not be equal to $\mathcal{K}_0 * \mathcal{LO}$, we do not necessarily have the strong uniqueness property of 9.3. But under certain conditions on $\mathcal{K}$ we can still strengthen 9.2. To motivate what we are looking for, let for each $\mathcal{K}$, in the context of 9.2,

$$\mathcal{K}^* = \{\langle \boldsymbol{A}_0, \prec^* \rangle : \langle A_0, \prec \rangle \in \mathcal{K}\},$$

where

$$a \prec^* b \Leftrightarrow a \succ b$$

is the reverse ordering of $\prec$. Clearly $\mathcal{K}^*$ is simply bi-definable with $\mathcal{K}$, so $\mathcal{K}$ satisfies the Ramsey (resp., ordering) property iff $\mathcal{K}^*$ does. We will formulate now a condition on $\mathcal{K}$ which implies that the only $\mathcal{K}'$ simply bi-definable with $\mathcal{K}$ are $\mathcal{K}$ and $\mathcal{K}^*$.

Given $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec\rangle \in \mathcal{K}, \boldsymbol{B} = \langle \boldsymbol{B}_0, \prec'\rangle \in \mathcal{K}$ and $a, b \in A_0, c, d \in B_0$ with $a \prec b, c \prec d$, we say that $(a, b), (c, d)$ *have the same type if* there is an isomorphism of the substructure of $\boldsymbol{A}$ generated by a, b with the substructure of $\boldsymbol{B}$ generated by c, d which sends a to c and b to d. Equivalently, this means that $(a, b), (c, d)$ satisfy (in $\boldsymbol{A}, \boldsymbol{B}$ resp.) exactly the same atomic formulas in L. We denote by $tp_{\boldsymbol{A}}(a, b)$ the *type of* (a, b), i.e., the set of all atomic formulas satisfied by (a, b) in $\boldsymbol{A}$. Thus $(a, b), (c, d)$ have the same type iff $tp_{\boldsymbol{A}}(a, b) = tp_{\boldsymbol{B}}(c, d)$.

We now say that $\mathcal{K}$ satisfies the *triangle condition* if for any two distinct types $\sigma \neq \tau$, there is $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec\rangle \in \mathcal{K}$ and $a \prec b \prec c$ in A_0 such that

$$tp_{\boldsymbol{A}}(a, b) = tp_{\boldsymbol{A}}(b, c) = \sigma, tp_{\boldsymbol{A}}(a, c) = \tau$$

or

$$tp_{\boldsymbol{A}}(a, b) = tp_{\boldsymbol{A}}(b, c) = \tau, tp_{\boldsymbol{A}}(a, c) = \sigma$$

(i.e., two "sides" of the triangle a, b, c have one of types σ, τ and the third one the other).

Corollary 9.4. *In the context of 9.2, if $\mathcal{K}$ satisfies the triangle condition, then $\mathcal{K}' = \mathcal{K}$ or $\mathcal{K}' = \mathcal{K}^*$.*

Proof. In the notation of the proof of 9.2, it is enough to show that $\prec_0' = \prec_0$ or $\prec_0' = \prec_0^*$. If this fails, then we can find a, b, c, d with $a \prec_0 b, c \prec_0 d, a \prec_0' b, d \prec_0' c$. Denote by σ the type of (a, b) (in the substructure of $\boldsymbol{F}$ generated by a, b) and by τ the type of (c, d). Then $\sigma \neq \tau$, since $\prec_0'$ is $\mathrm{Aut}(\boldsymbol{F})$-invariant. By the triangle condition, there is $\boldsymbol{A} \in \mathcal{K}$, which without loss of generality we can assume to be a substructure of $\boldsymbol{F}$, and $\bar{a}, \bar{b}, \bar{c}$ in A such that $\bar{a} \prec_0 \bar{b} \prec_0 \bar{c}$ and either $tp_{\boldsymbol{A}}(\bar{a}, \bar{b}) = tp_{\boldsymbol{A}}(\bar{b}, \bar{c}) = \sigma,\ tp_{\boldsymbol{A}}(\bar{a}, \bar{c}) = \tau$, in which case we have that $\bar{a} \prec_0' \bar{b} \prec_0' \bar{c}$ but $\bar{c} \prec_0' \bar{a}$, a contradiction, or $tp_{\boldsymbol{A}}(\bar{a}, \bar{b}) = tp_{\boldsymbol{A}}(\bar{b}, \bar{c}) = \tau,\ tp_{\boldsymbol{A}}(\bar{a}, \bar{c}) = \sigma$ in which case we have that $\bar{c} \prec_0' \bar{b} \prec_0' \bar{a}$ but $\bar{a} \prec_0' \bar{c}$, also a contradiction. ⊣

As an application, let us show, for example, that the class $\mathcal{K}$ of all convexly ordered finite equivalence relations is the only (reasonable order Fraïssé) expansion of the class $\mathcal{EQ}$ (of finite equivalence relations) that satisfies both the Ramsey and ordering properties. Indeed, if $\mathcal{K}'$ is any other such class then either $\mathcal{K}' \subseteq \mathcal{K}$, and then we are done by 9.3, or else $\mathcal{K}'$ contains a structure $\boldsymbol{A}' = \langle A_0', E', \prec'\rangle$, where $\prec'$ is such that there are $a, b, c \in A_0'$ with $(a, c) \in E', (a, b) \notin E', (b, c) \notin E'$ and $a \prec' b \prec' c$. Then we can use $\boldsymbol{A}'$ to witness the triangle condition for $\mathcal{K}'$, so $\mathcal{K} = \mathcal{K}'$ or $\mathcal{K} = (\mathcal{K}')^*$, by 9.4. Then $\mathcal{K}' = \mathcal{K}$ or $\mathcal{K}' = (\mathcal{K})^* = \mathcal{K}$ and we are done. (Note that $\mathcal{K}$ itself does not satisfy the triangle condition).

Another interesting example is the following. Let $\mathcal{K}_0$ = the class of finite posets and let $\mathcal{K}$ = the class of all finite $\langle A, \sqsubset, \prec \rangle$ where $\langle A, \sqsubset \rangle$ is a poset and $\prec$ is a linear ordering extending $\sqsubset$. Then (see "Addendum" at the end of this paper) $\mathcal{K}$ satisfies the Ramsey and ordering properties. Now it is easy to see that $\mathcal{K}$ also satisfies the triangle condition, so the only expansions that have both of these properties are $\mathcal{K}$ and $\mathcal{K}^*$.

Finally, let us point out that in other cases, again in the context of 9.2, there may be many $\mathcal{K}'$ bi-definable with $\mathcal{K}$. For example, in case $\mathcal{K}_0$ = all finite-dimensional vector spaces over a finite field F, we can see, by Section 6, ($\boldsymbol{C}$), that there are different $\mathcal{K}'$ corresponding to different orderings of F. In such cases one seeks to classify all expansions of a given $\mathcal{K}_0$ that satisfy the Ramsey and ordering properties. Of particular interest are, of course, the cases of finite-dimensional vector spaces and Boolean algebras.

For the case of Boolean algebras, we can show that there are exactly 12 reasonable order Fraïssé expansions of the class $\mathcal{K}_0 = \mathcal{BA}$ of finite Boolean algebras. These are defined as follows:

Let $\pi_1, \pi_2, \ldots, \pi_6$ be the six permutations of the symbols $0, 1, a$. For each such π_i, we define an expansion $\mathcal{K}_i$ of $\mathcal{K}_0$ as follows: Say, for example, $\pi_i = a, 1, 0$. Then $\langle \boldsymbol{A}_0, \prec \rangle \in \mathcal{K}_i$ iff $\boldsymbol{A}_0$ is a finite Boolean algebra, $\prec$ a linear ordering of A_0, and there is a natural ordering $\prec'$ on A_0 such that $\prec, \prec'$ agree on $A_0 \setminus \{0^{\boldsymbol{A}_0}, 1^{\boldsymbol{A}_0}\}$ but $a_0 \prec 1^{\boldsymbol{A}_0} \prec 0^{\boldsymbol{A}_0}$, for all $a_0 \in A_0 \setminus \{0^{\boldsymbol{A}_0}, 1^{\boldsymbol{A}_0}\}$. (Note of course that $0^{\boldsymbol{A}_0} \prec' a_0 \prec' 1^{\boldsymbol{A}_0}$, for all $a_0 \in A_0 \setminus \{0^{\boldsymbol{A}_0}, 1^{\boldsymbol{A}_0}\}$.) Clearly $\mathcal{K}_i$ is simply bi-definable with $\mathcal{OBA}$ so each $\mathcal{K}_i$ is a reasonable order Fraïssé expansion of $\mathcal{K}_0$ satisfying the Ramsey and ordering properties. (Note that if, e.g., $\pi_1 = 0, 1, a$, we have $\mathcal{K}_1 = \mathcal{OBA}$.) Thus the 12 classes $\mathcal{K}_i, \mathcal{K}_i^*, i = 1, \ldots, 6$, are distinct and satisfy both the Ramsey and ordering properties. Then a canonization argument based on the Dual Ramsey Theorem will give us the following result which is then used to show that every reasonable Fraïssé expansion of $\mathcal{K}_0$ that satisfies the Ramsey and ordering properties must be in this list:

Given a finite Boolean algebra $\boldsymbol{A}$, there is a finite Boolean algebra $\boldsymbol{B}$, with $\boldsymbol{A} \leq \boldsymbol{B}$, such that if $\prec$ is any linear ordering on $\boldsymbol{B}$, extending the partial ordering of $\boldsymbol{B}$, then there is $\boldsymbol{A}' \in \binom{\boldsymbol{B}}{\boldsymbol{A}}$ such that $\prec |A'$ is natural.

This order canonization theorem should be compared with those of Nešetřil-Prömel-Rödl-Voigt [55] (see also Prömel [70]) that deal with canonizing orderings of Boolean lattices rather than Boolean algebras.

Finally, we can also classify all reasonable order Fraïssé expansions of the class $\mathcal{K}_0 = \mathcal{V}_F$ of all finite-dimensional vector spaces over a fixed finite field F. It turns out that if $\mathrm{card}(F) = q$, then there are exactly $4(q-2)!$ many such classes described as follows:

Let F be a finite field of cardinality q, $F^* = F \setminus \{0\}$ its multiplicative group of non-0 elements. For each ordering $<$ on F, where $0 < F^*$, let $<'$ be the ordering of F such that $<' |F^* = < |F^*$ but $F^* <' 0$. (Here 0 is the "zero" of the field F).

For each such $<$, let $\mathcal{K}_1(<)$ be the class of all $\langle \boldsymbol{V}_0, \prec\rangle$, where $\boldsymbol{V}_0$ is a finite-dimensional vector space over F, and $\prec$ is an anti-lexicographical ordering on V_0 induced from an ordered basis of V_0, by using $<$ for ordering F. If we denote by $\mathbf{0}$ the "zero" of V_0, clearly $\mathbf{0} \prec V_0 \setminus \{\mathbf{0}\}$. Let also $\mathcal{K}_2(<)$ be the class of all $\langle \boldsymbol{V}_0, \prec'\rangle$ such that for some $\prec$ with $\langle \boldsymbol{V}_0, \prec\rangle \in \mathcal{K}_1(<)$, $\prec |V_0 \setminus \{\mathbf{0}\} = \prec' |V_0 \setminus \{\mathbf{0}\}$ but $V_0 \setminus \{\mathbf{0}\} \prec' \mathbf{0}$. Then $\mathcal{K}_1(\prec), \mathcal{K}_2(\prec)$ are simply bi-definable, so they have the Ramsey and ordering properties. For each $<'$ as before, define $\mathcal{L}_1(<')$ to be the class of all $\langle \boldsymbol{V}_0, \prec\rangle$ such that $\prec$ is an anti-lexicographical ordering of V_0 induced from an ordered basis of V_0 by using $<'$ for ordering F. Note now that $V_0 \setminus \{\mathbf{0}\} \prec \mathbf{0}$. Define, similarly to the above, $\mathcal{L}_2(\prec')$. Then

$$\mathcal{L}_1(<')^* = \mathcal{K}_1((<')^*), \mathcal{L}_2(<')^* = \mathcal{K}_2((<')^*),$$

and so $\mathcal{L}_1(<'), \mathcal{L}_2(<')$ also have the Ramsey and ordering properties. Now using an order canonization theorem based on the Ramsey theorem for vector spaces due to Graham-Leeb-Rothschild [32] we can show that every reasonable Fraïssé order expansion of $\mathcal{K}_0$ satisfying the Ramsey and ordering properties is one of

$$\mathcal{K}_1(<), \mathcal{K}_2(<), \mathcal{L}_1(<'), \mathcal{L}_2(<').$$

However these $4(q-1)!$ classes are not distinct.

Consider the action of F^* on F by multiplication and the induced action of F^* on the set of linear orderings on F with $0 < F^*$. This action is free, so there are exactly $\frac{(q-1)!}{(q-1)} = (q-2)!$ orbits. Let $\sim$ be the equivalence relation induced by this action. Then one can see that

$$\mathcal{K}_1(<_1) = \mathcal{K}_1(<_2)$$

iff $<_1 \sim <_2$, and so we get exactly $(q-2)!$ distinct classes of the form $\mathcal{K}_1(<)$. Similarly we get exactly $(q-2)!$ classes of each of the forms $\mathcal{K}_2(<), \mathcal{L}_1(<'), \mathcal{L}_2(<')$. One can finally show that the four collections

$$\{\mathcal{K}_1(<)\}, \{\mathcal{K}_2(<)\}, \{\mathcal{L}_1(<')\}, \{\mathcal{L}_2(<')\}$$

are pairwise disjoint, so we get exactly $4(q-2)!$ many distinct classes.

In connection with these classification problems, it is worth pointing out that, in the context of 9.2 again, when the signature L_0 is relational and finite, there are only finitely many reasonable Fraïssé order classes that expand a given class in L_0 and satisfy both the Ramsey and ordering properties. In fact, more generally, using the concept of type introduced before 9.4, it is easy to see that this is even true when there are only finitely many types. Indeed, in the notation of the proof of 9.2, note that $\prec_0'$ is completely determined (up to an action by an element of $\mathrm{Aut}(\boldsymbol{F}_0)$) by the set

$$T(\prec_0') = \{tp_{\boldsymbol{F}}(a, b) : a \prec_0' b\},$$

where

$$\boldsymbol{F} = \langle \boldsymbol{F}_0, \prec_0\rangle$$

(i.e., the set of all types of pairs on which $\prec'_0$ agrees with $\prec_0$). But there are only finitely many possibilities for $T(\prec'_0)$, so there are only finitely many possibilities for $\boldsymbol{F}' = \langle \boldsymbol{F}_0, \prec'_0\rangle$ and thus for $\mathcal{K}' = \mathrm{Age}(\boldsymbol{F}')$.

We finally show that among all possible Ramsey expansions $\mathcal{K}$ of a Fraïssé class $\mathcal{K}_0$, the ones that also have the ordering property, satisfy an important minimality property.

Theorem 9.5. *Let $\mathcal{K}_0$ be a Fraïssé class in a signature L_0, let $L = L_0 \cup \{<\}$ and let $\mathcal{K}, \mathcal{K}'$ be reasonable Fraïssé order classes in L that are expansions of $\mathcal{K}_0$. If $\mathcal{K}$ satisfies the Ramsey property and $\mathcal{K}'$ satisfies both the Ramsey and ordering properties, then $\mathcal{K}' \subseteq \mathcal{K}$ up to simple bi-definability, i.e., there is a reasonable Fraïssé order class $\mathcal{K}''$ in L which is an expansion of $\mathcal{K}_0$ such that $\mathcal{K}'' \subseteq \mathcal{K}$ and $\mathcal{K}''$ is simply bi-definable with $\mathcal{K}'$.*

Proof. Using the notation of the proof of 9.2, we have, by 7.5, that the G_0-space $X_{\mathcal{K}'}$ is isomorphic to a minimal subflow X of $X_{\mathcal{K}}$. Let $\pi : X_{\mathcal{K}'} \to X$ be an isomorphism and let $\pi(\prec'_0) = \prec''_0 \in X$. Then

$$\mathrm{Aut}(\langle \boldsymbol{F}_0, \prec'_0\rangle) = \mathrm{Aut}(\langle \boldsymbol{F}_0, \prec''_0\rangle),$$

so that there is a simple formula φ'' that defines $\prec''_0$ in $\langle \boldsymbol{F}_0, \prec'_0\rangle$ and a simple formula φ' that defines $\prec'_0$ in $\langle \boldsymbol{F}_0, \prec''_0\rangle$. From this it easily follows that $\langle \boldsymbol{F}_0, \prec''_0\rangle$ is ultrahomogeneous. Let $\mathcal{K}'' = \mathrm{Age}(\langle \boldsymbol{F}_0, \prec''_0\rangle)$. Then $\mathcal{K}'' \subseteq \mathcal{K}$ is a reasonable Fraïssé order class in L, which is an expansion of $\mathcal{K}_0$. As in the proof of 9.2, it is simply bi-definable with $\mathcal{K}'$. ⊣

Thus the expansions of a given $\mathcal{K}_0$ that have both the Ramsey and ordering properties are the smallest, up to simple bi-definability, expansions that have the Ramsey property. The preceding argument also suggests an approach to showing that, if there is an expansion $\mathcal{K}$ with the Ramsey property, then there is one with both Ramsey and ordering properties: Simply pick $\prec' \in X_{\mathcal{K}}$ such that $\overline{G_0 \cdot \prec'}$ is minimal and let $\mathcal{K}' = \mathrm{Age}(\langle \boldsymbol{F}_0, \prec'\rangle)$. Then try to show that $\mathcal{K}'$ has both the Ramsey and ordering properties. We will see in the next section that this approach indeed works.

10. Ramsey degrees

Consider the class of finite graphs $\mathcal{GR}$. Although this class does not satisfy the Ramsey property, there is still an important Ramsey-type result that holds for finite graphs: For each finite graph $\boldsymbol{A}_0 \in \mathcal{GR}$, there is a finite number t such that for all $\boldsymbol{A}_0 \leq \boldsymbol{B}_0 \in \mathcal{GR}$, there is $\boldsymbol{C}_0 \geq \boldsymbol{B}_0$ in $\mathcal{GR}$ with the following property: For any coloring $c : \binom{\boldsymbol{C}_0}{\boldsymbol{A}_0} \to \{1, \ldots, k\}$ (of any number of colors), there is $\boldsymbol{B}'_0 \in \binom{\boldsymbol{C}_0}{\boldsymbol{B}_0}$ such that c takes at most t colors on $\binom{\boldsymbol{B}'_0}{\boldsymbol{A}_0}$. Moreover, one can explicitly compute the least number t with that property.

Such results for graphs and other classes of finite structures have existed implicitly or explicitly, in the literature for some time now (see, e.g., Nešetřil-Rödl [56], [58], Abramson-Harrington [1], and Fouché [20], [21], [22], [23]). We will present here a version of this theory in the general context of our paper and notice some interesting connections with the results in Section 9 leading, in particular, to the proof of the result mentioned at the end of Sections 7 and 9.

Throughout this section, $\mathcal{K}_0$ *will be a Fraïssé class in a signature* L_0*, and* $\mathcal{K}$ *a hereditary order class in* $L = L_0 \cup \{<\}$ *with* $\mathcal{K}_0 = \mathcal{K}|L$*, i.e.,* $\mathcal{K}$ *is an expansion of* $\mathcal{K}_0$*.*

For $\boldsymbol{A}_0 \in \mathcal{K}_0$, let

$$X_{\mathcal{K}}^{\boldsymbol{A}_0} = \{\prec : \prec \text{ is a linear ordering on } A_0 \text{ and } \langle \boldsymbol{A}_0, \prec\rangle \in \mathcal{K}\}.$$

Thus $X_{\mathcal{K}}^{\boldsymbol{A}_0} \neq \emptyset$ by assumption and, since $\mathcal{K}$ is closed under isomorphism, $\mathrm{Aut}(\boldsymbol{A}_0)$ acts on $X_{\mathcal{K}}^{\boldsymbol{A}_0}$ in the obvious way. If $\prec \in X_{\mathcal{K}}^{\boldsymbol{A}_0}$, we call its orbit under this action the ($\boldsymbol{A}_0 - \mathcal{K}$–) *pattern* of $\prec$. Thus the pattern of $\prec$ is the set of all $\prec'$ such that $\langle \boldsymbol{A}_0, \prec'\rangle \cong \langle \boldsymbol{A}_0, \prec\rangle$. Put also

$$\begin{aligned} t_{\mathcal{K}}(\boldsymbol{A}_0) &= \text{the cardinality of the set of } \boldsymbol{A}_0 - \mathcal{K} - \text{patterns} \\ &= \mathrm{card}(X_{\mathcal{K}}^{\boldsymbol{A}_0}/\mathrm{Aut}(\boldsymbol{A}_0)) \\ &= \tfrac{\mathrm{card}(X_{\mathcal{K}}^{\boldsymbol{A}_0})}{\mathrm{card}(\mathrm{Aut}(\boldsymbol{A}_0))}, \end{aligned}$$

since clearly $\mathrm{Aut}(\boldsymbol{A}_0)$ acts freely on $X_{\mathcal{K}}^{\boldsymbol{A}_0}$, and thus every orbit has the same cardinality as $\mathrm{Aut}(\boldsymbol{A}_0)$.

Finally, for $\boldsymbol{A}_0 \leq \boldsymbol{B}_0 \leq \boldsymbol{C}_0$ in $\mathcal{K}_0$, let

$$\boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_{k,t}$$

mean that for every coloring $c : \binom{\boldsymbol{C}_0}{\boldsymbol{A}_0} \to \{1, \dots, k\}$ there is $\boldsymbol{B}'_0 \in \binom{\boldsymbol{C}_0}{\boldsymbol{B}_0}$ such that c on $\binom{\boldsymbol{B}'_0}{\boldsymbol{A}_0}$ takes at most t values. Thus

$$\boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_{k,1} \Leftrightarrow \boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_{k}.$$

Also for $\boldsymbol{A}_0 \leq \boldsymbol{B}_0 \in \mathcal{K}_0$, and $\langle \boldsymbol{B}_0, <_{\boldsymbol{B}_0}\rangle \in \mathcal{K}$, let

$$\begin{aligned} t_{\mathcal{K}}(\boldsymbol{A}_0, \boldsymbol{B}_0, <_{\boldsymbol{B}_0}) = \quad & \text{the number of } \boldsymbol{A}_0 - \mathcal{K} - \text{patterns of} \\ & \prec \in \boldsymbol{X}_{\mathcal{K}}^{\boldsymbol{A}_0} \text{ such that } \langle \boldsymbol{A}_0, \prec\rangle \leq \langle \boldsymbol{B}_0, <_{\boldsymbol{B}_0}\rangle. \end{aligned}$$

Clearly $t_{\mathcal{K}}(\boldsymbol{A}_0, \boldsymbol{B}_0, <_{B_0}) \leq t_{\mathcal{K}'}(\boldsymbol{A}_0)$, for any hereditary class $\mathcal{K}' \subseteq \mathcal{K}$ with $\mathcal{K}'|L_0 = \mathcal{K}_0, \langle \boldsymbol{B}_0, <_{\boldsymbol{B}_0}\rangle \in \mathcal{K}'$.

Proposition 10.1. *If* $\mathcal{K}$ *has the Ramsey property, then for* $\boldsymbol{A}_0 \leq \boldsymbol{B}_0$ *in* $\mathcal{K}_0$*, and* $\langle \boldsymbol{B}_0, <_{\boldsymbol{B}_0}\rangle \in \mathcal{K}, k \geq 2$*, there is* $\boldsymbol{C}_0 \geq \boldsymbol{B}_0$ *in* $\mathcal{K}_0$ *with*

$$\boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_{k, t_{\mathcal{K}}(\boldsymbol{A}_0, \boldsymbol{B}_0, <_{\boldsymbol{B}_0})},$$

and thus also

$$\boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_{k, t_{\mathcal{K}'}(\boldsymbol{A}_0)},$$

for any hereditary class $\mathcal{K}' \subseteq \mathcal{K}$ which is an expansion of $\mathcal{K}_0$.

Proof. Put $t = t_{\mathcal{K}}(\boldsymbol{A}_0, \boldsymbol{B}_0, <_{\boldsymbol{B}_0})$. Choose representatives $<_1, \dots, <_t$ for the $\boldsymbol{A}_0 - \mathcal{K}-$patterns realized by $\prec$ with $\langle \boldsymbol{A}_0, \prec \rangle \leq \langle \boldsymbol{B}_0, <_{\boldsymbol{B}_0} \rangle$. We will define inductively $\langle \boldsymbol{D}_i, <_{\boldsymbol{D}_i} \rangle \in \mathcal{K}, 0 \leq i \leq t$ as follows:

Let $\langle \boldsymbol{D}_0, <_{\boldsymbol{D}_0} \rangle = \langle \boldsymbol{B}_0, <_{\boldsymbol{B}_0} \rangle$. Then for $i \geq 1$, let $\langle \boldsymbol{D}_i, <_{\boldsymbol{D}_i} \rangle \in \mathcal{K}$ be such that

$$\langle \boldsymbol{D}_i, <_{\boldsymbol{D}_i} \rangle \to (\langle \boldsymbol{D}_{i-1}, <_{\boldsymbol{D}_{i-1}} \rangle)^{\langle \boldsymbol{A}_0, <_i \rangle}_k.$$

Take now $\boldsymbol{C}_0 = \boldsymbol{D}_t$. We claim that this works:

Let $c : \binom{\boldsymbol{C}_0}{A_0} \to \{1, \dots, k\}$ be a coloring. Then let

$$c_t : \binom{\langle \boldsymbol{C}_0, <_{\boldsymbol{D}_t} \rangle}{\langle \boldsymbol{A}_0, <_t \rangle} \to \{1, \dots, k\}$$

be defined by

$$c_t : (\langle \boldsymbol{A}_0', <' \rangle) = c(\boldsymbol{A}_0'),$$

where $\langle \boldsymbol{A}_0', <' \rangle = \langle \boldsymbol{A}_0', <_{\boldsymbol{D}_t} | A_0' \rangle \cong \langle \boldsymbol{A}_0, <_t \rangle$. There is a copy $\langle \boldsymbol{D}_{t-1}'$, $<_{\boldsymbol{D}_{t-1}'} \rangle$ of $\langle \boldsymbol{D}_{t-1}, <_{\boldsymbol{D}_{t-1}} \rangle$ in $\langle \boldsymbol{D}_t, <_{\boldsymbol{D}} \rangle$ and a color $1 \leq k_t \leq k$ such that the c_t-color of any copy of $\langle \boldsymbol{A}_0, <_t \rangle$ in $\langle \boldsymbol{D}_{t-1}', <_{\boldsymbol{D}_{t-1}'} \rangle$ is equal to k_t. Iterate now this process starting with $\langle \boldsymbol{D}_{t-1}', <_{\boldsymbol{D}_{t-1}'} \rangle$ and the coloring $c_{t-1} : \binom{\langle \boldsymbol{D}_{t-1}', <_{\boldsymbol{D}_{t-1}} \rangle}{\langle \boldsymbol{A}_0, <_{t-1} \rangle} \to \{1, \dots, k\}$, given by $c_{t-1}(\langle \boldsymbol{A}_0', <' \rangle) = c(\boldsymbol{A}_0')$, where

$$\langle \boldsymbol{A}_0', <' \rangle = \langle \boldsymbol{A}_0', <_{\boldsymbol{D}_{t-1}'} | A_0' \rangle \cong \langle \boldsymbol{A}_0, <_{t-1} \rangle,$$

to find a copy $\langle \boldsymbol{D}_{t-2}', <_{\boldsymbol{D}_{t-2}'} \rangle$ of $\langle \boldsymbol{D}_{t-2}, <_{\boldsymbol{D}_{t-2}} \rangle$ in $\langle \boldsymbol{D}_{t-1}', <_{\boldsymbol{D}_{t-1}'} \rangle$, and thus in the structure $\langle \boldsymbol{D}_t, <_{\boldsymbol{D}_t} \rangle$, and a color $1 \leq k_{t-1} \leq k$ such that the c_{t-1}-color of any copy of $\langle \boldsymbol{A}_0, <_{t-1} \rangle$ in $\langle \boldsymbol{D}_{t-2}', <_{\boldsymbol{D}_{t-2}'} \rangle$ is equal to k_{t-1}, etc. After t steps, we get a copy of $\langle \boldsymbol{D}_0, <_{\boldsymbol{D}_0} \rangle = \langle \boldsymbol{B}_0, <_{\boldsymbol{B}_0} \rangle$ in $\langle \boldsymbol{D}_1', <_{\boldsymbol{D}_1'} \rangle$ and a color $1 \leq k_1 \leq k$ that works for copies of $\langle \boldsymbol{A}_0, <_1 \rangle$. Call this copy $\langle \boldsymbol{B}_0', <_{\boldsymbol{B}_0'} \rangle$. Then clearly c on $\binom{\boldsymbol{B}_0'}{\boldsymbol{A}_0}$ takes at most the values $\{1, \dots, k_t\}$, because if $\boldsymbol{A}_0' \in \binom{\boldsymbol{B}_0'}{\boldsymbol{A}_0}$ and the pattern of $<_t | A_0' = <'_{\boldsymbol{B}_0} | A_0'$ is $<_i$, then

$$\langle \boldsymbol{A}_0', <_t | A_0' \rangle \in \binom{\langle \boldsymbol{D}_i', <_{\boldsymbol{D}_i'} \rangle}{\langle \boldsymbol{A}_0, <_i \rangle},$$

so $c_i(\langle \boldsymbol{A}_0', <' \rangle) = c(\boldsymbol{A}_0') = k_i$. ⊣

Actually the preceding proof also establishes the following.

Proposition 10.1′ *If $\mathcal{K}$ has the Ramsey property, then for $\boldsymbol{A}_0 \leq \boldsymbol{B}_0$ in $\mathcal{K}_0$, $k \geq 2$, there is $\boldsymbol{C}_0 \geq \boldsymbol{B}_0$ and a linear ordering $\prec$ on $\boldsymbol{C}_0$ with the property that $\langle \boldsymbol{C}_0, \prec \rangle \in \mathcal{K}$*

and for any coloring $c : \binom{\boldsymbol{C}_0}{\boldsymbol{A}_0} \to \{1, \ldots, k\}$*, there is* $\boldsymbol{B}'_0 \in \binom{\boldsymbol{C}_0}{\boldsymbol{B}_0}$ *so that the color* $c(\boldsymbol{A}'_0)$*, for* $\boldsymbol{A}'_0 \in \binom{\boldsymbol{B}'_0}{\boldsymbol{A}_0}$*, depends only on the pattern of* $\prec |A'_0$*.*

If now $\mathcal{K}$ has also the ordering property, the number $t_{\mathcal{K}}(\boldsymbol{A}_0)$ is best possible.

Proposition 10.2. *If* $\mathcal{K}$ *has the Ramsey and ordering properties, then for any* $\boldsymbol{A}_0 \in \mathcal{K}_0$*, there is* $\boldsymbol{B}_0 \geq \boldsymbol{A}_0$ *in* $\mathcal{K}_0$ *such that for all* $\boldsymbol{C}_0 \geq \boldsymbol{B}_0$ *in* $\mathcal{K}_0$ *there is a coloring* $c : \binom{\boldsymbol{C}_0}{\boldsymbol{A}_0} \to \{1, \ldots, t_{\mathcal{K}}(\boldsymbol{A}_0)\}$*, such that for any* $\boldsymbol{B}'_0 \in \binom{\boldsymbol{C}_0}{\boldsymbol{B}_0}$*,* c *takes all the values* $1, \ldots, t_{\mathcal{K}}(\boldsymbol{A}_0)$ *on* $\binom{\boldsymbol{B}'_0}{\boldsymbol{A}_0}$*.*

Proof. Let, by the ordering property, $\boldsymbol{B}_0 \in \mathcal{K}_0$ be such that for any ordering $\prec$ on A_0, and any ordering $\prec'$ on B_0 with $\langle \boldsymbol{A}_0, \prec\rangle, \langle \boldsymbol{B}_0, \prec'\rangle \in \mathcal{K}$ we have $\langle \boldsymbol{A}_0, \prec\rangle \leq \langle \boldsymbol{B}_0, \prec'\rangle$. Another way of saying this is that as $\boldsymbol{A}'_0$ varies over $\binom{\boldsymbol{B}_0}{\boldsymbol{A}_0}$, $\prec' |A_0$ realizes all possible patterns.

Let now $\boldsymbol{C}_0 \in \mathcal{K}_0, \boldsymbol{C}_0 \geq \boldsymbol{B}_0$, and $<_{\boldsymbol{C}_0}$, a linear ordering on C_0, be such that

$$\langle \boldsymbol{C}_0, <_{\boldsymbol{C}_0}\rangle \in \mathcal{K}.$$

Define $c : \binom{\boldsymbol{C}_0}{\boldsymbol{A}_0} \to \{1, \ldots, t_{\mathcal{K}}(\boldsymbol{A}_0\}$ by enumerating the set of patterns as $p_1, \ldots, p_{t_{\mathcal{K}}(\boldsymbol{A}_0)}$ and letting, for $\boldsymbol{A}'_0 \in \binom{\boldsymbol{C}_0}{\boldsymbol{A}_0}$, $c(\boldsymbol{A}'_0) = i$, where the pattern of $\langle \boldsymbol{A}'_0, <_{\boldsymbol{C}_0} |A'_0\rangle$ is p_i. Then, by the above, if $\boldsymbol{B}'_0 \in \binom{\boldsymbol{C}_0}{\boldsymbol{B}_0}$, clearly c takes all values $1, \ldots, t_{\mathcal{K}}(\boldsymbol{A}_0)$ on $\binom{\boldsymbol{B}'_0}{\boldsymbol{A}_0}$. ⊣

Corollary 10.3. *If* $\mathcal{K}$ *has the Ramsey and ordering properties, then for* $\boldsymbol{A}_0 \in \mathcal{K}_0, t_{\mathcal{K}}(\boldsymbol{A}_0)$ *is the least number* t *such that for any* $\boldsymbol{A}_0 \leq \boldsymbol{B}_0$ *in* $\mathcal{K}_0, k \geq 2$*, there is* $\boldsymbol{C}_0 \geq \boldsymbol{B}_0$ *in* $\mathcal{K}_0$ *with* $\boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_{k,t}$*.* ⊣

In particular, this shows that $t_{\mathcal{K}}(\boldsymbol{A}_0)$ and $\mathrm{card}(X^{\boldsymbol{A}_0}_{\mathcal{K}})$ are independent of $\mathcal{K}$, as long as $\mathcal{K}$ has the Ramsey and ordering properties. So one has a uniqueness property of expansions of $\mathcal{K}_0$ that have both the Ramsey and ordering properties. Theorem 9.2 provides a much stronger uniqueness property, which immediately implies this.

Also if $\mathcal{K}$ has the Ramsey property and $\mathcal{K}'$ has both the Ramsey and ordering properties, then $t_{\mathcal{K}'}(\boldsymbol{A}_0) \leq t_{\mathcal{K}}(\boldsymbol{A}_0)$ and $\mathrm{card}(X^{\boldsymbol{A}_0}_{\mathcal{K}'}) \leq \mathrm{card}(X^{\boldsymbol{A}_0}_{\mathcal{K}})$. Of course Theorem 9.5 provides a stronger minimality property of such $\mathcal{K}'$. In particular, if it happens, as in many examples that we have seen before, that $\mathcal{K} = \mathcal{K}_0 * \mathcal{LO}$ has both the Ramsey and ordering properties, then clearly $\mathcal{K}$ is also the unique expansion of $\mathcal{K}_0$ that has the Ramsey property.

For each class $\mathcal{K}_0$ and $\boldsymbol{A}_0 \in \mathcal{K}_0$, let $t(\boldsymbol{A}_0, \mathcal{K}_0)$ be the least t, if it exists, such that for any $\boldsymbol{A}_0 \leq \boldsymbol{B}_0$ in $\mathcal{K}_0, k \geq 2$, there is $\boldsymbol{C}_0 \geq \boldsymbol{B}_0$ in $\mathcal{K}_0$ with

$$\boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_{k,t}.$$

Otherwise, let $t(\boldsymbol{A}_0, \mathcal{K}_0) = \infty$. Following Fouché [21], call $t(\boldsymbol{A}_0, \mathcal{K}_0)$ the *Ramsey degree* of $\boldsymbol{A}_0$ (in $\mathcal{K}_0$). We have seen that if $\mathcal{K}_0$ admits an expansion $\mathcal{K}$ with the Ramsey and ordering properties, then $t(\boldsymbol{A}_0, \mathcal{K}_0) = t_{\mathcal{K}}(\boldsymbol{A}_0)$.

For example, if $\mathcal{K}_0 = \mathcal{GR}$, then $t(\boldsymbol{A}_0, \mathcal{K}_0) = \frac{\mathrm{card}(\boldsymbol{A}_0)!}{\mathrm{card}(\mathrm{Aut}(\boldsymbol{A}_0))}$. For instance, if $\boldsymbol{A}_0 = K_n$, the complete graph on n vertices, or $\boldsymbol{A}_0 =$ the complement of K_n, then $t(\boldsymbol{A}_0, \mathcal{K}_0) = 1$, i.e., $\boldsymbol{A}_0$ satisfies Ramsey's Theorem, and moreover the K_n and their complements are the only graphs that have this property (see Nešetřil-Rödl [56], [58]). For other calculations of this sort, see [20],[21],[22],[23].

Notice also that $t_{\mathcal{K}}(\boldsymbol{A}_0) = 1$ for all $\boldsymbol{A}_0 \in \mathcal{K}_0$ iff $\mathcal{K}$ is order forgetful (see 5.5).

Actually the preceding results admit appropriate converses. First, for Proposition 10.1 we have:

Proposition 10.4. *Suppose $\mathcal{K}$ is reasonable and has the ordering property and for any $\boldsymbol{A}_0 \leq \boldsymbol{B}_0$ in $\mathcal{K}_0, k \geq 2$, there is $\boldsymbol{C}_0 \geq \boldsymbol{B}_0$ in $\mathcal{K}_0$ with*

$$\boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_{k,t_{\mathcal{K}}(\boldsymbol{A}_0)}.$$

Then $\mathcal{K}$ has the Ramsey property. In particular, any reasonable $\mathcal{K}$ with the ordering property, that is contained in an expansion of $\mathcal{K}_0$ with the Ramsey property, also has the Ramsey property.

Proof. Suppose $\langle \boldsymbol{A}_0, \prec\rangle, \langle \boldsymbol{B}_0, \prec'\rangle \in \mathcal{K}$ are given with

$$\langle \boldsymbol{A}_0, \prec\rangle \leq \langle \boldsymbol{B}_0, \prec'\rangle.$$

We need to find $\langle \boldsymbol{C}_0, \prec''\rangle \geq \langle \boldsymbol{B}_0, \prec'\rangle$ in $\mathcal{K}$ such that

$$\langle \boldsymbol{C}_0, \prec''\rangle \to (\langle \boldsymbol{B}_0, \prec'\rangle))^{\langle \boldsymbol{A}_0, \prec\rangle}_2.$$

First, by the ordering property, we can find $\boldsymbol{B}_1 \in \mathcal{K}_0$ such that for any ordering $\prec_0$ on B_0 and any ordering $\prec_1$ on B_1 with $\langle \boldsymbol{B}_0, \prec_0\rangle, \langle \boldsymbol{B}_1, \prec_1\rangle \in \mathcal{K}$, we have $\langle \boldsymbol{B}_0, \prec_0\rangle \leq \langle \boldsymbol{B}_1, \prec_1\rangle$. Since $\mathcal{K}$ is reasonable, this implies that for any ordering $\prec_0$ on A_0 and any ordering $\prec_1$ on B_1 we also have $\langle \boldsymbol{A}_0, \prec_0\rangle \leq \langle \boldsymbol{B}_1, \prec_1\rangle$.

By hypothesis, there is $\boldsymbol{C}_0 \geq \boldsymbol{B}_1$ in $\mathcal{K}_0$ such that

$$\boldsymbol{C}_0 \to (\boldsymbol{B}_1)^{\boldsymbol{A}_0}_{t_{\mathcal{K}}(\boldsymbol{A}_0)+1, t_{\mathcal{K}}(\boldsymbol{A}_0)}.$$

Fix an ordering $\prec''$ on $\boldsymbol{C}_0$ with $\langle \boldsymbol{C}_0, \prec''\rangle \in \mathcal{K}$. We claim that $\langle \boldsymbol{C}_0, \prec''\rangle$ works. Clearly, $\langle \boldsymbol{B}_0, \prec'\rangle \leq \langle \boldsymbol{C}_0, \prec''\rangle$.

Consider now a coloring

$$c : \binom{\langle \boldsymbol{C}_0, \prec''\rangle}{\langle \boldsymbol{A}_0, \prec\rangle} \to \{1, 2\}.$$

Use this to define a coloring

$$c' : \binom{\boldsymbol{C}_0}{\boldsymbol{A}_0} \to \{0\} \cup \{p_1, \dots, p_{t_{\mathcal{K}}(\boldsymbol{A}_0)}\},$$

where $p_1, \dots, p_{t_{\mathcal{K}}(A_0)}$ enumerate the $\boldsymbol{A}_0 - \mathcal{K}-$patterns, as follows: Let $\boldsymbol{A}_0' \in \binom{\boldsymbol{C}_0}{\boldsymbol{A}_0}$. Then

$$\begin{aligned} c'(\boldsymbol{A}_0') \; &= \text{the pattern of } \prec'' |A_0', \text{if this pattern is different from that of} \\ &\prec, \text{ or if } c(\langle \boldsymbol{A}_0', \prec'' |A_0'\rangle) = 1, \\ &= 0, \text{ otherwise} \end{aligned}$$

Let now $\boldsymbol{B}_1' \in \binom{\boldsymbol{C}_0}{\boldsymbol{B}_1}$ be such that c' takes at most $t_{\mathcal{K}}(\boldsymbol{A}_0)$ values on $\binom{\boldsymbol{B}_1'}{\boldsymbol{A}_0}$. Since any pattern is realized among the $\prec'' |A_0'$, where $\boldsymbol{A}_0' \in \binom{\boldsymbol{B}_1'}{\boldsymbol{A}_0}$, it follows that c is constant on $\binom{\langle \boldsymbol{B}_1', \prec'' |B_1'\rangle)}{\langle \boldsymbol{A}_0, \prec\rangle}$. Since

$$\langle B_0, \prec'\rangle \leq \langle \boldsymbol{B}_1', \prec'' |B_1'\rangle,$$

we are done.

The last assertion follows by also using 10.1. ⊣

To formulate a converse to Corollary 10.3, let us first define a local version of the ordering property.

We say that $\mathcal{K}$ satisfies the *ordering property at* $\boldsymbol{A}_0 \in \mathcal{K}_0$, if there is $\boldsymbol{B}_0 \geq \boldsymbol{A}_0$ in $\mathcal{K}$ such that for every orderings $\prec, \prec'$ on A_0, B_0, resp., with $\langle \boldsymbol{A}_0, \prec\rangle, \langle \boldsymbol{B}_0, \prec'\rangle \in \mathcal{K}$, we have $\langle \boldsymbol{A}_0, \prec\rangle \leq \langle B_0, \prec'\rangle$. Thus $\mathcal{K}$ has the ordering property iff $\mathcal{K}$ has the ordering property at each $\boldsymbol{A}_0 \in \mathcal{K}_0$.

We now have the following converse to Corollary 10.3.

Proposition 10.5. *Assume $\mathcal{K}$ has the Ramsey property. Then the following are equivalent:*

(i) $\mathcal{K}$ has the ordering property at $\boldsymbol{A}_0 \in \mathcal{K}$.

(ii) $t(\boldsymbol{A}_0, \mathcal{K}_0) = t_{\mathcal{K}}(\boldsymbol{A}_0)$.

Proof. (i) ⇒ (ii): As in the proof of Proposition 10.2.

(ii) ⇒ (i): Assume (i) fails, so that for every $\boldsymbol{B}_0 \geq \boldsymbol{A}_0$ in $\mathcal{K}_0$ there are $\prec, \prec'$ with $\langle \boldsymbol{A}_0, \prec\rangle \in \mathcal{K}, \langle \boldsymbol{B}_0, \prec'\rangle \in \mathcal{K}$ but $\langle \boldsymbol{A}_0, \prec\rangle \not\leq \langle \boldsymbol{B}_0, \prec'\rangle$. Let $t = t_{\mathcal{K}}(\boldsymbol{A}_0)$. Then $t_{\mathcal{K}}(\boldsymbol{A}_0, \boldsymbol{B}_0, \prec') \leq t - 1$, so we are done by 10.1. ⊣

Corollary 10.6. *If $\mathcal{K}$ has the Ramsey property, then the following are equivalent:*

(i) $\mathcal{K}$ has the ordering property.

(ii) For every $\boldsymbol{A}_0 \in \mathcal{K}, t(\boldsymbol{A}_0, \mathcal{K}_0) = t_{\mathcal{K}}(\boldsymbol{A}_0)$. ⊣

We apply these facts to prove the result mentioned in the last paragraph of Section 9.

Theorem 10.7. *Let $\mathcal{K}_0$ be a Fraïssé class in a signature L_0, and assume that $\mathcal{K}$ is a reasonable Fraïssé order class in $L = L_0 \cup \{<\}$ which is an expansion of $\mathcal{K}_0$ and satisfies the Ramsey property. Then there is a reasonable Fraïssé order class $\mathcal{K}' \subseteq \mathcal{K}$, which is an expansion of $\mathcal{K}_0$, and satisfies both the Ramsey and ordering properties.*

Proof. We will use the notation of the proof of 9.2. Let $X' \subseteq X_{\mathcal{K}}$ be a minimal G_0-subflow of the G_0-flow $X_{\mathcal{K}}$, and let $\prec' \in X'$, so that $X' = \overline{G_0 \cdot \prec'}$. Let $\mathcal{K}' = \mathrm{Age}(\langle \boldsymbol{F}_0, \prec' \rangle) \subseteq \mathcal{K}$. We will show that this works. Clearly $\mathcal{K}'|L_0 = \mathcal{K}_0$, i.e., $\mathcal{K}'$ is an expansion of $\mathcal{K}_0$, and $\mathcal{K}'$ is hereditary and satisfies JEP. It is also easy to see that $\mathcal{K}'$ is reasonable. Note that $X' = X_{\mathcal{K}'} = \{\prec : \prec$ is a linear ordering on F_0 and for every finite $\boldsymbol{B}_0 \subseteq \boldsymbol{F}_0, \langle \boldsymbol{B}_0, \prec |B_0\rangle \in \mathcal{K}'\}$. It follows then from the proof of 7.4, (i) $\Rightarrow$ (ii), that $\mathcal{K}'$ satisfies the ordering property. Thus to verify that $\mathcal{K}'$ has the Ramsey property it is enough, by 10.4, to check that for any $\boldsymbol{A}_0 \leq \boldsymbol{B}_0$ in $\mathcal{K}_0, k \geq 2$, there is $\boldsymbol{C}_0 \geq \boldsymbol{B}_0$ in $\mathcal{K}_0$ with $\boldsymbol{C}_0 \to (\boldsymbol{B}_0)^{\boldsymbol{A}_0}_{k, t_{\mathcal{K}'}(\boldsymbol{A}_0)}$. This follows from 10.1. Finally, from Section 3 (second to last paragraph) it follows that $\mathcal{K}'$ has the amalgamation property, and this completes the proof. ⊣

We can also use similar ideas to prove the converse of 7.5 (ii).

Theorem 10.8. *Let $L \supseteq \{<\}$ be a signature, $L_0 = L \setminus \{<\}, \mathcal{K}$ a reasonable Fraïssé order class in L, and let $\mathcal{K}_0 = \mathcal{K}|L_0$ and $\boldsymbol{F} = \mathrm{Flim}(\mathcal{K}), \boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0) = \boldsymbol{F}|L_0$. Let $G_0 = \mathrm{Aut}(\boldsymbol{F}_0), G = \mathrm{Aut}(\boldsymbol{F})$ and let $X_{\mathcal{K}}$ be the set of linear orderings on $F (= F_0)$ which are $\mathcal{K}$-admissible. Then the following are equivalent:*

(i) $\mathcal{K}$ has the Ramsey and ordering properties.

(ii) $X_{\mathcal{K}}$ is the universal minimal flow of G_0.

Proof. (i) $\Rightarrow$ (ii) is 7.5 (ii).

(ii) $\Rightarrow$ (i): Since $X_{\mathcal{K}}$ is a minimal flow, $\mathcal{K}$ has the ordering property by 7.4. So it is enough to verify the hypothesis of 10.4. By the usual ultrafilter argument, as in the proof of 4.5, it is enough to show for given $\boldsymbol{A}_0 \leq \boldsymbol{B}_0$ in $\mathcal{K}, k \geq 2$, and $c : \binom{\boldsymbol{F}_0}{\boldsymbol{A}_0} \to \{1, \ldots, k\}$, that there is $\boldsymbol{B}'_0 \in \binom{\boldsymbol{F}_0}{\boldsymbol{B}_0}$ such that c on $\binom{\boldsymbol{B}'_0}{\boldsymbol{A}_0}$ obtains at most $t_{\mathcal{K}}(\boldsymbol{A}_0)$ many values. To see this, consider the G_0-flow $\{1, \ldots, k\}^{\binom{\boldsymbol{F}_0}{\boldsymbol{A}_0}}$, where G_0 acts on this space in the usual way: $g \cdot \gamma(\boldsymbol{A}'_0) = \gamma(g^{-1}(\boldsymbol{A}'_0))$. Let $X = \overline{G_0 \cdot c}$. Then there is a homomorphism $\pi : X_{\mathcal{K}} \to X$. Put $\pi(\prec_0) = c_0$, where $\boldsymbol{F} = \langle \boldsymbol{F}_0, \prec_0 \rangle$. Since G stabilizes $\prec_0, G$ also stabilizes c_0. ¿From this it easily follows that the color $c_0(\boldsymbol{A}'_0)$ for any $\boldsymbol{A}'_0 \in \binom{\boldsymbol{F}_0}{\boldsymbol{A}_0}$ depends only on the pattern of $\prec_0 |A'_0$, thus c_0 takes at most $t_{\mathcal{K}}(\boldsymbol{A}_0)$

values. Fix now $\boldsymbol{B}_0'' \in \binom{\boldsymbol{F}_0}{\boldsymbol{B}_0}$. Then there is $g \in G_0$ such that $g \cdot c \Big| \binom{\boldsymbol{B}_0''}{\boldsymbol{A}_0} = c_0 \Big| \binom{\boldsymbol{B}_0''}{\boldsymbol{A}_0}$. Let $\boldsymbol{B}_0' = g^{-1}(\boldsymbol{B}_0'')$. Then c on $\binom{\boldsymbol{B}_0'}{\boldsymbol{A}_0}$ obtains at most $t_{\mathcal{K}}(\boldsymbol{A}_0)$ many values and we are done. ⊣

There are some natural questions that are suggested by the preceding facts. First, recall the question that we raised in Section 7 of understanding when $\mathcal{K}_0$ has an expansion $\mathcal{K}$ with the Ramsey property. A necessary condition for the existence of such a $\mathcal{K}$ is that $t(\boldsymbol{A}_0, \mathcal{K}_0) < \infty$, for all $\boldsymbol{A}_0 \in \mathcal{K}_0$. Is that actually a necessary and sufficient condition? We should point out that we do not know an example of a $\mathcal{K}_0$ for which there is an $\boldsymbol{A}_0 \in \mathcal{K}_0$ with $t(\boldsymbol{A}_0, \mathcal{K}_0) = \infty$, although one should surely exist.

Second, in the context of 10.8, le us say that a G_0-flow is *universal* if it can be mapped homomorphically to any other G_0-flow. Thus the universal minimal flow of G_0 is a minimal and universal G_0-flow. We have seen in 7.4 that $X_{\mathcal{K}}$ is a minimal G_0-flow iff $\mathcal{K}$ has the ordering property. Can we strengthen 10.8 by showing that $X_{\mathcal{K}}$ is a universal G_0-flow iff $\mathcal{K}$ has the Ramsey property?

Finally, recall that if $\mathcal{K}_0$ has an expansion $\mathcal{K}$ with the Ramsey and ordering properties, then the automorphism group $G_0 = \mathrm{Aut}(\boldsymbol{F}_0)$, where $\boldsymbol{F}_0 = \mathrm{Flim}(\mathcal{K}_0)$, has universal minimal flow $X_{\mathcal{K}}$, which is the inverse limit of the family $\{X_{\mathcal{K}}^{\boldsymbol{A}_0}\}$, where $\boldsymbol{A}_0$ varies over finite substructures of $\boldsymbol{F}_0$ ordered under inclusion. Thus if $X_{\mathcal{K}}$ is finite, say of cardinality n, then clearly $\mathrm{card}(X_{\mathcal{K}}^{\boldsymbol{A}_0}) \leq n$, so, in particular, $\mathrm{card}(\mathrm{Aut}(\boldsymbol{A}_0)) \leq n$. It is easy to find examples where $\mathrm{card}(X_{\mathcal{K}}) = 1$, i.e., G_0 is extremely amenable. Take, for instance, $\mathcal{K}_0$ to be a Fraïssé order class with the Ramsey and ordering properties in a language $L_0 \supseteq \{<\}$ and let $\mathcal{K}$ consist of all structures of the form $\boldsymbol{A} = \langle \boldsymbol{A}_0, \prec' \rangle$, with $\prec' = <^{\boldsymbol{A}_0}$. However, we do not know examples of $\mathcal{K}_0, \mathcal{K}$ as above with $2 \leq \mathrm{card}(X_{\mathcal{K}}) < \infty$. Also note that if $\mathcal{K}$ is closed under products, then $X_{\mathcal{K}}$ cannot be finite, as $\sup_{\boldsymbol{A}_0 \in \mathcal{K}_0} \mathrm{card}(\mathrm{Aut}(\boldsymbol{A}_0)) = \infty$.

11. Concluding remarks and problems

(A) One of the two main ingredients in our proofs of extreme amenability of automorphism groups are the results of the corresponding structural Ramsey theory. It is therefore natural to pose the following problem a solution of which could enhance the already existing tradition of using the methods of topological dynamics to prove results of Ramsey theory.

Problem 11.1. Find alternate proofs (that use the methods of topological dynamics itself as well as the intrinsic geometry of the acting groups) that the automorphism groups of any of the following structures are extremely amenable:

(i) The rationals with the usual ordering.
(ii) The random ordered graph.
(iii) The random K_n-tree ordered graph, $(n = 3, 4, \dots)$.

(iv) The random $\mathcal{A}$-free ordered hypergraph of type L_0, for any class $\mathcal{A}$ of irreducible finite hypergraphs of type L_0.
(v) The ordered rational Urysohn space.
(vi) The $\aleph_0$-dimensional vector space over a finite field with the canonical ordering.
(vii) The countable atomless Boolean algebra with the canonical ordering.

(B) In connection with the extreme amenability of $U(\ell_2)$ and the group of isometries of ℓ_2 (see the end of Section 6), we would like to ask whether a new proof of these results can be found based on Ramsey theory, as was done in Section 6, (**E**) for the isometry group of the Urysohn space.

(C) Topological groups G isomorphic to closed subgroups of the infinite symmetric group S_∞, or, equivalently, those G whose open subgroups form a nbhd basis at the identity (groups with small open subgroups) have played a leading role in this article. One can meaningfully extend to homogeneous spaces of such groups the concept of Ramsey degree as follows.

For each bounded nonempty set A in a Euclidean space and $\epsilon > 0$, let $N(\epsilon, A)$ be the *covering number,* that is, the smallest number of sets of diameter $\leq \epsilon$ that can cover A. Similarly, for a bounded function f from some set X into a Euclidean space, we define $N(\epsilon, f) = N(\epsilon, \text{range}(f))$.

Let G be a topological group and let H be a subgroup. We define the *small oscillation degree* $n(G, H)$ of G, H to be the smallest number t such that:

(*) for every finite subset F of G , $\epsilon > 0$, and bounded left uniformly continuous f from G/H to some Euclidean space, we can find some h in G with $N(\epsilon, f'|hF) \leq t$, where f' is the lift of f to G.

If no such t exists, we put $n(G, H) = \infty$. Put $n(G) = n(G, \{1\})$. Note that

$$n(G, H) \leq n(G, H') \leq n(G),$$

if H' is a subgroup of H.

If H is open in G, then $n(G, H)) \leq t$ if and only if for every coloring c of G/H with any finite number of colors and every finite F included in G, there is an h in G such that c on hF takes at most t colors. One can prove that for a group G with small open subgroups, $n(G) = \sup\{n(G, H) : H \text{ is an open subgroup of } G\}$. Consequently, such a group G is extremely amenable iff $n(G) = 1$. (In the case of a discrete semigroup G this was established by Mitchell [49], while in a general case of a group acting on a metric space the latter equivalence is due to Gromov and Milman [38], [47], where a suitable extension of the condition $n(G) = 1$ was studied under the name of *concentration property.*)

Let now $\mathcal{K}$ be a Fraïssé class, $\boldsymbol{F}$ its Fraïssé limit and $G = \text{Aut}(\boldsymbol{F})$. For each finite substructure $\boldsymbol{A}$ of $\boldsymbol{F}$, let $H(\boldsymbol{A})$ be the (setwise) stabilizer of the domain of $\boldsymbol{A}$ in the action of G on the finite substructures of $\boldsymbol{F}$. Then one can see that

$$n(G, H(\boldsymbol{A})) = t(\boldsymbol{A}, \mathcal{K}) = \text{ the Ramsey degree of } \boldsymbol{A} \text{ (in the class } \mathcal{K}).$$

(D) We have seen that the extreme amenability of the automorphism group of an ultrahomogeneous ordered countable structure is equivalent to a corresponding finite Ramsey-theoretic result. This leads us to the following natural problem.

Problem 11.2. In each of the cases (i)–(vii) of Problem 11.1, find the topological dynamics analog of a corresponding infinite Ramsey-theoretic result.

We will sketch some possible approaches to Problem 11.2 in (E) and (F) below. But let us first explain what we mean by 'the corresponding *infinite* Ramsey-theoretic result'. First of all recall that in the arrow notation the infinite Ramsey theorem can be stated as,

$$\mathbb{N} \to (\mathbb{N})^k_l$$

for finite numbers k and l. This is what one calls *finite-dimensional* Ramsey theorem for $\mathbb{N}$. There is also an *infinite-dimensional* Ramsey theorem for $\mathbb{N}$ which states

$$\mathbb{N} \to_* (\mathbb{N})^{\mathbb{N}}_l,$$

where $*$ signifies some restriction on the colorings such as for example the restriction on Borel colorings in the well known Galvin-Prikry theorem [25]. Recall that in (i)-(vii) we really deal with groups of the form $\mathrm{Aut}(\boldsymbol{F})$, where $\boldsymbol{F}$ is a Fraïssé limit of a countable Fraïssé class $\mathcal{K}$. The corresponding *finite-dimensional* Ramsey theoretic results deal with arrow-relations of the form

$$\boldsymbol{F} \to (\boldsymbol{F})^{\boldsymbol{A}}_{l,t} \text{ and } \boldsymbol{F} \to (\boldsymbol{F})^{\boldsymbol{A}}_l$$

for $\boldsymbol{A} \in \mathcal{K}$. In other words, for $\boldsymbol{A} \in \mathcal{K}$, one is interested in the existence and computation of the *big Ramsey degree* $T(\boldsymbol{A}, \mathcal{K})$, the minimal integer t such that $\boldsymbol{F} \to (\boldsymbol{F})^{\boldsymbol{A}}_{l,t}$ for every positive integer l. Of course one is interested also in analogs of the infinite-dimensional Ramsey theorem such as, for example, the arrow-relations of the form $\boldsymbol{F} \to_* (\boldsymbol{F})^{\boldsymbol{F}}_{l,t}$ but at this stage in our knowledge even the theory of arrow-relations of the form $\boldsymbol{F} \to (\boldsymbol{F})^{\boldsymbol{A}}_{l,t}$ is far from being fully developed. The theory, however, does have substantial results of this form. We mention one quite old but not so widely known result due to D. Devlin [11] (see also Todorcevic [77]) that deals with the class $\mathcal{LO}$ of all finite linear orderings. More precisely, Devlin's theorem says that for every positive integer k there is a positive integer t such that

$$\mathbb{Q} \to (\mathbb{Q})^k_{l,t}$$

for all positive integers l, and that the minimal integer t satisfying this arrow-relation for all l is equal to the $(2k+1)$st *tangent number* T_{2k+1} given by the formula $\tan z = \sum_{n=0}^{\infty} T_n z^n/n!$. Thus if $\boldsymbol{A}_k$ denotes a linearly ordered set of size k then the sequence $t_k = T(\boldsymbol{A}_k, \mathcal{LO})$ is a well studied sequence of numbers which starts as $t_1 = 1, t_2 = 2, t_3 = 16, t_4 = 272, \ldots$ (see Knuth-Buckholtz [41]). We note that the existence of

$T(\boldsymbol{A}_k, \mathcal{LO})$ was known to R. Laver (see Erdös-Hajnal [17]) before Devlin's work and that the existence follows also rather directly from results of K. Milliken [45] on which Devlin's work was based. For results about other Fraïssé structures such as for example the random graph the reader is referred to Pouzet-Sauer [69] and Sauer [75].

One can ask similar questions for other kinds of extremely amenable topological groups not directly covered by the list 11.1 (i)–(vii). A particularly important example is the unitary group $U(\ell_2)$ of the Hilbert space equipped with the strong operator topology. The result by Gromov and Milman [38] that the unitary group $U(\ell_2)$ is extremely amenable implies the following property: If f is a uniformly continuous function on the unit sphere $\mathbb{S}^\infty$ of ℓ_2 with values in some $\mathbb{R}^n$, then for every $\epsilon > 0$ and every compact subset K of $\mathbb{S}^\infty$ there is $u \in U(\ell_2)$ such that the oscillation of f on $u(K)$ is $< \epsilon$. (See Milman and Schechtman [48], Milman [47] or Gromov [37].) The exact infinite-dimensional analog of the above property of spheres is impossible as demonstrated by Odell and Schlumprecht [62] in their solution of the famous distortion problem for ℓ_2: there exists a bounded uniformly continuous f on $\mathbb{S}^\infty$ such that f has oscillation 1 in every unit sphere of an infinite-dimensional subspace of ℓ_2.

(E) It seems that the above phenomena can be described within the following framework. Recall that the *left uniformity*, $\mathcal{U}_L(G)$, of a topological group G has as basic entourages of the diagonal the sets

$$V_L = \{(x, y) \in G \times G \colon x^{-1}y \in V\},$$

where V is a neighborhood of identity in G. In particular, every topological group admits the *completion with regard to the left uniformity,* also known as the *Weil completion.* (See, e.g., Chapter 10 in [73].) For example, in the case where G is a metrizable group, the left completion of G is just the metric completion of (G, d), where d is any left-invariant compatible metric. We will denote the left completion by $\hat{G}^L$. While in many cases — for instance, when G is locally compact, or abelian, or has small invariant neighborhoods — the left completion $\hat{G}^L$ is again a topological group, in general it is not the case (Dieudonné [13]), and the left completion of a topological group is only a topological semigroup (with jointly continuous multiplication), see Proposition 10.2(a) in [73].

For example, the left completion of the unitary group $U(\ell_2)$ with the strong operator topology can be identified with the semigroup of all linear isometries from ℓ_2 to its subspaces, with the composition of maps as the semigroup operation and the strong operator topology. The semigroup $\widehat{\mathrm{Aut}(\langle \mathbb{Q}, < \rangle)}^L$ is formed by all order-preserving injections from $\mathbb{Q}$ to itself, equipped with the composition operation and the topology of pointwise convergence on $\mathbb{Q}$ viewed as discrete.

If H is a subgroup of G, then the left uniform structure on G/H is, by definition, the finest uniform structure making the factor-map $\pi \colon G \to G/H$ uniformly continuous with regard to $\mathcal{U}_L(G)$.

Here are two examples. Fix a point ξ in the unit sphere $\mathbb{S}^\infty$ of the Hilbert space ℓ_2 ('north pole'), and denote by $H = \mathrm{St}_\xi$, the isotropy subgroup of ξ:

$$\mathrm{St}_\xi = \{u \in U(\ell_2)\colon u(\xi) = \xi\}.$$

This is a closed subgroup, isomorphic to $U(\ell_2)$ itself. There is a natural identification

$$U(\ell_2)/\mathrm{St}_\xi \ni u\mathrm{St}_\xi \mapsto u(\xi) \in \mathbb{S}^\infty,$$

as topological G-spaces. The left uniform structure on $\mathbb{S}^\infty$ viewed as a factor-space of the unitary group is the norm uniformity. In other words, basic entourages of diagonal in $\mathcal{U}_L(\mathbb{S}^\infty)$ are of the form

$$V_\epsilon = \{(\xi,\zeta)\colon \|\xi - \zeta\| < \epsilon\}.$$

Similarly, fix any finite set $F \subseteq \mathbb{Q}$, and denote by St_F the isotropy subgroup of F, that is, the set of all bijections $\tau \in \mathrm{Aut}(\langle \mathbb{Q}, <\rangle)$ that leave F (and therefore each element of F) fixed. This is an open subgroup of $\mathrm{Aut}(\langle \mathbb{Q}, <\rangle)$. The factor-space $\mathrm{Aut}(\mathbb{Q}, \leq)/\mathrm{St}_F$ can be identified with the set $[\mathbb{Q}]^n$ of all n-subsets of $\mathbb{Q}$, where $n = |F|$, under the correspondence

$$\mathrm{Aut}(\langle \mathbb{Q}, <\rangle)/\mathrm{St}_F \ni \tau\mathrm{St}_F \mapsto \tau(F) \in [\mathbb{Q}]^n.$$

The left uniformity on the factor-space $\mathrm{Aut}(\langle \mathbb{Q}, <\rangle)/\mathrm{St}_F \cong [\mathbb{Q}]^n$ is discrete.

If f is a real-valued function on a set X, the *oscillation* of f is

$$\mathrm{Osc}(f) = \sup_{x,y\in X} |f(x) - f(y)|.$$

The following definition is modeled on a classical concept from geometric functional analysis, first introduced by Milman [46] in the language of non-emptiness of the spectrum $\mathfrak{S}(f)$ of a function f.

Let $f\colon G \to \mathbb{R}$ be a left uniformly continuous function on a topological group G. Say that f is *oscillation stable* if for every $\epsilon > 0$ and every right ideal $\mathcal{I}$ of $\hat{G}^L$ there is a right ideal $\mathcal{J} \subseteq \mathcal{I}$ with the property

$$\mathrm{Osc}(f \mid \mathcal{J}) < \epsilon.$$

Here we have denoted by the same letter f the (unique) extension of f by continuity over the left completion $\hat{G}^L$.

If H is a subgroup of a topological group G, we say that a left uniformly continuous function $f\colon G/H \to \mathbb{R}$ is *oscillation stable* if the composition $\tilde{f} = f \circ \pi$ with the factor-map $\pi\colon G \to G/H$ is oscillation stable.

Say that the pair G, H, where H is a topological subgroup of a topological group G, is *oscillation stable* if every bounded left uniformly continuous function $f\colon G/H \to \mathbb{R}$ is oscillation stable. One can show that a pair G, H is oscillation stable if and only if for every bounded left uniformly continuous function $f\colon G/H \to \mathbb{R}$ and every $\epsilon > 0$ there is a right ideal $\mathcal{I}$ of $\hat{G}^L$ such that $\mathrm{Osc}\,(f \mid \mathcal{I}) < \epsilon$.

If $G = U(\ell_2)$ and $H = \mathrm{St}_\xi$, then a function f on the unit sphere $\mathbb{S}^\infty \cong U(\ell_2)/\mathrm{St}_\xi$ is oscillation stable in the sense of our definition if and only if it is oscillation stable in the classical sense, see e.g. Definition 13.1 in [5]. The result by Odell and Schlumprecht [62] that the infinite-dimensional Hilbert space has the distortion property is equivalent, in our language, to saying that the pair $U(\ell_2), \mathrm{St}_\xi$ is not oscillation stable for some (any) $\xi \in \mathbb{S}^\infty$.

Also, it follows from Devlin's theorem that the pair $\mathrm{Aut}(\langle\mathbb{Q}, <\rangle)$, St_F is oscillation stable if and only if $|F| = 1$.

One question that remains unanswered, is: *does there exist a non-trivial oscillation stable topological group,* that is, a topological group $G \neq \{e\}$ for which the pair $G, \{e\}$ is oscillation stable?[1]

Oscillation stability for topological groups is a strictly stronger concept than extreme amenability, as, for instance, both topological groups $U(\ell_2)$ and $\mathrm{Aut}(\langle\mathbb{Q}, <\rangle)$ are extremely amenable but not oscillation stable.

(F) For topological groups with small open neighborhoods one is able to capture the quantitative, as well as qualitative, content of results from Ramsey theory similar to Devlin's theorem considered above in (**C**).

Let G be a topological group, H be a subgroup. Define the *big oscillation degree* $N(G, H)$ of G to be the smallest t such that:

(**) for any bounded left uniformly continuous function f from G/H to some Euclidean space and any $\epsilon > 0$, there is a right ideal $\mathcal{I}$ in the left completion $\hat{G}^L$ of G such that (denoting also by f the extension of the lift of f to $\hat{G}^L$) $N(\epsilon, f|\mathcal{I}) \leq t$.

If no such t exists, we say again that $N(G)$ is infinite. We have $n(G, H) \leq N(G, H) \leq N(G)$ $(= N(G, \{1\})$.

If H is a subgroup of G, then the condition $N(G, H) = 1$ is equivalent to the oscillation stability of G, H, and it turns out that a group G with small open subgroups is oscillation stable iff $N(G, H) = 1$, for all open subgroups H of G.

Let G be a non-trivial group with small open subgroups. Any such G is the automorphism group of a Fraïssé structure $\boldsymbol{F}$. It turns out now that if the signature of $\boldsymbol{F}$ is relational and finite (or even more generally if there are only finitely many, up to isomorphism, 2-generated structures in $\mathcal{K}$), then G is not oscillation stable. On the other hand, we have that in general for any $\mathcal{K}$,$\boldsymbol{F}$,and G as above and $\boldsymbol{A}$ in $\mathcal{K}$,

$$N(G, H(\boldsymbol{A})) = T(\boldsymbol{A}, \mathcal{K}).$$

Thus, if $\mathcal{K}$ is the class of finite linear orderings and if $\boldsymbol{A}_n$ is a finite linear ordering of size n, then applying Devlin's theorem we get that $N(G, H(\boldsymbol{A}_n))$ is equal to the nth odd tangent number t_n.

Addendum. We have recently received the preprint Nešetřil [53], which discusses many concepts and results of structural Ramsey theory relevant to our paper. In

[1]Recently Hjorth informed the authors that he can prove that no non-trivial Polish group is oscillation stable.

particular, some new examples of classes with the Ramsey property are presented concerning posets, directed graphs, etc. One very interesting case is the class of all structures $\langle A, \sqsubset, \prec \rangle$, where $\langle A, \sqsubset \rangle$ is a finite poset and $\prec$ is a linear extension of $\sqsubset$ (see Nešetřil–Rödl [59], Fouché [20]). The Fraïssé limit of this class is of the form $\boldsymbol{F} = \langle F_0, \sqsubset_0, \prec_0 \rangle$, where $\langle F_0, \sqsubset_0 \rangle$ is the random poset (i.e., the Fraïssé limit of the class of finite posets) and $\prec_0$ is an appropriate linear extension of $\sqsubset_0$. In particular, it follows from our results here that $\mathrm{Aut}(\boldsymbol{F})$ is extremely amenable and that the universal minimal flow of the automorphism group of the random poset $\boldsymbol{F}_0 = \langle F_0, \sqsubset_0 \rangle$ is the space of all linear extensions of $\sqsubset_0$.

The proof of the result announced in Nešetřil [52] is contained in the recent preprint Nešetřil [54]. Finally, Nguyen Van The [61] has shown that the class of finite convexly ordered ultrametric spaces (where an ordered metric space is convexly ordered if each metric ball is an interval) has the Ramsey and ordering properties and uses this to compute the universal minimal flow of the isometry group of the Baire space.

Appendix 1. A new proof of Veech's theorem

Veech's theorem (Theorem 2.2.1 in [83]) is an important result of abstract topological dynamics, asserting that every locally compact group acts freely on a suitable compact space. Alternative proofs of this result can be found in [2] (in the second countable case) and in [71]. The latter author notes that his proof is 'really the same,' but it emphasizes different features of the original idea. The same applies to our proposed proof, which is, we hope, more accessible.

Lemma A1.1. *Let G be a locally compact group, and let $g \in G$, $g \neq e$. There exists a right invariant continuous pseudometric d on G, bounded by 1 and such that $0 < d(e, g) < 1$ and the closure of the open ball of unit radius is compact.*

Proof. Let ν be a left-invariant Haar measure on G. For a $f \in L^2(G, \nu)$ and $h \in H$, define ${}^h f \in L^2(G, \nu)$ via ${}^h f(x) := f(h^{-1}x)$.

Case 1: $g^2 \neq e$. Choose a symmetric compact neighborhood of the identity, V, in G, with the property $g, g^2 \notin V^2$, and a function $f \in L^2(G, \nu)$ supported on V and such that the L^2-norm $\|f\| = 1$. Let $\phi = f + {}^{g^{-1}}f$. Clearly, $\|\phi\| = \sqrt{2}$. For each $x, y \in G$, define

$$\rho(x, y) := \|{}^{x^{-1}}\phi - {}^{y^{-1}}\phi\| \equiv \|\phi - {}^{yx^{-1}}\phi\|.$$

This ρ is a right-invariant continuous pseudometric on G, bounded by $2\sqrt{2}$. If translates of ϕ by x^{-1} and by y^{-1} are orthogonal, then $\rho(x, y) = 2$. It follows that, if $h \in G$ and $\rho(e, h) < 2$, then $(V \cup g^{-1}V) \cap (h^{-1}V \cup h^{-1}gV) \neq \emptyset$. This is equivalent to $h \in V^2 \cup gV^2 \cup V^2 g \cup gV^2 g$, and so the open ball $\mathcal{O}_2(e)$ of radius 2 has compact closure. Also, $\rho(e, g) = \|f - {}^{g^{-2}}f\| = \sqrt{2}$, because the supports of f and ${}^{g^{-2}}f$ are disjoint. The required pseudometric d is now defined by

$$d(x, y) = \frac{1}{2} \min\{\rho(x, y), 2\}.$$

Case 2: $g^2 = e$. Let V be a compact symmetric neighborhood of identity with $g \notin V^2$. Let f be of L^2-norm one and supported on V, and let $\phi = f + 2 \cdot {}^g f$. Define the right-invariant continuous pseudometric ρ via $\rho(x,y) := \|{}^{x^{-1}}\phi - {}^{y^{-1}}\phi\| \equiv \|\phi - {}^{yx^{-1}}\phi\|$. Similarly to Case 1, the closure of the open ball of radius $\sqrt{10}$ is compact. Finally, set

$$d(x,y) = \frac{1}{\sqrt{10}} \min\{\rho(x,y), \sqrt{10}\}.$$

$\dashv$

Veech's Theorem. *Every locally compact group G acts freely on the greatest ambit $S(G)$.*

Proof. Let G be a locally compact group. Let $g \in G$ and $g \neq e$. We will show that g has no fixed points in the greatest ambit of G.

(A) Choose a a pseudometric d on G as in Lemma A1.1.

(B) Choose an $\varepsilon > 0$ satisfying

$$9\varepsilon < d(e,g) < 1 - 4\varepsilon. \tag{11.1}$$

By Zorn's lemma, there exists a maximal subset $A \subseteq G$ with the property that whenever $a, b \in A$ and $a \neq b$, one has $\mathcal{O}_\varepsilon(a) \cap \mathcal{O}_\varepsilon(b) \neq \emptyset$. Such an A is a 2ε-net in G.

(C) Define a graph, Γ, whose vertex set is A and such that two vertices, $a, b \in A$, are adjacent if and only if $a \neq b$ and $ab^{-1} \in K^2$.

Let $\varkappa$ denote the cardinality of an arbitrary finite family, γ, of open balls of radius ε covering the compact set K^2. Any family δ of pairwise disjoint open balls of radius ε with centers in K^2 has cardinality not exceeding $\varkappa$, because every mapping assigning to every $B \in \delta$ a ball B' in the family γ containing the center of B is an injection. If $a, b_1, \ldots, b_n \in A$ and a is adjacent to each b_i, then $b_i a^{-1} \in K^2$ for all i and the ε-balls centered at $b_i a^{-1}$, $i = 1, 2, \ldots, n$, are pairwise disjoint (the metric d is right-invariant). It follows that $n \leq \varkappa$ and Γ has a finite degree $\leq \varkappa$.

(D) As a consequence, the vertices of Γ can be colored with at most $\varkappa + 1$ colors in such a way that no two adjacent vertices have the same color. Let $A = \bigsqcup_{i=1}^m A_i$, $A_i \neq \emptyset$, be such a coloring, where $m \leq \varkappa + 1$.

(E) For each $i = 1, 2, \ldots, m$ define a function $f_i \colon G \to \mathbb{R}$ via

$$f_i(x) := d(x, A_i),$$

the distance from $x \in G$ to A_i. This f_i is a 1-Lipschitz function, bounded by 1.

(F) Let $a, b \in A$ be such that $d(a,b) < 1$. For some $i = 1, 2, \ldots, m$, one has $a \in A_i$ and, since a and b are adjacent, also $b \notin A_i$. Moreover, a is the only element of A_i at a distance < 1 from b and thus the nearest neighbor to b in A_i. Indeed, assuming that there is a $c \in A_i$ with $c \neq a$ and $d(c,b) < 1$, one has $ba^{-1} \in K$, $cb^{-1} \in K$, and thus $ca^{-1} = cb^{-1}ba^{-1} \in K^2$, meaning that c and a are adjacent, in contradiction with the choice of the coloring.

Since each function f_j, $j = 1, 2, \ldots, m$ is 1-Lipschitz,

$$\max_{j=1}^{m} |f_j(a) - f_j(b)| = d(a, b) \text{ whenever } a, b \in A \text{ and } d(a, b) < 1. \tag{11.2}$$

(G) Define the mapping $f\colon G \to \ell^\infty(m)$, where $\ell^\infty(m) = \mathbb{R}^m$ with the max norm $\|.\|_\infty$, as $f = (f_1, f_2, \ldots, f_m)$. Then (11.2) is equivalent to

$$\forall a, b \in A,\ (d(a, b) < 1) \Rightarrow \|f(a) - f(b)\|_\infty = d(a, b). \tag{11.3}$$

(H) Let $x \in G$ be arbitrary. There are $a, b \in A$ such that $d(x, a) < 2\varepsilon$ and $d(gx, b) < 2\varepsilon$. Since $d(gx, x) = d(g, e)$, it follows by the triangle inequality that $5\varepsilon < d(a, b) < 1$, and Eq. (11.3) implies that

$$\|f(a) - f(b)\|_\infty = d(a, b) > 5\varepsilon.$$

By the triangle inequality

$$\begin{aligned} \|f(x) - f(gx)\|_\infty &\geq \|f(a) - f(b)\|_\infty - 4\varepsilon \\ &> \varepsilon. \end{aligned}$$

(I) The mapping $f\colon G \to \mathbb{R}^m$, being right uniformly continuous and bounded, admits a unique continuous extension, $\bar{f}$, over the greatest ambit $S(G)$ of G. By continuity,

$$\forall x \in S(G),\ \|f(x) - f(g \cdot x)\|_\infty \geq \varepsilon > 0.$$

In particular, the action by g on the greatest ambit is fixed point-free. ⊣

Appendix 2. Non-metrizability of the universal minimal flow for non-compact locally compact groups

A subset A of a group G is called (*discretely*) *left syndetic* if finitely many left translates of A cover G. Here is a simple and well-known fact from abstract topological dynamics.

Lemma A2.1. *Let G be a topological group, and let M be a minimal compact G-flow. Let $W \subseteq M$ be a non-empty open subset, and let $\xi \in M$. Then the set*

$$\widetilde{W} := \{g \in G\colon g \cdot \xi \in W\}$$

is discretely left syndetic in G.

Proof. The translates $h \cdot W$, $h \in G$ form an open cover of M, because otherwise there would be a point $\zeta \in M$ whose G-orbit misses W, in contradiction with the assumed minimality of M. Choose finitely many elements, $h_1, h_2, \ldots, h_n \in G$ with the property that $h_i \cdot W$, $i = 1, 2, \ldots, n$, cover M. It remains to notice that for every $h \in G$, $\widetilde{h \cdot W} = h\widetilde{W}$, and so the left translates $h_i\widetilde{W}$, $i = 1, 2, \ldots, n$, cover G. ⊣

Theorem A2.2. *The universal minimal flow $M(G)$ of a non-compact locally compact group G is non-metrizable.*

Proof. Let G be a locally compact group. Let U be a neighborhood of identity whose closure is compact. Use Zorn's lemma to choose a maximal subset $X \subseteq G$ with the property that $\{Ux\colon x \in X\}$ is a disjoint family.

According to Pym [71] (the Local Structure Theorem on p. 172), the closure $\overline{X}$ of X in the greatest ambit $S(G)$ is homeomorphic to βX, the Stone-Čech compactification of the discrete space X. Also, if V is an open subset of G with $\overline{V} \subseteq U$, then the subspace $V \cdot \overline{X}$ is open in $S(G)$ and homeomorphic with $V \times \overline{X}$ (and consequently with $V \times \beta X$) under the map $(v, \xi) \mapsto v \cdot \xi$, $v \in V$, $\xi \in \overline{X}$. Finally, given any $\xi \in S(G)$, U and X can be chosen so that $\xi \in \overline{X}$.

Denote by M an isomorphic copy of the universal minimal flow $M(G)$ sitting inside the greatest ambit $S(G)$. Assume from now on that U and X as above are chosen in such a way that $\overline{X} \cap M \neq \emptyset$. Let also V and V_1 be open neighborhoods of identity in G with the property $V \subseteq \overline{V} \subseteq V_1 \subseteq \overline{V_1} \subseteq U$. Since $V \cdot \overline{X}$ is open in $S(G)$, it follows that $(V \cdot \overline{X}) \cap M$ is (non-empty and) open in M.

Assume now that M is metrizable, in order to deduce that G is compact.

The closed subspace $(\overline{V} \cdot \overline{X}) \cap M$ of M is also metrizable and compact. The second coordinate projection, proj_2, from $V_1 \cdot \overline{X} \cong V_1 \times \beta X$ to $\overline{X} \cong \beta X$ is continuous, and therefore the image $K = \mathrm{proj}_2((\overline{V} \cdot \overline{X}) \cap M)$ is a compact metrizable subspace of the extremally disconnected space βX.

Since an extremally disconnected space does not contain any nontrivial convergent sequences (see e.g. [16], Exercise 6.2.G.(a) on p. 456), it follows that K is finite. Consequently, for each $\kappa \in K$ the subset $(V \cdot \kappa) \cap M$ is open in $(V \cdot \overline{X}) \cap M$, and we conclude that for some $\kappa' \in \overline{X}$ the set $W = (V \cdot \kappa') \cap M$ is non-empty and open in $(V \cdot \overline{X}) \cap M$ and therefore in M itself.

Let $\xi \in W = (V \cdot \kappa') \cap M$ be arbitrary. By Lemma A2.1, the set

$$\widetilde{W} = \{g \in G\colon g \cdot \xi \in W\}$$

is discretely left syndetic in G. For some $v \in V$, one has $\xi = v \cdot \kappa'$. The set

$$W^{\ddagger} := \{g \in G\colon gv \cdot \kappa' \in V \cdot \kappa'\}$$

is bigger than $\widetilde{W}$ and therefore also discretely left syndetic. Since the action of G on the greatest ambit $S(G)$ is free by Veech's theorem, the condition $gv \cdot \kappa' \in V \cdot \kappa'$ is equivalent to $gv \in V$ and, in its turn, implies $g \in V^2$. It follows that $W^{\ddagger} \subseteq V^2$.

The compact set $(\overline{V})^2$ contains V^2 and is therefore discretely left syndetic as well. Consequently, G is compact. $\dashv$

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