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AMENABLE VERSUS HYPERFINITE BOREL EQUIVALENCE RELATIONS

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Let X be a standard Borel space (i.e., a Polish space with the associated Borel structure), and let E be a *countable* Borel equivalence relation on X, i.e., a Borel equivalence relation E for which every equivalence class $[x]_E$ is countable. By a result of Feldman-Moore [FM], E is induced by the orbits of a Borel action of a countable group G on X.

The structure of general countable Borel equivalence relations is very little understood. However, a lot is known for the particularly important subclass consisting of hyperfinite relations. A countable Borel equivalence relation is called *hyperfinite* if it is induced by a Borel $\mathbb{Z}$-action, i.e., by the orbits of a single Borel automorphism. Such relations are studied and classified in [DJK] (see also the references contained therein). It is shown in Ornstein-Weiss [OW] and Connes-Feldman-Weiss [CFW] that for every Borel equivalence relation E induced by a Borel action of a countable amenable group G on X and for every (Borel) probability measure μ on X, there is a Borel invariant set $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \upharpoonright Y$(= the restriction of E to Y) is hyperfinite. (Recall that a countable group G is *amenable* if it carries a finitely additive translation invariant probability measure defined on all its subsets.) Motivated by this result, Weiss [W2] raised the question of whether every E induced by a Borel action of a countable amenable group is hyperfinite. Later on Weiss (personal communication) showed that this is true for $G = \mathbb{Z}^n$. However, the problem is still open even for abelian G. Our main purpose here is to provide a weaker affirmative answer for general amenable G (and more—see below). We need a definition first. Given two standard Borel spaces X, Y, a *universally measurable* isomorphism between X and Y is a bijection $f: X \to Y$ such that both f, f^{-1} are universally measurable. (As usual, a map $g: Z \to W$, with Z and W standard Borel spaces, is called *universally measurable* if it is μ-measurable for every probability measure μ on Z.) Notice now that to assert that a countable Borel equivalence relation on X is hyperfinite is trivially equivalent to saying that there is a standard Borel space Y and a hyperfinite Borel equivalence relation F on Y, which is *Borel* isomorphic to E, i.e., there is a Borel bijection $f: X \to Y$ with $xEy \Leftrightarrow f(x)Ff(y)$. We have the following theorem.

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THEOREM 1^-. *Assume the Continuum Hypothesis (CH). Let E be an equivalence relation induced by a Borel action of a countable amenable group G on a standard Borel space X. Then there is a hyperfinite Borel equivalence relation F on a standard Borel space Y and a universally measurable isomorphism $f: X \to Y$ of E with F, i.e., $xEy \Leftrightarrow f(x)Ff(y)$.*

It follows, as a corollary, that any such E is induced by the orbits of a universally measurable automorphism, aassuming CH again, but this is much easier to prove directly using the results in [CFW] (see §3).

As it turns out, one can formulate and prove a more general result, which is the best result possible along these lines. To do this, we need to recall the notion of amenability of a countable Borel equivalence relation that was introduced in [K1] by appropriately adapting, in this context, the measure theoretic notions of Zimmer [Z] and Connes-Feldman-Weiss [CFW]. Let E be a countable Borel equivalence relation on X. We say that E is *amenable* if there is a map assigning to each E-equivalence class C a finitely additive probability measure φ_C, defined on all subsets of C, such that $C \mapsto \varphi_C$ is universally measurable; i.e., for each Borel bounded $F: X^2 \to \mathbb{R}$, the function $f: X \to \mathbb{R}$ given by $f(x) = \int_{[x]_E} F(x, y)\, d\varphi_{[x]_E}(y)$ is universally measurable.

It can be shown, assuming CH (see [K1]), that if E is induced by a Borel action of a countable amenable group, then E is amenable. It follows that if E is universally measurable isomorphic to a hyperfinite F, then E is amenable. Thus, the following is optimal in this context.

THEOREM 1. *Assume CH. Let E be an amenable countable Borel equivalence relation. Then there is a hyperfinite Borel equivalence relation F, which is universally measurable isomorphic to E.*

Using the classification up to Borel isomorphism of hyperfinite Borel equivalence relations in [DJK], one can also classify amenable Borel equivalence relations up to universally measurable isomorphism. To do this, recall that, given a countable Borel equivalence relation E on X, a probability measure μ on X is called *E-invariant* if it is invariant for the Borel action of a countable group G which induces E. μ is called *E-ergodic* if the invariant Borel sets have measure 0 or 1. Denote by $\mathscr{E}(E)$ the set of E-invariant, ergodic probability measures. Finally, we say that E is *aperiodic* if all its equivalence classes are infinite and that E is *smooth* if there is a Borel set that meets every equivalence class in exactly one point. We have the following corollary.

COROLLARY 2. *Assume CH. Let E and F be amenable Borel equivalence relations. Denote by $\cong^{\mathrm{um}}$ the relation of universally measurable isomorphism. Then for E and F aperiodic and nonsmooth, we have*

$$E \cong^{\mathrm{um}} F \Leftrightarrow \mathrm{card}(\mathscr{E}(E)) = \mathrm{card}(\mathscr{E}(F)).$$

The proof of Theorem 1 is based, on the one hand, on the result of Connes-Feldman-Weiss [CFW], which asserts the equivalence of the notions of amenability and hyperfiniteness in the measure theoretic context, and on the other hand, on the work in [DJK]. We do not know if Theorem 1 can be proved in ZFC alone.

One can view Theorem 1 as providing some evidence for a positive answer to the following problem (see [K2]), which extends Weiss's question.

Let E be an amenable countable Borel equivalence relation. Is E hyperfinite?

If this is true, then the most likely attempt for a proof would seem to be through a dichotomy theorem of the following form. If E is a countable Borel equivalence relation, then either E is hyperfinite or else there is some canonical type of non-amenable countable Borel equivalence relation which embeds into E, where F *embeds* into E if F is Borel isomorphic to the restriction of E to a Borel set. Since when E is amenable and F embeds into E, F is also amenable (see [K1]), this would show that amenability implies hyperfiniteness.

Canonical examples of nonamenable countable Borel equivalence relations come from free Borel actions of the free group with two generators F_2 with an invariant probability measure (see, e.g., [K1]). (An action $(g.x) \mapsto g.x$ is *free* if $g.x = x \Rightarrow g = 1$.) One nice such class of examples that may be manageable combinatorially, so that it could be useful to such a dichotomy result, comes from free actions of F_2 by Lipschitz automorphisms of $2^{\mathbb{N}}$. A *Lipschitz automorphism* of $2^{\mathbb{N}}$ is a homeomorphism $\pi: 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $2^{\mathbb{N}}$ onto itself such that for some sequence of permutations π_n on 2^n (= the set of finite binary sequences of length n) which is *coherent*, i.e., $\pi_m(s) \upharpoonright n = \pi_n(s \upharpoonright n)$ for any $m \geq n$, $s \in 2^m$, we have $\pi(x) = \bigcup_n \pi_n(x \upharpoonright n)$ for any $x \in 2^{\mathbb{N}}$. (For some information on Lipschitz automorphisms, see [DJK] and [SS].) It can be shown that there is a free action $(g, x) \mapsto g.x$ of F_2 on $2^{\mathbb{N}}$ with each $x \mapsto g.x$ being a Lipschitz automorphism (see [SS]). Since Lipschitz automorphisms leave invariant the canonical Lebesque measure on $2^{\mathbb{N}}$ (the product of the $\{1/2, 1/2\}$-measure on $\{0, 1\}$), the equivalence relations induced by such actions are nonamenable. So one can raise the following question.

Let E be a countable Borel equivalence relation, which is not hyperfinite. Can one embed into E the equivalence relation induced by a free action of F_2 by Lipschitz automorphisms on $2^{\mathbb{N}}$?

§1. Smooth equivalence relations and sets. In this and the next two sections, we review some concepts and results needed in the proof of Theorem 1.

Let X be a *standard Borel space*, i.e., a set equipped with a σ-algebra (its Borel sets), which is Borel isomorphic to the σ-algebra of the Borel sets in a Polish space. A *Borel equivalence relation* E on X is an equivalence relation, which is Borel as a subset of X^2 (with the product Borel structure). For each Borel subset $Y \subseteq X$, we denote by $E \upharpoonright Y = E \cap Y^2$ the *restriction* of E to Y. Finally, for each $A \subseteq X$, we denote by $[A]_E = \{x: \exists y[y \in A \ \& \ xEy]\}$ the *(E-)saturation* of A, and we denote by $[x]_E = [\{x\}]_E$ the *(E-)equivalence class* of x. A is called *(E-)invariant* if $A = [A]_E$.

A Borel equivalence relation E on X is *countable* if, for all $x \in X$, $[x]_E$ is countable. For countable E, if $A \subseteq X$ is Borel, then so is $[A]_E$ (since projections of Borel sets with countable sections are Borel). A Borel equivalence relation E is called *smooth* if there is a standard Borel space Y and a Borel map $f: X \to Y$ such that $xEy \Leftrightarrow f(x) = f(y)$. When E is countable, since every Borel relation with countable sections can be uniformized by a Borel function, this is equivalent to asserting the existence of a Borel set A which meets every E-equivalence class in exactly one point.

We denote by E_0 the equivalence relation on $2^{\mathbb{N}}$ defined by

$$xE_0y \Leftrightarrow \exists n\forall m \geq n(x(m) = y(m)).$$

This is not smooth and we have the following result.

THEOREM 1.1 (Harrington-Kechris-Louveau [HKL]). *Let E be a Borel equivalence relation. Then the following are equivalent*:

(1) *E is not smooth*;

(2) $E_0 \sqsubseteq E$.

In the above, for Borel equivalence relations E and F on X and Y, respectively, we set

$$E \sqsubseteq F \Leftrightarrow \text{there is a Borel injection } f\colon X \to Y \text{ with } xEy \Leftrightarrow f(x)Ff(y).$$

Given any countable Borel equivalence relation E on X and a Borel set $A \subseteq X$, we call *A smooth for E* if $E \restriction A$ is smooth. It is easy to check that this is equivalent to saying that $E \restriction [A]_E$ is smooth, so A is smooth iff $[A]_E$ is smooth. It is also straightforward to verify that the smooth sets form a σ-ideal, which is proper iff E is not smooth.

The following simple fact will be needed later on. Recall that a σ-ideal has the *countable chain condition* (*ccc*) if there is no uncountable family of pairwise disjoint sets not belonging to the σ-ideal.

PROPOSITION 1.2. *Let E be a Borel equivalence relation. If E is not smooth, then the σ-ideal of smooth for E sets does not have the ccc.*

PROOF. Since $E_0 \sqsubseteq E$, it is sufficient to take $E = E_0$. For $x, y \in 2^{\mathbb{N}}$ let $\langle x, y\rangle = \langle x(0), y(0), x(1), y(1), \ldots\rangle$. Set $A_x = \{\langle x, y\rangle : y \in 2^{\mathbb{N}}\}$. Clearly, $x \neq y \Rightarrow A_x \cap A_y = \emptyset$, and each A_x is not smooth for E_0, as the map $y \mapsto \langle x, y\rangle$ embeds E_0 into $E_0 \restriction A_x$. ⊣

The smooth sets can be characterized alternatively as follows.

> Given a Borel equivalence relation E on X and a (Borel) probability mea- μ on X, we say that μ is *E-ergodic* if every Borel E-invariant set has μ-measure 0 or 1.

We also say that μ is *E-nonatomic* if $\mu([x]_E) = 0$, $\forall x \in X$. Then we have

THEOREM 1.3 (Harrington-Kechris-Louveau [HKL]). *Let E be a Borel equivalence relation on X, and let $A \subseteq X$ be a Borel set. The following are equivalent*:

(1) *A is smooth for E*;

(2) *A is μ-null for every E-ergodic, nonatomic probability measure μ.*

When E is countable, this has a further equivalent. Call a probability measure μ *E-quasi-invariant* if the saturation of any μ-null Borel set is μ-null. Given any countable Borel equivalence relation E and a probability measure μ, there is an E-quasi-invariant probability measure μ^* so that $\mu \ll \mu^*$ and μ and μ^* agree on the E-invariant sets (so μ is E-ergodic iff μ^* is E-ergodic). To see this, recall the following basic representation theorem.

THEOREM 1.4 (Feldman-Moore [FM]. *Let E be a countable Borel equivalence relation on X. Then there is a countable group G and a Borel action $(g, x) \in G \times X \mapsto g.x \in X$ such that E is the equivalence relation induced by the orbits of this action, i.e., $xEy \Leftrightarrow \exists g \in G(g.x = y)$.*

Using this theorem and letting $G = \{g_1, g_2, \ldots\}$, we can define μ^* by $\mu^*(A) = \Sigma 2^{-i}\mu(g_i \cdot A)$ for any Borel set A.

We can now add the following equivalence to Theorem 1.3.

THEOREM 1.3 (continued). *When E is a countable Borel equivalence relation,* (1) *is also equivalent to*

(3) *A is μ-null for every E-ergodic, nonatomic, quasi-invariant probability measure.*

Denote by $\cong$ the relation of *Borel isomorphism* between equivalence relations, i.e.,

$$E \cong F \Leftrightarrow \exists f\colon X \to Y \ (f \text{ is a Borel bijection and } xEy \Leftrightarrow f(x)Ff(y)).$$

Also let

$$E \sqsubseteq^i F \Leftrightarrow \exists A \subseteq Y \ (A \text{ is } F\text{-invariant and } E \cong F \upharpoonright A).$$

Finally, call E *aperiodic* if every E-equivalence class is infinite. We then have the following simple fact.

PROPOSITION 1.5. *Let E and F be smooth aperiodic countable Borel equivalence relations on uncountable standard Borel spaces X and Y. Then $E \cong F$. If R is an aperiodic, nonsmooth Borel equivalence relation on the standard Borel space Z, then $E \sqsubseteq^i R$.*

PROOF. Let $A \subseteq X$ (resp. $B \subseteq Y$) be Borel sets meeting every E (resp. F)-equivalence class in exactly one point. Then A and B are uncountable, so let $\pi\colon A \to B$ be a Borel isomorphism. Since E, F are aperiodic, let $f_n\colon A \to X$, $g_n\colon B \to Y$ be such that $f_n(x) \neq f_m(x)$, $g_n(x) \neq g_m(x)$ if $n \neq m$ and $[x]_E = \{f_n(x)\colon n \in \mathbb{N}\}$, $\forall x \in A$, $[y]_F = \{g_n(y)\colon n \in \mathbb{N}\}$, $\forall y \in B$. Then define $\rho\colon X \to Y$ by $\rho(f_n(x)) = g_n(\pi(x))$. Clearly, ρ is a Borel isomorphism of E with F.

For the second assertion, it is enough to find an uncountable smooth Borel invariant subset for R. Since $E_0 \sqsubseteq R$, it is enough to prove this for E_0.

For each $x \in 2^{\mathbb{N}}$, let

$$\hat{x} = \{\bar{x}(0), \bar{x}(1), \bar{x}(2), \ldots\},$$

where $\bar{x}(n) = p_0^{x(0)+1} \cdots p_{n-1}^{x(n-1)+1}$ with $p_n = (n+1)$th prime number. By identifying $\hat{x}$ with its characteristic function, we have $x \neq y \Rightarrow \neg(\hat{x}E_0\hat{y})$. Then $[\{\hat{x}\colon x \in 2^{\mathbb{N}}\}]_{E_0}$ works. ⊣

§2. Hyperfiniteness. A Borel equivalence relation E on X is called *hyperfinite* if it is induced by a Borel $\mathbb{Z}$-action, i.e., if there is a Borel automorphism T of X such that $xEy \Leftrightarrow \exists n \in \mathbb{Z}\ (T^n x = y)$. Trivially, smooth $\Rightarrow$ hyperfinite.

For the basic theory and classification of such relations, see [DJK]. We will recall here some results of this paper. First denote by $\approx$ the relation of *biembeddability*, i.e.,

$$E \approx F \Leftrightarrow E \sqsubseteq F \ \& \ F \sqsubseteq E.$$

Then we have

THEOREM 2.1 (Dougherty-Jackson-Kechris [DJK]). *Let E, F be nonsmooth, hyperfinite Borel equivalence relations. Then $E \approx F$.*

Next we have a classification up to Borel isomorphism. For each countable Borel equivalence relation E on X, we call a probability measure μ on X *E-invariant* if, for a Borel group action $(g, x) \mapsto g.x$ of a countable group G inducing E, we have that μ is invariant under this action, i.e., $g.\mu = \mu \ \forall g \in G$. (This is easily seen to be independent of the choice of the action—see [DJK].) Denote by $\mathscr{E}(E)$ the set of E-invariant, ergodic probability measures. Then we have

THEOREM 2.2 (Dougherty-Jackson-Kechris [DJK]). *Let E, F be aperiodic, nonsmooth, hyperfinite Borel equivalence relations. Then*

$$E \cong F \Leftrightarrow \operatorname{card}(\mathscr{E}(E)) = \operatorname{card}(\mathscr{E}(F)).$$

The possible values for $\operatorname{card}(\mathscr{E}(E))$ are $0, 1, 2, \ldots, \aleph_0, 2^{\aleph_0}$ (see [DJK]). Examples of equivalence relations obtaining these values in that order are E_t on $2^{\mathbb{N}}$ (where $xE_ty \Leftrightarrow \exists n \exists m \forall k (x_{n+k} = y_{m+k})$, $E_0, E_0 \times \Delta(n)$, $2 \leq n \leq \aleph_0$ (where $\Delta(n)$ is the equality relation on n elements); products are defined as usual by

$$E \times F = \{((x, y), (x', y')) \colon xEx' \ \& \ yFy'\},$$

$E^*(\mathbb{Z}, 2)$ ($=$ is the restriction of the equivalence relation induced by the shift on $2^{\mathbb{Z}}$ on the aperiodic part of $2^{\mathbb{Z}}$).

The equivalence relations with $\mathscr{E}(E) = \varnothing$ can be characterized as follows. We call a countable Borel equivalence relation E on X *compressible* if there is a Borel injection $f \colon X \to X$ with $f(x)Ex, \forall x \in X$ and $f([x]_E) \neq [x]_E, \forall x \in X$. Then we have the following theorem.

THEOREM 2.3 (Nadkarni [N]). *Let E be a countable Borel equivalence relation. Then $\mathscr{E}(E) = \varnothing$ iff E is compressible.*

Compressible equivalence relations have another basic property (see, e.g., [DJK]). Namely, if E, F are countable Borel equivalence relations and E is compressible, then $E \sqsubseteq F$ iff $E \sqsubseteq^i F$. In particular, from Theorems 2.1 and 1.1, it follows that if E is a nonsmooth, aperiodic, compressible, hyperfinite Borel equivalence relation, then for any nonsmooth, countable Borel equivalence F, we have $E \sqsubseteq^i F$. Also, if E is smooth and aperiodic (thus, compressible), then the same conclusion holds by Proposition 1.5 and Theorem 1.1. Finally, note that for E and F countable and compressible, $E \approx F$ iff $E \cong F$.

The equivalence relation E_0 has a unique invariant probability measure. This is of course the standard Lebesgue measure $\boldsymbol{m}$ on $2^{\mathbb{N}}$, i.e., the product of the $(1/2, 1/2)$ measure on $\{0, 1\}$. It follows that if $A \subseteq 2^{\mathbb{N}}$ is a Borel E_0-invariant set with $\boldsymbol{m}(A) = 0$, then $E_0 \upharpoonright A$ has no invariant probability measure. Note also, that if $A \subseteq 2^{\mathbb{N}}$ is Borel with $\boldsymbol{m}(A) > 0$, then A is not E_0-smooth (since $E_0 \upharpoonright [A]_{E_0}$ admits a nonatomic, ergodic, invariant measure).

Finally, we recall the following classical result of ergodic theory. (See, e.g., [W1].)

THEOREM 2.4 (Dye's Theorem). *Let E and F be hyperfinite Borel equivalence relations on X and Y, respectively, and let μ and ν be nonatomic probability measures in $\mathscr{E}(E)$ and $\mathscr{E}(F)$, respectively. Then there are invariant Borel sets $X_0 \subseteq X$, $Y_0 \subseteq Y$ with $\mu(X_0) = \nu(Y_0) = 1$ and $E \upharpoonright X_0 \cong F \upharpoonright Y_0$ via a Borel isomorphism that sends μ to ν.*

§3. Amenability.

First, we recall from [K1] the notion of an amenable countable Borel equivalence relation.

Let X be a standard Borel space, and let E be a countable Borel equivalence relation on X. We call E *amenable* if there is a map $C \mapsto \Phi_C$, assigning to each

E-equivalence class $C = [x]_E$ of E a *mean* Φ_C on C (i.e., a continuous linear functional on $l^\infty(C)$, the Banach space of bounded real functions on C, such that $\inf(f) \le \Phi_C(f) \le \sup(f)$), with the property that $C \mapsto \Phi_C$ is universally measurable, i.e., for each bounded Borel $F: X^2 \to \mathbb{R}$ the function $f: X \to \mathbb{R}$ given by $f(x) = \Phi_{[x]_E}(F_x)$ is universally measurable.

A countable group G is *amenable* if there is a mean Φ on G with $\Phi(f) = \Phi(h \mapsto f(gh))$ for all $g \in G$, $f \in l^\infty(G)$. If E is an equivalence relation induced by a Borel action of an amenable group G on X, then (see [K1], 2.3) E is amenable, assuming CH. In particular, this is true when G is abelian, solvable, etc.

This notion of amenability for countable Borel equivalence relations comes from a concept of amenability relative to a given probability measure on X due to Zimmer [Z], which has been reformulated in Connes-Feldman-Weiss [CFW] as follows.

Let X be a standard Borel space, let μ be a (Borel) probability measure on X, and let E be a countable Borel equivalence relation on X. We say that E is *μ-amenable* if there is a map $C \mapsto \Phi_C$ assigning to each E-equivalence class C of E a mean Φ_C on C such that for each bounded Borel map $F: X^2 \to \mathbb{R}$, if $f(x) = \Phi_{[x]_E}(F_x)$, then $f: X \to \mathbb{R}$ is μ-measurable.

The equivalence of this definition, from [CFW], to the original one in [Z] is given, for example, in [AL].

The following fundamental result on amenability will be crucial in our proof below.

THEOREM 3.1 (Connes-Feldman-Weiss [CFW]). *Let X be a standard Borel space, let μ be a probability measure on X, and let E be a countable Borel equivalence relation on X. If E is μ-amenable, then there is a Borel E-invariant set $Y \subseteq X$ with $\mu(Y) = 1$ such that $E \restriction Y$ is hyperfinite.*

In [CFW], this result is stated for μ E-quasi-invariant, but it is easily seen to hold for any μ. To see this, given any μ, let μ^* be the E-quasi-invariant measure defined just after Theorem 1.4. Then E is μ^*-amenable. This follows from the fact that a function $f: X \to \mathbb{R}$ is μ^*-measureable iff for all $g \in G$ (notation as in Theorem 1.4) the function $f_g(x) = f(g.x)$ is μ-measurable. If $C \mapsto \Phi_C$ shows that E is μ-amenable, then given any bounded Borel map $F: X^2 \to \mathbb{R}$, if $f(x) = \Phi_{[x]_E}(F_x)$, then for all $g \in G$, $f_g(x) = \Phi_{[g.x]_E}(F_{g.x}) = \Phi_{[x]_E}(F^g_x)$, where $F^g(x, y) = F(g.x, y)$, so F^g is still Borel bounded, and thus, f_g is μ-measurable.

Let us note the following fact. By a *universally measurable automorphism* of X, we mean a bijection T of X such that both T and T^{-1} are universally measurable.

PROPOSITION 3.2. *Assume CH. Let E be a countable Borel equivalence relation on a standard Borel space X. Then the following are equivalent*:

(i) *E is amenable*;

(ii) $\forall \mu$ (*E is μ-amenable*);

(ii)* $\forall \mu$(*there is a Borel E-invariant set $Y \subseteq X$ with $\mu(Y) = 1$ and $E \restriction Y$ hyperfinite*);

(iii) *E is "universally measurable" hyperfinite, i.e., there is a universally measurable automorphism T of X such that $xEy \Leftrightarrow \exists n \in \mathbb{Z}(T^n(x) = y)$.*

PROOF. Clearly (i) $\Rightarrow$ (ii).

To prove (ii) $\Rightarrow$ (iii) we argue as follows. By the CH, enumerate all probability measures on X in a sequence $\{\mu_\alpha\}_{\alpha < \omega_1}$ of length ω_1 = the first uncountable ordinal.

By Theorem 3.1, find an invariant Borel set B_0 and a Borel automorphism T_0 of B_0 such that $\mu_0(B_0) = 1$ and T_0 induces $E_0 \upharpoonright B_0$. Let α_1 be the least index such that $\mu_{\alpha_1}(B_0) < 1$. Then by Theorem 3.1 again, find an invariant Borel set B_1 disjoint from B_0 and a Borel automorphism T_1 of B_1 such that $\mu_{\alpha_1}(B_0 \cup B_1) = 1$ and T_1 induces $E \upharpoonright B_1$. Proceed this way by transfinite induction to find $\{B_\alpha\}_{\alpha<\omega_1}$, $\{T_\alpha\}_{\alpha<\omega_1}$, where the B_α are Borel and pairwise disjoint, and each T_α is a Borel automorphism of B_α such that $E \upharpoonright B_\alpha$ is induced by T_α and $\mu_\alpha\left(\bigcup_{\beta\leq\alpha} B_\beta\right) = 1$. Then $\bigcup_{\alpha<\omega_1} B_\alpha = X$ (by looking at the Dirac measures), and so we can define the bijection T on X by $T \upharpoonright B_\alpha = T_\alpha \upharpoonright B_\alpha$. Clearly, T is universally measurable (as it is equal to $\bigcup_{\beta\leq\alpha} T_\beta$ μ_α-a.e.), and T induces E.

Finally, we show that (iii) $\Rightarrow$ (i). By Mokobodzki's Theorem (see, e.g., [K1]), which uses CH, fix a universally measurable mean on $\mathbb{Z}$ such that $\Phi(p) = \Phi(m \mapsto p(m+n))$ for any $n \in \mathbb{Z}$. (To say that Φ is universally measurable means that $\Phi_{\mathbb{Z}} \upharpoonright [-1,1]^{\mathbb{Z}}$ is universally measurable as a map from $[-1,1]^{\mathbb{Z}}$ into $[-1,1]$.) Then define, for any E-equivalence class C, the mean on C given by

$$\varphi_C(p) = \Phi_{\mathbb{Z}}(n \mapsto p(T^n(x)))$$

for any $x \in C$. It is easy to see that this is independent of x. To see that $C \mapsto \varphi_C$ is universally measurable, note that for $F\colon X^2 \to \mathbb{R}$ Borel bounded, we have $f(x) = \Phi_{[x]_E}(F_x) = \Phi_{\mathbb{Z}}(n \mapsto F_x(T^n(x))) = \Phi_{\mathbb{Z}}(n \mapsto F(x, T^n(x)))$, so f is universally measurable, since universally measurable functions are closed under composition.

The equivalence of (ii), (ii)* follows from 3.1. ⊣

Finally, for later use, we compute (an upper bound for) the complexity of the concept of μ-amenability. Without loss of generality, we will work with the Cantor space $X = 2^{\mathbb{N}}$. A Borel probability measure on this space can be identified with a function $\mu\colon 2^{<\mathbb{N}} \to [0,1]$ ($2^{<\mathbb{N}}$ = the set of finite binary sequences) such that $\mu(s) = \mu(s^\frown 0) + \mu(s^\frown 1)$ and $\mu(\emptyset) = 1$. Call this set M. It is clearly a Π^0_1 set in $[0,1]^{2^{<\mathbb{N}}}$. Fix (see, e.g., Moschovakis [M]) a Π^1_1 set $C \subseteq \mathbb{N}^{\mathbb{N}}$ and Σ^1_1, resp., Π^1_1 sets S, resp., $R \subseteq \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that for $\alpha \in C$, $R_\alpha = S_\alpha$ $(= B_\alpha)$ and, for every Borel set $B \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, there is an $\alpha \in C$ with $B_\alpha = B$. We view C and the map $\alpha \in C \mapsto B_\alpha$ as a coding of the Borel binary relations on $2^{\mathbb{N}}$. It is easy to check that

$$E = \{\alpha \in C\colon B_\alpha \text{ is a countable equivalence relation}\}$$

is Π^1_1.

PROPOSITION 3.3. *The set*

$$\{(\mu, \alpha) \in M \times \mathbb{N}^{\mathbb{N}}\colon B_\alpha \textit{ is a countable equivalence which is } \mu\textit{-amenable}\}$$

is Π^1_1.

REMARK. Actually, one can improve Π^1_2 to Π^1_1 by using Theorem 3.1 and deep results of Connes on operator algebras. In fact, the above set is Δ^1_1 on $M \times E$.

PROOF. We will use Zimmer's definition of μ-amenability for a countable Borel equivalence relation (see [Z] and also [AL] for a proof of its equivalence with the one used above).

Fix a standard Borel space X, a countable Borel equivalence relation E on X, and a probability measure μ on X. Let B be a (complex) separable Banach space, and let $\mathrm{Iso}(B)$ be the group of isometries of B with the strong operator topology. Let B^*_1 be the closed unit ball of B^* with the weak*-topology. For each $T \in \mathrm{Iso}(B)$, denote (by abuse of notation) by T^* the adjoint restricted to B_1^*, so that T^* is a homeomorphism of B_1^*.

A map $\alpha\colon X^2 \to \mathrm{Iso}(B)$ is called a *Borel cocycle* if it is Borel and $\alpha(x,y)\alpha(y,z) = \alpha(x,z)$ for all $xEyEz$. Its *adjoint cocycle* α^* is defined by $\alpha^*(x,y) = (\alpha(x,y)^{-1})^*$. A Borel map $x \mapsto K_x$ from X into the space of compact subsets of B_1^* (with the usual Hausdorff topology) is called a *Borel field* if each K_x is nonempty, compact, and convex. A μ-measurable map $\varphi\colon X \to B_1^*$ is called a *section* of $\{K_x\}$ if $\varphi(x) \in K_x$ μ-a.e. A Borel field $\{K_x\}$ is called *α-invariant* if for μ-a.e. x

$$y \in [x]_E \Rightarrow \alpha^*(x,y)K_y = K_x,$$

and then a section φ is called *α-invariant* if for μ-a.e. x

$$y \in [x]_E \Rightarrow \alpha^*(x,y)\varphi(y) = \varphi(x).$$

Finally, E is *μ-amenable* (*according to Zimmer*) if for every complex separable Banach space B and every Borel cocycle $\alpha\colon X^2 \to \mathrm{Iso}(B)$, every α-invariant Borel field $\{K_x\}$ has an α-invariant section.

The translation of this definition into a Π^1_2 formula is a straightforward, but tedious, coding exercise. We make some comments about one (of the many) possible ways of encoding the various objects involved in this definition and leave the details of the verification to the reader.

A separable Banach space B is a closed subspace of $C(2^{\mathbb{N}})$ and, hence, can be coded by a countable dense subset of it, i.e., a member of $C(2^{\mathbb{N}})^{\mathbb{N}}$. An isometry of B can be coded by its restriction to this countable dense subset of B. One can view B_1^* as a closed subset of $\Delta^{\mathbb{N}}$, where Δ is the closed unit disc in $\mathbb{C}$, identifying $b^* \in B_1^*$, with its restriction to the countable dense subset of B intersected with the unit ball. Thus, Borel fields can be viewed as Borel maps of X $(=2^{\mathbb{N}}$ in our case) into the space of compact subsets of $\Delta^{\mathbb{N}}$. Finally, sections, being μ-measurable, can be coded as sequences of continuous functions from X $(=2^{\mathbb{N}})$ into $\Delta^{\mathbb{N}}$ which converge pointwise μ-a.e. ⊣

§4. The main theorem. We prove here the main result of this paper.

THEOREM 4.1. *Assume CH. Let E be an amenable countable Borel equivalence relation on X. Then there is a hyperfinite Borel equivalence relation F, which is universally measurable isomorphic to E.*

PROOF. We will need the following technical lemma, whose proof is postponed to §5.

LEMMA 4.2. *There is a sequence $\{T_\theta\}_{\theta<\omega_1}$ of pairwise disjoint, nonsmooth, invariant Borel sets for E_0 such that if $T = \bigcup_{\theta<\omega_1} T_\theta$, then T is Borel and $\mathbf{m}(T) = 0$ (where $\mathbf{m}$ is the Lebesgue measure on $2^{\mathbb{N}}$), and for any Borel probability measure μ on $2^{\mathbb{N}}$ with $\mu(T) = 1$, we have $\mu\left(\bigcup_{\theta<\alpha} T_\theta\right) = 1$ for some $\alpha < \omega_1$ (depending on μ).*

Similarly, there is a sequence $\{S_\theta\}_{\theta<\omega_1}$ of pairwise disjoint, uncountable, smooth invariant Borel sets for E_0, such that if $S = \bigcup_{\theta<\omega_1} S_\theta$, then S is Borel smooth, and for any Borel probability measure μ on $2^{\mathbb{N}}$ with $\mu(S) = 1$, we have $\mu\left(\bigcup_{\theta<\alpha} S_\theta\right) = 1$ for some $\alpha < \omega_1$ (depending on μ).

Let F be a countable Borel equivalence relation on the standard Borel space Y. Let $A \subseteq Y$ be a universally measurable invariant set. We say that A is *inner smooth* if every Borel $B \subseteq A$ is smooth. This is equivalent to saying that there is no Borel embedding of E_0 into Y with range contained in A and is also equivalent to saying that $\mu(A) = 0$ for all nonatomic, ergodic, probability measures μ (by Theorems 1.1 and 1.3). We call A *inner compressible* if every invariant Borel subset of A is compressible, or equivalently $\mu(A) = 0$, for all ergodic, invariant, probability measures μ (by Theorem 2.3).

LEMMA 4.3. *Assume CH. Let F be an aperiodic, countable Borel equivalence relation on the standard Borel space Y. Let $A \subseteq Y$ be universally measurable and inner smooth. Then there is a smooth Borel equivalence relation R on a standard Borel space Z and a universally measurable isomorphism of $F \restriction A$ with R.*

PROOF. By Proposition 1.5, if μ is a nonatomic Borel probability measure on Y with $\mu(A) > 0$ and U is an uncountable smooth invariant Borel set for E_0, then there is Borel invariant $V \subseteq A$ with $\mu(A \setminus V) = 0$ such that $F \restriction V \cong E_0 \restriction U$.

Now enumerate all nonatomic probability measures μ on Y with $\mu(A) > 0$ in a sequence $\{\mu_\alpha\}_{\alpha<\omega_1}$, and define by induction pairwise disjoint Borel invariant sets $A_\alpha \subseteq A$ such that $F \restriction A_\alpha \cong E_0 \restriction S_\alpha$, where $\{S_\alpha\}$ are as in Lemma 4.2 and $\mu_\alpha(A \setminus \bigcup_{\beta \leq \alpha} A_\beta) = 0$. (This can be done because if A_β, $\beta < \alpha$, have been defined, then we can use $A \setminus \bigcup_{\beta<\alpha} A_\beta$ instead of A in the preceding claim.) But then, by piecing together the Borel isomorphisms of A_α, S_α, we obtain a universally measurable isomorphism of $F \restriction A$ with $E_0 \restriction S$. Thus, we can take $Z = S$ and $R = E_0 \restriction S$. ⊣

We can prove now the following key lemma for the proof of Theorem 4.1.

LEMMA 4.4. *Assume CH. Let F be an aperiodic, amenable, countable Borel equivalence relation on the standard Borel space Y. Let $A \subseteq Y$ be universally measurable and inner compressible. Then there is a hyperfinite Borel equivalence relation R on a standard Borel space Z and a universally measurable isomorphism of $F \restriction A$ with R.*

PROOF. Assume this fails. Then we claim that if μ is a Borel probability measure on Y with $\mu(A) > 0$ and U is a nonsmooth invariant Borel set for E_0 with $\underset{\sim}{m}(U) = 0$, there is a Borel invariant set $V \subseteq A$ with $\mu(A \setminus V) = 0$ such that $F \restriction V \cong E_0 \restriction U$.

Granting this, we obtain a contradiction, exactly as in the proof of Lemma 4.3, by using the sequence $\{T_\alpha\}$ of Lemma 4.2.

We now prove this claim. We will attempt to define transfinite sequences of pairwise disjoint Borel invariant sets $V_\alpha \subseteq A$, $U_\alpha \subseteq U$ for $\alpha < \omega_1$, such that if $\{\mu_\alpha\}_{\alpha<\omega_1}$, with $\mu_0 = \mu$, $\{\nu_\alpha\}_{\alpha<\omega_1}$ enumerate, respectively, the probability Borel measures with $\mu_\alpha(A) > 0$, $\nu_\alpha(U) > 0$, then $\mu_\alpha(A \setminus \bigcup_{\beta\leq\alpha} V_\beta) = 0$, $\nu_\alpha(U \setminus \bigcup_{\beta\leq\alpha} U_\beta) = 0$, and $F \restriction V_\alpha \cong E_0 \restriction U_\alpha$. If this succeeds for each α, then we get a universally measurable isomorphism of $F \restriction A$ with $E_0 \restriction U$, which is a contradiction.

We start with an invariant Borel set $V'_0 \subseteq A$ such that $\mu(A \setminus V'_0) = 0$. By the Connes-Feldman-Weiss Theorem 3.1, we can also assume that $F \restriction V'_0$ is hyperfinite. Then by Proposition 1.5, Theorem 2.1, and the fact that $F \restriction V'_0$ is aperiodic and compressible, there is a Borel invariant set $U'_0 \subseteq U$ with $F \restriction V'_0 \cong E_0 \restriction U'_0$. Then find a Borel invariant set U_0 with $U'_0 \subseteq U_0 \subseteq U$ and $\nu_0(U \setminus U_0) = 0$. By Lemma 4.3, $A \setminus V'_0$ is not inner smooth, so there is a Borel invariant V_0, $V'_0 \subseteq V_0 \subseteq A$ such that $F \restriction V_0 \cong E_0 \restriction U_0$ (notice here that $E_0 \restriction U_0$ is compressible, since $\boldsymbol{m}(U_0) = 0$).

Assume now that all V_β, U_β for $\beta < \alpha$ have been constructed. Clearly, $A \setminus \bigcup_{\beta<\alpha} V_\beta$ is still not inner smooth (by Lemma 4.3). Find V'_α Borel invariant such that $\mu_\alpha(A \setminus (\bigcup_{\beta<\alpha} V_\beta \cup V'_\alpha)) = 0$, $F \restriction V'_\alpha$ is hyperfinite, and $V'_\alpha \cap V_\beta = \phi$, if $\beta < \alpha$. Then

$A \setminus (\bigcup_{\beta<\alpha} V_\beta \cup V'_\alpha)$ is not inner smooth. If $U \setminus \bigcup_{\beta<\alpha} U_\beta$ is not smooth, then we can proceed, as in the case $\alpha = 0$, to find U_α, V_α. It follows that for some least $\alpha > 0$, $U \setminus \bigcup_{\beta<\alpha} U_\beta$ is smooth. But then in this case, since $\bigcup_{\beta<\alpha} V_\beta \cong \bigcup_{\beta<\alpha} U_\beta$ and $A \setminus \bigcup_{\beta<\alpha} V_\beta$ is not inner smooth, it is straightforward that $F \restriction V \cong E_0 \restriction U$ for some Borel invariant set $V \supseteq \bigcup_{\beta<\alpha} V_\beta$, and we are done, since $\mu(A \setminus V) = 0$. ⊣

We now finish the proof of Theorem 4.1. Given E, as in the statement of the theorem, we can, of course, assume that E is aperiodic and nonsmooth. We consider two cases: $\mathscr{E}(E) = \varnothing$ or $\mathscr{E}(E) \neq \varnothing$.

If $\mathscr{E}(E) = \varnothing$, then, by Theorem 2.3, E is compressible, and we are done by Lemma 4.4.

So assume $\mathscr{E}(E) \neq \varnothing$. We will need here an Ergodic Decomposition Theorem of Varadarajan [V].

THEOREM 4.5 (Varadarajan [V]). *Let E be a countable Borel equivalence relation on a standard Borel space X. Assume $\mathscr{E}(E) \neq \varnothing$. The set $\mathscr{E}(E)$ is Borel (in the standard Borel space of probability Borel measures on X), and there is a Borel surjection $x \mapsto e_x$ from X onto $\mathscr{E}(E)$ such that*

(1) $xEy \Rightarrow e_x = e_y$;

(2) *if $X_e = \{x: e_x = e\}$, then $e(X_e) = 1$ and e is the unique invariant, ergodic probability measure for $E \restriction X_e$;*

(3) *if μ is an invariant probability Borel measure for E, then $\mu(A) = \int e_x(A)\, d\mu(x)$.*

Since $\mathscr{E}(E) \neq \varnothing$ is a Borel set in a standard Borel space, it has cardinality $1, 2, \ldots, \aleph_0$ or $2^{\aleph_0}$. Choose a hyperfinite, nonsmooth, aperiodic Borel equivalence relation F on a standard Borel space Y with $\mathscr{E}(F)$ having the same cardinality as $\mathscr{E}(E)$. Let π: $\mathscr{E}(E) \to \mathscr{E}(F)$ be a Borel isomorphism, and set $\pi(e) = e^*$. Finally, let $x \mapsto e_x$, $y \mapsto f_y$, $(X_e)_{e \in \mathscr{E}(E)}$, $(Y_f)_{f \in \mathscr{E}(F)}$ be the ergodic decompositions of E and F as in Theorem 4.5.

We need here a further technical lemma, whose proof we postpone to §5.

Fix a Π^1_1 set $C^X \subseteq \mathbb{N}^\mathbb{N}$ and Π^1_1, resp., Σ^1_1 sets $R^X \subseteq \mathbb{N}^\mathbb{N} \times X$, resp., $S^X \subseteq \mathbb{N}^\mathbb{N} \times X$, such that $\alpha \in C^X \Rightarrow R^X_\alpha = S^X_\alpha\ (= B^X_\alpha)$ and for every Borel set $B \subseteq X$, there is $\alpha \in C^X$ with $B = B^X_\alpha$. We view such an α as a code for B. Similarly, we choose (C^Y, R^Y, S^Y) for Y. Finally, we fix a Π^1_1 set $C^{X,Y} \subseteq \mathbb{N}^\mathbb{N}$ and Π^1_1, resp., Σ^1_1 sets $R^{X,Y} \subseteq \mathbb{N}^\mathbb{N} \times X \times Y$, resp., $S^{X,Y} \subseteq \mathbb{N}^\mathbb{N} \times X \times Y$ such that $\alpha \in C^X \Rightarrow (R^{X,Y}_\alpha = S^{X,Y}_\alpha$ and is the graph of a partial function from X into Y) and such that for every Borel function f with domain a Borel subset of X and values in Y, we have graph$(f) = R^{X,Y}_\alpha = S^{X,Y}_\alpha$ for some $\alpha \in C^{X,Y}$. In this case we write $f = f^{X,Y}_\alpha$. Again, we view α as a code of f. (For the existence of these codings, see, e.g., Moschovakis [M].)

LEMMA 4.6. *There are universally measurable functions a, a^*, b: $\mathscr{E}(E) \to \mathbb{N}^\mathbb{N}$ such that for each $e \in \mathscr{E}(E)$, $a(e)$ codes a Borel invariant subset $B^X_{a(e)} \subseteq X_e$, $a^*(e)$ codes a Borel invariant subset $B^Y_{a^*(e)} \subseteq Y_{e^*}$ with $e(B^X_{a(e)}) = e^*(B^Y_{a^*(e)}) = 1$, and $b(e)$ codes a Borel isomorphism of $E \restriction B^X_{a(e)}$ with $F \restriction B^Y_{a^*(e)}$ such that $f^{X,Y}_{b(e)}$ sends e to e^*.*

Granting this, let

$$x \in X_0 \Leftrightarrow \exists e[x \in B^X_{a(e)}] \Leftrightarrow x \in B^X_{a(e_x)} \Leftrightarrow R^X(a(e_x), x),$$
$$x \in Y_0 \Leftrightarrow \exists e[y \in B^Y_{a^*(e)}] \Leftrightarrow y \in B^Y_{a^*(\pi^{-1}(f_y))} \Leftrightarrow R^Y(a^*(\pi^{-1}(f_y)), y).$$

Then, since Π^1_1 sets are universally measurable and universally measurable functions are closed under composition, X_0 and Y_0 are universally measurable. Define

$f_0: X_0 \to Y_0$ by

$$f_0(x) = f_{b(e_x)}(x).$$

Clearly, f_0 is an isomorphism of $E \restriction X_0$ with $F \restriction Y_0$. We claim that f_0 is universally measurable. Define $\tilde{f}: \mathbb{N}^{\mathbb{N}} \times X \to Y$ by

$$\tilde{f}(\alpha, x) = \begin{cases} f_\alpha^{X,Y}(x) & \text{if } \alpha \in C^{X,Y} \text{ and } x \in \operatorname{dom}(f_\alpha^{X,Y}), \\ y_0 & \text{(some fixed element of } Y\text{) otherwise.} \end{cases}$$

Then $\tilde{f}$ is universally measurable (the $\tilde{f}$-inverse image of a Borel set is in the σ-algebra generated by the Σ_1^1 sets) and for $x \in X_0$, $f_0(x) = \tilde{f}(b(e_x), x)$, so f_0 is universally measurable. Similarly, we deal with f_0^{-1}.

Let $X_1 = X \setminus X_0$, $Y_1 = Y \setminus Y_0$. It is enough to show that $E \restriction X_1$ is universally measurable isomorphic to $F \restriction Y_1$. First notice that, by property (3) of Theorem 4.5 and Theorem 2.3, X_1 is inner compressible and so is Y_1. Next, we can assume that X_1, Y_1 are not inner smooth. To see this, fix $e_0 \in \mathscr{E}(E)$. By Proposition 1.2, since $E \restriction B_{a(e_0)}^X$ is not smooth (since it admits an ergodic nonatomic measure), we can find a Borel set $B_0 \subseteq B_{a(e_0)}^X$ which is not smooth and has e_0-measure 0. Replace $B_{a(e_0)}^X$ by $B_{a(e_0)}^X \setminus B_0$, $B_{a^*(e_0)}^X$ by $f_{b(e_0)}^{X,Y}[B_{a(e_0)}^X \setminus B_0]$, and $f_{b(e_0)}^{X,Y}$ by $f_{b(e_0)}^{X,Y} \restriction [B_{a(e_0)}^X \setminus B_0]$.

By Lemma 4.4, $E \restriction X_1$ is universally measurably isomorphic to a hyperfinite Borel equivalence relation R_1 on some standard Borel Z_1. Then R_1 is aperiodic, nonsmooth, and compressible (by Theorems 1.1, 1.3, and 2.3). Similarly, find R_2 for $F \restriction Y_1$. Then by Theorems 2.2 and 2.3, $R_1 \cong R_2$, so $E \restriction X_1$ are universally measurable isomorphic. ⊣

§5. Proof of Lemmas 4.2 and 4.6.

Proof of Lemma 4.2. For each $x \in 2^{\mathbb{N}}$, let $\hat{x} \subseteq \mathbb{N}$ be the set $\hat{x} = \{\bar{x}(0), \bar{x}(1), \bar{x}(2), \ldots\}$, where $\bar{x}(n) = p_0^{x(0)+1} \cdot p_1^{x(1)+1} \cdots p_{n-1}^{x(n-1)+1}$, with $p_n =$ the $(n+1)$th prime number. Identify $\hat{x}$ with its characteristic function, so $\hat{x} \in 2^{\mathbb{N}}$ as well. Clearly, $x \neq y \Rightarrow \neg(xE_0\hat{y})$. For $a, b \in 2^{\mathbb{N}}$, let $\langle a, b\rangle = (a(0), b(0), a(1), b(1), \ldots) \in 2^{\mathbb{N}}$, and define $T \subseteq 2^{\mathbb{N}}$ by

$$x \in T \Leftrightarrow \exists y, z, w \in 2^{\mathbb{N}}[xE_0 y \wedge y = \langle \hat{z}, w\rangle].$$

Clearly, T is Borel and E_0-invariant. Moreover, $\boldsymbol{m}(T) = 0$, as can be seen as follows. The map $(a, b) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} \mapsto \langle a, b\rangle \in 2^{\mathbb{N}}$ is a measure-preserving Borel bijection of $(2^{\mathbb{N}} \times 2^{\mathbb{N}}, \boldsymbol{m} \times \boldsymbol{m})$ with $(2^{\mathbb{N}}, \boldsymbol{m})$. The set $\{\hat{z}: z \in 2^{\mathbb{N}}\} \times 2^{\mathbb{N}}$ has $\boldsymbol{m} \times \boldsymbol{m}$-measure 0, since $\{\hat{z}: z \in 2^{\mathbb{N}}\}$ (being smooth for E_0) has $\boldsymbol{m}$-measure 0. Thus, $\{\langle \hat{z}, w\rangle: z, w \in 2^{\mathbb{N}}\}$ has $\boldsymbol{m}$-measure 0 and hence, so does T (being its E_0-saturation).

For $x \in T$, notice that there is a unique z with $\langle \hat{z}, w\rangle E_0 x$ for some w. This is because $\langle \hat{z}, w\rangle E_0 \langle \hat{v}, t\rangle \Rightarrow \hat{z} E_0 \hat{v} \Rightarrow z = v$. So define $\varphi: T \to \omega_1$ by

$$\varphi(x) = \omega_1^z \qquad (= \text{the } \textit{Church-Kleene ordinal} \text{ of } z),$$

where $\langle \hat{z}, w\rangle E_0 x$ for some w.

Now, enumerate in an increasing sequence $\{\alpha_\theta\}_{\theta < \omega_1}$, the ordinals of the form ω_1^z for $z \in 2^{\mathbb{N}}$ (i.e., by Sacks' Theorem the countable admissibles). Set

$$T_\theta = \{x \in T: \varphi(x) = \alpha_\theta\}.$$

We claim that this works.

(a) To show that T_θ is Borel, note that

$$x \in T_\theta \Leftrightarrow \exists z \leq_T x \exists w \leq_T x[xE_0\langle \hat{z}, w\rangle \ \& \ \omega_1^z = \theta],$$

where "$a \leq_T b$" means that "a is recursive in b". Since for each θ the set

$$z \in R_\theta \Leftrightarrow \omega_1^z = \theta$$

is Borel, we are done.

(b) T_θ is E_0-invariant. This is obvious from its definition.

(c) $\{T_\theta\}$ are pairwise disjoint. This is also obvious from the definition.

(d) T_θ is not smooth. Let $x_0 \in T_\theta$ and let z_0, w_0 be such that $x_0 E_0 \langle \hat{z}_0, w_0\rangle$. Then $\langle \hat{z}_0, w\rangle \in T_\theta$ for all $w \in 2^{\mathbb{N}}$. Let $f\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be defined by $f(w) = \langle \hat{z}_0, w\rangle$. Then $wE_0v \Leftrightarrow f(w)(E_0 \restriction T_\theta)f(v)$, so $E_0 \restriction T_\theta$ is not smooth, i.e., T_θ is not smooth.

(e) Now let μ be a probability measure with $\mu(T) = 1$. We will first show that there is some $\theta < \omega_1$ with $\mu(T_\theta) > 0$. Otherwise, $\mu(T_\theta) = 0$ for all $\theta < \omega_1$. Consider then the following relation $\leq_\varphi$ on T:

$$x \leq_\varphi y \Leftrightarrow \varphi(x) \leq \varphi(y) \Leftrightarrow \exists z, w \exists t, v[xE_0\langle \hat{z}, w\rangle \ \& \ yE_0\langle \hat{t}, v\rangle \ \& \ \omega_1^z \leq \omega_1^t].$$

Since the relation "$\omega_1^z \leq \omega_1^t$" is Σ_1^1, $\leq_\varphi$ is Σ_1^1, and hence, is universally measurable. By applying Fubini to it, we conclude that $\mu(T) = 0$, a contradiction.

There are clearly only countably many θ with $\mu(T_\theta) > 0$. Suppose $\theta_0 < \omega_1$ is large enough so that $\mu(T_\theta) = 0$, $\forall \theta \geq \theta_0$. By the preceding argument, $\mu(\bigcup_{\theta \geq \theta_0} T_\theta) = 0$, so $\mu(\bigcup_{\theta < \theta_0} T_\theta) = 1$.

For S, we take the following subset of T:

$$x \in S \Leftrightarrow \exists y, z \in 2^{\mathbb{N}}[xE_0 y \wedge y = \langle \hat{z}, 0\rangle]$$

with 0 the constant 0 sequence, and then we define $S_\theta = S \cap T_\theta$. ⊣

Proof of 4.6. We can assume without loss of generality that $X = Y = 2^{\mathbb{N}}$. In this case the sets C^X and R^X are Π_1^1, S_X is Σ_1^1, and similarly, for $C^{X,Y}$, $R^{X,Y}$, and $S^{X,Y}$. Consider the relation

$$\begin{aligned} R(e, a, a^*, b) \Leftrightarrow\ & e \in \mathscr{E}(E),\ a \text{ codes a Borel invariant subset } B_a^X \text{ of} \\ & X_e,\ a^* \text{ codes a Borel invariant subset } B_{a^*}^Y \text{ of } Y_{e^*}, \\ & e(B_a^X) = e^*(B_{a^*}^Y) = 1 \ \& \ b \text{ codes a Borel} \\ & \text{isomorphism } f_b^{X,Y} \text{ of } E \restriction B_a^X \text{ with } F \restriction B_{a^*}^Y \text{ which sends} \\ & e \text{ to } e^*. \end{aligned}$$

Then R is $\underset{\sim}{\Pi}_1^1$. By the Connes-Feldman-Weiss Theorem and Dye's Theorem 2.4, we have that $\forall e \in \mathscr{E}(E) \exists a, a^*, bR(e, a, a^*, b)$. Thus, by the Uniformization Theorem for $\underset{\sim}{\boldsymbol{\Pi}}_1^1$ sets, we can find functions $a(e), a^*(e), b(e)$ with $R(e, a(e), a^*(e), b(e))$ such that $e \mapsto (a(e), a^*(e), b(e))$ has $\underset{\sim}{\boldsymbol{\Pi}}_1^1$ graph. If every $\underset{\sim}{\boldsymbol{\Sigma}}_2^1$ set is universally measurable, then this function is universally measurable, and we are done. This works, for example, if we assume that there is a measurable cardinal. If we want to work in ZFC alone, we will use a metamathematical argument.

Denote by $p_0 \in \mathbb{N}^{\mathbb{N}}$ a parameter that encodes Borel codes for $E, F, \mathscr{E}(E), \mathscr{E}(F)$, $x \mapsto e_x$, $y \mapsto f_y$. Then (viewing e as a member of M as in Proposition 3.3) we have

$$R(e, a, a^*, b) \Leftrightarrow \psi(p_0, e, a, a^*, b)$$

with ψ a Π^1_1 formula. Denote by ψ^* the canonical Π^1_1 formula uniformizing ψ on a, a^*, b. Denote by $a(e), a^*(e), b(e)$ the functions resulting from ψ^*. By a result of Solovay, in order to show that a, a^*, b are universally measurable, it is enough to show that, for a sufficiently large finite part Σ of ZFC, if N is a countable standard model of Σ containing p_0, then $N \models \forall e \in \mathscr{E}(E) \exists a, a^*, b\psi^*(p_0, e, a, a^*, b)$, or equivalently, $N \models \forall e \in \mathscr{E}(E) \exists a, a^*, b\psi(p_0, e, a, a^*, b)$. This will follow if we have that $N \models \forall e \in \mathscr{E}(E)$ (E is e-amenable). Thus, fix $e \in N$, $e \in \mathscr{E}(E)$, in order to show that $N \models E$ is e-amenable. Since Π^1_2 formulas are downward absolute for countable standard models of sufficiently large finite parts of ZFC, it is sufficient to have that the property "E is e-amenable" is Π^1_2 in e, p_0. But this is given by Proposition 3.3. ⊣

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