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Author(s): Alexander S. Kechris and Alain Louveau
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THE CLASSIFICATION OF HYPERSMOOTH BOREL EQUIVALENCE RELATIONS

ALEXANDER S. KECHRIS AND ALAIN LOUVEAU

This paper is a contribution to the study of Borel equivalence relations in standard Borel spaces, i.e., Polish spaces equipped with their Borel structure. A class of such equivalence relations which has received particular attention is the class of hyperfinite Borel equivalence relations. These can be defined as the increasing unions of sequences of Borel equivalence relations all of whose equivalence classes are finite or, as it turns out, equivalently those induced by the orbits of a single Borel automorphism. Hyperfinite equivalence relations have been classified in [DJK], under two notions of equivalence, Borel bi-reducibility, and Borel isomorphism.

An equivalence relation E on X is **Borel reducible** to an equivalence relation F on Y if there is a Borel map $f : X \to Y$ with $xEy \Leftrightarrow f(x)Ff(y)$. We write then $E \leq F$. If $E \leq F$ and $F \leq E$ we say that E, F are **Borel bi-reducible**, in symbols $E \approx^* F$. When $E \approx^* F$ the quotient spaces $X/E, Y/F$ have the same "effective" or "definable" cardinality. We say that E, F are **Borel isomorphic** if there exists a Borel bijection $f\colon X \to Y$ with $xEy \Leftrightarrow f(x)Ff(y)$. Below we denote by E_0, E_t the equivalence relations on the Cantor space $2^{\mathbb{N}}$ given by: $xE_0y \Leftrightarrow \exists n \forall m \geq n(x_m = y_m)$, $xE_ty \Leftrightarrow \exists n \exists k \forall m(x_{n+m} = y_{k+m})$. We denote by Δ_X the equality relation on X, and finally we call E **smooth** if $E \leq \Delta_{2^{\mathbb{N}}}$. This just means that elements of X can be classified up to E-equivalence by concrete invariants which are members of some Polish space.

It is shown now in [DJK] that up to Borel bi-reducibility there is exactly one non-smooth hyperfinite Borel E, namely E_0, and up to Borel isomorphism there are exactly countably many non-smooth hyperfinite aperiodic (i.e., having no finite equivalence classes) Borel E, namely E_t, $E_0 \times \Delta_n$ $(1 \leq n \leq \aleph_0)$, $E_0 \times \Delta_{2^{\mathbb{N}}}$ (where $\Delta_n = \Delta_X$, with $\mathrm{card}(X) = n$, if $1 \leq n \leq \aleph_0$).

In this paper we investigate and classify the class of Borel equivalence relations which are the "continuous" analogs of the hyperfinite ones. We call a Borel equivalence relation E **hypersmooth** if it can be written as $E = \bigcup_n E_n$, where $E_0 \subseteq E_1 \subseteq \cdots$ is an increasing sequence of smooth Borel equivalence relations. These have been also studied (in a measure theoretic context) in the Russian literature under the name *tame equivalence relations*. They include many interesting examples such as: The increasing union of a sequence of closed or even G_δ equivalence relations (like for example the coset equivalence relation of a Polish group modulo a subgroup, which is the increasing union of a sequence of closed subgroups), the hyperfinite equivalence relations, the "tail" equivalence relations

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$E_0(U)$, $E_t(U)$ of a Borel map $U\colon X \to X$ given by $xE_0(U)y \Leftrightarrow \exists n(U^n(x) = U^n(y))$ and $xE_t(U)y \Leftrightarrow \exists n \exists m(U^n(x) = U^m(y))$, the equivalence relations induced by the orbits of a Borel action of a Polish locally compact group which is compactly generated of polynomial growth (e.g., $\mathbb{R}^n$), the equivalence relation induced by the composants of an indecomposable continuum, etc.

Denote by E_1 the equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ given by $xE_1y \Leftrightarrow \exists n \forall m \geq n$ $(x_m = y_m)$. This is the "continuous" analog of E_0 and is clearly hypersmooth. It is well-known that $E_0 < E_1$ (i.e., $E_0 \leq E_1$, but $E_1 \not\leq E_0$) and it is easy to see that $E \leq E_1$ for any Borel hypersmooth E. The main result in this paper is now the following dichotomy, which was motivated by results in the measure theoretic context, see [V], [VF], [VG].

Theorem 1. *If E is a hypersmooth Borel equivalence relation, then exactly one of the following holds*:

(I) $E \leq E_0$;

(II) $E_1 \leq E$.

(Actually in (II) the reducing function can be taken to be injective, i.e., an embedding.)

From this it follows that up to Borel bi-reducibility there are exactly two non-smooth hypersmooth Borel equivalence relations, namely E_0 and E_1. With some further work one can obtain also results on classification up to Borel isomorphism. For example, up to Borel isomorphism there are only two non-smooth hypersmooth Borel E, satisfying some mild natural conditions, that have equivalence classes of size $2^{\aleph_0}$, namely $E_0 \times I_{2^{\mathbb{N}}}$ and E_1 (where $I_{2^{\mathbb{N}}} = 2^{\mathbb{N}} \times 2^{\mathbb{N}}$).

Despite the fact that our main result involves only notions of classical descriptive set theory, the proof makes heavy use of effective descriptive set theory, as was the case with the proof of the Glimm-Effros type dichotomy for Borel equivalence relations proved in [HKL].

Although the dichotomy expressed in Theorem 1 is of a "local" nature, as it refers only to hypersmooth Borel equivalence relations, it turns out surprisingly to have also global consequences concerning the structure of arbitrary Borel equivalence relations. Consider the partial (pre-)order $\leq$ on Borel equivalence relations. A **node** is a Borel equivalence relation E such that for any Borel $F, E \leq F$ or $F \leq E$, i.e., E is comparable to any Borel equivalence relation. It is trivial that each Δ_n $(n = 1, 2, \dots)$ is a node and by Silver's Theorem in [S], which implies that for any Borel E either $E \leq \Delta_{\aleph_0}$ or $\Delta_{2^{\mathbb{N}}} \leq E$, we have that $\Delta_{\aleph_0}$, $\Delta_{2^{\mathbb{N}}}$ are also nodes. We now have:

Theorem 2. *The only nodes in the partial order $\leq$ on Borel equivalence relations are Δ_n $(1 \leq n \leq \aleph_0)$, $\Delta_{2^{\mathbb{N}}}$, and E_0.*

This has the following immediate implication. Say that a pair of Borel equivalence relations (E, E^*) with $E < E^*$ has the **dichotomy property** if for any Borel equivalence relation F we have $F \leq E$ or $E^* \leq F$. Clearly (Δ_n, Δ_{n+1}), $n = 1, 2, \dots$, have this property. By Silver's Theorem so does $(\Delta_{\aleph_0}, \Delta_{2^{\mathbb{N}}})$, and by the result in [HKL] the same holds for $(\Delta_{2^{\mathbb{N}}}, E_0)$. It follows from Theorem 2 that these are the only such pairs, i.e., except for the trivial case of (Δ_n, Δ_{n+1}), the only global dichotomy theorems for Borel equivalence relations are Silver's Theorem and the general Glimm-Effros Dichotomy established in [HKL].

The paper is organized as follows: Section 0 contains preliminaries on descriptive set theory and equivalence relations. Section 1 discusses the basic properties of hypersmooth relations and several examples. In Section 2 we prove the main theorem. Section 3 contains consequences concerning isomorphism classifications. In Section 4 we discuss results and examples relating to the possibility of reducing E_1 to other Borel equivalence relations. Finally, Section 5 contains the "global" consequences of our main results mentioned above.

0. Preliminaries

A) A **standard Borel space** is a set X equipped with a σ-algebra $\mathcal{S}$ such that for some Polish (i.e., separable completely metrizable) topology τ on X, $\mathcal{S}$ in the class of Borel sets of τ. We call the members of $\mathcal{S}$ the **Borel sets** in X. Every uncountable standard Borel space is Borel isomorphic to the **Baire space** $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ and to the **Cantor space** $\mathcal{C} = 2^{\mathbb{N}}$.

We use the customary notation and terminology concerning descriptive set theory, see, e.g., [Mo]. In particular $\mathbf{\Sigma}^1_1$ denotes the class of analytic sets, $\mathbf{\Pi}^1_1$ the class of co-analytic sets and $\mathbf{\Delta}^1_1$ the class of bi-analytic sets, i.e., these which are both analytic and co-analytic. By Souslin's Theorem the bi-analytic sets are exactly the Borel sets.

The use of effective descriptive set theory is crucial for the proof of our main result. Again we use standard terminology and notation as in [Mo]. Thus $\Sigma^1_1, \Pi^1_1, \Delta^1_1$ denote resp. the classes of effectively analytic, co-analytic and bi-analytic sets. We denote by ω_1^x the first ordinal not recursive in x and by ω_1^{CK} the first non-recursive ordinal.

The results from (both classical and effective) descriptive set theory that we will use can be found in [Mo], and in [HKL] in regards to the Gandy-Harrington topology, with the exception of two reflection theorems that we will now state. Their proofs can be found in [HMS], [K3].

0.1. First Reflection Theorem. *Let $\Phi \subseteq \mathcal{P}(\mathcal{N})$ (= the power set of $\mathcal{N}$) be Π^1_1 on Σ^1_1, i.e., for $B \subseteq \mathcal{N} \times \mathcal{N}$ in Σ^1_1, $\{y\colon \Phi(B_y)\}$ is in Π^1_1. Then if $\Phi(A)$ holds for $A \in \Sigma^1_1$, there is $A' \supseteq A$, $A' \in \Delta^1_1$ such that $\Phi(A')$ holds.*

0.2. Burgess Reflection Theorem. *Let $R \subseteq \mathcal{N}^{\mathbb{N}} \times \mathcal{N}^n$ ($n \in \mathbb{N}$) be Π^1_1 and let $\Phi \subseteq \mathcal{P}(\mathcal{N})$ be given by*

$$\begin{aligned}\Phi(A) \Leftrightarrow \forall x \in \mathcal{N}^{\mathbb{N}} \forall y \in \mathcal{N}^n \{\forall n(x_n) \in A) \;\&\\ \forall i < n(y_i \notin A) \Rightarrow R(x,y)\}.\end{aligned}$$

If $A \subseteq \mathcal{N}$ is Σ^1_1 and $\Phi(A)$ holds, then there is $A' \supseteq A$, $A' \in \Delta^1_1$ such that $\Phi(A')$ holds.

B) By a **Polish group** we mean a topological group whose topology is Polish. If X is a standard Borel space, a Borel action of G on X is an action $(g,x) \mapsto g \cdot x$ of G on X which is Borel as a function from $G \times X$ into X.

C) If X is a set and E an equivalence relation on X, we denote by $[x]_E$ the **equivalence class** of x, by $X/E = \{[x]_E\colon x \in X\}$ the **quotient space** of X by E, and by $[A]_E = \{x : \exists y \in A(xEy)\}$ the E-**saturation** of $A \subseteq X$. If $[A]_E = A$ we say that A is E-**invariant**.

A **transversal** for E is a subset $T \subseteq X$ which meets every equivalence class in exactly one point. A **selector** for E is a map $s\colon X \to X$ with $xEy \Rightarrow s(x) = s(y)Ey$.

We denote by Δ_X, I_X respectively the smallest and largest equivalence relations on X, i.e., Δ_X is equality on X and $I_X = X^2$.

If $A \subseteq X$, we denote by $E|A$ the restriction of E to A, i.e., $E|A = E \cap A^2$. If F is also an equivalence relation on X, $E \subseteq F$ means that E is a **subequivalence relation** of F, i.e., $xEy \Rightarrow xFy$.

Suppose now E, F are equivalence relations on X, Y resp. A **reduction** of E into F is a map $f\colon X \to Y$ with $xEy \Leftrightarrow f(x)Ff(y)$. Note that this induces an injection $f^*\colon X/E \to Y/F$ given by $f^*([x]_E) = [f(x)]_F$. If f is 1-1 we call this an **embedding**. If f is 1-1 and onto it is called an **isomorphism** of E, F. If f is an embedding and $f[X] = B$ is F-invariant, then we say that it is an **invariant embedding**. It is clearly an isomorphism of E with $F|B$. Invariant embeddings of E into F and F into E give rise, via the standard Schroeder-Bernstein argument, to an isomorphism of E and F.

The product of E, F is the equivalence relation $E \times F$ on $X \times Y$ defined by

$$(x, y)E \times F(x', y') \Leftrightarrow xEx' \ \&\ yFy'.$$

D) Assume now E, F are equivalence relations on standard Borel spaces X, Y. We write

$$\begin{aligned}
E \leq F &\Leftrightarrow \exists \text{ a Borel reduction of } E \text{ into } F,\\
E < F &\Leftrightarrow E \leq F \ \&\ F \not\leq E;\\
E \approx^* F &\Leftrightarrow E \leq F \ \&\ F \leq E;\\
E \sqsubseteq F &\Leftrightarrow \exists \text{ a Borel embedding of } E \text{ into } F;\\
E \approx F &\Leftrightarrow E \sqsubseteq F \ \&\ F \sqsubseteq E;\\
E \sqsubseteq^i F &\Leftrightarrow \exists \text{ a Borel invariant embedding of } E \text{ into } F;\\
E \cong F &\Leftrightarrow \exists \text{ a Borel isomorphism of } E, F.
\end{aligned}$$

Note that

$$E \cong F \Leftrightarrow E \sqsubseteq^i F \& \ F \sqsubseteq^i E.$$

Now let E be a Borel equivalence relation on a standard Borel space X. We call E **smooth** if E has a **countable Borel separating family**, i.e., a sequence (A_n) of Borel sets in X with

$$xEy \Leftrightarrow \forall n(x \in A_n \Leftrightarrow y \in A_n).$$

This is easily equivalent to saying that $E \leq \Delta_X$, for some standard Borel space X. If E admits a Borel transversal (equivalently a Borel selector), then E is smooth. The converse is in general false (see, e.g., [K3, 18.D]), but holds for most natural examples.

The following dichotomy result was proved in [HKL]. Let E_0 be the equivalence relation on $2^{\mathbb{N}}$ given by

$$xE_0y \Leftrightarrow \exists n \forall m \geq n(x_m = y_m).$$

Then for any Borel E, exactly one of the following holds: E is smooth or $E_0 \sqsubseteq E$. In fact the following effective version is proved in [HKL]: If E is a Δ^1_1 equivalence relation on $\mathcal{N}$, then exactly one of the following holds: $E \leq \Delta_{2^{\mathbb{N}}}$ via a Δ^1_1 reduction or $E_0 \sqsubseteq E$.

E) A Borel equivalence relation E on X is called **finite**, resp. **countable**, if every equivalence class $[x]_E$ is finite, resp. countable. It is called **hyperfinite** if $E = \bigcup_n E_n$, with $E_0 \subseteq E_1 \subseteq \cdots$ an increasing sequence of finite Borel equivalence relations. Clearly hyperfinite equivalence relations are countable. For more about their structure, see [DJK]. For example, they can be characterized as those that are induced by the orbits of a Borel action of $\mathbb{Z}$ on X, i.e., which are of the form $E = \{(x, T^n(x)) : n \in \mathbb{Z}\}$ with T a Borel automorphism of X. Also they turn out to be exactly those that can be written as $E = \bigcup_n E_n$, with $E_0 \subseteq E_1 \subseteq \cdots$ an increasing sequence of smooth countable Borel equivalence relations.

1. Basic facts and examples

Let X be a standard Borel space and E a Borel equivalence relation on X. We call E **hypersmooth** if $E = \bigcup_n F_n$, where $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ is an increasing sequence of smooth Borel equivalence relations. Such equivalence relations are called **tame** in the Russian literature; see [V], [VF], [VG].

Let us note some simple closure properties of hypersmooth relations.

Proposition 1.1. (i) *If F is hypersmooth and $E \leq F$, then E is hypersmooth;*

(ii) *If E is hypersmooth and A is Borel, $E|A$ is hypersmooth;*

(iii) *If E, F are hypersmooth, so is $E \times F$.*

The proofs are straightforward.

The following is a basic open problem.

Problem 1.2. If $E = \bigcup_n F_n$, where $F_0 \subseteq F_1 \subseteq \cdots$ is an increasing sequence of Borel hypersmooth equivalence relations, is E hypersmooth?

We next discuss examples:

0) It is well-known (see, e.g., [K1, 2.2]) that ever closed equivalence relation is smooth, and in [HKL] this is extended to G_δ equivalence relations. So if $E = \bigcup_n E_n$, $E_0 \subseteq E_1 \subseteq \cdots$ an increasing sequence of closed or even G_δ equivalence relations, then E is hypersmooth. Conversely, it follows from [K3, 13.11] that if E is Borel hypersmooth on the standard Borel space X, there is a Polish topology τ giving the Borel structure of X, such that $E = \bigcup_n E_n$, with $E_0 \subseteq E_1 \subseteq \cdots$ closed in (X^2, τ^2) equivalence relations.

1) Every Borel hyperfinite equivalence relation (see [DJK]) is hypersmooth. In fact, we view hypersmooth relations as "continuous" analogs of the hyperfinite ones.

2) For any standard Borel space Ω, let $E_0(\Omega), E_t(\Omega)$ be the following equivalence relations on $X = \Omega^{\mathbb{N}}$:

$$xE_0(\Omega)y \Leftrightarrow \exists n \forall m \geq n (x_m = y_m),$$
$$xE_t(\Omega)y \Leftrightarrow \exists n \exists m \forall k (x_{n+k} = y_{m+k}).$$

It is clear that $E_0(\Omega)$ is hypersmooth, and it is shown in [DJK] that so is $E_t(\Omega)$.

Put

$$E_0 = E_0(2),$$
$$E_1 = E_0(2^{\mathbb{N}}).$$

3) We can generalize the examples in 2) as follows:

Let X be a standard Borel space and $U : X \to X$ a Borel map. Put

$$xE_0(U)y \Leftrightarrow \exists n (U^n(x) = U^n(y)),$$
$$xE_t(U)y \Leftrightarrow \exists n \exists m (U^n(x) = U^m(y)).$$

Then $E_0(U), E_t(U)$ are hypersmooth (see [DJK]). If we take $X = \Omega^{\mathbb{N}}$ and $U((x_n)) = (x_{n+1})$, the **shift** on $\Omega^{\mathbb{N}}$, we obtain the examples in 2).

4) Let G be a Polish group and $H \subseteq G$ a subgroup. Let $G/H = \{xH : x \in G\}$ be the (left) coset space of H in G and put

$$xE_H y \Leftrightarrow xH = yH$$

for the associated equivalence relation. If H is closed, then it is well-known that E_H is smooth, in fact has a Borel transversal. Conversely (see [Mi]), if H is Borel and E_H is smooth, then H is closed.

If now $H = \bigcup_n H_n$, with $H_0 \subseteq H_1 \subseteq \cdots$ an increasing sequence of closed subgroups of G, then E_H is clearly hypersmooth. Both E_0, E_1 are of this form. For E_0, we take $G = \mathbb{Z}_2^{\mathbb{N}}$, $H_n = \mathbb{Z}_2^n$ (viewed as a subgroup of $\mathbb{Z}_2^{\mathbb{N}}$ by identifying $(x_1, \ldots, x_n) \in \mathbb{Z}_2^{\mathbb{N}}$ with $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$). For E_1 let $G = \mathbb{T}^{\mathbb{N}}$ ($\mathbb{T}$ the unit circle), $H_n = \mathbb{T}^n$. (This does not give literally E_1, which lives on $2^{\mathbb{N}}$, but a Borel isomorphic copy of it.)

5) If G is a Polish locally compact group and $(g, x) \mapsto g \cdot x$ a Borel action of G on X, we denote by E_G the (Borel) equivalence relation induced by the orbits of this action, i.e.,

$$xE_G y \Leftrightarrow \exists g \in G(g \cdot x = y).$$

It is shown in [W] and [K1] that $E_{\mathbb{R}} \leq E_0$, so $E_{\mathbb{R}}$ is hypersmooth. Thus the orbit equivalence relation of a flow (i.e., an $\mathbb{R}$-action) is hypersmooth. This was extended in [JKL] to show that $E_G \leq E_0$ for any G which is compactly generated of polynomial growth (e.g., $\mathbb{R}^n$); thus all such E_G are hypersmooth.

6) The following interesting example was discovered recently by Solecki: Let X be a **continuum** (i.e., a compact connected metric space). It is called **indecomposable** if it is not the union of two proper subcontinua. For any indecomposable continuum X and $x \in X$, the **composant** of x is the union of all proper subcontinua containing x. The composants form a partition of X (into $2^{\aleph_0}$ pieces), and let us denote by E_X the corresponding equivalence relation. By a result of Rogers [R], E_X is F_σ. Solecki has in fact shown that $E = \bigcup_n E_n$, $E_0 \subseteq E_1 \subseteq \cdots$ an increasing sequence of closed equivalence relations, so E is hypersmooth.

The equivalence relation E_1 is universal among hypersmooth Borel equivalence relations.

Proposition 1.3. *Let E be a hypersmooth Borel equivalence relation. Then $E \sqsubseteq E_1$.*

Proof. Let $E = \bigcup_n F_n$, with F_n an increasing sequence of smooth Borel equivalence relations on X. Let $f_n : X \to 2^{\mathbb{N}}$ be Borel with $xF_n y \Leftrightarrow f_n(x) = f_n(y)$ and assume that $F_0 = \Delta_X$, so that f_0 is injective. Define $f : X \to (2^{\mathbb{N}})^{\mathbb{N}}$ by

$$f(x) = (f_0(x), f_1(x), \cdots).$$

Then f is Borel injective and $xEy \Leftrightarrow f(x)E_1 f(y)$, so $E \sqsubseteq E_1$. □

The universal relation E_1 also has the following important property which has been known for some time (see, e.g., [FR], [K1, §5]).

Proposition 1.4. *If F is a countable Borel equivalence relation, then $E_1 \not\leq F$. In particular, $E_0 < E_1$.*

In fact 1.4 is also a consequence of the following stronger result, which also has other applications.

Theorem 1.5. *Let X be a standard Borel space, F a countable Borel equivalence relation on X and $f : (2^{\mathbb{N}})^{\mathbb{N}} \to X$ a Borel map such that $xE_1y \Rightarrow f(x)Ff(y)$. Then there are $(x_n), (y_n) \in (2^{\mathbb{N}})^{\mathbb{N}}$ with $n \mapsto x_n, n \mapsto y_n$ injective, $x_n \neq y_m$ for all n, m and $f((x_n)) = f((y_n))$.*

Proof. We can identify $(2^{\mathbb{N}})^{\mathbb{N}}$ with $2^{\mathbb{N}\times\mathbb{N}}$. It has the usual product topology, whose basic nbhds are given by $N_p = \{x \in 2^{\mathbb{N}\times\mathbb{N}}\colon x|(m \times n) = p\}$, where $p \in 2^{m\times n}$, $m, n \in \mathbb{N}$. Similarly identity $(2^{\mathbb{N}})^m$ with $2^{m\times\mathbb{N}}$ with the product topology, whose basic nbhds are $N_p^{(m)} = \{x \in 2^{m\times\mathbb{N}}\colon x|(m \times n) = p\}$, for $p \in 2^{m\times n}$, $n \in \mathbb{N}$. We call $p \in 2^{m\times n}$ $(m, n \in \mathbb{N})$ **conditions**.

We use below the following general notation:

$\forall^* xP(x)$ means "$P(x)$ holds on a comeager set",
$\forall^+ xP(x)$ means "$P(x)$ holds on a non-meager set",

and for U open,

$\forall^* x \in UP(x)$ means "P holds on a comeager in U set",
$\forall^+ x \in UP(x)$ means "P holds on a non-meager in U set".

In this notation, the Kuratowski-Ulam Theorem asserts that if P has the property of Baire, then

$$\forall^*(x, y)P(x, y) \Leftrightarrow \forall^* x\forall^* yP(x, y) \Leftrightarrow \forall^* y\forall^* xP(x, y).$$

Assume now f is as in the theorem. Then f is Baire measurable, so f is continuous on a dense G_δ set $G = \bigcap_n G_n \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$, where the G_n are open, dense and decreasing.

We will construct inductively for $n \in \mathbb{N}$:

1) conditions $p_n, q_n \in 2^{(l_n\times k_n)}$, with l_n, k_n strictly increasing;

2) $x_i, y_i \in 2^{\mathbb{N}}$ for $i < l_n$, such that $i \mapsto x_i, i \mapsto y_i$ are injective, $x_i \neq y_j$, $\forall j \leq i$, $y_i \neq x_j$, $\forall j \leq i$ and $p_n = (x_0, \ldots, x_{l_n-1})|k_n$, $q_n = (y_0, \ldots, y_{l_n-1})|k_n$, which moreover satisfy:

(a) $N_{p_n}, N_{q_n} \subseteq G_n$;
(b) $\forall^* \alpha \in (2^{\mathbb{N}})^{\mathbb{N}}((x_0, \ldots, x_{l_n-1})^\frown\alpha \in G \;\&\; (y_0, \ldots, y_{l_n-1})^\frown\alpha \in G)$;
(c) $\forall^+ \alpha \in (2^{\mathbb{N}})^{\mathbb{N}}(f((x_0, \ldots, x_{l_n-1})^\frown\alpha) = f((y_0, \ldots, y_{l_n-1})^\frown\alpha))$.

We will write below $\overline{x}_k$ for $(x_0, \ldots, x_k)$ and similarly for the y's.

Assuming this can be done, by (b), (c) we can find $\{\alpha^n\}$ such that

$$\overline{x}_{l_n-1}{}^\frown\alpha^n, \overline{y}_{l_{n-1}}{}^\frown\alpha^n \in G$$

and

$$f(\overline{x}_{l_n-1}{}^\frown\alpha^n) = f(\overline{y}_{l_n-1}{}^\frown\alpha^n).$$

If $x = (x_n)$, $y = (y_n)$, by (a) we have $x, y \in G$ and since $\overline{x}_{l_n-1}{}^\frown\alpha^n \to x$, $\overline{y}_{l_n-1}{}^\frown\alpha^n \to y$ and $\overline{x}_{l_n-1}{}^\frown\alpha^n$, $\overline{y}_{l_n-1}{}^\frown\alpha^n \in G$, we have $f(x) = f(y)$ by continuity. But also $n \mapsto x_n$, $n \mapsto y_n$ are injective and $x_n \neq y_m$ for all n, m, so we are done.

To show that this construction is possible, we use the following lemma:

Lemma. *Let $p \in 2^{m\times n}$, φ a Borel function defined on a comeager in N_p set, such that on its domain*

$$xE_1y \Rightarrow \varphi(x)F\varphi(y).$$

Then we can find $q \in 2^{m\times n'}$ with $q \supseteq p$ and a condition r such that

$$\forall^* \delta \in N_q^{(m)} \forall^* \varepsilon \in N_q^{(m)} \forall^* x \in N_r (\varphi(\delta^\frown x) = \varphi(\varepsilon^\frown x)).$$

Proof. Fix $x \in (2^{\mathbb{N}})^{\mathbb{N}}$. Define a partial function $\xi_x : (2^{\mathbb{N}})^m \to X$ by $\xi_x(\delta) = \varphi(\delta^\frown x)$, for $\delta \in (2^{\mathbb{N}})^m$ such that $\varphi(\delta^\frown x)$ is defined. By the Kuratowski-Ulam Theorem ξ_x is defined on a comeager in $N_p^{(m)}$ set of δ's, for a comeager in $(2^{\mathbb{N}})^{\mathbb{N}}$ set of x's.

Since for δ, $\varepsilon \in (2^{\mathbb{N}})^m$, $\delta^\frown x E_1 \varepsilon^\frown x$, the image of ξ_x is contained in some F-equivalence class, so is countable. Thus we can find some $q_x \supseteq p$, $q_x \in 2^{m\times n_x}$, such that ξ_x is constant on a comeager set in $N_{q_x}^{(m)}$. Then find conditions r and $q \in 2^{m\times n'}$ with $q \supseteq p$ such that on a comeager in N_r set of x's, $q_x = q$. Then we have

$$\forall^* x \in N_r \forall^* \delta \in N_q^{(m)} \forall^* \varepsilon \in N_q^{(m)} (\varphi(\delta^\frown x) = \varphi(\varepsilon^\frown x)),$$

and so, by Kuratowski-Ulam,

$$\forall^* \delta \in N_1^{(m)} \forall^* \varepsilon \in N_q^{(m)} \forall^* x \in N_r (\varphi(\delta^\frown x) = \varphi(\varepsilon^\frown x)).$$

□

We now construct the p_n, q_n, x_n, y_n. Assume the construction has been completed up to n. By (b), (c), find a condition p' such that

$$\forall^* \alpha \in N_{p'} [f(\overline{x}_{l_n - 1}{}^\frown \alpha) = f(\overline{y}_{l_n-1}{}^\frown \alpha) \ \&\ \overline{x}_{l_n-1}{}^\frown \alpha, \overline{y}_{l_n-1}{}^\frown \alpha \in G].$$

Fix such an α. As $\overline{x}_{l_n-1}{}^\frown \alpha$, $\overline{y}_{l_n-1}{}^\frown \alpha \in G$, we can find conditions $p'_{n+1}, q'_{n+1} \in 2^{l'_{n+1}\times k'_{n+1}}$ with $l'_{n+1} > l_n$, $k'_{n+1} > k_n$ and $p'_{n+1} \supseteq p_n \sqcup p'$, $q'_{n+1} \supseteq q_n \sqcup p'$ (where $p_n \sqcup p' | (l_n \times k_n) = p_n$ and $p_n \sqcup p'(l_n + i, j) = p'(i,j)$ and similarly for $q_n \sqcup p'$), $p'_{n+1} \subseteq \overline{x}_{l_n-1}{}^\frown \alpha$, $q'_{n+1} \subseteq \overline{y}_{l_n-1}{}^\frown \alpha$, $N_{p'_{n+1}}, N_{q'_{n+1}} \subseteq G_{n+1}$. Notice that

$$p'_{n+1} | ([l_n, l'_{n+1}) \times k'_{n+1}) = q'_{n+1} | ([l_n, l'_{n+1}) \times k'_{n+1}) = \alpha | ([0, l'_{n+1} - l_n) \times k'_{n+1}).$$

Define $p \in 2^{(l'_{n+1} - l_n)\times k'_{n+1}}$ by $p(i,j) = p'_{n+1}(l_n + i, j)$. Thus $p \supseteq p'$.

Use the lemma for this p and $\varphi(x) = f(\overline{x}_{l_n-1}{}^\frown x) = f(\overline{y}_{l_n-1}{}^\frown x)$, to obtain q and r as in the lemma.

We have then, if $m = l'_{n+1} - l_n$:

$$\forall^* \delta \in N_q^{(m)} \forall^* \varepsilon \in N_q^{(m)} \forall^* x \in N_r$$
$$[f(\overline{x}_{l_n-1}{}^\frown \varepsilon^\frown x) = f(\overline{y}_{l_n-1}{}^\frown \delta^\frown x) = f(\overline{x}_{l_n-1}{}^\frown \delta^\frown x) = f(\overline{y}_{l_n-1}{}^\frown \varepsilon^\frown x)],$$

and by (b)

$$\forall^* \delta \in N_q^{(m)} \forall^* \varepsilon \in N_q^{(m)} \forall^* x \in (2^{\mathbb{N}})^{\mathbb{N}}$$
$$(\overline{x}_{l_n-1}{}^\frown \varepsilon^\frown x \in G \ \&\ \overline{x}_{l_n-1}{}^\frown \delta^\frown x \in G$$
$$\&\ \overline{y}_{l_n-1}{}^\frown \varepsilon^\frown x \in G \ \&\ \overline{y}_{l_n-1}{}^\frown \delta^\frown x \in G).$$

Since the set of all $(\delta, \varepsilon) \in N_q^{(m)} \times N_q^{(m)}$ which satisfy at least one of the following conditions:

$$\begin{aligned} \delta_i &= x_j \quad \text{for some } i < m, j < l_n, \\ \delta_i &= \delta_j \quad \text{for some } j \neq i < m, \\ \varepsilon_i &= y_j \quad \text{for some } i < m, j < l_n, \\ \varepsilon_i &= \varepsilon_j \quad \text{for some } j \neq i < m, \\ \delta_i &= \varepsilon_j \quad \text{for some } j \leq i < m, \\ \varepsilon_i &= \delta_j \quad \text{for some } j \leq i < m, \\ \delta_i &= y_j \quad \text{for some } i < m, j < l_n, \\ \varepsilon_i &= x_j \quad \text{for some } i < m, j < l_n, \end{aligned}$$

is meager in $N_q^{(m)} \times N_q^{(m)}$, we can find $x_{l_n}, \ldots, x_{l_n+m-1}$, $y_{l_n}, \ldots, y_{l_n+m-1}$ with $i \mapsto x_i$, $i \mapsto y_i$ injective for $i < l_n + m$ and $x_i \neq y_j$, $y_i \neq x_j$ for $j \leq i < l_n + m$ and such that for $l_{n+1} = l_n + m$

$$\forall^* \alpha \in N_r[f(\overline{x}_{l_{n+1}-1}{}^\frown\alpha) = f(\overline{y}_{l_{n+1}-1}{}^\frown\alpha)]$$

and

$$\forall^* \alpha \in (2^{\mathbb{N}})^{\mathbb{N}}[\overline{x}_{l_{n+1}-1}{}^\frown\alpha \in G \ \&\ \overline{y}_{l_{n+1}-1}{}^\frown\alpha \in G].$$

Finally, choose k_{n+1} large enough. □

2. The main theorem

Our main result is that up to Borel bireducibility, E_0 and E_1 are the only non-smooth Borel hypersmooth equivalence relations. More precisely we have the following:

Theorem 2.1. *Let E be a hypersmooth Borel equivalence relation. Then exactly one of the following holds:*
(I) $E \leq E_0$;
(II) $E_1 \sqsubseteq E$.

Since by [HKL], if E is non-smooth Borel, then $E_0 \sqsubseteq E$, it follows that for any hypersmooth Borel E, exactly one of the following holds:
(i) E is smooth;
(ii) $E \approx^* E_0$;
(iii) $E \approx E_1$.

The proof of 2.1 uses the methods of effective descriptive set theory. In fact we prove the following effective result.

Theorem 2.2. *Let $\{F_n\}$ be a sequence of equivalence relations on the Baire space $\mathcal{N}$ such that $F_0 \subseteq F_1 \subseteq \cdots$ and each F_n is Π^0_1, uniformly on n. Let $E = \bigcup_n F_n$. Then exactly one of the following holds:*
(I) $E \leq E_0$ *via a Δ^1_1 map;*
(II) $E_1 \sqsubseteq E$ *via a continuous embedding.*

Before we prove 2.2, let us argue that it, and its obvious relativization, implies 2.1. Indeed by the relativized version of 2.2, if $F_0 \subseteq F_1 \subseteq \cdots$ is an increasing sequence of closed equivalence relations in $\mathcal{N}$ and $E = \bigcup_n F_n$, then either $E \leq E_0$ or $E_1 \sqsubseteq E$ (via a continuous function). Assume now that E is an arbitrary hypersmooth Borel equivalence relation on the standard Borel space X. Then (see

[K3, 13.11]) there is a Polish topology on X generating its Borel structure and closed relations in this topology $F_0 \subseteq F_1 \subseteq \cdots$, with $E = \bigcup_n F_n$. Let $C \subseteq \mathcal{N}$ be closed and $\pi\colon C \to X$ a continuous injective map from C onto X.

Define F'_n on $\mathcal{N}$ by

$$xF'_n y \Leftrightarrow (x = y) \text{ or } [x, y \in C \ \&\ \pi(x) F_n \pi(y)].$$

Then $F'_0 \subseteq F'_1 \subseteq \cdots$ and each F'_n is a closed equivalence relation on $\mathcal{N}$. Let $E' = \bigcup_n F'_n$. Then $E' \leq E_0$ or $E_1 \sqsubseteq E'$ (via a continuous function). If $E' \leq E_0$ via f, then $E \leq E_0$ via $f \circ \pi^{-1}$. If $E_1 \sqsubseteq E'$ via a continuous embedding g, then $g[(2^{\mathbb{N}})^{\mathbb{N}}] \subseteq C$, so $\pi \circ g$ is a continuous embedding of $(2^{\mathbb{N}})^{\mathbb{N}}$ into X, which witnesses that $E_1 \sqsubseteq E$.

Proof of 2.2. For each $n < m$, put

$$Y_{n,m} = \bigcup \{A \in \Sigma^1_1 \colon A^2 \cap F_m \subseteq F_n\}.$$

By the First Reflection Theorem, if $A \in \Sigma^1_1$ and $A^2 \cap F_m \subseteq F_n$, there is $B \in \Delta^1_1$, $B \supseteq A$ with $B^2 \cap F_m \subseteq F_n$, so

$$Y_{n,m} = \bigcup \{A \in \Delta^1_1 \colon A^2 \cap F_m \subseteq F_n\}$$

and in particular $Y_{n,m}$ is Π^1_1, uniformly in n, m. Put

$$X_{n,m} = \mathcal{N} \setminus Y_{n,m}, \qquad X^* = \bigcap_n \bigcup_{m>n} X_{n,m}.$$

Thus $X^* \in \Sigma^1_1$.

Case I. $X^* = \varnothing$.

We show then that (I) holds. Since $\mathcal{N} = \bigcup_n \bigcap_{m>n} Y_{n,m}$, by effective reduction we can find a pairwise disjoint sequence $\{S_n\}$ of Δ^1_1 sets, uniformly in n, such that $\bigcup_n S_n = \mathcal{N}$ and $S_n \subseteq \bigcap_{m>n} Y_{n,m}$, i.e.,

$$\forall x \in S_n \forall m > n \exists A \in \Delta^1_1 (x \in A \ \&\ A^2 \cap F_m \subseteq F_n).$$

For equivalence relations $R \subseteq S$, we say that S/R is **countable**, if every S-equivalence class contains only countably many R-equivalence classes. We claim now that $(E|S_n)/(F_n|S_n)$ is countable: It is clearly enough to show that $(F_m|S_n)/(F_n|S_n)$ is countable for any $m > n$. But if C is an $F_m|S_n$-equivalence class and $D \subseteq C$ an $F_n|S_n$-equivalence class, then there is a Δ^1_1 nonempty set A such that $A \cap C \subseteq D$, so clearly there are only countably many such D in C.

Now define a new equivalence relation F'_0 on $\mathcal{N}$ by

$$xF'_0 y \Leftrightarrow \exists n (x, y \in S_n \ \&\ xF_n y).$$

Clearly $F'_0 \subseteq E$, F'_0 is Δ^1_1 and smooth. Moreover, E/F'_0 is countable. Put also, for $n > 0$,

$$xF'_n y \Leftrightarrow (x, y \in \bigcup_{n' \leq n} S_{n'} \ \&\ xF_n y) \text{ or }$$
$$\exists m > n (x, y \in S_m \ \&\ xF_m y).$$

Then F'_n is smooth Δ^1_1, uniformly on n, $F'_0 \subseteq F'_1 \subseteq \cdots$ and $E = \bigcup_n F'_n$.

Now let $\varphi\colon \mathcal{N} \to 2^{\mathbb{N}}$ be Δ^1_1 such that

$$xF'_0 y \Leftrightarrow \varphi(x) = \varphi(y).$$

Put $\varphi[\mathcal{N}] = A \subseteq 2^{\mathbb{N}}$, so that $A \in \Sigma^1_1$.

Let $\{C_k^{(n)}\}_{n,k\in\mathbb{N}}$ be a uniformly Δ_1^1 family of sets such that for each n, $\{C_k^{(n)}\}_{k\in\mathbb{N}}$ is a separating family for F'_n. Define the equivalence relation $\overline{F}_n$ on A and subsets $\overline{C}_k^{(n)}$ of A by

$$\begin{aligned}\alpha\overline{F}_n\beta &\Leftrightarrow \exists x\exists y[\varphi(x)=\alpha \ \&\ \varphi(y)=\beta \ \&\ xF'_ny],\\ \alpha\in\overline{C}_k^{(n)} &\Leftrightarrow \exists x[\varphi(x)=\alpha \ \&\ x\in C_k^{(n)}].\end{aligned}$$

Since $F'_0\subseteq F'_n$ and each $C_k^{(n)}$ is F'_n-invariant, we also have for $\alpha,\beta\in A$:

$$\begin{aligned}\alpha\overline{F}_n\beta &\Leftrightarrow \forall x\forall y[\varphi(x)=\alpha \ \&\ \varphi(y)=\beta \Rightarrow xF'_ny],\\ \alpha\in\overline{C}_k^{(n)} &\Leftrightarrow \forall x[\varphi(x)=\alpha \Rightarrow x\in C_k^{(n)}].\end{aligned}$$

Thus $\overline{F}_n$, $\{\overline{C}_k^{(n)}\}$ are uniformly Δ_1^1 on A. Clearly $\{\overline{C}_k^{(n)}\}$ is a separating family for $\overline{F}_n$. Also if $\overline{E}=\bigcup_n\overline{F}_n$, then $\overline{E}$ is a countable Δ_1^1 equivalence relation on A.

Now let $\{\overline{\overline{C}}_k^{(n)}\}$ be uniformly Δ_1^1 such that $\overline{C}_k^{(n)}=A\cap\overline{\overline{C}}_k^{(n)}$. Consider then the statements (1)–(6) below, in variables $\widetilde{A}\subseteq 2^{\mathbb{N}}$ and $\widetilde{F}=\{\widetilde{F}_n\}_{n\in\mathbb{N}}$, $\widetilde{F}\subseteq\mathbb{N}\times 2^{\mathbb{N}}\times 2^{\mathbb{N}}$:

(1) $\widetilde{F}_n$ is an equivalence relation on $\widetilde{A}$;

(2) $\forall x\in\widetilde{A}\forall y\in\widetilde{A}(x\widetilde{F}_ny\Rightarrow y\in\Delta_1^1(x))$;

(3) $\widetilde{F}_n\subseteq\widetilde{F}_{n+1}$;

(4) $\forall n\forall k[\overline{\overline{C}}_k^{(n)}\cap\widetilde{A}$ is $\widetilde{F}_n$-invariant];

(5) $\forall x\forall y[x\in\widetilde{A} \ \&\ y\in\widetilde{A} \ \&\ \neg x\widetilde{F}_ny\Rightarrow\exists k(x\in\overline{\overline{C}}_k^{(n)} \ \&\ y\notin\overline{\overline{C}}_k^{(n)})]$;

(6) $\forall x\forall y[x\widetilde{F}_ny \ \&\ x\in\widetilde{A} \ \&\ y\in\widetilde{A}\Rightarrow x\overline{F}_ny]$.

These are clearly satisfied by A and $\overline{F}=\{\overline{F}_n\}_{n\in\mathbb{N}}$. They also have the form for applying the Burgess Reflection Theorem. (It is understood here that in (6) we use a Π_1^1 definition for $\overline{F}_n$.) So we can find Δ_1^1 sets $A^*\supseteq A$, $F^*=\{F_n\}$, $F_n^*\supseteq\overline{F}_n$, still satisfying (1)–(6). By (1) and (3), F_n^* are increasing equivalence relations on A^*, while each F_n^* is countable by (2), thus so is $E^*=\bigcup_n F_n^*$. Also (4), (5) imply that $\{\overline{\overline{C}}_k^{(n)}\cap A\}_{k\in\mathbb{N}}$ is a separating family for F_n^*, so F_n^* is smooth. Finally, (6) shows that $E^*|A=\overline{E}$.

Since E^* is hypersmooth and countable, by [DJK] E^* can be reduced by a Δ_1^1 function to E_0. Since φ reduces E to E^* as well, we have that $\psi\circ\varphi$ is a Δ_1^1 reduction of E to E_0.

This completes Case I.

Case II. $X^*\neq\varnothing$.

We will show then that (II) holds. Since X^* is nonempty Σ_1^1, the set

$$X=X^*\cap\{x\in\mathcal{N}\colon\omega_1^x=\omega_1^{\mathrm{CK}}\}$$

is also Σ_1^1 and nonempty.

Note that the Gandy-Harrington topology when restricted to $\{x\in\mathcal{N}\colon\omega_1^x=\omega_1^{\mathrm{CK}}\}$ has a clopen basis (since the intersection of a Π_1^1 set with this set is a countable union of Δ_1^1 sets), so regular; thus by the Choquet Criterion (see, e.g., [K3, 8.18]) it is Polish. Denote the Gandy-Harrington topology restricted to X by τ. Fix also a complete metric d for τ on X. We can of course assume that $d\geq\delta$, where δ is the ordinary metric on $\mathcal{N}$. We will embed E_1 into $E|X$ (continuously for the ordinary topology on $\mathcal{N}$.)

Fix the canonical bijection $\langle\ \rangle$ of $\mathbb{N}^2$ with $\mathbb{N}$ given by the Cantor diagonal enumeration, i.e.,

$$\langle n,k\rangle = \frac{(n+k)(n+k+1)}{2} + k.$$

For $s \in 2^p$, where $p = \langle n,k\rangle$, and $j \in \mathbb{N}$ we let $s_j(i) = s(\langle j,i\rangle)$, provided $\langle j,i\rangle < p$. This associates to s a sequence $\langle s_j : j \in \mathbb{N}\rangle$ of finite sequences, which are eventually $\varnothing$. Put

$$L(p) = \min\{j : s_j = \varnothing\} = \min\{j : \langle j,0\rangle \geq p\}.$$

Define also for $s,t \in 2^p$, $j \leq L(p)$,

$$s \sim_j t \Leftrightarrow \forall j' \geq j(s_{j'} = t_{j'}).$$

Then $\sim_j$ is an equivalence relation on 2^p and $\sim_0 \subseteq \sim_1 \subseteq \cdots \subseteq \sim_{L(p)}$. Moreover, $\sim_0$ is equality and $\sim_{L(p)} = 2^p \times 2^p$.

For $\alpha \in 2^{\mathbb{N}}$, let also $\alpha_m(k) = \alpha(\langle m,k\rangle)$. Then, identifying $\alpha \in 2^{\mathbb{N}}$ with $\{\alpha_m\}_{m\in\mathbb{N}} \in (2^{\mathbb{N}})^{\mathbb{N}}$, we have that

$$\begin{aligned}\alpha E_1 \beta &\Leftrightarrow \exists n \forall m \geq n(\alpha_m = \beta_m)\\ &\Leftrightarrow \exists n \forall p(n \leq L(p) \Rightarrow \alpha|p \sim_n \beta|p).\end{aligned}$$

For an equivalence relation E (on some set S) and sets A, B ($\subseteq S$) let

$$AEB \Leftrightarrow \forall x \in A \exists y \in B(xEy) \ \&\ \forall y \in B \exists x \in A(xEy).$$

Note that $AEB \Leftrightarrow [A]_E = [B]_E$, and this is an equivalence relation too.

We first claim that in order to embed E_1 into $E|X$ (continuously), it is enough to build a family $\{U_s\}_{s\in 2^{<\mathbb{N}}}$ and a strictly increasing function $N\colon \mathbb{N} \to \mathbb{N}$ satisfying:

(i) U_s is a nonempty Σ^1_1 subset of X, $t \supsetneqq s \ \to \overline{U}^{\tau}_t \subset U_s$, $U_{s^\frown 0} \cap U_{s^\frown 1} = \varnothing$ and $d(U_{s^\frown i}) \leq 2^{-lh(s)}$.

(ii) If $s,t \in 2^p$, $p = \langle n,k\rangle$ and $s \sim_n t$, then $(U_{s^\frown 0} \times U_{t^\frown 1}) \cap F_{n+1} = \varnothing$.

(iii) If $s,t \in 2^p$, $p = \langle n,k\rangle$, $j \leq L(p)$ and $s \sim_j t$, then $U_s F_{N(j)} U_t$.

Indeed, assume this can be done. For $\alpha \in 2^{\mathbb{N}}$, define $f(\alpha)$ by $\{f(\alpha)\} = \bigcap_n U_{\alpha|n}$. This is clearly well-defined and 1-1 by (i). It is also continuous for the ordinary topology on $\mathcal{N}$ as $d \geq \delta$. We argue that f embeds E_1 into E.

Suppose that $\alpha E_1 \beta$, say $\alpha_m = \beta_m$ for $m \geq n$. Then for p such that $n \leq L(p)$, $\alpha|p \sim_n \beta|p$. By (iii) $U_{\alpha|p} F_{N(n)} U_{\beta|p}$, so there are $\alpha_p \in U_{\alpha|p}$, $\beta_p \in U_{\beta|p}$ with $\alpha_p F_{N(n)} \beta_p$. Since $\alpha_p \to f(\alpha)$, $\beta_p \to f(\beta)$ in the ordinary topology on $\mathcal{N}$ and $F_{N(n)}$ is closed in that topology, $f(\alpha) F_{N(n)} f(\beta)$, so $f(\alpha) E f(\beta)$.

Assume now $(\alpha,\beta) \notin E_1$. Let $p = \langle n,k\rangle$ be such that $\alpha(p) \neq \beta(p)$. Let n_0 be smallest with $\alpha|(p+1) \sim_{n_0+1} \beta|(p+1)$. Clearly $n_0 \geq n$. Let k_0 be least with $\alpha(p_0) \neq \beta(p_0)$ for $p_0 = \langle n_0, k_0\rangle$. Thus $p_0 \leq p$. Now $\alpha|p_0 \sim_{n_0} \beta|p_0$, so by (ii) $(U_{\alpha|(p_0+1)} \times U_{\beta|(p_0+1)}) \cap F_{n_0+1} = \varnothing$ and thus $(U_{\alpha|(p_0+1)} \times U_{\beta|(p_0+1)}) \cap F_n = \varnothing$, so $\neg f(\alpha) F_n f(\beta)$. Since this happens for infinitely many n, $\neg f(\alpha) E f(\beta)$.

For the rest of the proof, let us introduce the following terminology:

Given $n < m$ and $\varnothing \neq A \in \Sigma^1_1$, $A \subseteq X$, we will say that F_n is **meager in** F_m **on** A if F_n is meager in F_m on A^2 with the $\tau \times \tau$-topology.

Since F_n, F_m are both closed in the product of the ordinary topology and thus in the $(\tau \times \tau)$-topology, this means that there are no nonempty Σ^1_1 subsets $C, D \subseteq A$ with $\varnothing \neq (C \times D) \cap F_m \subseteq F_n$. We claim that this is equivalent to saying that there is no nonempty Σ^1_1 set $B \subseteq A$ with $B^2 \cap F_m \subseteq F_n$. Indeed, if such C, D exist, we can assume first that $C F_m D$ by replacing them by $C \cap [D]_{F_m}$, $D \cap [C]_{F_m}$. We claim then that $C^2 \cap F_m \subseteq F_n$. Indeed, if $x, y \in C$ and xF_my, find $z \in D$ with xF_mz, hence xF_nz. Also yF_mz, hence yF_nz and thus xF_ny.

We can use this argument to prove immediately the following basic lemma.

Lemma 2.3. *Let $A \subseteq X$ be a nonempty Σ^1_1 set and $x_1, \dots, x_k \in A$. For any $n \in \mathbb{N}$ there is $m > n$ and a Σ^1_1 set $A^* \subseteq A$ such that $x_1, \dots, x_k \in A^*$ and F_n is meager in F_m on A^*.*

Proof. Recall that $X \subseteq X^* = \bigcap_n \bigcup_{m>n} X_m$. So $A \subseteq \bigcup_{m>n} X_{m,n}$. So find $m > n$ with $x_1, \dots, x_k \in X_{m,n}$. This can be done as $\{X_{n,m}\}_{m>n}$ is increasing. Put $A^* = A \cap X_{n,m}$. If F_n is not meager in F_m on A^*, then by the preceding argument, there is a nonempty Σ^1_1 set $B \subseteq A^*$ with $B^2 \cap F_m \subseteq F_n$, so by the definition of $Y_{n,m}$, $B \subseteq Y_{n,m}$, so $B = \varnothing$, a contradiction. $\square$

In order to construct the family $\{U_s\}$ and the function N satisfying (i)–(iii) above, we will impose the following requirements:

- $R(0)$: $U_\varnothing$ will be a nonempty Σ^1_1 subset of X and $N(0) > 0$ will be such that F_0 is meager in $F_{N(0)}$ on $U_\varnothing$.
- $R(1)$ (as (i) before): For $s \in 2^p$, $i = 0$ or 1, $\overline{U}^{\mathcal{T}}_{s^\frown i} \subseteq U_s$, $U_{s^\frown 0} \cap U_{s^\frown 1} = \varnothing$ and $d(U_{s^\frown i}) \leq 2^{-p}$.
- $R(2)$ (as (ii) before): For $s, t \in 2^p$, $p = \langle n, k\rangle$, if $s \sim_n t$, then $(U_{s^\frown 0} \times U_{t^\frown 1}) \cap F_{n+1} = \varnothing$.
- $R(3)$: For $j \leq L(p+1)$, F_j is meager in $F_{N(j)}$ on $\bigcup_{s \in 2^{p+1}} U_s$.
- $R(4)$: (a) For $s, t \in 2^p$, $j \leq L(p)$, $i = 0$ or 1:
$$s \sim_j t \Rightarrow U_{s^\frown i} F_{N(j)} U_{t^\frown i};$$

  (b) for $s \in 2^p$, $p = \langle n, k\rangle$,
$$U_{s^\frown 0} F_{N(n+1)} U_{s^\frown 1}.$$

We claim that these are enough, i.e., they imply (i)–(iii).

Clearly $R(0)$–$(2) \Rightarrow$(i), (ii). We will verify that $R(4) \Rightarrow$(iii):

Assume $R(4)$. We have to show that for all p, $s, t \in 2^p$, $j \leq L(p)$, if $s \sim_j t$, then $U_s F_{N(j)} U_t$. This is clear for $p = 0$. Suppose it holds for $p = \langle n, k\rangle$ and consider $p + 1$, $s, t \in 2^{p+1}$, say $s = \bar{s}^\frown i$, $t = \bar{t}^\frown i'$ with $\bar{s}, \bar{t} \in 2^p$, $j \leq L(p+1)$. First let $i = i'$: If $j \leq L(p)$, then $\bar{s}^\frown i \sim_j \bar{t}^\frown i$ implies $\bar{s} \sim_j \bar{t}$, so by $R(4)$(a) $U_s F_{N(j)} U_t$. If $j = L(p+1)$, since $\bar{s} \sim_{L(p)} \bar{t}$, by $R(4)$(a) again we have $U_s F_{N(L(p))} U_t$, so $U_s F_{N(j)} U_t$, since $L(p+1) \geq L(p)$ and N is increasing. Consider now the case $i \neq i'$, say $i = 0$, $i' = 1$. If $\bar{s}^\frown 0 \sim_j \bar{t}^\frown 1$, then $\bar{s}^\frown 1 \sim_j \bar{t}^\frown 1$, so, by the first case ($i = i'$), we have $U_{\bar{s}^\frown 1} F_{N(j)} U_{\bar{t}^\frown 1}$. By $R(4)$(b) also, $U_{\bar{s}^\frown 0} F_{N(n+1)} U_{\bar{s}^\frown 1}$. But $\bar{s}^\frown 0$, $\bar{t}^\frown 1$ differ at $p = \langle n, k\rangle$, so clearly $j \geq n + 1$ and, since N is increasing, $N(j) \geq N(n+1)$, so $U_{\bar{s}^\frown 0} F_{N(j)} U_{\bar{s}^\frown 1}$. By transitivity $U_{\bar{s}^\frown 0} F_{N(j)} U_{\bar{t}^\frown 1}$, and we are done.

We construct now, by induction on $p \in \mathbb{N}$, $\{U_s\}_{s \in 2^p}$ and $N(j)$, for $j \leq L(p)$, satisfying $R(0)$–(4).

For $p = 0$, we choose a nonempty Σ^1_1 set $U_\varnothing \subseteq X$ and $N(0) > 0$, so that F_0 is meager in $F_{N(0)}$ on $U_\varnothing$. This can be done by Lemma 2.3.

For the inductive step we will need some new concepts and a few combinatorial lemmas.

A **tree** is a finite undirected graph which is connected and has no loops. A **labelled tree** is a tree T together with an assignment $(s, t) \mapsto n(s, t)$ which gives for each edge (s,t) of T a natural number $n(s,t)$ (its **label**). We usually write $s \overset{n}{—} t$ if $n(s,t) = n$.

By a **tree structure** we mean a triple (T, U, M), where

(i) T is a labelled tree;

(ii) U is a map assigning to each vertex s of T a nonempty Σ^1_1 set $U(s) = U_s \subseteq X$;

(iii) M is a mapping from the set of labels of T into $\mathbb{N}$.

A tree structure (T, U, M) is **good** if moreover

(iv) $s \overset{n}{—} t \Rightarrow U_s F_{M(n)} U_t$.

A tree structure (T, U', M) **refines** (T, U, M) if $U'_s \subseteq U_s$ for every vertex s of T.

Lemma 2.4. (i) *Let U, V be Σ^1_1 nonempty sets and $x \in U$, $y \in V$. If F is a Σ^1_1 equivalence relation and xFy, then there are nonempty Σ^1_1 sets $U' \subseteq U$, $V \subseteq V$ with $x \in U'$, $y \in V'$ and $U'FV'$.*

(ii) *If U, V are nonempty Σ^1_1 sets, F a Σ^1_1 equivalence relation and UFV, then for any nonempty Σ^1_1 set $A \subseteq U$, we can find a nonempty Σ^1_1 set $B \subseteq V$ with AFB. Moreover, if $x \in A$, $y \in V$ and xFy then $y \in B$.*

Proof. (i) Let $U' = U \cap [V]_F$ and $V' = V \cap [U]_F$.

(ii) Let $B = V \cap [A]_F$. □

Lemma 2.5. *Let (T, U, M) be a good tree structure. Let s_0 be a vertex of T and A a nonempty Σ^1_1 subset of U_{s_0}. Then there is a refinement (T, U', M) of (T, U, M) which is good, such that $U'_{s_0} = A$. Moreover, if $x_s \in U_s$ for all vertices and $s \overset{n}{—} t \Rightarrow x_s F_{M(n)} x_t$, then if $x_{s_0} \in A$ we can insure that $x_s \in U'_s$, for all s.*

Proof. Let $l(s,t)$ be the distance function of T, i.e., the length of the unique path from s to t. Let $l(s) = l(s, s_0)$. We will define U'_s by induction on $l(s)$. For $l(s) = 0$, i.e., $s = s_0$, we have $U'_{s_0} = A$. Assume now $l(s) > 0$ and let t be the vertex following s on the unique path from s to s_0, so that $l(t) = l(s) - 1$. Thus U'_t has been defined. If $s \overset{n}{—} t$, then $U_s F_{M(n)} U_t$ and, since $U'_t \subseteq U_t$, we can find $U'_s \subseteq U_s$ so that $U'_s F_{M(n)} U'_t$, by 2.4(ii). □

Lemma 2.6. *Let (T, U, M) be a tree structure and $x_s \in U_s$ for every vertex s of T. If*

$$s \overset{n}{—} t \Rightarrow x_s F_{M(n)} x_t,$$

then there is a refinement (T, U', M) of (T, U, M) which is good and $x_s \in U'_s$ for all s.

Proof. By induction on the cardinality of the set of vertices of T. Let s be a terminal vertex of T, i.e., one which belongs to a unique edge. Let t be the other vertex of this edge. If we delete s and this unique edge (but not t), we obtain a new tree T^* with one fewer vertex. Let (U^*, M^*) be $(U, M)|T^*$. By the induction hypothesis, there is a good refinement $(T^*, U^{*'}, M^*)$ of (T^*, U^*, M^*) satisfying $x_{s^*} \in U^{*'}_{s^*}$ for $s^* \neq s$. Applying 2.4(i) to $F_{M(n)}$, where $s \overset{n}{—} t$, $U_s, U^{*'}_t$ and the points x_s, x_t, we can find $U'_s \subseteq U_s$, $U'_t \subseteq U^{*'}_t$ with $x_s \in U'_s$, $x_t \in U'_t$ and $U'_s F_{M(n)} U'_t$. Now apply 2.5 to (T^*, U^*, M^*) with $s_0 = t$, $A = U'_t$, to define U'_{s^*} for all vertices s^* of T^*. This defines U' for all vertices of T. □

Lemma 2.7. *Let (T, U, M) be a good tree structure. Let $n \in \mathbb{N}$. Then there is $m > n$ and a refinement (T, U', M) of (T, U, M) which is good and F_n is meager in F_m on $\bigcup_{s \in V} U'_s$, where $V =$ set of vertices of T.*

Proof. Since (T, U, M) is good, if s_0 is a fixed vertex of T, we can define by induction on $l(s, s_0)$ a sequence of points $x_s \in U_s$ with $s \overset{n}{—} t \Rightarrow x_s F_{M(n)} x_t$. By Lemma 2.3 applied to $A = \bigcup_{s \in V} U_s$ and the points $\{x_s\}_{s \in V}$, we can find $m > n$ and $A^* \subseteq A$ Σ_1^1 such that F_n is meager in F_m on A^* and $x_s \in A^*$, $\forall s \in V$. Put $U_s^* = A^* \cap U_s$. Then $x_s \in U_s^*$ and, by 2.6 applied to (T, U^*, M), we can find a good refinement (T, U', M) with $x_s \in U_s'$. If $A' = \bigcup_{s \in V} U_s'$, then F_n is meager in F_m on A', since $A' \subseteq A^*$. $\square$

We now come to the final and key lemma. First we need a definition.

Let T be a labelled tree. Given $n \in \mathbb{N}$, we say that two vertices s, t of T are n-**connected** if all the labels in the path from s to t are $\leq n$.

Lemma 2.8. *Let (T, U, M) be a good tree structure with M monotone. Let L be the largest label of T and $n \leq L$. Let N be such that F_{n+1} is meager in F_N on $\bigcup_{s \in V} U_s$ (V = the set of vertices of T), where $M(n) \leq N \leq M(n')$ for any label $n' > n$ (if such exists.) Then there are two refinements (T, U^0, M), (T, U^1, M) of (T, U, M) which are good and*

(i) *$U_s^0 F_N U_s^1$, for any $s \in V$;*

(ii) *$(U_s^0 \times U_t^1) \cap F_{n+1} = \varnothing$, if s, t are n-connected.*

Proof. Clearly n-connectedness is an equivalence relation on V, dividing it into components which are subtrees of T. Enumerate these as $C_1, \dots, C_K$.

We will consider first the case $K = 1$, i.e., $n = L$, in which case the requirement $N \leq M(n')$, $\forall n' > n$ is vacuous. So we must have $U_s^0 F_N U_s^1$, $\forall s \in V$ and $(U_s^0 \times U_t^1) \cap F_{n+1} = \varnothing$, $\forall s, t \in V$.

Enumerate in a sequence $(s_1, t_1), \dots, (s_p, t_p)$ the set $V \times V$. We will define by induction on $0 \leq j \leq p$ good tree structures $(T, U^{i,j}, M)$ for $i = 0, 1$ such that

A) $U^{i,0} = U$, $i \in \{0, 1\}$;

and for $j + 1 \leq p$:

B) $(T, U^{i,j+1}, M)$ refines $(T, U^{i,j}, M)$;

C) $(U^{0,j+1}(s_{j+1}) \times U^{1,j+1}(t_{j+1})) \cap F_{n+1} = \varnothing$;

D) $U^{0,j+1}(s_{j+1}) F_N U^{1,j+1}(t_{j+1})$.

If this can be done, put $U_s^0 = U^{0,p}(s)$, $U_s^1 = U^{1,p}(s)$. By C), if $(s, t) = (s_{j+1}, t_{j+1})$, we have $U_s^0 \times U_t^1 = U^{0,p}(s) \times U^{1,p}(t) \subseteq U^{0,j+1}(s_{j+1}) \times U^{1,j+1}(t_{j+1})$, which is disjoint from F_{n+1}. So (ii) is satisfied. For (i), notice that we have $U_{s_p}^0 F_N U_{t_p}^1$. Given any $s \in V$, there is a path from s to s_p with labels $\leq L = n$, so by transitivity and the fact that M is monotone and $M(n) \leq N$, we have $U_s^0 F_N U_{s_p}^0$. Similarly, $U_s^1 F_N U_{t_p}^1$, so $U_s^0 F_N U_s^1$.

For the inductive construction of $U^{i,j}$, note that $U^{i,0}$ is given. Assume $U^{i,j}$ is given for both $i = 0, 1$ in order to construct $U^{i,j+1}$. Since F_{n+1} is meager in F_N on $(U^{0,j}(s_{j+1}) \cup U^{1,j}(t_{j+1}))$, we can shrink $U^{0,j}(s_{j+1})$, $U^{1,j}(t_{j+1})$ to nonempty Σ_1^1 sets A, B resp., so that $(A \times B) \cap F_{n+1} = \varnothing$ but $A F_N B$. (Notice that, by the induction hypothesis, $U^{0,j}(s_{j+1}) F_N U^{1,j}(t_{j+1})$.) Then apply 2.5 to $(T, U^{0,j}, M)$, $A \subseteq U^{0,j}(s_{j+1})$ and $(T, U^{1,j}, M)$, $B \subseteq U^{1,j}(t_{j+1})$ to obtain good refinements $(T, U^{0,j+1}, M)$, $(T, U^{1,j+1}, M)$ resp., with $U^{0,j+1}(s_{j+1}) = A$, $U^{1,j+1}(t_{j+1}) = B$.

Consider finally the case when $K > 1$, i.e., $n < L$. We will define by induction on $1 \leq j \leq K$ good tree structures $(T, U^{0,j}, M)$, $(T, U^{1,j}, M)$ such that:

A) $U^{0,0} = U^{1,0} = U$;

B) $(T, U^{i,j+1}, M)$ refines $(T, U^{i,j}, M)$ for $i = 0, 1, j < K$;

C) if $s, t \in C_{j'}$, $j' \leq j$, then $U^{0,j}(s) F_N U^{1,j}(t)$;

D) if $s, t \in C_j$, then $(U^{0,j}(s) \times U^{1,j}(t)) \cap F_{n+1} = \varnothing$;

E) if $j' > j$, then for $s \in C_{j'}$, we have $U^{0,j}(s) = U^{1,j}(s)$.

We then put $U^0_s = U^{0,K}(s)$, $U^1_s = U^{1,K}(s)$. This clearly works.

We are given $U^{0,0}, U^{1,0}$ by A). Assume now $U^{i,j}$ has been defined for $i = 0, 1$. We will define $U^{i,j+1}$. Let $C = C_{j+1}$. Then by E) $U^{0,j}|C = U^{1,j}|C$. Since F_{n+1} is meager in F_N on $\bigcup\{U^{i,j}(s) \colon s \in C\}$, we can apply the previous case (i.e., $K = 1$) to define $U^{0,j+1}(s)$, $U^{1,j+1}(s)$ for $s \in C$, which are good refinements of $U^{0,j}|C$, $U^{1,j}|C$ resp., and satisfy C), D) for $s, t \in C$.

We can now use the same argument as in 2.5 to define $U^{0,j+1}(s)$, $U^{1,j+1}(s)$ for $s \notin C$. For such an s there is a unique shortest path to some point in C of length $l(s, C)$. We define $U^{i,j+1}(x)$ inductively on $l(s, C)$: If s' is the next vertex in the shortest path from s to C, we can assume by induction that $U^{i,j+1}(s')$ has been defined and we let

$$U^{i,j+1}(s) = U^{i,j}(s) \cap [U^{i,j+1}(s')]_{F_{M(k)}}$$

if $s \overset{k}{—} s'$.

Clearly B) is satisfied, and so is D), for $j + 1$.

To prove C), we note that it is clear if $s, t \in C_{j+1}$ by construction. So assume $s, t \in C_{j'}$, $j' \leq j$. Since $(T, U^{i,j+1}, M)$ are good and $N \geq M(n)$, we have, by transitivity, $U^{i,j+1}(s) F_N U^{i,j+1}(t)$. So it is enough to show that

$$U^{0,j+1}(s) F_N U^{1,j+1}(s), \ \forall s \in V.$$

So we prove, by induction on $l(s, C)$, that $U^{0,j+1}(s) F_N U^{1,j+1}(s)$. We let $l(s, C) = 0$, if $s \in C$. This is clear then for $l(s, C) = 0$, by construction. Else let s' be as before, so by the induction hypothesis, $U^{0,j+1}(s') F_N U^{1,j+1}(s')$. If $k \leq n$ (where $s \overset{k}{—} s'$), then

$$U^{0,j+1}(s) F_{M(k)} U^{0,j+1}(s') F_N U^{1,j+1}(s') F_{M(k)} U^{0,j+1}(s'),$$

so we are done as $M(k) \leq M(n) \leq N$, by transitivity. Else $k > n$. Then let $x \in U^{0,j+1}(s)$. Since $U^{0,j}(s) F_N U^{1,j}(s)$ (by C), E) for j), let $y \in U^{1,j}(s)$ be such that $x F_N y$. Let also $x' \in U^{0,j+1}(s')$ with $x' F_{M(k)} x$ and $y' \in U^{1,j+1}(s')$ with $x' F_N y'$. Then $y F_{M(k)} y'$, as $N \leq M(k)$, so $y \in U^{1,j+1}(s)$ by definition. So, reversing also the roles of $U^{0,j+1}(s)$ and $U^{1,j+1}(s)$, we have $U^{0,j+1}(s) F_N U^{1,j+1}(s)$.

Finally, we prove E), i.e.,

$$(*) \qquad s \in C_{j'}, j' > j + 1 \Rightarrow U^{0,j+1}(s) = U^{1,j+1}(s).$$

This is again by induction on $l(s, C)$. Let s' be as before, $s \overset{k}{—} s'$, and assume $(*)$ holds for s'. If $s' \in C_{j'}$, $j' > j + 1$, then we are clearly done, since

$$\begin{aligned} U^{0,j+1}(s) &= U^{0,j}(s) \cap [U^{0,j+1}(s')]_{F_{M(k)}} \\ &= U^{1,j}(s) \cap [U^{1,j+1}(s')]_{F_{M(k)}} \\ &\qquad \text{(by the induction hypothesis for } s' \text{ and E) for } j) \\ &= U^{1,j+1}(s). \end{aligned}$$

Otherwise, $s \in C_{j'}$, for $j' \leq j + 1$, so that in particular $k > n$, and thus $M(k) \geq N$. Then, by C),

$$U^{0,j+1}(s') F_N U^{1,j+1}(s'),$$

so

$$U^{0,j+1}(s')F_{M(k)}U^{1,j+1}(s'),$$

i.e.,

$$[U^{0,j+1}(s')]_{F_{M(k)}} = [U^{1,j+1}(s')]_{F_{M(k)}},$$

and we are done as above. $\square$

We are now ready to proceed to the construction of U_s for $s \in 2^{p+1}$ satisfying $R(1)$–(4), assuming U_s, for $s \in \bigcup_{p' \leq p} 2^{p'}$, are given satisfying $R(1)$–(4) for all $p' < p$.

Lemma 2.9. *Let A be a finite set, $\sim_0 \subseteq \sim_1 \subseteq \cdots \subseteq \sim_k$ a sequence of equivalence relations on A, with $\sim_0 =$ equality and $\sim_k = A \times A$. Then there is a labelled tree T with set of vertices A and labels in the set $\{0, \dots, k\}$ such that*

$$a \sim_j b \Leftrightarrow a, b \text{ are } j\text{-connected.}$$

Proof. By induction on k. For $k = 0$, this is obvious. Assume it true for $k = p$. Let $k = p + 1$. Pick a point $a_i \in C_i$, where $\{C_i\}_{i=1}^q$ are the $\sim_p$-equivalence classes. For each C_i there is, by the induction hypothesis, a labelled tree T_i with set of vertices C_i and labels $\{0, \dots, p\}$ satisfying the above for $\sim_j |C_i$, $0 \leq j \leq p$. Define T, with set of vertices A, by adding to the edges of the T_i's the edges (a_i, a_{i+1}) for $i = 1, \dots, q - 1$ with label $p + 1 = k$. This clearly works. $\square$

Apply this lemma now to $A = 2^p$ and $\sim_0, \sim_1, \dots, \sim_{L(p)}$. Call the resulting labelled tree T. Consider the tree structure (T, U, M), where U is as given by the induction hypothesis and $M(n) = N(n)$ for $n \leq L(p)$, which again is given by the induction hypothesis. Note that M is monotone (in fact strictly increasing).

Note now that condition $R(4)$ for $p' = p - 1$ implies that (T, U, M) is good: Indeed, let $p' = \langle n', k' \rangle$. Let $\bar{s}, \bar{t} \in 2^p$ be such that $\bar{s} = s^\frown i$, $\bar{t} = t^\frown i'$, $s, t \in 2^{p'}$. Let $j \leq L(p) = L(p' + 1)$ be such that $\bar{s} \sim_j \bar{t}$. There are two cases: (A) $k' = 0$, so that $L(p') = n'$, $L(p' + 1) = L(p) = n' + 1$; (B) $k' > 0$, so that $n' < L(p') = L(p' + 1)$. In case (A), if $j = L(p) = n' + 1$ then $U_{\bar{s}} F_{N(j)} U_{\bar{t}}$ by $R(4)$ (a), (b) for p', transitivity, and the monotonicity of N. If $j < L(p)$, i.e., $j \leq n'$, then $i = i'$, so since also $j \leq L(p')$, we have by $R(4)$ (a) for p' that $U_{\bar{s}} F_{N(j)} U_{\bar{t}}$. In case (B), if $j = L(p)$ we are done as before. If $j < L(p) = L(p')$, then either $i = i'$, and since $s \sim_j t$ as well, we are done, by $R(4)$ (a) for p'; or else $i \neq i'$ in which case $j > n'$, so $N(j) \geq N(n' + 1)$. Again $s \sim_j t$, so $U_{s^\frown i} F_{N(j)} U_{t^\frown i}$ by $R(4)$ (a) for p' and $U_{t^\frown i'} F_{N(n'+1)} U_{t^\frown i}$ by $R(4)$ (b) for p', thus $U_{s^\frown i} F_{N(j)} U_{t^\frown i'}$ again.

We now have two cases for p.

Case $(\alpha) : p = \langle n, 0 \rangle$, so that $L(p) = n$ and $L(p + 1) = n + 1$.

In this case we have to define also $N(n + 1)$. For that we apply 2.7: We can find a refinement (T, U^*, M) of (T, U, M) and $N(n + 1) > N(n)$ such that F_{n+1} is meager in $F_{N(n+1)}$ on $\bigcup_{s \in 2^p} U^*_s$. (Note that if F_{n+1} is meager in F_m on A, it is also meager in $F_{m'}$ on A for $m' \geq m$.) By applying also 2.5, repeatedly, we can assume that $d(U^*_s) \leq 2^{-p}$ and $\overline{U^*_s}^{\mathcal{T}} \subseteq U_s$.

Case $(\beta) : p = \langle n, k \rangle$ for $k > 0$, so that $n < L(p) = L(p + 1)$.

In this case, we do not have to define a new value of N. Also, by the induction hypothesis, $R(3)$ implies that F_{n+1} is meager in$F_{N(n+1)}$ on $\bigcup_{s \in 2^p} U_s$. So let (T, U^*, M) be a good refinement of (T, U, M) so that $d(U^*_s) \leq 2^{-p}$ and $\overline{U^*_s}^{\mathcal{T}} \subseteq U_s$ (by 2.5 again).

So in either case we have a good tree structure (T, U^*, M) refining (T, U, M) with F_{n+1} meager in $F_{N(n+1)}$ on $\bigcup_{s \in 2^p} U_s^*$ and $d(U_s^*) \leq 2^{-p}$, $\overline{U_x^*}^\tau \subseteq U_s$. Let $N = N(n+1)$. Thus $M(n) \leq N \leq M(n')$ for any $n' > n$, $n' \leq L(p)$ (if such exists). By 2.8 then, there are good refinements (T, U^0, M), (T, U^1, M) of (T, U^*, M) satisfying $U_s^0 F_N U_s^1$, $\forall s \in 2^p$ and $(U_s^0 \times U_t^1) \cap F_{n+1} = \varnothing$, for any two vertices s, t of T which are n-connected, i.e., by 2.9, $s \sim_n t$. We put now

$$U_{s^\frown i} = U_s^i.$$

Clearly $R(1)$, (2) are satisfied (note that $U_{s^\frown 0} \cap U_{s^\frown 1} = \varnothing$ follows from $R(2)$). Moreover, $R(3)$ holds by the choice of $N(n+1)$ and the induction hypothesis. Next, $R(4)$ (a) is true since (T, U_s^0, M), (T, U_s^1, M) are good and M is monotone. Finally, $R(4)$ (b) holds because of (i) of 2.8.

Thus the construction is complete, and so is the proof of 2.2 □

Let us note also the following corollary of the main result, which points out another interesting property of E_1.

Theorem 2.10. *Let E be a Borel equivalence relation. Then*

$$E_1 \leq E \Leftrightarrow E_1 \sqsubseteq^i E.$$

Proof. Let $f\colon (2^{\mathbb{N}})^{\mathbb{N}} \to X$ be such that $x E_1 y \Leftrightarrow f(x) E f(y)$. Let $X^* = f[(2^{\mathbb{N}})^{\mathbb{N}}]$ and $X^{**} = [X^*]_E$. We claim first that X^{**} is Borel. Indeed, let

$$R(x, y) \Leftrightarrow f(y) E x.$$

Then R is Borel with K_σ sections, so since $x \in X^{**} \Leftrightarrow \exists y R(x, y)$, X^{**} is Borel. Moreover, there is a Borel function $\varphi\colon X^{**} \to (2^{\mathbb{N}})^{\mathbb{N}}$ such that $R(x, \varphi(x))$, i.e.,$f \circ \varphi(x) E x$. Then φ is a reduction of $E|X^{**}$ into E_1, so from 2.1 it follows that $E_1 \sqsubseteq E|X^{**}$; thus $E_1 \sqsubseteq E$. So we can assume above that f is 1-1. We can also suppose that $X = 2^{\mathbb{N}}$.

Now define $g\colon X^{**} \to (2^{\mathbb{N}})^{\mathbb{N}}$ by $g(x) = (x, \varphi(x)_1, \varphi(x)_2, \ldots)$. Then g is 1-1 and $g(x) E_1 \varphi(x)$, so $f \circ g(x) E x$. Also $g \circ f(y) E_1 y$. Now apply Schroeder-Bernstein to f, g to show that $E_1 \cong E_1 | X^{**}$, thus $E_1 \sqsubseteq^i E$. □

3. Stable equivalence and isomorphism

As an application of the result in Section 2 we can also classify hypersmooth Borel equivalence relations, at least under further mild regularity assumptions, with respect to stable equivalence, where we call two Borel equivalence relations E, F **stably equivalent** if

$$E \times I_{2^{\mathbb{N}}} \cong F \times I_{2^{\mathbb{N}}}.$$

Let us say that a Borel equivalence relation E on X is **strongly smooth** if it admits a Borel selector, and **strongly hypersmooth** if $E = \bigcup_n F_n$, with $F_0 \subseteq F_1 \subseteq \cdots$ and F_n strongly smooth.

There are easy examples of smooth Borel E which are not strongly smooth (see, e.g., [K3, 18.D]), and from 3.8 below these are not even strongly hypersmooth. However most natural examples of smooth E are actually strongly smooth. Also every smooth E with K_σ equivalence classes is strongly smooth.

We have

Theorem 3.1. *Let E be a non-smooth strongly hypersmooth Borel equivalence relation. Then E is stably equivalent to exactly one of $E_0 \times I_{2^{\mathbb{N}}}, E_1$.*

The same conclusion holds if E is hypersmooth with K_σ equivalence classes, in the sense that E is a Borel equivalence relation on a Borel set B in some Polish space Y and all E-equivalence classes are K_σ in Y.

Proof. We will need the following two lemmas:

Lemma 3.2. *Let E be a Borel equivalence relation with K_σ classes, i.e., E is a Borel equivalence relation on a Borel set B in a Polish space Y and each E-equivalence class is K_σ in Y. Let X be a standard Borel space and F a Borel equivalence relation on X. If $E \times I_{2^{\mathbb{N}}} \sqsubseteq F$, then $E \times I_{2^{\mathbb{N}}} \sqsubseteq^i F$.*

Proof. Let the Borel function f embed $E \times I_{2^{\mathbb{N}}}$ into F and define X^*, X^{**} as in the proof of 2.10. As in that proof, X^{**} is Borel and there is a Borel function $\varphi: X^{**} \to B \times 2^{\mathbb{N}}$ with $f \circ \varphi(x) F x$. Assuming, without loss of generality, that $X = 2^{\mathbb{N}}$, define $g: X^{**} \to B \times 2^{\mathbb{N}}$ as follows: If $\varphi(x) = (b, z)$, then $g(x) = (b, x)$. Then $\varphi(x) E \times I_{2^{\mathbb{N}}} g(x)$, so $f \circ g(x) F x$. Also g is 1-1. Now apply Schroeder-Bernstein. $\square$

Lemma 3.3. *Let E be Borel and strongly hypersmooth. Then $E \times I_{2^{\mathbb{N}}} \sqsubseteq^i E_1$.*

Proof. Put $\widetilde{E} = E \times I_{2^{\mathbb{N}}}$, so that $\widetilde{E}$ is also strongly hypersmooth. We can of course assume that the space of $\widetilde{E}$ is $2^{\mathbb{N}}$. Let $\widetilde{E} = \bigcup_n F_n$, $F_0 \subseteq F_1 \subseteq \cdots$, with $F_0 = \Delta(2^{\mathbb{N}})$, F_n strongly smooth with Borel selector f_n. Consider the canonical embedding $f(x) = (f_n(x))$ of $\widetilde{E}$ into E_1, and define X^*, X^{**} for this f as in the proof of 2.10. We claim that X^{**} is Borel and there is Borel $\varphi: X^{**} \to 2^{\mathbb{N}}$ with $f \circ \varphi((x_m)) E_1 (x_m)$. The proof then can be completed as in 3.2.

To see that X^{**} is Borel, we verify that

$$(x_m) \in X^{**} \Leftrightarrow \exists n \forall m \geq n (x_m = f_m(x_n)).$$

$\Rightarrow$: Pick x so that $(x_m) E_1 f(x)$ and n so that $\forall m \geq n(x_m = f_m(x))$. Then $x_n = f_n(x)$ and $x_n F_n x$, so $x_n F_m x$ for $m \geq n$ and $f_m(x_n) = f_m(x)$ for $m \geq n$; thus $x_m = f_m(x_n)$, $\forall m \geq n$.

$\Leftarrow$: Fix n with $x_m = f_m(x_n)$ for $m \geq n$. Then $f(x_n) E_1 (x_m)$ and $(x_m) \in X^{**}$.

Finally, if $(x_m) \in X^{**}$, let

$$\varphi((x_m)) = x_n,$$

where n is least with $\forall m \geq n (x_m = f_m(x_n))$. Then $f \circ \varphi((_m)) E_1 (x_m)$. $\square$

We now complete the proof of 3.1. Since E is not smooth, $E_0 \sqsubseteq E$, so $E_0 \times I_{2^{\mathbb{N}}} \sqsubseteq E \times I_{2^{\mathbb{N}}}$. Since $E_0 \times I_{2^{\mathbb{N}}}$ has K_σ equivalence classes and $E_0 \times I_{2^{\mathbb{N}}} \times I_{2^{\mathbb{N}}} \cong E_0 \times I_{2^{\mathbb{N}}}$, we have by 3.2 that $E_0 \times I_{2^{\mathbb{N}}} \sqsubseteq^i E \times I_{2^{\mathbb{N}}}$.

Since E is hypersmooth, $E \leq E_0$ or else $E_1 \sqsubseteq E$. In the first case we have that $E \times I_{2^{\mathbb{N}}} \sqsubseteq E_0 \times I_{2^{\mathbb{N}}}$. (Indeed, if g reduces E to E_0, the function

$$f(x, \alpha) = (g(x), \langle x, \alpha \rangle),$$

where $\langle \, \rangle : X \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel injection and X the space of E, embeds $E \times I_{2^{\mathbb{N}}}$ into $E_0 \times I_{2^{\mathbb{N}}}$.) If now E has K_σ equivalence classes, $E \times I_{2^{\mathbb{N}}} \sqsubseteq^i E_0 \times I_{2^{\mathbb{N}}}$ by 3.2, so $E \times I_{2^{\mathbb{N}}} \cong E_0 \times I_{2^{\mathbb{N}}}$. On the other hand, if E is strongly hypersmooth, $E \times I_{2^{\mathbb{N}}} \sqsubseteq^i E_1$ by 3.3, so, replacing $E \times I_{2^{\mathbb{N}}}$ by an isomorphic copy, we can assume that it has K_σ equivalence classes, so, by 3.2 again, $E \times I_{2^{\mathbb{N}}} \sqsubseteq^i E_0 \times I_{2^{\mathbb{N}}}$ and we have shown that $E \times I_{2^{\mathbb{N}}} \cong E_0 \times I_{2^{\mathbb{N}}}$.

In the case when $E_1 \sqsubseteq E$, we have by 2.10 that $E_1 \sqsubseteq^i E \times I_{2^{\mathbb{N}}}$. By 1.3 and 3.2 or 3.3, depending on whether E has K_σ equivalence classes or is strongly hypersmooth, we also have $E \times I_{2^{\mathbb{N}}} \sqsubseteq^i E_1$, so $E_1 \cong E \times I_{\mathbb{R}}$. □

Let us call a Borel equivalence relation E **uniformly continuous** if $E \cong E \times I_{2^{\mathbb{N}}}$. In view of 3.1, the only uniformly continuous, non-smooth, strongly hypersmooth Borel equivalence relations are $E_0 \times I_{2^{\mathbb{N}}}$ and E_1. This should be compared with analogous measure theoretic results of Vershik and Vinokurov-Ganikhodzhaev (see [V], [VF], [VG].)

The following criterion can be useful in verifying whether a given E is uniformly continuous.

Proposition 3.4. *Let E be a Borel equivalence relation. Then the following are equivalent:*

(i) *E is uniformly continuous.*

(ii) *There is a smooth Borel $F \subseteq E$ which has uniformly continuum-size equivalence classes, i.e., there is a Borel function $f\colon X \times 2^{\mathbb{N}} \to X$ such that $xFy \to f(x,\alpha) = f(y,\alpha)Fx$ and $\alpha \neq \beta \Rightarrow f(x,\alpha) \neq f(x,\beta)$ (where X is the space of E).*

Proof. (i)⇒(ii): It is enough to show that $E \times I_{2^{\mathbb{N}}}$ satisfies (ii). Indeed, let $F \subseteq E \times I_{2^{\mathbb{N}}}$ be given by

$$(x,\alpha)F(y,\beta) \Leftrightarrow x = y.$$

Clearly F is smooth. Put also

$$f((x,\alpha),r) = (x,r).$$

(ii)⇒(i): Let $g(x) = (x,\overline{0})$, so that g embeds E into $E \times I_{2^{\mathbb{N}}}$ (where $\overline{0} = 000\cdots$). Let f be given now by (ii). Fix a Borel injection $\langle\ \rangle\colon X \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ and define $h\colon X \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$h(x,\alpha) = f(x,\langle x,\alpha\rangle).$$

Then h is 1-1 and, since $h(x,\alpha)Ex$,

$$g \circ h(x,\alpha) E \times I_{2^{\mathbb{N}}} (x,\alpha),$$

while

$$h \circ g(x) E x,$$

so by applying Schroeder-Bernstein to g, h we have that $E \cong E \times I_{2^{\mathbb{N}}}$. □

As an application, we see that

$$E_1 (= E_0(2^{\mathbb{N}})) \cong E_t(2^{\mathbb{N}}).$$

This is because $E_t(2^{\mathbb{N}})$ is hypersmooth, uniformly continuous (as it contains F, where $xFy \Leftrightarrow \forall n \geq 1 (x_n = y_n)$) and $E_1 \sqsubseteq E_t(2^{\mathbb{N}})$ (via the map $x \in (2^{\mathbb{N}})^{\mathbb{N}} \mapsto (\langle \overline{n}, x_n\rangle) \in (2^{\mathbb{N}})^{\mathbb{N}}$, where $\overline{n} = 0^{n\frown}1^\infty, \langle\ \rangle\colon (2^{\mathbb{N}})^2 \to 2^{\mathbb{N}}$ is a Borel injection); thus $E_t(2^{\mathbb{N}}) \not\leq E_0$. In fact, more generally, if E is Borel hypersmooth with $E_1 \subseteq E \subseteq E'_C$, where E'_C on $(2^{\mathbb{N}})^{\mathbb{N}}$ is defined by

$$(x_n)E'_C(y_n) \Leftrightarrow \{x_n : n \in \mathbb{N}\}\Delta\{y_n : n \in \mathbb{N}\} \text{ is finite},$$

then $E \cong E_1$. This is because any such E is not reducible to a countable Borel equivalence relation, by 1.5, and is uniformly continuous, since it contains F as above.

As another application, let $U\colon X \to X$ be Borel and uniformly $2^{\aleph_0}$-to-1, i.e., assume there is a function $V\colon U[X] \times 2^{\mathbb{N}} \to X$ with analytic graph such that $U(V(y,\alpha)) = y$ for all $\alpha \in 2^{\mathbb{N}}$, and $\alpha \neq \beta \Rightarrow V(y,\alpha) \neq V(y,\beta)$. Then $E_0(U), E_t(V)$ are either smooth or else Borel isomorphic to one of $E_1, E_0 \times I_{2^{\mathbb{N}}}$. This is because $F \subseteq E_0(U) \subseteq E_t(U)$, where $xFy \Leftrightarrow U(x) = U(y)$, and F together with $f(x,\alpha) = V(U(x),\alpha)$ satisfies 3.4, (ii). (Of course, all these cases can occur, as we can see by taking U to be the restriction of the shift on $(2^{\mathbb{N}})^{\mathbb{N}}$ to various Borel invariant subsets.)

In [K2] it is shown that for any Borel equivalence relation of the form $E_{\mathbb{R}}$ (i.e., induced by a Borel flow) none of whose equivalence classes is a singleton, we have that either $E_{\mathbb{R}}$ is smooth or else $E_{\mathbb{R}} \cong E_0 \times I_{2^{\mathbb{N}}}$. By the results in [JKL] it follows also that if G is compactly generated of polynomial growth and E_G has all equivalence classes uncountable, then again E_G is smooth or else $E_G \cong E_0 \times I_{2^{\mathbb{N}}}$.

Finally, we can apply also the preceding methods to classify E_H, and therefore the coset spaces G/H, for subgroups H of Polish groups G, which can be written as unions of increasing sequences of closed subgroups. The result is as follows:

Theorem 3.5. *Let G be a Polish group, $H = \bigcup_n H_n$, where (H_n) is a increasing sequence of closed subgroups. Denote by E_H the equivalence relation*

$$xE_Hy \Leftrightarrow xH = yH.$$

Then exactly one of the following holds:

(i) *H is closed and E_H is smooth.*

(ii) *H is not closed but, for sufficiently large n, H_{n+1}/H_n is countable. Then if H is uncountable, $E_H \cong E_0 \times I_{2^{\mathbb{N}}}$, while if H is countable $E_H \approx E_0$.*

(iii) *For infinitely many n, H_{n+1}/H_n is uncountable and $E_H \cong E_1$.*

Proof. Assume that H is not closed, so E_H is not smooth. Then $E_0 \sqsubseteq E_H$.

Consider first the case where for sufficiently large n, H_{n+1}/H_n is countable. If all H_n are countable, so is H and thus E_H is countable. Since E_H is hypersmooth, by [DJK], E_H is then hyperfinite and $E_H \approx E_0$. So assume some H_n is uncountable. Renumber so that H_0 is uncountable and H_{n+1}/H_n is countable for all n, i.e., H_n/H_0 is countable for all n and so H/H_0 is countable. Then E_{H_0} is strongly smooth and E_H/E_{H_0} is countable. It follows that $E_H \leq E_0$. Let X_0 be a Borel transversal for E_{H_0}. Define F_n on X_0 by

$$F_n = E_{H_n} \restriction X_0$$

and $F = \bigcup_n F_n$, so $F = E_H \restriction X_0$. Since F is countable Borel and hypersmooth, it is hyperfinite, so $F \leq E_0$. But $E_H \leq F$ by the map: $h(x) =$ unique element of $X_0 \cap [x]_{E_{H_0}}$.

By 3.1 then, $E_H \times I_{\mathbb{R}} \cong E_0 \times I_{\mathbb{R}}$. But we claim that E_H is uniformly continuous, so $E_H \cong E_0 \times I_{\mathbb{R}}$. For that we use 3.4. Let $F = E_{H_0}$. Since H_0 is uncountable, let $\varphi\colon \mathbb{R} \to H_0$ be a Borel bijection. Let X_0 be a Borel transversal for E_{H_0} and let $h(x)$ be defined as above. Put $f(x,\alpha) = h(x) \cdot \varphi(\alpha)$.

Now consider the case when for infinitely many n, H_{n+1}/H_n is uncountable. By renumbering, we can assume that H_0 is uncountable and H_{n+1}/H_n is uncountable for each n. We will show then that $E_1 \sqsubseteq E_H$. As before E_H is strongly hypersmooth and uniformly continuous, so $E_H \cong E_1$.

Since H_{n+1}/H_n is uncountable, we claim that H_n is meager in H_{n+1}. Indeed, otherwise $H_n = H_n \cdot H_n^{-1}$ would contain an open nbhd of the identity in H_{n+1}, so H_n would be open in H_{n+1} and H_{n+1}/H_n would be countable.

From this we can derive the following. □

Lemma. *For each* n, E_{H_n} *is meager in* $E_{H_{n+1}}$ *(equipped with the relativized product topology from* G^2; $E_{H_{n+1}}$ *is closed in* G^2*).*

Proof. Put $F_n = E_{H_{n+1}}$. Let U, V be open in G with $(U \times V) \cap F_{n+1} \neq \varnothing$. We will find open $U' \subseteq U$, $V' \subseteq V$ such that $(U' \times V') \cap F_{n+1} \neq \varnothing$, but $(U' \times V') \cap F_n = \varnothing$.

Consider $L = \{g \in G \colon \exists x \in U(xg \in V)\}$. This is open in G and $L \cap H_{n+1} \neq \varnothing$. So find open $L' \subseteq L$ with $L' \cap H_{n+1} \neq \varnothing$, but $L' \cap H_n = \varnothing$. Fix $g_0 \in H_{n+1}$, $g_0 \in L'$ and $x_0 \in U$ with $x_0 g_0 = y_0 \in V$. Let

$$T = \{(x, y) \colon x \in U \ \&\ y \in V \ \&\ x^{-1}y \in L'\}.$$

This is an open nbhd of (x_0, y_0), so we can find $U' \subseteq U$, $V' \subseteq V$ with $(x_0, y_0) \in U' \times V'$, $U' \times V' \subseteq T$. Since $(x_0, y_0) \in F_{n+1}$, we have $(U' \times V') \cap F_{n+1} \neq \varnothing$. If $(U' \times V') \cap F_n \neq \varnothing$, let $x' \in U'$, $y' \in V'$, $g' \in H_n$ be such that $x'g' = y'$. Then $(x')^{-1}y' = g' \in L' \cap H_n$, a contradiction. So $(U' \times V') \cap F_n = \varnothing$.

We can now repeat the argument of Case II in the proof of 2.1. Instead of the Gandy-Harrington topology we work with the Polish topology of G. The two relevant points are:

(i) E_{H_n} is meager in E_{H_m} for any $n < m$.

(ii) The E_{H_n}-saturation of an open set in G is also open, being a union of translates of this open set.

Thus in the construction of CaseII we can take U_s to be open sets in G. □

If we are wiling to allow a wider class of isomorphisms than Borel, we actually have a simpler formulation of the preceding results. Recall that a function is **C-measurable** if it is measurable with respect to the smallest σ-algebra containing the Borel sets and closed under the Souslin operation $\mathcal{A}$. We also call sets in this class **C-measurable**.

Theorem 3.6. *Let* E *be a nonsmooth, hypersmooth Borel equivalence relation. If every* E*-equivalence class is uncountable, then* E *is isomorphic by a* C*-measurable isomorphism to exactly one of* E_1 *or* $E_0 \times I_{2^{\mathbb{N}}}$.

(Here, to say that an isomorphism f between standard Borel spaces is C-measurable, means that both f and f^{-1} are C-measurable.)

Proof. By 3.1 it is enough to show that if E is as in the hypothesis of 3.6, then E is isomorphic to $E \times I_{2^{\mathbb{N}}}$ by a C-measurable isomorphism.

Let X, Y be standard Borel spaces. A bijection $f \colon A \to B$, where $A \subseteq X, B \subseteq Y$ and $f[A] = B$, will be called C-**measurable** if f, f^{-1}, A, B are C-measurable. Note that by the usual Schroeder-Bernstein argument if $f \colon X \to A \subseteq Y$, $g \colon Y \to B \subseteq X$ are C-measurable bijections, then there is a C-measurable bijection $h \colon X \to Y$.

Consider now E (on X) and $E \times I_{2^{\mathbb{N}}}$ (on $X \times 2^{\mathbb{N}}$). Clearly the map $f(x) = (x, \overline{0})$ is a C-measurable bijection of X with $X \times \{\overline{0}\}$. If we can find a C-measurable bijection $g \colon X \times 2^{\mathbb{N}} \rightarrowtail A \subseteq X$ such that $g(x, \alpha) E x$, then by applying Schroeder-Bernstein to f, g we obtain a C-measurable isomorphism of E with $E \times I_{2^{\mathbb{N}}}$.

Let $E = \bigcup_n F_n$, $F_0 \subseteq F_1 \subseteq \cdots$, where F_0 is equality and F_n is smooth. Let $g_n \colon X \to X$ be a C-measurable selector for F_n. Define a new C-measurable selector f_n of F_n by letting $f_0 =$ identity, $f_{n+1} = f_n \circ g_{n+1}$. If $T_n = \{x \colon f_n(x) = x\}$ is the corresponding transversal for f_n, then T_n is C-measurable and $T_{n+1} \subseteq T_n$.

Lemma 3.7. *Let $A_n = \{x\colon x \in T_n \ \&\ [x]_{F_n}$ is uncountable$\}$. Then there is a C-measurable bijection $R_n\colon A_n \times 2^{\mathbb{N}} \to B_n \subseteq X$ such that*
(i) $R_n(x,\alpha)F_n x$,
(ii) $B_n \cap B_m = \varnothing$, *if* $n \neq m$.

Granting this lemma, we complete the proof as follows: For each $x \in X$, let $n(x) =$ least n such that $[x]_{F_n}$ is uncountable. This exists as $[x]_E = \bigcup_n [x]_{F_n}$ and $[x]_E$ is uncountable. Put

$$g(x,\alpha) = R_{n(x)}(f_{n(x)}(x), \langle x,\alpha\rangle)$$

(where $\langle\ \rangle\colon X \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel bijection.)

First we check that g is 1-1: If $g(x,\alpha) = g(x',\alpha')$, then as the ranges of R_n are disjoint, $n(x) = n(x')$ $(= n.)$ So $R_n(f_n(x), \langle x,\alpha\rangle) = R_n(f_n(x'), \langle x',\alpha'\rangle) = y$. Then $\langle x,\alpha\rangle = \langle x',\alpha'\rangle$.

Clearly G is C-measurable—note that $x \mapsto n(x)$ is C-measurable. Let $B =$ range(g). We check that B is C-measurable: For $\alpha \in 2^{\mathbb{N}}$, let $\alpha = \langle(\alpha)_0, (\alpha)_1\rangle$. Then if π_0, π_1 are the two projections of $X \times 2^{\mathbb{N}}$,

$$\begin{aligned} y \in \mathrm{range}(g) \Leftrightarrow \exists n[y \in \mathrm{range}(R_n)\ \&\ & \\ & n((\pi_1(R_n^{-1}(y)))_0) = n\ \& \\ & \pi_0(R_n^{-1}(y)) = f_n((\pi_1(R_n^{-1}(y)))_0)]. \end{aligned}$$

Also for $y \in$ range(g), $A \subseteq X \times \mathbb{R}$, A a Borel set,

$$\begin{aligned} g^{-1}(y) \in A \Leftrightarrow \exists n[y \in \mathrm{range}(R_n)\ \&\ & \\ & n((\pi_1(R_n^{-1}(y)))_0) = n\ \& \\ & \pi_0(R_n^{-1}(y)) = f_n((\pi_1(R_n^{-1}(y)))_0)\ \& \\ & ((\pi_1(R_n^{-1}(y)))_0, (\pi_1(R_n^{-1}(y)))_1) \in A]. \end{aligned}$$

So g^{-1} is also C-measurable.

Finally, $g(x,\alpha)F_{n(x)}f_{n(x)}(x)F_{n(x)}x$, so $g(x,\alpha)Ex$.

Proof of Lemma 3.7. Clearly $R_0 = \varnothing$. By a standard result of descriptive set theory, we can find R_1^* satisfying the conditions of the lemma for F_1. Put $R_1(x,\alpha) = R_1^*(x, \langle 1,\alpha\rangle)$, where $\langle\ \rangle$ is a Borel bijection of $\mathbb{N} \times 2^{\mathbb{N}}$ with $2^{\mathbb{N}}$. This works as well.

Now consider A_2 and split it in two parts: $A_2' = \{x \in A_2\colon \exists y[y \in [x]_{F_2}\ \&\ [y]_{F_1}$ is uncountable$\}$, $A_2'' = A_2 \backslash A_2'$. So A_2' is $\mathbf{\Sigma}_1^1$ and A_2'' is in $\mathbf{\Pi}_1^1\ \&\ \mathbf{\Sigma}_1^1$, so certainly both are C-measurable. Let g_2 be a C-measurable function with domain A_2', such that $g_2(x)$ is a y witnessing that $x \in A_2'$. Let $f_1(y) = z \in T_1$. Thus $[z]_{F_1}$ is uncountable. Put $R_2^{**}(x,\alpha) = R_1^*(z, \langle 2,\alpha\rangle)$. Then R_2^{**} is a C-measurable bijection with domain A_2'. (Note that given z, x can be determined as $x = f_2(z)$.) Let R_2^{***} be a C-measurable injection with domain A_2'' satisfying (i) of the lemma for $x \in A_2''$, $n = 2$. Put $R_2^* = R_2^{**} \cup R_2^{***}$. Clearly R_2^* satisfies all the conditions of the lemma for F_2. Put

$$R_2(x,\alpha) = R_2^*(x, \langle 1,\alpha\rangle).$$

Next, split A_3 into two parts: $A_3' = \{x \in A_3\colon \exists y[y \in [x]_{F_2}\ \&\ [y]_{F_2}$ is uncountable$]\}$, $A_3'' = A_3 \backslash A_3'$. For A_3'', define R_3^{***} as before. For A_3', let g_3 be C-measurable choosing a witness $y = g_3(x)$ to the fact that $x \in A_3'$. Let $f_2(y) = z \in T_2$. Thus $z \in A_2$. If $z \in A_2''$, let $R_3^{**}(x,\alpha) = R_2^{***}(z, \langle 2,\alpha\rangle)$. If $z \in A_2'$, let $f_1(g_2(z)) = w \in T_1$

and let $R_3^{**}(x,\alpha) = R_1^*(w, \langle 3, \alpha \rangle)$. Finally, put $R_3^* = R_3^{**} \cup R_3^{***}$ and $R_3(x,\alpha) = R_3^*(x, \langle 1, \alpha \rangle)$.

Proceed this way by induction on n. □

We conclude this section with an additional result that further clarifies the role of strong hypersmoothness.

Theorem 3.8. *If the Borel equivalence relation E is smooth and strongly hypersmooth, then E is strongly smooth.*

Remark. Note that this also implies that the assumption that E is strongly hypersmooth is essential in 3.3, because by 3.8 (and the remarks preceding 3.1) there is a smooth but not strongly hypersmooth E. Then $E \times I_{2^{\mathbb{N}}} \sqsubseteq^i E_1$ fails, since otherwise $E \times I_{2^{\mathbb{N}}}$ would be strongly hypersmooth, as E_1 is, and so would be E.

Proof (of 3.8). By relativization, it is enough to assume that E is a Δ_1^1 equivalence relation on $\mathcal{N}$, (E_n) is a Δ_1^1-sequence of Π_1^0 equivalence relations on $\mathcal{N}$ with $E_0 \subseteq E_1 \subseteq \cdots$ and $\bigcup_n E_n = E$, $f\colon \mathcal{N} \to \mathcal{N}$ is Δ_1^1 such that $xEy \Leftrightarrow f(x) = f(y)$, and (f_n) is a Δ_1^1-sequence of Δ_1^1 functions $f_n\colon \mathcal{N} \to \mathcal{N}$ so that f_n is a selector for E_n. Let $f(\mathcal{N}) = A$, which is a Σ_1^1 subset of $\mathcal{N}$. To show that E is strongly smooth, it is enough to show that for each $z \in A$ there is $x \in \Delta_1^1(z)$ with $f(x) = z$.

Let $C = f^{-1}[\{z\}]$. If $y \in C$, then $[y]_E = C$, so $C = \bigcup_n [y]_{E_n}$. Apply the Baire Category Theorem in the Gandy-Harrington topology relativized to z (i.e., the topology τ^z whose basic nbhds are the $\Sigma_1^1(z)$ subsets of $\mathcal{N}$), noticing that $C \in \tau^z$ and $[y]_{E_n}$ are closed in $\mathcal{N}$, so τ^z-closed. We can then find $S \subseteq [y]_{E_n}$, for some n, with $S \neq \varnothing$, $S \in \Sigma_1^1(z)$. Now $f_n(x') = f_n(y)$, $\forall x' \in S$, so $x = f_n(y) \in \Delta_1^1(z) \cap C$ and we are done. □

4. Embedding E_1

It is a delicate question to decide, for a given Borel equivalence relation E, whether $E_1 \leq E$. The only obstruction we know is given in 4.1 below.

Let E be a Borel equivalence relation on X. We call E **idealistic** if there is a map $C \in X/E \mapsto I_c$, assigning to each E-equivalence class C a σ-ideal I_C of subsets of C, with $C \notin I_C$, such that I_C satisfies the ccc (countable chain condition), i.e., any collection of pairwise disjoint subsets of C which are not in I_C is countable, and moreover the map $C \mapsto I_C$ is Borel in the following sense:

For each Borel $A \subseteq X^2$ the set A_I defined by

$$x \in A_I \Leftrightarrow \{y \in [x]_E \colon A(x,y)\} \in I_{[x]_E}$$

is Borel.

Examples of idealistic E include those induced by Borel actions of Polish groups and the **measured** ones, i.e., those for which there is a Borel assignment $x \mapsto \mu_x$ of probability measures such that $\mu_x([x]_E) = 1$ and $xEy \Rightarrow \mu_x \sim \mu_y$ (see for example [K1]).

Theorem 4.1. *Let E be an idealistic Borel equivalence relation. Then $E_1 \not\leq E$.*

Proof. If $E_1 \leq E$, then by 2.10, $E_1 \sqsubseteq^i E$, so E_1 must be idealistic too. But E_1 is the union of a sequence of smooth Borel equivalence relations, so by Theorem 1.5 of [K1], $E_1 \leq F$ for a countable Borel equivalence relation F, contradicting 1.4. □

If G is a Polish group acting in a Borel way on a standard Borel space X and E_G is the corresponding equivalence relation, then $E_1 \not\leq E_G$ by 4.1 provided E_G is Borel. However by a modification of the proof of 4.1 we do not need to impose this restriction.

Theorem 4.2. *Let E_G be the equivalence relation induced by a Borel action of a Polish group. Then $E_1 \not\leq E_G$.*

Proof. Assume $E_1 \leq E_G$ via the Borel function $f\colon (2^{\mathbb{N}})^{\mathbb{N}} \to X$, where E_G lives on X. Let $f((2^{\mathbb{N}})^{\mathbb{N}}) = Y$ and $Z = [Y]_{E_G}$, so that Y, Z are $\mathbf{\Sigma}^1_1$.

Let $g\colon Y \to (2^{\mathbb{N}})^{\mathbb{N}}$ be a C-measurable inverse for f and define the equivalence relation F on Y by

$$yFy' \Leftrightarrow g(y)E_1 g(y') \ (\Leftrightarrow yE_G y').$$

Then $F = \bigcup_n F_n$, where $F_0 \subseteq F_1 \subseteq \cdots$ are equivalence relations on Y which are C-measurable smooth, i.e., for each n there is a C-measurable function $s_n\colon Y \to 2^{\mathbb{N}}$ with $yF_n y' \Leftrightarrow s_n(y) = s_n(y')$.

Now for each $D \in X/E_G$ let I_D be the canonical σ-ideal on D (see, e.g., [K1]) given by

$$A \in I_D \Leftrightarrow \{g \in G\colon g\cdot x \in A\} \text{ is meager in } G,$$

where $(g, x) \mapsto g\cdot x$ is the action and $x \in D$. (This definition is independent of x.) Clearly I_D is ccc. Fix a C-measurable function $h\colon Z \to Y$ such that $h(z)E_G z$. For each $C \in Y/F$, define the following σ-ideal J_C on C :

$$A \in J_C \Leftrightarrow h^{-1}(A) \in I_{[C]_{E_G}}.$$

It is clearly ccc, and $C \notin J_C$. Also $C \mapsto J_C$ satisfies the following:

For each C-measurable $C \subseteq Y^2$, the set C_J defined by

$$y \in C_J \Leftrightarrow \{y' \in [y]_F\colon C(y, y')\} \in J_{[y]_F}$$

is $\mathbf{\Delta}^1_2$.

We can now repeat the argument for the proof of 1.5, (ii)$\Rightarrow$(i) in [K1], to show that there is a $\mathbf{\Sigma}^1_2$ set $A \subseteq Y$ which meets every F-equivalence class in a countable nonempty set. Let $F' = F|A$. It follows that there is a $\mathbf{\Delta}^1_2$ function $H\colon (2^{\mathbb{N}})^{\mathbb{N}} \to X$ such that $xE_1 y \Leftrightarrow H(x)F'H(y)$. Then we can repeat the proof of 1.5 (with H replacing f there) to reach a contradiction. The only additional fact that is needed in the present case is that H is Baire measurable, i.e., we need to know that $\mathbf{\Sigma}^1_2$ sets have the Baire property. However, since the result we want to prove (i.e., 4.2) is equivalent (in ZFC) to a Π^1_3 sentence, it is enough, by standard metamathematical results, to prove it assuming additionally $\mathrm{MA} + \neg\,\mathrm{CH}$, which implies that all $\mathbf{\Sigma}^1_2$ sets have the property of Baire, and we are done. □

We can use these results to discuss various classes of examples. Let us consider first equivalence relations generated by filters on $\mathbb{N}$; see [L]. For E a Borel equivalence relation on X and $\mathcal{F}$ a Borel filter on $\mathbb{N}$, denote by $E^{\mathcal{F}}$ the following Borel equivalence relation on $X^{\mathbb{N}}$:

$$(x_n)E^{\mathcal{F}}(y_n) \Leftrightarrow \{n\colon x_n E y_n\} \in \mathcal{F}.$$

If $E = \Delta_2$ is the equality relation on $2 = \{0, 1\}$, we write $2^{\mathcal{F}}$ instead of $\Delta_2^{\mathcal{F}}$.

If E has uncountably many equivalence classes, then $\Delta_{2^{\mathbb{N}}} \leq E$, so $E_1 = \Delta_{2^{\mathbb{N}}}^{\mathcal{N}_0} \leq E^{\mathcal{N}_0}$, where $\mathcal{N}_0 =$ the **Fréchet filter** $= \{A \subseteq \mathbb{N}\colon A \text{ is cofinite}\}$. Given two filters

$\mathcal{F}, \mathcal{G}$, let $\mathcal{F} \leq \mathcal{G}$ iff there is $\varphi\colon \mathbb{N} \to \mathbb{N}$ with $\mathcal{F} = \varphi\mathcal{G} = \{A \subseteq \mathbb{N}\colon \varphi^{-1}[A] \in \mathcal{G}\}$. It is well-known that if $\mathcal{F}$ is a free Borel filter, then $\mathcal{N}_0 \leq \mathcal{F}$, and it is easy to see that

$$\mathcal{F} \leq \mathcal{G} \Rightarrow E^{\mathcal{F}} \leq E^{\mathcal{G}}.$$

So $E^{\mathcal{N}_0} \leq E^{\mathcal{F}}$ for any free Borel $\mathcal{F}$, and thus for any Borel E with uncountably many equivalence classes we have $E_1 \leq E^{\mathcal{F}}$.

The situation with $2^{\mathcal{F}}$ is quite different. Clearly $E_1 \not\leq 2^{\mathcal{N}_0} = E_0$. We denote by $\mathbb{N}$ also the filter $\{\mathbb{N}\}$. Then $E_1 \cong 2^{\mathbb{N}\times\mathcal{N}_0}$, so $E_1 \leq 2^{\mathcal{F}}$ for any $\mathcal{F}$ with $\mathbb{N} \times \mathcal{N}_0 \leq \mathcal{F}$. Since $\mathbb{N} \leq \mathcal{N}_0$, it follows that $\mathbb{N} \times \mathcal{N}_0 \leq \mathcal{N}_0 \times \mathcal{N}_0 = \mathcal{N}_1$ and so $E_1 \leq 2^{\mathcal{N}}$ and thus $E_1 \leq 2^{\mathcal{N}_\xi}$, where $\mathcal{N}_\xi$ are the iterated Fréchet filters; see [L].

On the other hand, every ideal on $\mathbb{N}$ is also a subgroup of the compact group $(\mathbb{Z}_2^{\mathbb{N}}, +)$, where $+$ is coordinatewise addition. If $\mathcal{F}$ is a Borel filter and $\mathcal{I}$ its dual ideal, then $\mathcal{I}$ is a Borel subgroup of $\mathbb{Z}_2^{\mathbb{N}}$ and $2^{\mathcal{F}}$ is exactly the Borel equivalence relation given by the cosets of $\mathcal{I}$ in $\mathbb{Z}_2^{\mathbb{N}}$, thus is generated by a Borel $\mathcal{I}$-action. Recall now that a standard Borel group G (i.e., a group which is a standard Borel space for which multiplication and inverse are Borel) is called **Polishable** if there is a (necessarily unique) Polish topology on G with the same Borel structure, under which G becomes a topological group. Thus if $\mathcal{I}$ is Polishable, $E_1 \not\leq 2^{\mathcal{F}}$ by 4.1. This has an interesting application concerning ideals:

If $\mathcal{I}$ is a Borel ideal, $\mathcal{F}$ its dual filter and $\mathbb{N} \times \mathcal{N}_0 \leq \mathcal{F}$, then $\mathcal{I}$ is not Polishable.

On the other hand, there are interesting Polishable ideals $\mathcal{I}$ (for which therefore $E_1 \not\leq 2^{\mathcal{F}}$). For instance, let $k_0 = 0 < k_1 < k_2 < \cdots$ be such that $k_{n+1} = k_n + n$ and let $I_n = [k_n, k_{n+1})$, so that $\mathrm{card}(I_n) = n$. Define, for $p \in [1, \infty)$, the ideal $\mathcal{I}_p$ by

$$A \in \mathcal{I}_p \Leftrightarrow (\mathrm{card}(A \cap I_n)/n)_{n\geq 1} \in l^p.$$

Another class of E for which $E_1 \not\leq E$ comes from model theory. If E is the isomorphism relation on a Borel (invariant under isomorphism) class of countable structures, then by 4.2, $E_1 \not\leq E$. In particular, this applies to the examples discussed in [FS]. For each Borel equivalence relation E on X, we denote, as in [L], by E^+ the following equivalence relation on $X^{\mathbb{N}}$:

$$(x_n)E^+(y_n) \Leftrightarrow \forall n \exists m(x_n E y_m) \ \&\ \forall n \exists m(x_m E y_n).$$

Then $\Delta_{2^{\mathbb{N}}}^{+}, \Delta_{2^{\mathbb{N}}}^{++}, \ldots$ (to the transfinite) can be Borel reduced to the equivalence relation of isomorphism of trees discussed in [FS], so E_1 embeds in none of them either.

Another important Borel equivalence relation is measure equivalence $\sim$. (Here X is the space of probability measures on an uncountable standard Borel space, and $\mu \sim \nu \Leftrightarrow \mu \ll \nu \ \&\ \nu \ll \mu$.) By the Spectral Theorem, $\sim \leq E$, where E is the equivalence relation of unitary isomorphism of normal operators on Hilbert space. By 4.2, $E_1 \not\leq E$, so $E_1 \not\leq \sim$. We can reduce to $\sim$ various other Borel equivalence relations and thus use this to give alternative proofs that E_1 cannot be reduced to them. For example, consider $E_C = \Delta_{2^{\mathbb{N}}}^{+} \leq \sim$ (use $(x_n) \mapsto \sum 2^{-n}\delta_{x_n}$, with $\delta_x =$ the Dirac measure on x). Now if F is a countable Borel equivalence relation, then $F \leq E_C$ (send any x to an enumeration of $[x]_F$ in a Borel way) and easily $E_C^{\mathbb{N}} \leq E_C$. It follows that $E_1 \not\leq E_C$ and $E_1 \not\leq F^{\mathbb{N}}$, for any countable F.

Finally, note that in [K1] an example is given of a K_σ subgroup H of $\mathbb{T}^{\mathbb{N}}$ which is Polishable but E_H is not comparable in $\leq$ with E_1. Another example would be $\mathcal{I}_p$. So 3.5 does not extend to F_σ subgroups.

We conclude with the following problems:

Problem. If E is a Borel equivalence relation, is it true that either $E_1 \leq E$ or E is idealistic?

Problem. Let $\mathcal{F}$ be a Borel filter on $\mathbb{N}$ and $\mathcal{I}$ its dual ideal. Is it true that either $E_1 \leq 2^{\mathcal{F}}$ or $\mathcal{I}$ is Polishable? (This has been recently solved affirmatively by Solecki.)

5. Global effects

Although the preceding results are "local", being concerned with Borel equivalence relations which are $\leq E_1$, they have a surprising "global" consequence about the structure of the class of all Borel equivalence relations.

Given a pair (E, E^*) of Borel equivalence relations with $E < E^*$, we say that (E, E^*) satisfies the **dichotomy property** if for any Borel equivalence relation F we have $F \leq E$ or $E^* \leq F$.

Thus Silver's Theorem (see [S]) asserts that $(\Delta_{\mathbb{N}}, \Delta_{2^{\mathbb{N}}})$ satisfies the dichotomy property, and the Glimm-Effros type dichotomy proved in [HKL] implies that $(\Delta_{2^{\mathbb{N}}}, E_0)$ satisfies the dichotomy property. Notice also that trivially (Δ_n, Δ_{n+1}) $(n = 1, 2, \dots)$ satisfy the dichotomy property. If (E, E^*) satisfies the dichotomy property, then E, E^* are **nodes** in $\leq$, i.e., every Borel equivalence relation is comparable in $\leq$ to each one of them.

We have, now,

Theorem 5.1. *Let E be a Borel equivalence relation which is a node for $\leq$, i.e., for any Borel equivalence relation F we have $E \leq F$ or $F \leq E$. Then $E \approx^* \Delta_n$ $(n = 1, 2, \dots)$, $\Delta_{\mathbb{N}}$, $\Delta_{2^{\mathbb{N}}}$ or E_0.*

In particular, the only pairs (E, E^*) satisfying the dichotomy property are (Δ_n, Δ_{n+1}) $(n = 1, 2, \dots)$, $(\Delta_{\mathbb{N}}, \Delta_{2^{\mathbb{N}}})$, $(\Delta_{2^{\mathbb{N}}}, E_0)$.

Proof. Call a class U of Borel equivalence relations **unbounded** (in $\leq$) if there is no Borel equivalence relation E such that $\forall F \in U (F \leq E)$. We will use the following result.

Theorem 5.2 (Harrington, unpublished). *There is a nonempty class U of Borel equivalence relations which is unbounded, and every $F \in U$ is induced by a Borel action of a Polish group (so in particular is idealistic).*

Let E be a Borel equivalence relation which is such that $E \leq F$ or $F \leq E$ for all Borel equivalence relations F. By applying 5.2, we obtain that $E \leq F$ for some $F \in U$. We also have $E_1 \leq E$ or $E \leq E_1$. If $E_1 \leq E$ then $E_1 \leq F$, which violates 4.1, so $E \leq E_1$. Then by 2.1, and the remarks following it, either $E \approx^* E_1$, which is impossible, or $E \approx^* E_0$ or E is smooth. In the last case, clearly (by Silver's Theorem) $E \approx^* \Delta_{2^{\mathbb{N}}}$ or $E \leq E_{\mathbb{N}}$, and if $E \leq E_{\mathbb{N}}$, either $E \approx^* E_n$, $n = 1, 2, \dots$, or $E \approx^* E_{\mathbb{N}}$. □

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Department of Mathematics, A. P. Sloan Laboratory of Mathematics and Statistics, California Institute of Technology, Pasadena, California 91125

E-mail address: `kechris@caltech.edu`

Equipe d'Analyse, Université Paris VI, 4, Place Jussieu, 75230 Paris Cedex 05, France

E-mail address: `louveau@ccr.jussieu.edu`