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THE CLASS OF SYNTHESIZABLE PSEUDOMEASURES

BY

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In this paper we study descriptive set theoretic questions related to concepts of harmonic synthesis on the unit circle $\mathbf{T}$, and their relationship with the structure of uniqueness sets.

We denote by $A = A(\mathbf{T})$ the space of functions on $\mathbf{T}$ with absolutely convergent Fourier series, by PM the space of pseudomeasures on $\mathbf{T}$ and by PF the space of pseudo-functions on $\mathbf{T}$. Thus $PF^* = A$, $A^* = PM$. Finally $K(\mathbf{T})$ denotes the compact space of closed subsets of $\mathbf{T}$ with the Hausdorff metric. The three basic notions associated with harmonic synthesis are the following:

(i) A function $f \in A$ *satisfies synthesis* if $\langle f, S \rangle = 0$ for all $S \in PM$ with $f = 0$ on $\operatorname{supp}(S)$.

(ii) A pseudomeasure $S \in PM$ *satisfies synthesis* if $\langle f, S \rangle = 0$ for all $f \in A$ with $f = 0$ on $\operatorname{supp}(S)$. This is equivalent to saying that $S \in N(\operatorname{supp}(S))$, where for each $E \in K(\mathbf{T})$, we let

$$M(E) = \text{space of (Borel complex) measures whose (closed) support is contained in } E,$$

$$N(E) = \text{weak}^*\text{-closure of } M(E).$$

For simplicity, if $S \in PM$ satisfies synthesis, we will call it a *synthesizable* pseudomeasure.

(iii) A set $E \in K(\mathbf{T})$ is a *set of synthesis* if for all $f \in A, S \in PM$ with $\operatorname{supp}(S) \subseteq E$ and $f = 0$ on E we have $\langle f, S \rangle = 0$. Equivalently, if

$$I(E) = \{f \in A: f = 0 \text{ on } E\},$$

$$J(E) = \{f \in A: f = 0 \text{ on an (open) nbhd of } E\},$$

E is of synthesis iff the strong closure of $J(E)$ in A is equal to $I(E)$. Also equivalently, E is of synthesis iff $N(E) = PM(E)$ (= the space of pseudomeasures supported by E).

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We wish to classify the descriptive complexity of the above notions. For the first one, we look at the separable Banach space A and set

$$X = \{f \in A \colon f \text{ satisfies synthesis}\}.$$

We claim that this set is G_δ. For that notice that if $Z(f) = \{x \in \mathbf{T}\colon f(x) = 0\}$,

$$f \in X \Leftrightarrow f \in \overline{J(Z(f))}$$

(where $\overline{Z}$ denotes the strong closure of Z in A). Now

$$f \in X \Leftrightarrow \forall m > 0\, \exists g \in A\left(g \in J(Z(f)) \wedge \|f - g\|_A < \frac{1}{m}\right).$$

Let $g_0 \in A$. It is enough to show that for any $m > 0$,

$$L(g_0, m) = \left\{f \colon g_0 \in J(Z(f)) \wedge \|f - g_0\|_A < \frac{1}{m}\right\}$$

is open in A. If $f_0 \in L(g_0, m)$, it is not hard to find a small neighborhood of f_0 contained in $L(g_0, m)$. This proves that X is a G_δ set.

Remark. One could also ask for the complexity of

$$Y = \{f \in A \colon \|f\|_A \leq 1 \text{ and } f \text{ satisfies synthesis}\}$$

in the compact, metrizable with the weak*-topology unit ball of A, denoted by $B_1(A)$. Since the identity map from $B_1(A)$ into A is a function of first class between these two spaces, and maps Y onto $X \cap B_1(A)$, which is a G_δ subset of A, it follows that Y is a $F_{\sigma\delta}$ subset of $B_1(A)$

For the third notion, we look at the space $K(\mathbf{T})$ and the set

$$S = \{E \in K(\mathbf{T}) \colon E \text{ is a set of synthesis}\}.$$

It was shown by Kechris and Solovay, using a result of Katznelson-McGehee [4], that this is a $\mathbf{\Pi}^1_1$ (coanalytic) not Borel set; see [6], p. 346.

It thus remains only to classify the complexity of the second notion. We look here at the compact, metrizable with the weak*-topology unit ball $B_1(PM)$ of PM and the set

$$\mathscr{S} = \{S \in B_1(PM) \colon S \text{ is synthesizable}\}.$$

Our first main result is then the following

THEOREM 1. *The class $\mathscr{S}$ of synthesizable pseudomeasures (in the unit ball of PM with the weak*-topology) is $\mathbf{\Pi}_1^1$ but not Borel.*

The proof of Theorem 1 is based on a rank argument (see [6], pp. 110, 148). Given any closed set $E \in K(\mathbf{T})$ define, by transfinite induction on α, a subspace $M^{(\alpha)}(E) \subseteq PM$ as follows:

$$M^{(0)}(E) = M(E),$$

$$M^{(\alpha+1)}(E) = \left(M^{(\alpha)}(E)\right)^{(1)}$$

$$= \text{the set of limits of weak*-converging sequences from } M^{(\alpha)}(E),$$

$$M^{(\lambda)}(E) = \bigcup_{\alpha<\lambda} M^{(\alpha)}(E), \quad \lambda \text{ limit.}$$

For technical reasons we actually work with

$$M^{[\alpha]}(E) = \overline{M^{(\alpha)}(E)}$$

$$= \text{the strong closure in } (PM) \text{ of } M^{(\alpha)}(E).$$

For each E, there is a countable ordinal α with $M^{[\alpha]}(E) = N(E)$ ($= M^{[\beta]}(E)$ for all $\beta \geq \alpha$). So for each $S \in \mathscr{S}$ define its *order* by

$$\mathrm{ORD}(S) = \text{least } \alpha \text{ such that } S \in M^{[\alpha]}(\mathrm{supp}(S)).$$

Thus ORD: $\mathscr{S} \to \omega_1$ is a rank function on $\mathscr{S}$. For each $S \in \mathscr{S}$, ORD(S) is the smallest (transfinite) number of iterations of weak*-sequential limits that is needed to generate S from measures on its support.

After checking the easy fact that $\mathscr{S}$ is $\mathbf{\Pi}_1^1$, we show that ORD: $\mathscr{S} \to \omega_1$ is a $\mathbf{\Pi}_1^1$-rank (see [6] for this notion). This is done by developing an alternative "tree-rank" ψ: $\mathscr{S} \to \omega_1$, which is clearly a $\mathbf{\Pi}_1^1$-rank, and then showing that ORD, ψ are equivalent. Finally, by the boundedness theorem for $\mathbf{\Pi}_1^1$-ranks (see e.g. [6], p. 148) it will be enough to show that ORD is unbounded in ω_1. This is done by using again the Katznelson-McGehee [4] result.

We concentrate next on the class of synthesizable pseudofunctions $\mathscr{S}_0 = \mathscr{S} \cap PF$. The main problem is whether they form a Borel class or not. The statement that they are Borel is equivalent to the statement that every synthesizable pseudofunction has order $< \alpha_0$, for some fixed $\alpha_0 < \omega_1$. We conjecture that this fails:

Conjecture. The class $\mathscr{S}_0$ is not Borel.

Although we do not know the answer to this conjecture, we show in the second part of this paper that it is crucially related to many interesting definability and structural problems concerning (closed) sets of uniqueness.

Let U, U_0 be the classes of closed sets of uniqueness (resp. extended uniqueness) in **T**, i.e., those not supporting non-0 pseudofunctions (resp. measures which are pseudofunctions). Therefore U is characterized as the class of those E for which $J(E)$ is weak*-dense in A, i.e., $PF \cap PM(E) = \{0\}$. Piatetski-Shapiro [10] has also defined the intermediate class U_1, $U \subseteq U_1 \subseteq U_0$ as the class of those E for which $I(E)$ is weak*-dense in A, i.e., $PF \cap N(E) = \{0\}$. As opposed to U, U_0 this class is not a σ-ideal, so we denote by U_1^* the σ-ideal of closed sets it generates, $U \subseteq U_1^* \subseteq U_0$. The connection with synthesis is explained in [6]:

$$E \in U_1^* \Leftrightarrow E \text{ does not support non-0 pseudofunctions of synthesis.}$$

By results of Körner [7], Piatetski-Shapiro [10] the inclusions $U \subseteq U_1^* \subseteq U_0$ are proper. In [6] it is shown that in some sense U_1^* is structurally very close to U. However, the gap (U_1^*, U_0) remains more mysterious. Lyons [8] takes a first step in analyzing it by introducing a further intermediate class U_2, $U \subseteq U_1 \subseteq U_2 \subseteq U_0$. One of its characterizations is that

$$E \in U_2 \Leftrightarrow E \text{ does not support a non-0 pseudofunction in } \overline{M(E)} = M^{[0]}(E).$$

Then if U_2^* is the σ-ideal of closed sets generated by U_2, we have

$$E \in U_2^* \Leftrightarrow E \text{ does not support non-0 synthesizable pseudofunctions of order 0.}$$

It turns out of course that $U_2^* \subseteq U_0$ is also proper as Lyons [8] shows by using the Piatetski-Shapiro method for the strictness of the inclusion $U_1^* \subseteq U_0$. In some sense, which perhaps some structural theorems can make precise, U_2^* seems close to U_0.

We concentrate here on the gap (U_1^*, U_2^*). We introduce a transfinite decreasing sequence of classes $U_{1,\alpha}$ and $U_{1,\alpha}^*$ (the σ-ideal generated by $U_{1,\alpha}$) defined for $0 \le \alpha \le \omega_1$ by

$$E \in U_{1,\alpha} \Leftrightarrow M^{[\alpha]}(E) \cap PF = \{0\}.$$

It turns out again that

$$E \in U_{1,\alpha}^* \Leftrightarrow E \text{ does not support a non-0 synthesizable pseudofunction of order } \le \alpha.$$

Thus $U_{1,0} = U_2$, $U_{1,0}^* = U_2^*$ are Lyons' classes and $U_{1,\omega_1} = U_1$, $U_{1,\omega_1}^* = U_1^*$, so we have the picture

$$U \subseteq U_1^* \subseteq \dots U_{1,\beta}^* \subseteq \dots \subseteq U_{1,\alpha}^* \subseteq \dots \subseteq U_{1,1}^* \subseteq U_{1,0}^* \subseteq U_0 \quad (\alpha \leq \beta)$$

(similarly without the stars). We prove for these classes results analogous to those of Lyons [8] for $U_{1,0}$, $U_{1,0}^*$. For example $U_{1,\alpha}$, $U_{1,\alpha}^*$ are $\mathbf{\Pi}_1^1$ (and non-Borel) and $U_{1,\alpha}^*$ has a canonical Borel basis $U_{1,\alpha}' \subseteq U_{1,\alpha}$ (i.e. $U_{1,\alpha}'$ generates also $U_{1,\alpha}^*$ as a σ-ideal of closed sets).

The main question concerning this hierarchy is of course whether it collapses, i.e., for some $\alpha_0 < \omega_1$, $U_{1,\alpha_0}^* = U_{1,\omega_1}^* = U_1^*$. An important question left open in [6] is the level of definability of U_1^*. It follows from results of Solovay [11], Kaufman [3] (see also [6]) that U_1^* cannot be $\mathbf{\Sigma}_1^1$. On the other hand it is $\mathbf{\Delta}_2^1$, i.e., both $\mathbf{\Sigma}_2^1$ and $\mathbf{\Pi}_2^1$, in fact (for readers familiar with generalized recursion theory) it belongs to the class $\mathbf{\Sigma}_1^1$-IND, which is a much smaller subclass of $\mathbf{\Delta}_2^1$. (The class $\mathbf{\Sigma}_1^1$-IND coincides, by a result of Solovay, with the class $\mathfrak{D}\,\mathbf{\Sigma}_2^0$; see [9], 7C and 6D). Thus the first main question is whether U_1^* is $\mathbf{\Pi}_1^1$. (A similar question can be raised for U_1). Our second main result states that all these problems are equivalent to our earlier conjecture.

THEOREM 2. *The following are equivalent.*
(i) *The class of synthesizable pseudofunctions* $\mathscr{S}_0$ *is Borel,*
(ii) *The hierarchy* $U_{1,\alpha}^*$ *collapses, i.e., for some* $\alpha_0 < \omega_1$, $U_{1,\alpha_0}^* = U_1^*$,
(iii) *The class* U_1^* *is* $\mathbf{\Pi}_1^1$.

Thus a proof of the conjecture will establish that U_1^* is neither $\mathbf{\Sigma}_1^1$ nor $\mathbf{\Pi}_1^1$ (in fact by a result of Dougherty-Kechris it would not be even a union of a $\mathbf{\Sigma}_1^1$ and a $\mathbf{\Pi}_1^1$ set). This would establish the first natural example in analysis of a set lying between levels of the projective hierarchy.

Of course a disproof of the conjecture would be also extremely interesting. At this stage it is not even known whether $U_1^* \neq U_{1,0}^*$ (a question already raised in Lyons [8]). This is equivalent to asking whether there is a synthesizable pseudofunction which is not a *strong* limit of measures on it support. Lyons' result that $U_{1,0}^* \neq U_0$ means that there are pseudofunctions which are strong limits of measures in their support but not of such measures which are pseudofunctions.

Addendum. R. Kaufman (private communication) has recently showed that indeed $U_1^* \neq U_{1,0}^*$, in fact that there is a synthesizable *PF* which is not a weak*-limit of a sequence of measures on its support. It would seem that this is a major step in a proof of the above conjecture.

Before we proceed to the proofs of the results discussed in this introduction we would like to point out in general that the reader will find helpful material in the following references (listed at the end of the paper): [2], especially Ch. 3 and 4 and [6], especially Ch. V, VIII, X.

1. The order of a synthesizable pseudomeasure

Let X denote a separable Banach space and X^* its dual. For each subspace $Y \subseteq X^*$ we let

$$Y^{(1)} = \{x^* \in X^*: \exists\{x_n^*\}(x_n^* \in Y \wedge x_n^* \xrightarrow{w^*} x^*)\}$$

be the weak*-sequential closure of Y. Define then by transfinite induction, $Y^{[\alpha]}$, as follows

$$\begin{aligned} Y^{[0]} &= \overline{Y}, \\ Y^{[\alpha+1]} &= \overline{(Y^{[\alpha]})^{(1)}}, \\ Y^{[\lambda]} &= \overline{\cup_{\alpha<\lambda} Y^{[\alpha]}}, \quad \lambda \text{ limit,} \end{aligned}$$

where $\overline{Z}$ denotes strong closure in X^*. By a theorem of Banach (see [6], pp. 153–156) there is countable ordinal α_0 such that $Y^{[\alpha_0]} = Y^{[\beta]}$ for all $\beta \geq \alpha_0$ and

$$Y^{[\alpha_0]} = \overline{Y}^{w^*} = \text{the weak*-closure of } Y.$$

Given now any $x^* \in X^*$, let

$$\mathrm{ORD}(Y, x^*) = \begin{cases} \text{least } \alpha \text{ such that } x^* \in Y^{[\alpha]} & \text{if } x^* \in \overline{Y}^{w^*}, \\ \omega_1 & \text{otherwise.} \end{cases}$$

Then for any fixed Y, ORD(Y, $\cdot$) is a rank on $\overline{Y}^{w^*}$.

Remark. One could also use the transfinite sequence $Y^{(\alpha)}$ given by

$$Y^{(0)} = Y, \quad Y^{(\alpha+1)} = (Y^{(\alpha)})^{(1)}, \quad Y^{(\lambda)} = \cup_{\alpha<\lambda} Y^{(\alpha)}, \lambda \text{ limit.}\rangle$$

However, the sequence $Y^{[\alpha]}$ is more appropriate for our purposes here, as will become clear in §2.

Specializing now to $X = A$, $X^* = PM$, let $x^* = S \in PM$, $\|S\|_{PM} \leq 1$, $Y = M(\mathrm{supp}(S))$. Then since

$$S \text{ is synthesizable } \Leftrightarrow S \in \overline{M(\mathrm{supp}(S))}^{w^*}$$

define

$$\mathrm{ORD}(S) = \mathrm{ORD}(M(\mathrm{supp}(S)), S).$$

Then ORD: $\mathscr{S} \to \omega_1$ is a rank on $\mathscr{S}$ (and $\mathrm{ORD}(S) = \omega_1$ if $S \notin \mathscr{S}$).

Using the result of Katznelson-McGehee [4], we will now show that ORD is unbounded in ω_1.

THEOREM 1. *For each countable ordinal, α, there is a synthesizable pseudomeasure S (with $\|S\|_{PM} \leq 1$) such that* $\mathrm{ORD}(S) > \alpha$.

Proof. For each set $E \in K(\mathbf{T})$ let $|E|$ be the least countable ordinal α such that

$$M^{[\alpha]}(E) =^{\mathrm{def}} M(E)^{[\alpha]}$$

is equal to $\overline{M(E)}^{w^*}$($= N(E)$). In Katznelson-McGehee [4], the authors show that for each $\alpha \leq \omega_1$ there is $E \in K(\mathbf{T})$ such that $|E| > \alpha$. (Moreover these E are of synthesis themselves, i.e., $N(E) = PM(E)$($=$ the class of pseudomeasures supported by E). We will not need this extra information below).

Fix now $\alpha < \omega_1$ and let $E \in K(\mathbf{T})$ be such that $|E| > \alpha$. Then $M^{[\alpha]}(E) \neq N(E)$, so there is $S \in N(E)$, $S \notin M^{[\alpha]}(E)$, i.e., $\mathrm{ORD}(M(E), S) > \alpha$. We will construct $T \in N(E)$ with $\mathrm{supp}(T) = E$, so that $T \in \mathscr{S}$, and such that $T \notin M^{[\alpha]}(E)$ as well. Then clearly

$$\mathrm{ORD}(M(\mathrm{supp}(T)), T) > \alpha$$

and we are done.

Let $\{x_n : n = 1, 2, 3, \dots\}$ be a countable set dense in $E \setminus \mathrm{supp}(S)$. Let

$$T' = \sum \frac{1}{2^n} \delta_{x_n},$$

where δ_x is the Dirac measure at x. Then $\mathrm{supp}(T') = \overline{\{x_1, x_2, \dots\}} = K$. For each $\varepsilon > 0$, let

$$T_\varepsilon = S + \varepsilon T'.$$

Clearly $T_\varepsilon \in N(E)$. We claim that $\mathrm{supp}(T_\varepsilon) = E$. Clearly $K \subseteq \mathrm{supp}(T_\varepsilon)$. Also $E \setminus K \subseteq \mathrm{supp}(S)$, so $E \setminus K \subseteq \mathrm{supp}(T_\varepsilon)$. Thus $E = \mathrm{supp}(T_\varepsilon)$.

Now we argue that for some ε_0, $T = T_{\varepsilon_0}$ is such that $T \notin M^{[\alpha]}(E)$. Otherwise, for each $\varepsilon > 0$, $T_\varepsilon \in M^{[\alpha]}(E)$. But then $S = \lim_{n\to\infty} T_{1/n}$ (where convergence is in the strong sense), so $S \in M^{[\alpha]}(E)$ (since this is a closed subspace), i.e., $\mathrm{ORD}(M(E), S) \leq \alpha$, a contradiction. □

Sections 2 and 3 are devoted to showing that ORD is a $\mathbf{\Pi}_1^1$-rank on the $\mathbf{\Pi}_1^1$ set $\mathscr{S}$. First let us verify this last assertion.

PROPOSITION 2. *The set $\mathscr{S}$ of synthesizable pseudomeasures is $\mathbf{\Pi}_1^1$ in $B_1(PM)$.*

Proof. For $S \in B_1(PM)$ we have

$$S \notin \mathscr{S} \Leftrightarrow \exists f \in A[f = 0 \text{ on } \operatorname{supp}(S) \wedge \langle f, S\rangle \neq 0].$$

It is therefore enough to check that the set

$$P = \{(f, S) : f \in A \;\&\; S \in B_1(PM) \;\&\; f = 0 \text{ on } \operatorname{supp}(S) \;\&\; \langle f, S\rangle \neq 0\}$$

is Borel in $A \times B_1(PM)$. Now

$$P_1 = \{(f, S) : \langle f, S\rangle \neq 0\}$$

is clearly open. Also

$$P_2 = \{(f, S) : f = 0 \text{ on } \operatorname{supp}(S)\}$$

is Borel, since if $\{V_n\}$ is an open basis for $\mathbf{T}$,

$$\begin{aligned}(f, S) \in P_2 &\Leftrightarrow \forall n[V_n \cap \operatorname{supp}(S) \neq \varnothing \Rightarrow \exists x(x \in \overline{V_n} \wedge f(x) = 0)]\\ &\Leftrightarrow \forall n[\exists g \in A(\operatorname{supp}(g) \subseteq V_n \wedge \langle g, S\rangle \neq 0)\\ &\qquad \Rightarrow \exists x(x \in \overline{V_n} \wedge f(x) = 0)]. \quad \square\end{aligned}$$

To show now that ORD is a $\mathbf{\Pi}_1^1$-rank on $\mathscr{S}$ we will describe an alternative "tree rank" on $\mathscr{S}$, for which it is easy to show that it is a $\mathbf{\Pi}_1^1$-rank, and then we will complete the proof by showing the equivalence of ORD and this "tree rank".

2. A "tree-rank" on $\mathscr{S}$

Going back to the general context, let X be a separable Banach space and let D be a countable set dense in the open unit ball of X and closed under multiplication by elements of $\mathbf{Q} + i\mathbf{Q}$. Given a subspace $Y \subseteq X^*$, $x^* \in X^*$, $\varepsilon \in \mathbf{Q} \cap (0, 1)$ we define a tree T_{Y, x^*}^{ε} on $\operatorname{Seq} D$ (= the set of all finite sequences from D) as follows

$$T^{\varepsilon}_{Y,x^*} = \{\varnothing\} \cup \Big\{(x_0,\dots,x_n)\colon \forall j \le n\big(x_j \in D \wedge |\langle x_j, x^*\rangle| \ge \varepsilon\|x^*\|\big) \\ \wedge \forall j < n\big(\|x_j - x_{j+1}\| \le 2^{-(j+3)}\big) \\ \wedge \forall j \le n\big(\|x_j\|_Y \le 2^{-(j+1)}\big)\Big\}$$

where $\|x\|_Y = \sup\{(|\langle x, y^*\rangle|\colon \|y^*\| \le 1, y^* \in Y\}$. (Notice that $\|x\|_Y$ does not mean that $x \in Y$).

This tree is a local version of the tree T^{ε}_Y associated to Y as in [6], p. 161. We prove first a local version of (part of) Proposition 1 in [6], p. 161.

Proposition 1. *The following are equivalent, for Y a subspace of X^* and $x^* \in X^*$:*

(i) $x^* \in \overline{Y}^{w^*}$;

(ii) $\forall \varepsilon \in \mathbf{Q} \cap (0,1)(T^{\varepsilon}_{Y,x^*}$ *is well-founded*).

Proof. First suppose $x^* \notin \overline{Y}^{w^*}$. Then there exist $\varepsilon \in \mathbf{Q} \cap (0,1)$, $x \in X$ with $\|x\| < 1$ such that

$$|\langle x, x^*\rangle| \ge 2\varepsilon\|x^*\| \quad \text{and} \quad \langle x, y^*\rangle = 0 \text{ for all } y^* \in Y.$$

Any sequence from D converging to x fast enough gives an infinite branch in T^{ε}_{Y,x^*}.

Conversely, assume T^{ε}_{Y,x^*} has an infinite branch $\{x_n\}$, for some $\varepsilon \in \mathbf{Q} \cap (0,1)$. Then $\{x_n\}$ is a Cauchy sequence which converges (strongly) to some $x \in X$. Then $\|x\|_Y = 0$, i.e., $\langle x, y^*\rangle = 0$ for all $y^* \in Y$, and $|\langle x, x^*\rangle| \ge \varepsilon\|x^*\| > 0$, so $x^* \notin \overline{Y}^{w^*}$. □

Definition. For $Y \subseteq X^*$ a subspace and $x^* \in \overline{Y}^{w^*}$, let

$$\beta(Y, x^*) = \sup\{\mathrm{ht}(T^{\varepsilon}_{Y,x^*}) + 1\colon \varepsilon \in \mathbf{Q} \cap (0,1)\} \\ \Big(= \lim_{\varepsilon\to 0}\big(ht(T^{\varepsilon}_{Y,x^*}) + 1\big)\Big)$$

where ht(T) is the height of a well-founded tree T(see [6], p. 141). If $x^* \notin \overline{Y}^{w^*}$ we let $\beta(Y, x^*) = \omega_1$. Note that $\varepsilon < \varepsilon' \Rightarrow T^{\varepsilon'}_{Y,x^*} \subseteq T^{\varepsilon}_{Y,x^*}$ so that $\mathrm{ht}(T^{\varepsilon'}_{Y,x^*}) \le h(T^{\varepsilon}_{Y,x^*})$, which justifies the expression of $\beta(Y, x^*)$ as $\lim_{\varepsilon\to 0}$.

Lemma 2. *For $Y \subseteq X^*$ a subspace, $x^* \in \overline{Y}^{w^*}$, $\beta(Y, x^*)$ is a limit ordinal.*

Proof. We show that $\beta(Y, x^*) \ge \omega$ and $\beta(Y, x^*) > \omega\cdot\alpha \Rightarrow \beta(Y, x^*) \ge \omega\cdot(\alpha + 1)$.

Fix $N \in \mathbf{N}$. Let $x \in D$ be such that

$$\|x^*\| \cdot 2^{-(N+2)} \leq |\langle x, x^* \rangle|, \quad \|x\| \leq 2^{-(N+1)}.$$

Then $s = (x, x, \ldots, x)(N+1 \text{ times})$ is in T^{ε}_{Y,x^*} with $\varepsilon = 2^{-(N+2)}$, hence

$$h(T^{\varepsilon}_{Y,x^*}) \geq N+1 \quad \text{and} \quad \beta(Y, x^*) \geq \omega.$$

Let now $\alpha > 0$ and $\beta(Y, x^*) > \omega \cdot \alpha$. Then for some $\varepsilon > 0$, $h(T^{\varepsilon}_{Y,x^*}) > \omega \cdot \alpha$. Let $N \in \mathbf{N}$ and put $\varepsilon' = \varepsilon \cdot 2^{-(N+4)}$. The tree

$$T' = 2^{-(N+4)} \cdot T^{\varepsilon, N}_{Y,x^*}$$

is a subtree of $T^{\varepsilon'}_{Y,x^*}$, where

$$\begin{aligned} T^{\varepsilon,N}_{Y,x^*} = \{\varnothing\} \cup \Big\{(x_0, \ldots, x_n) \colon & x_j \in D \wedge |\langle x_j, x^* \rangle| \geq \varepsilon \cdot \|x^*\| \\ & \wedge \forall j < n\big(\|x_j - x_{j+1}\| \leq 2^{-(N+j+3)}\big) \\ & \wedge \forall j \leq n\big(\|x_j\| \leq 2^{-(N+j+1)}\big)\Big\}. \end{aligned}$$

If $(x) \in T'$ and $s = (x_0, \ldots, x_n) \in T'$ then $\|x - x_0\| \leq 2^{-(N+3)}$; hence

$$(x, \ldots, x, x_0, \ldots, x_n)(N+1 \text{ times}) \in T^{\varepsilon'}_{Y,x^*}.$$

Since $\mathrm{ht}(T') = \mathrm{ht}(T^{\varepsilon}_{Y,x^*}) \geq \omega \cdot \alpha$ (see for example the argument in pp. 162–163 of [6]) it follows that

$$\mathrm{ht}(T^{\varepsilon'}_{Y,x^*}) \geq \omega \cdot \alpha + N + 1,$$

so $\beta(Y, x^*) \geq \omega \cdot (\alpha + 1)$. □

DEFINITION. For $Y \subseteq X^*$ a subspace, and $x^* \in X^*$ let $RK_T(Y, x^*)$ be defined by

$$\beta(Y, x^*) = \begin{cases} \omega \cdot RK_T(Y, x^*) & \text{if } x^* \in \overline{Y}^{w^*} \\ \omega_1 & \text{otherwise.} \end{cases}$$

The main result is now;

THEOREM 3. *Let X be a separable Banach space, $Y \subseteq X^*$ a subspace and $x^* \in X^*$. Then if $x^* \notin \overline{Y}$,*

$$\mathrm{ORD}(Y, x^*) = RK_T(Y, x^*).$$

For the proof of that theorem we will need the following lemmas.

LEMMA 4. *Let $Y \subseteq X^*$, $x^* \in X^*$. Then the following are equivalent:*
(i) $x^* \notin Y^{[1]}$,
(ii) $\exists \varepsilon > 0 \ \exists \{x_n\} \forall n [x_n \in X \wedge \|x_n\| \leq 1 \wedge |\langle x_n, x^* \rangle| \geq \varepsilon \cdot \|x^*\| \wedge \|x_n\|_Y \leq 2^{-(n+1)}]$

LEMMA 5. *Let $Y \subseteq X^*$, $x^* \in X^*$. Let λ be a limit ordinal and $\alpha_n \to \lambda$, α_n increasing. Then the following are equivalent:*
(i) $x^* \notin Y^{[\lambda]}$,
(ii) $\exists \varepsilon > 0 \ \exists \{x_n) \forall n [x_n \in X \wedge \|x_n\| \leq 1 \wedge |\langle x_n, x^* \rangle| \geq \varepsilon \|x^*\| \wedge \|x_n\|_{Y^{[\alpha_n]}} \leq 2^{-(n+1)}]$.

LEMMA 6. *Let $Y \subseteq X^*$, $x^* \in X^*$. Suppose*

$$u \in T^{\varepsilon}_{Y, x^*} \quad \textit{and} \quad \mathrm{ht}(u, T^{\varepsilon}_{Y, x^*}) \geq \omega \cdot \alpha.$$

Then $u \in T^{\varepsilon}_{Y^{[\alpha]}, x^*}$. (Here $\mathrm{ht}(u, T)$ is the height of a sequence $u \in T$ in the well-founded tree T; see [6], *p.* 141).

LEMMA 7. *Let $Y \subseteq X^*$, $x^* \in X^*$. Let $u = (x_0, \dots, x_n) \in T^{\varepsilon}_{Y^{[\alpha]}, x^*}$. Assume moreover that*

$$\|x_n\| \cdot (1 + 2^{-(n+4)}) < 1, \quad |\langle x_n, x^* \rangle| \cdot (1 - 2^{-(n+4)}) > \varepsilon \cdot \|x^*\|$$

and

$$\|x_n\|_{Y^{[\alpha]}} \leq 2^{-(n+4)} \cdot \|x_n\|.$$

Then $\mathrm{ht}(u, T^{\varepsilon}_{Y, x^*}) \geq \omega \cdot \alpha$.

Proof of Theorem 3 (assuming the lemmas). If $x^* \notin \overline{Y}^{w^*}$ then

$$\mathrm{ORD}(Y, x^*) = RK_T(Y, x^*) = \omega_1.$$

So assume $x^* \in \overline{Y}^{w^*}$. We will first show that for $x^* \notin \overline{Y}$,

(A) $\beta(Y, x^*) > \omega \cdot \beta \Rightarrow \mathrm{ORD}(Y, x^*) > \beta$.

Since $x^* \notin \overline{Y}$, $x^* \notin \overline{Y} = Y^{[0]}$; thus $\mathrm{ORD}(Y, x^*) \geq 1$, so (A) with $\beta = 0$ is automatically true. Let us prove it now in the case $\beta = \alpha + 1$ is a successor. Thus let

$$\beta(Y, x^*) > \omega \cdot (\alpha + 1).$$

Then for some $\varepsilon > 0$,

$$u = (x_0, \ldots, x_n) \in T^{\varepsilon}_{Y, x^*}, \qquad \mathrm{ht}(u, T^{\varepsilon}_{Y, x^*}) \geq \omega \cdot (\alpha + 1).$$

Then for each $N \in \mathbf{N}$, we can find $\nu_N = (x^N_{n+1}, \ldots, x^N_{n+N})$ with $u\hat{\ }\nu_N \in T^{\varepsilon}_{Y,x^*}$ such that

$$\mathrm{ht}(u\hat{\ }\nu_N, T^{\varepsilon}_{Y, x^*}) \geq \omega \cdot \alpha.$$

By Lemma 6, $u\hat{\ }\nu_N \in T^{\varepsilon}_{Y^{[\alpha]}, x^*}$. Since the sequence $(x^N_{n+N})_{N \in \mathbf{N}}$ is such that

$$|\langle x^N_{n+N}, x^* \rangle| \geq \varepsilon \cdot \|x^*\| \quad \text{and} \quad \|x^N_{n+N}\|_{Y^{[\alpha]}} \leq 2^{-(n+N+1)},$$

Lemma 4 gives $x^* \notin Y^{[\alpha+1]}$, i.e.,

$$\mathrm{ORD}(Y, x^*) > \alpha + 1.$$

Finally, let $\beta = \lambda$ be a limit ordinal. Since $\beta(Y, x^*) > \omega \cdot \lambda$, there is $\varepsilon > 0$ and

$$u = (x_0, \ldots, x_n) \in T^{\varepsilon}_{Y, x^*}$$

with

$$\mathrm{ht}(u, T^{\varepsilon}_{Y, x^*}) \geq \omega \cdot \lambda.$$

Choose $\alpha_n \to \lambda$, α_n increasing. Then

$$\mathrm{ht}(u, T^{\varepsilon}_{Y, x^*}) \geq \omega \cdot (\alpha_N + 1) \quad \text{for all } n \in \mathbf{N}.$$

So for all N, $p \in \mathbf{N}$, there exists

$$\nu_{N,p} = \left(x^{N,p}_{n+1}, \ldots, x^{N,p}_{n+p}\right)$$

with

$$\mathrm{ht}\left(u\hat{\ }\nu_{N,p}, T^{\varepsilon}_{Y, x^*}\right) \geq \omega \cdot \alpha_N;$$

thus by Lemma 6, $u\hat{\ }\nu_{N,p} \in T^{\varepsilon}_{Y^{[\alpha_N]}, x^*}$. *Now consider the sequence* $(x^{N,N}_{n+N})_{n \in N}$. *Then*

$$|\langle x^{N,N}_{n+N}, x^* \rangle| \geq \varepsilon \cdot \|x^*\| \quad \text{and} \quad \|x^{N,N}_{n+N}\|_{Y^{[\alpha_N]}} \leq 2^{-(N+n+1)}$$

so now, by Lemma 5, $x^* \notin Y^{[\lambda]}$, i.e. $\mathrm{ORD}(Y, x^*) > \lambda$.

We complete now the proof by showing the converse:

(B) $\mathrm{ORD}(Y, x^*) > \beta \Rightarrow \beta(Y, x^*) > \omega \cdot \beta$.

Again if $\beta = 0$, since $\beta(Y, x^*) \geq \omega$, (B) is automatically satisfied. Now let $\beta = \alpha + 1$ be a successor. Thus $x^* \notin Y^{[\alpha+1]}$. Then, by Lemma 4, we can find $\varepsilon > 0$ and $\{x_n\}$ with $x_n \in D$ and

$$|\langle x_n, x^*\rangle| \cdot (1 - 2^{-(n+4)}) > \varepsilon \cdot \|x^*\|,$$
$$\|x_n\|(1 + 2^{-(n+4)}) < 1,$$
$$\|x_n\|_{Y^{[\alpha]}} \leq 2^{-(n+4)}\|x_n\|.$$

Let $u_n = (x_n, \dots, x_n)$ (repeated n times). Then $u_n \in T^{\varepsilon}_{Y^{[\alpha]}, x^*}$ and satisfies the hypotheses of Lemma 7, so $\mathrm{ht}(u_n, T^{\varepsilon}_{Y, x^*}) \geq \omega \cdot \alpha$. Thus $\mathrm{ht}(T^{\varepsilon}_{Y, x^*}) \geq \omega \cdot \alpha + n$ for each n, i.e.,

$$\mathrm{ht}(T^{\varepsilon}_{Y, x^*}) \geq \omega \cdot (\alpha + 1)$$

and so

$$\beta(Y, x^*) > \omega \cdot (\alpha + 1).$$

Next suppose $\beta = \lambda$ is limit in (B). Choose $\alpha_n \to \lambda$, α_n increasing. Since $x^* \notin Y^{[\lambda]}$ by Lemma 5 we can find a sequence $\{x_n\}$ and an $\varepsilon > 0$ such that $x_n \in D$ and

$$|\langle x_n, x^*\rangle| \cdot (1 - 2^{-(n+4)}) > \varepsilon \cdot \|x^*\|,$$
$$\|x_n\|_{Y^{[\alpha_n]}} \leq 2^{-(n+4)} \cdot \|x_n\|$$

and

$$\|x_n\| \cdot (1 + 2^{-(n+4)}) < 1.$$

Again let $u_n = (x_n, \dots, x_n)$ (repeat n times). Then $u_n \in T^{\varepsilon}_{Y^{[\alpha_n]}, x^*}$ and satisfies the hypotheses of Lemma 7 so $\mathrm{ht}(u_n, T^{\varepsilon}_{Y, x^*}) \geq \omega \cdot \alpha_n$. Then $\mathrm{ht}(T^{\varepsilon}_{Y, x^*}) \geq \omega \cdot \alpha_n$ for all n, i.e.,

$$\mathrm{ht}(T^{\varepsilon}_{Y, x^*}) \geq \omega \cdot \lambda \quad \text{and} \quad \beta(Y, x^*) > \omega \cdot \lambda$$

so we are done.

It remains to prove the lemmas.

Proof of Lemma 4. (i) $\Rightarrow$ (ii). Suppose $x^* \notin Y^{[1]}$, $\|x^*\| = 1$ (without loss of generality). Put

$$\varepsilon = \inf_n \operatorname{dist}\left(x^*, \overline{Y \cap B_n(X^*)}^{w^*}\right),$$

where $B_n(X^*)$ is the closed ball of X^* of radius n. We claim that $\varepsilon > 0$. Otherwise, for each n there is $z_n^* \in \overline{Y \cap B_n(X^*)}^{w^*}$ with $\|x^* - z_n^*\| \to 0$. Since $\overline{Y \cap B_n(X^*)}^{w^*} \subseteq Y^{(1)}$ we have

$$x^* \in \overline{Y^{(1)}} = Y^{[1]},$$

a contradiction.

Now, by Hahn-Banach, for each n we can find $x_n \in X$, $\|x_n\| = 1$ with

$$\mathrm{Re}\langle x_n, z^*\rangle \geq \mathrm{Re}\langle x_n, y^*\rangle$$

for all z^* in the open ball of radius ε around x^* and all $y^* \in B_n(x^*) \cap Y$. Thus

$$\mathrm{Re}\langle x_n, z^*\rangle \geq |\langle x_n, y^*\rangle|$$

for all such z^*, y^* and so

$$\mathrm{Re}\langle x_n, z^*\rangle \geq n|\langle x_n, y^*\rangle|$$

for all such z^* and all $y^* \in B_1(x^*) \cap Y$; i.e.,

$$\mathrm{Re}\langle x_n, z^*\rangle \geq n \cdot \|x_n\|_Y.$$

In particular, since $1 \geq \mathrm{Re}\langle x_n, x^*\rangle$, we have $\|x_n\|_Y \leq 1/n$ and, since $\mathrm{Re}\langle x_n, z^*\rangle \geq 0$ for all such z^*, we have $|\langle x_n, x^*\rangle| \geq \varepsilon$.

(As pointed out by the referee, these arguments are related to those concerning the BX-topology in Dunford-Schwartz, Linear operators, part I, Ch. V).

(ii) $\Rightarrow$ (i). Fix ε and $\{x_n\}$ and suppose $x^* \in Y^{[1]}$ towards a contradiction. For each $\delta > 0$ find $z^* \in Y^{(1)}$ with $\|x^* - z^*\| < \delta$. Then find $\{z_p^*\}$ with $z_p^* \in Y$ and

$$z_p^* \xrightarrow{w^*} z^*.$$

Fix M with $\|z_p^*\| \leq M$, all p. Then

$$\begin{aligned} |\langle x^*, x_n\rangle| &\leq |\langle x^* - z^*, x_n\rangle| + |\langle z^* - z_p^*, x_n\rangle| + |\langle z_p^*, x_n\rangle| \\ &\leq \|x^* - z^*\| + \|\langle z^* - z_p^*, x_n\rangle| + 2^{-(n+1)} \cdot M. \end{aligned}$$

Now choose $\delta < \frac{1}{3}\varepsilon \cdot \|x^*\|$. This gives $\{z_p^*\}$ and M. Choose then n_0 with

$$2^{-(n_0+1)} \cdot M < \frac{\varepsilon}{3} \cdot \|x^*\|$$

and p with

$$|\langle z^* - z_p^*, x_{n_0}\rangle| < \frac{\varepsilon}{3} \cdot \|x^*\|.$$

Then $|\langle x^*, x_n\rangle| < \varepsilon \cdot \|x^*\|$, a contradiction. □

Proof of Lemma 5. (i) ⇒ (ii). Let $x^* \notin Y^{[\lambda]}$, $\|x^*\| = 1$. Let

$$\varepsilon = \inf_n \operatorname{dist}\left(x^*, \overline{B_n(X^*) \cap Y^{[\alpha_n]}}^{w^*}\right).$$

As before, $\varepsilon > 0$. Then apply Hahn-Banach to find x_n with $\|x_n\| = 1$ and

$$\operatorname{Re}\langle x_n, z^*\rangle \geq \operatorname{Re}\langle x_n, y^*\rangle$$

for all z^* in the open ball of radius ε around x^* and all $y^* \in B_n(X^*) \cap Y^{[\alpha_n]}$.

(ii) ⇒ (i). Fix again ε and $\{x_n\}$, and suppose $x^* \in Y^{[\lambda]}$ towards a contradiction. Then for each $\delta > 0$ we can find arbitrary large $n > 0$ and $z^* \in Y^{[\alpha_n]}$ with $\|x^* - z^*\| \leq \delta$. Then

$$\begin{aligned} |\langle x^*, x_n\rangle| &\leq |\langle x^* - z^*, x_n\rangle| + |\langle z^*, x_n\rangle| \\ &\leq \|x^* - z^*\| + 2^{-(n+1)} \cdot (\delta + \|x^*\|). \end{aligned}$$

So choose $\delta < \frac{1}{2}\varepsilon \cdot \|x^*\|$ and then n_0 with $2^{-(n_0+1)} \cdot (\delta + \|x^*\|) < \frac{1}{2}\varepsilon \cdot \|x^*\|$. Then

$$|\langle x^*, x_{n_0}\rangle| < \varepsilon \cdot \|x^*\|,$$

a contradiction. □

The proof of Lemmas 6, 7 are very similar to the proofs of the corresponding Lemmas 6, 7 in [6], p. 163, so we omit them here. We take this opportunity, however, to correct some misprints in the statement and proof of these results in [6]. These are as follows: On p. 165, line 4 replace

$$\text{“}\|x_n\|_{Y^{(\alpha)}} \leq 2^{-(n+3)}\text{”}$$

by

$$\text{“}\|x_n\|_{Y^{(\alpha)}} < \varepsilon \cdot 2^{-(n+3)}\text{”},$$

and on line 6^- replace

$$\text{“}\|x_n\|_{Y^{(\beta)}} \leq 2^{-(n+3)}\text{”}$$

by

$$\text{“}\|x_n\|_{Y^{(\beta)}} < 2^{-(n+4)}\text{”}.$$

On p. 167, line 7 replace

$$\text{“}\|x_n\|_{Y^{(\alpha+1)}} \leq 2^{-(n+3)}\text{”}$$

by

$$\text{“}\|x_n\|_{Y^{(\alpha+1)}} < \varepsilon \cdot 2^{-(n+3)}\text{”},$$

and

$$\text{“Let } \ldots b \cdot \|x_n\|_{Y^{(\alpha+1)}}\text{”}$$

by

$$\text{“Let } a = \varepsilon \cdot 2^{-(n+3)}\text{”};$$

then replace line 6^- by

$$\text{“}\geq \|x_n\| - a\text{”}$$

and finally on line 3^- replace

$$\text{“}\|y_k\|_{Y^{(\alpha)}} \leq 2^{-(n+k+3)}\text{”}$$

by

$$\text{“}\|y_k\|_{Y^{(\alpha)}} < \varepsilon \cdot 2^{-(n+k+3)}\text{”}.$$

3. The class $\mathscr{S}$ is not Borel

We use now the results in §2 to show the following:

THEOREM 1. *The rank* ORD: $\mathscr{S} \to \omega_1$ *is a* $\mathbf{\Pi}_1^1$*-rank on* $\mathscr{S}$.

COROLLARY 2. *The class* $\mathscr{S}$ *of synthesizable pseudomeasures* (*in the unit ball of PM with the weak*-topology*) *is* $\mathbf{\Pi}_1^1$ *but not Borel.*

This follows from the theorem in §1 by the boundedness theorem for $\mathbf{\Pi}_1^1$-ranks (see e.g. [6], p. 148).

Proof. Notice first that for $S \in B_1(PM)$,

$$\begin{aligned}
\mathrm{ORD}(S) = 0 &\Leftrightarrow \mathrm{ORD}(M(\mathrm{supp}(S)), S) = 0 \\
&\Leftrightarrow S \in \overline{M(\mathrm{supp}(S))} \\
&\Leftrightarrow \forall E \in K(\mathbf{T})\Big[\mathrm{supp}(S) \subseteq E \Rightarrow \\
&(*) \forall n\, \exists N\, \exists \mu \in B_N(M)\Big[\mathrm{supp}(\mu) \subseteq E \wedge \|\mu - S\|_{PM} \le \frac{1}{n}\Big]\Big]
\end{aligned}$$

where $B_N(M)$ is the closed ball of radius N in the space of measures on $\mathbf{T}$. Since the relation "$\mathrm{supp}(S) \subseteq E$" is closed in $B_1(PM) \times K(\mathbf{T})$ and similarly for "$\mathrm{supp}(\mu) \subseteq E$" in $B_N(M) \times K(\mathbf{T})$ while

$$\|\mu - S\|_{PM} \le \frac{1}{n} \Leftrightarrow \forall k\left(|\hat{\mu}(k) - S(k)| \le \frac{1}{n}\right)$$

is closed in $B_N(M) \times B_1(PM)$ it follows that

$$\{S: \mathrm{ORD}(S) = 0\}$$

is $\mathbf{\Pi}_1^1$. Similarly

$$\mathrm{ORD}(S) = 0 \Leftrightarrow \exists E \in K(\mathbf{T})[E \subseteq \mathrm{supp}(S) \wedge (*)]$$

and "$E \subseteq \mathrm{supp}(S)$" is Borel in $K(\mathbf{T}) \times B_1(PM)$, being equivalent to

$$\forall n[V_n \cap E \neq \varnothing \Rightarrow V_n \cap \mathrm{supp}(S) \neq 0]$$

(see the proof of Proposition 2 in §1), so

$$\{S: \mathrm{ORD}(S) = 0\}$$

is $\mathbf{\Sigma}_1^1$, i.e., $\mathbf{\Delta}_1^1$. So it is enough to show $\mathrm{ORD}(S)$ is a $\mathbf{\Pi}_1^1$-rank on

$$\mathscr{S}' = \mathscr{S} \setminus \{S \in B_1(PM): \mathrm{ORD}(S) = 0\}.$$

But for $S, T \in \mathscr{S}'$, by the theorem in §2 we have

$$\mathrm{ORD}(S) \le \mathrm{ORD}(T) \Leftrightarrow RK_T(M(\mathrm{supp}(S)), S) \le RK_T(M(\mathrm{supp}(T)), T),$$

so it is enough to show that

$$RK_T(M(\mathrm{supp}(S)), S)$$

or equivalently

$$\beta(M(\mathrm{supp}(S)), S)$$

is a $\mathbf{\Pi}_1^1$-rank on $\mathscr{S}'$. As in the proof in p. 175 of [6] it is enough to check that the set

$$\{S \in B_1(PM) \colon \|f\|_{M(\mathrm{supp}(S))} \leq \delta\}$$

for each $f \in A$, $\delta > 0$ is Borel. Since

$$\|f\|_{M(\mathrm{supp}(S))} \leq \delta \Leftrightarrow \forall N \forall \mu \in B_N(M) \cap B_1(PM), \\ (\mathrm{supp}(\mu) \subseteq \mathrm{supp}(S) \Rightarrow |\langle f, \mu \rangle| \leq \delta),$$

this follows as in the preceding computation. □

Remark. In the proof of the non-Borelness of $\mathscr{S}$, which in particular implies the existence of non-synthesizable pseudomeasures and therefore of sets which are not of synthesis, one is only using (see the proof of the Theorem in §1) that for each ordinal $\alpha < \omega_1$ there is $E \in K(\mathbf{T})$ with least $\beta(M(E)^{(\beta)} = N(E)) > \alpha$. In Katznelson-McGehee [4] such sets are constructed (which are also of synthesis) using the existence of sets which are not of synthesis (Malliavin's Theorem). Is it possible to construct E as above without making use of non-synthesis sets? If so, one would have a new proof of Malliavin's Theorem (in a much stronger form).

4. Synthesizable pseudofunctions—The problem of their classification

We will look now at the subclass of $\mathscr{S}$ consisting of the synthesizable *pseudofunctions*. We denote this class by

$\mathscr{S}_0 = \mathscr{S} \cap PF =$ the class of synthesizable pseudofunctions (with $\|S\|_{PM} \leq 1$).

Clearly, $\mathscr{S}_0$ is a $\mathbf{\Pi}_1^1$ set (in $B_1(PM)$ with the weak*-topology). However we do not know whether or not $\mathscr{S}_0$ is Borel. Since ORD restricted to $\mathscr{S}_0$ is also a $\mathbf{\Pi}_1^1$-rank on the $\mathbf{\Pi}_1^1$ set $\mathscr{S}_0$, it follows by the boundedness theorem for $\mathbf{\Pi}_1^1$-ranks, that the following are equivalent:

(i) $\mathscr{S}_0$ is Borel;

(ii) For some countable ordinal α_0, every synthesizable pseudofunction S has $\mathrm{ORD}(S) \leq \alpha_0$, i.e., can be synthesized from measures on its support in at most α_0 iterations of sequential weak*-limits:

It is this reformulation of (i) that makes much more plausible the non-Borelness of $\mathscr{S}_0$, so we will formulate this as a conjecture

Conjecture. The class of synthesizable pseudofunctions $\mathscr{S}_0$ is not Borel (in the unit ball of *PM* with the weak*-topology).

We will devote most of the rest of this paper to showing the connections of this conjecture to the structure theory of sets of uniqueness. It will be seen that either a proof or a disproof of this conjecture has interesting consequences.

Remark. One can also formulate the problem of the classification of the class of synthesizable pseudofunctions as the question of whether the set $\mathscr{S}_0' = \{S \in PF\colon S \text{ is synthesizable}\}$ is Borel in the separable Banch space *PF*. (It is clearly $\mathbf{\Pi}_1^1$). However this is easily seen to be equivalent to the above, since the injection of $B_1(PF)$ into $B_1(PM)$ is continuous (from the norm-topology of $B_1(PF)$ into the weak*-topology of $B_1(PM)$).

5. The new classes $U_{1,\alpha}$, $U_{1,\alpha}^*$ of uniqueness sets and their relationship with synthesis of pseudofunctions

Recall that U, U_0 denote respectively the classes of closed uniqueness, extended uniqueness sets. Piatetski-Shapiro [10] introduced the intermediate class U_1,

$$U \subseteq U_1 \subseteq U_0,$$

consisting of those E for which $\overline{I(E)}^{w*} = A$. As opposed to U, U_0 this class is not a σ-ideal (i.e, closed under countable unions which are closed) so, as in [6], let U_1^* denote the class of all closed sets which are countable unions of U_1-sets. Again

$$U \subseteq U_1^* \subseteq U_0.$$

These inclusions are proper from results of Körner [7] and Piatetski-Shapiro [10]; see also [6]. It has been shown in [6] that U_1^* is in some sense structurally very close to U. The following fact proved in [6] shows also the relationship of U_1^* with synthesis. Denote by M_1^p the class of closed sets E which are locally not in U_1^* (equivalently not in U_1); i.e., for each open set $V \subseteq \mathbf{T}$ with $V \cap E \neq \varnothing$, $\overline{V \cap E} \notin U_1^*$. Then we have

$$E \in M_1^p \Leftrightarrow E \text{ is the support of a synthesizable pseudofunction.}$$

(Note also (see again [6]) that if M^p is analogously defined for U, then $E \in M^p \Leftrightarrow E$ is the support of a pseudofunction.) In particular,

$$E \in U_1^* \Leftrightarrow E \text{ does not support a non-0 synthesizable pseudofunction}$$

(while $E \in U \Leftrightarrow E$ does not support a non-0 pseudofunction; and

$$E \in U_1 \Leftrightarrow E \text{ does not support a non-0 pseudofunction in } N(E)).$$

Although the relationship of U, U_1^* is reasonably well understood, that of U_1^*, U_0 is less clear. In Lyons [8] a first step was taken by introducing a new class U_2,

$$U \subseteq U_1 \subseteq U_2 \subseteq U_0.$$

One of the equivalent characteristics of U_2 sets is

$$E \in U_2 \Leftrightarrow \begin{array}{l} E \text{ supports no non-0 pseudofunction which is a strong limit} \\ \text{(in the PM-norm) of measures on } E. \end{array}$$

Again U_2 is not closed under countable unions (which are closed), so let us denote by U_2^* the class of such unions so that

$$U \subseteq U_1^* \subseteq U_2^* \subseteq U_0.$$

Again, as Lyons [8] shows, $U_2^* \subseteq U_0$ is proper, but whether the same is true for $U_1^* \subseteq U_2^*$ is left open.

In some sense the class U_2^* seems close to U_0 (perhaps some structural theorems relating the classes U_2^*, U_0 may make this more precise). However the relationship between U_1^*, U_2^* is not so clear.

To clarify this relationship we will introduce a natural transfinite decreasing hierarchy of classes $U_{1,\alpha}$, $U_{1,\alpha}^*$ $(0 \le \alpha \le \omega_1)$ whose first level is Lyons' $U_{1,0} = U_2$, $U_{1,0}^* = U_2^*$, and last level is U_1, U_1^*. It is also canonically associated with the hierarchy of synthesizable pseudofunctions and as we will see the conjecture of §4 is equivalent to the properness of this hierarchy.

DEFINITION 1. A closed set $E \in \mathbf{T}$ belongs to the class $U_{1,\alpha}$, $0 \le \alpha \le \omega_1$, if $M^{[\alpha]}(E) \cap PF = \{0\}$, i.e., if E supports no non-0 pseudofunction in $M^{[\alpha]}(E)$. Recall from §1 that $M^{[\alpha]}(E)$ is defined inductively by

$$\begin{aligned} M^{[0]}(E) &= \overline{M(E)}, \\ M^{[\alpha+1]}(E) &= \overline{M^{[\alpha]}(E)^{(1)}}, \\ M^{[\lambda]}(E) &= \overline{\textstyle\bigcup_{\alpha<\lambda} M^{[\alpha]}(E)}, \quad \lambda \text{ limit.} \end{aligned}$$

Since for all large enough countable α, $M^{[\alpha]}(E) = N(E)$ it follows that

$$U_{1,\omega_1} = \bigcap_{\alpha<\omega_1} U_{1,\alpha} = U_1.$$

We denote by $U_{1,\alpha}^*$ the class of closed sets which are countable unions of $U_{1,\alpha}$-sets. (We will see later on that $U_{1,\alpha}$ is not closed under countable unions.) Thus we have

$$U \subseteq U_1^* \subseteq \cdots \subseteq U_{1+\alpha}^* \subseteq \cdots \subseteq U_{1+\beta}^* \subseteq \cdots \subseteq U_{1,1}^* \subseteq U_{1,0}^* \subseteq U_0, \alpha \geq \beta$$

and $U_1^* = U_{1,\omega_1}^* = \bigcap_{\alpha<\omega_1} U_{1,\alpha}^*$.

Also, denote by $M_{1,\alpha}^p$ the class of closed sets which are locally not in $U_{1,\alpha}^*$ (or equivalently not in $U_{1,\alpha}$), i.e., those E such that for every open $V \subseteq \mathbf{T}$,

$$E \cap V \neq \varnothing \Rightarrow \overline{E \cap V} \notin U_{1,\alpha}^* \quad (\text{or } U_{1,\alpha}).$$

Thus $M_{1,\omega_1}^p = M_1^p$.

Let use first provide a characterization of $M_{1,\alpha}^p$ analogous to that of M_1^p that clearly illustrates the connection with the hierarchy of synthesizable pseudofunctions.

PROPOSITION 2. *The following are equivalent for $E \subseteq \mathbf{T}$, E closed*:

(i) $E \in M_{1,\alpha}^p$,

(ii) $E = \operatorname{supp}(S)$, *where* $S \in PF \cap M^{[\alpha]}(E)$, *i.e.*, $\operatorname{ORD}(S) \leq \alpha$.

In particular, $E \in U_{1,\alpha}^*$ *iff E does not support a non-*0 *synthesizable pseudofunction of* ORD $\leq \alpha$.

Proof. To prove (ii) $\Rightarrow$ (i) we need the following lemma whose proof by induction on α we leave to the reader.

LEMMA. *Let $E \subseteq \mathbf{T}$ be closed. If $S \in M^{[\alpha]}(E)$, $f \in A$ then $f \cdot S \in M^{[\alpha]}(E \cap \operatorname{supp}(f))$.*

So let

$$E = \operatorname{Supp}(S), \quad S \in PF \cap M^{[\alpha]}(E).$$

Let V be open with $V \cap E \neq \varnothing$. Then there is $f \in A$, $\operatorname{supp}(f) \subseteq V$ and $f \cdot S \neq 0$. Then

$$f \cdot S \in PF \cap M^{[\alpha]}(\overline{E \cap V}),$$

so $\overline{E \cap V} \notin U_{1,\alpha}$, i.e., $E \in M_{1,\alpha}^p$.

For the converse, repeat the proof given in [6, p. 229] for M^p noting that each $M^{[\alpha]}(E)$ is a (strongly) closed subspace of PM.

It is clear that the first important question about the hierarchy $\{U_{1,\alpha}^*\}$ is whether it collapses at some countable ordinal α_0, thereby $U_1^* = U_{1,\omega_1}^* = U_{1,\alpha_0}^*$. It seems again plausible to conjecture that it does not, although as we mentioned in §4 we do not even know if $U_1^* \neq U_{1,0}^*$. Our next result however establishes the equivalence of this conjecture with that of §4 and rather surprisingly ties this up with the question left open in [6] of whether U_1^* (and U_1) are $\mathbf{\Pi}_1^1$ sets.

THEOREM 3. *The following are equivalent:*

(i) *The class* $\mathscr{S}_0 = \mathscr{S} \cap PF$ *of synthesizable pseudofunctions is Borel (in the unit ball of PM with the weak*-topology);*

(ii) *The hierarchy* $\{U_{1,\alpha}^*\}$ *collapses, i.e., for some countable* α_0, $U_{1,\alpha_0}^* = U_1^*$;

(iii) *The class* U_1^* *is* $\mathbf{\Pi}_1^1$.

Moreover, if U_1 *is* $\mathbf{\Pi}_1^1$ *or if* U_1' *(the class of E for which I(E) is sequentially weak*-dense in A) is* $\mathbf{\Pi}_1^1$, *these equivalent conditions hold.*

Proof. The last assertion follows from the fact that U_1 and U_1' (by Piatetski-Shapiro's [10] Theorem; see also [6]) are hereditary bases for U_1^*, so if they are $\mathbf{\Pi}_1^1$ so is U_1^*, by the argument in VI.I.2 of [6].

The implication (i) ⇒ (ii) is clear, since if $\mathscr{S}_0$ is Borel, then for some countable α_0 all synthesizable pseudofunctions have order $\leq \alpha_0$, thus by Proposition 2, $M_{1,\alpha_0}^p = M_1^p$ and so $U_{1,\alpha_0}^* = U_1^*$ since

$$E \notin U_{1,\alpha}^* \Leftrightarrow \exists F\left[\varnothing \neq F \wedge F \subseteq E \text{ closed} \wedge F \in M_{1,\alpha}^p\right].$$

The implication (ii) ⇒ (iii) is clear from the following lemma.

LEMMA 4. *The classes* $U_{1,\alpha}$, $U_{1,\alpha}^*$ *are* $\mathbf{\Pi}_1^1$.

Proof. Since $U_{1,\alpha}$ is a hereditary basis for $U_{1,\alpha}^*$ it is enough to check that $U_{1,\alpha}$ is $\mathbf{\Pi}_1^1$. But

$$E \notin U_{1,\alpha} \Leftrightarrow \exists S \in B_1(PM)\left[S \neq 0 \wedge S \in PF \wedge S \in M^{[\alpha]}(E)\right].$$

Now it is easy to check by induction on α that $M^{[\alpha]}(E) \cap B_1(PM)$ is a $\mathbf{\Sigma}_1^1$ subset of $B_1(PM)$ (with the weak*-topology) so we are done. □

It remains to prove the main implication (iii) ⇒ (i). Since, if $\{V_n\}$ is an open basis in $\mathbf{T}$,

$$E \in M_1^p \Leftrightarrow \forall n\left(V_n \cap E \neq \varnothing \Rightarrow \overline{V_n \cap E} \notin U_1^*\right)$$

it follows that if U_1^* is $\mathbf{\Pi}_1^1$ then M_1^p is $\mathbf{\Sigma}_1^1$ and thus Borel, since M_1^p is also $\mathbf{\Pi}_1^1$ by the equivalence

$$E \in M_1^p \Leftrightarrow \forall n\left[V_n \cap E \neq \varnothing \Rightarrow \overline{V_n \cap E} \notin U_1'\right],$$

and the simple fact (see [6]) that U_1' is $\mathbf{\Sigma}_1^1$.

The following is the main lemma.

LEMMA 5. *If M_1^p is Borel, there is a Borel function F: $M_1^p \to B_1(PM)$ such that*

$$\forall E \in M_1^p(F(E) \in \mathscr{S}_0 \,\&\, \operatorname{supp}(F(E)) = E).$$

Granting this we complete the proof of (iii) $\Rightarrow$ (i) as follows: If U_1^* is $\mathbf{\Pi}_1^1$, then M_1^p is Borel so let F be as in Lemma 5. Then $\{F(E)$: $E \in M_1^p\}$ is a $\mathbf{\Sigma}_1^1$ subset of $\mathscr{S}_0$ and thus bounded in the $\mathbf{\Pi}_1^1$-rank ORD, i.e., for some α_0, ORD$(F(E)) \leq \alpha_0$. It follows that if $S \in \mathscr{S}_0$ there is $T \in \mathscr{S}_0$ with

$$\mathrm{ORD}(T) \leq \alpha_0 \quad \text{and} \quad \operatorname{supp}(S) = \operatorname{supp}(T).$$

(Recall here that M_1^p is exactly the set of supports of $S \in \mathscr{S}_0$). We will show that this implies that *every* $T \in \mathscr{S}_0$ has order $\leq \alpha_0$ which, since ORD is a $\mathbf{\Pi}_1^1$-rank on $\mathscr{S}_0$, shows that $\mathscr{S}_0$ is Borel.

Indeed let $T \in \mathscr{S}_0$, supp$(T) = E$. Then for every open V with $E \cap V \neq \varnothing$ there is $S \in PF$, $S \neq 0$ with $S \in M^{[\alpha_0]}(\overline{V \cap E})$. It follows (see Lemma VIII.4.2 of [6]) that $PF \cap M^{[\alpha_0]}(E)$ is weak*-dense in $N(E)$ and in particular $T \in \overline{PF \cap M^{[\alpha_0]}(E)}^{w^*}$, so

$$T \in \overline{PF \cap M^{[\alpha_0]}(E)}^{w} \quad \left(\text{the weak-closure of } PF \cap M^{[\alpha_0]}(E) \text{ in } PF\right).$$

By Mazur's Theorem, $T \in \overline{M^{[\alpha_0]}(E)} = M^{[\alpha_0]}(E)$, so ORD$(T) \leq \alpha_0$.

It remains to proves Lemma 5. Assume M_1^p is Borel. First note that we have the following characterization of M_1^p based on ideas of Piatetski-Shapiro:

$$E \in M_1^p \Leftrightarrow I(E) \text{ is weak*-closed}.$$

(See [6, VI.3.8] and the remark following it). Now consider the map

$$E \in M_1^p \mapsto I(E) \cap B_1(A) = I_1(E)$$

viewed as a function from M_1^p (a Borel set in $K(\mathrm{T})$) into $K(B_1(A))$, the space of closed subsets of $B_1(A)$ (with the weak*-topology)). We claim it is a Borel map. For that and for further arguments later on we will need some classical descriptive set theoretic facts concerning compact sets. These can be all found in [6], Chapter 4 but a non-logician reader may have trouble digging them out. A nice exposition with proofs or complete references can be found in Section 2 of [0].

THEOREM 6. *Let X, Y be compact metric spaces. Let $f\colon X \to K(Y)$ be a map such that the relation*

$$R(x, y) \Leftrightarrow y \in f(x)$$

is Borel in $X \times Y$. Then the map f is Borel.

So it is enough to show that the map $E \in M_1^p \mapsto I_1(E)$ (which can be viewed as mapping from all of $K(\mathbf{T})$ into $K(B_1(A))$ by defining it to be $\varnothing$ if $E \notin M_1^p$) has the property that the relation

$$R = \{(f, E): f \in B_1(A) \wedge E \in M_1^p \wedge f \in I(E)\}$$

is Borel in $K(\mathbf{T}) \times B_1(A)$. But for $f \in B_1(A)$, $E \in M_1^p$,

$$(f, E) \notin R \Leftrightarrow \exists x[x \in E \wedge f(x) \neq 0].$$

Note now that

$$\begin{aligned}\{(x, f) &\in \mathbf{T} \times B_1(A): f(x) \neq 0\} \\ &= \left\{(x, f) \in \mathbf{T} \times B_1(A): \exists \varepsilon > 0 \exists n \forall m \geq n \Big| \sum_{-m}^{m} \hat{f}(k) e^{ikx} \Big| \geq \varepsilon\right\}\end{aligned}$$

is an F_σ in $\mathbf{T} \times B_1(A)$ and thus so is $(K(\mathbf{T}) \times B_1(A)) \setminus R$ and we are done.

It is now a classical fact (see again [0]) that in each compact metric space X there is a Borel map $s\colon K(X) \to X$ such that $s(E) \in E$ for $E \neq \varnothing$. Fixing a countable subset $\{f_n\}$ of $B_1(A)$ which is *norm-dense* in $B_1(A)$ and applying this to the weak*-closed sets

$$B_1(A) \cap B\left(f_n, \frac{1}{m}\right)$$

(where $B(f, \varepsilon)$ is the closed ball in A with center f and radius ε) we can easily see that there is a Borel map $E \in M_1^p \mapsto \{d_n(E)\}$ from M_1^p into $B_1(A)^{\mathbf{N}}$ such that for $E \in M_1^p$, $E \neq \varnothing$, $\{d_n(E)\}$ is a *norm-dense* subset of $I_1(E)$.

We use that to show that the map $E \in M_1^p \mapsto N_1(E) = N(E) \cap B_1(PM)$ is also Borel from M_1^p into $K(B_1(PM))$. This follows from Theorem 6 again since for $E \in M_1^p$, $S \in B_1(PM)$ the relation

$$S \in N(E) \Leftrightarrow \forall f \in I(E)(\langle S, f\rangle = 0)$$
$$\Leftrightarrow \forall n(\langle S, d_n(E)\rangle = 0)$$

is clearly Borel.

If $E \in M_1^p$, $\varepsilon > 0$ and $N^\varepsilon(E) = \{S \in N(E)\colon R(S) =^{\mathrm{def}} \overline{\lim}|S(n)| < \varepsilon\}$ it follows from Lemma VIII.4.9 of [6] that for any $S \in N(E)$ with $\|S\|_{PM} < a$ there is $T_n \in N^\varepsilon(E)$ with

$$T_n \xrightarrow{w^*} S \quad \text{and} \quad \|T_n\|_{PM} < a + \varepsilon.$$

Following the proof on p. 308 of [6] we construct inductively a sequence of Borel functions $S_1(E)$, $S_2(E), \ldots$ from M_1^p into $B_1(PM)$ and $n_1(E)$, $n_2(E), \ldots$ from M_1^p into $\mathbf{N}$ such that

$$S_1(E), S_2(E), \ldots \in N_1(E), \quad 0 < n_1(E) < n_2(E) < \ldots,$$
$$\|S_1(E)\|_{PM} < \frac{1}{2}$$
$$\|S_k(E)\|_{PM} < \frac{1}{2} + \sum_{i \le k-1} 2^{-i-1} \quad \text{if } k \ge 2,$$
$$\|S_k(E)\|_{PM}^{n_i(E)} < 2^{-i} \text{ if } k \ge i$$

and

$$S_k(E)(0) > \tfrac{1}{4}.$$

(Here $\|S\|_{PM}^n = \sup_{|m| \ge n}|S(n)|$).

The main fact that we use in making $S_1, S_2, \ldots$ Borel is the following uniformization result of Arsenin and Kunugui (see [0]).

THEOREM 7 (Arsenin, Kunugui). *If X, Y are compact, metric spaces, $P \subseteq X \times Y$ is Borel and for each $x \in X$ the section $P_x = \{y\colon P(x, y)\}$ is F_σ, then there is a Borel function $f\colon X \to Y$ such that $P_x \neq \varnothing \Rightarrow f(x) \in P_x$.*

Looking at the relation

$$P(E, S_1) \Leftrightarrow E \in M_1^p \wedge S_1 \in N_1(E) \wedge \|S_1\|_{PM}$$
$$< \tfrac{1}{2} \wedge \exists n \forall |m| \ge n |S_1(m)| < \tfrac{1}{4} \wedge S(0) > \tfrac{1}{4}$$

which is Borel (in $K(\mathbf{T}) \times B_1(PM)$) with F_σ sections we can find $S_1(E)$ Borel with $P(E, S_1(E))$ for $E \in M_1^p$, $E \neq \varnothing$. Then let $n_1(E) = n_1$ be least with $\forall |m| \ge n_1 |S_1(E)(m)| \le \frac{1}{4}$. We define now $S_2(E)$, $n_2(E)$ as follows (the construction of $S_3(E)$, $n_3(E), \ldots$ is analogous): Note that from the above mentioned property of $N^\varepsilon(E)$, for each $m \geq n_1(E) = n_1$ there is $m' > m$ and $S \in N(E)$ with

$$\|S\|_{PM} < \tfrac{1}{2} + \tfrac{1}{4}, \ \|S\|_{PM}^{n_1, m} < \|S_1(E)\|_{PM}^{n_1} + \varepsilon < \tfrac{1}{2}, \ \|S\|_{PM}^{m'} < \tfrac{1}{8} \text{ and } S(0) > \tfrac{1}{4}$$

(here $\|S\|_{PM}^{n_1, m} = \sup_{n_1 \leq |n| \leq m} |S(n)|$). By the uniformization Theorem 7 all these can be found in a Borel way from E so by the usual "iterating and averaging" argument (see [6], p. 276), $S_2(E)$, $n_2(E)$ can be defined in a Borel way satisfying the required conditions.

Finally we use the following standard fact.

LEMMA 8. *Let X be a compact metric space. There is a Borel function f: $X^{\mathbf{N}} \to X$ such that $f(\{x_n\})$ is a limit of a converging subsequence $\{x_{n_i}\}$ of $\{x_n\}$.*

Applying this to the sequence $\{S_n(E)\}$ for $E \in M_1^p$ we obtain a Borel function $E \in M_1^p \mapsto T(E)$ which assigns to each $E \in M_1^p$ a weak*-limit of a subsequence of $\{S_n(E)\}$. By the properties of $\{S_i(E), n_i(E)\}$ it follows that if $E \neq \varnothing$, $E \in M_1^p$ then $T(E) \neq 0$ and $T(E) \in N_1(E) \cap PF$.

To complete the proof notice that from what we have just shown it follows that there is a sequence $E \in M_1^p \mapsto T_n(E)$ of Borel functions which assigns to each $E \in M_1^p$ and each n with $V_n \cap E \neq \varnothing$, $T_n(E) \in PF \cap N_1(\overline{V_n \cap E})$, $T_n(E) \neq 0$. Combining this with the procedure in p. 229 of [6], one easily constructs from the $\{T_n(E)\}$ a Borel function $E \in M_1^p \mapsto F(E)$ such that for $E \in M_1^p$, $F(E)$ is a synthesizable pseudofunction with support E. This completes the proof of Lemma 5 and the theorem. □

Remark. The "correct" way to formulate Lemma 5 is in the following stronger form as a basis theorem in the language of effective descriptive set theory:

LEMMA 5′. *If $E \in M_1^p$ (so that there is $S \in \mathscr{S}_0$ with* supp$(S) = E$), *there is $S \in \Delta_1^1(E)$, $S \in \mathscr{S}_0$ with* supp$(S) = E$.

To avoid the necessary logical background required in this formulation, we preferred the weaker version stated in Lemma 5. However the reader familiar with effective descriptive set theory will have no problem to view the proof above as establishing Lemma 5′ as well. We note here a rather subtle issue which the formulation of Lemma 5′ brings forward: If $E \in M^p$, i.e., E is the support of pseudofunction, then in general there is no $\Delta_1^1(E)$ pseudofunction with support equal to E. Otherwise

$$E \in M^p \Leftrightarrow \exists S \in \boldsymbol{\Delta}_1^1(E)(S \in PF \wedge S \in B_1(PM) \wedge \operatorname{supp}(S) = E),$$

so that M^p would be $\boldsymbol{\Pi}_1^1$ and thus Borel, since M^p is also $\boldsymbol{\Sigma}_1^1$ by the definition

$$E \in M^p \Leftrightarrow \forall n(V_n \cap E \neq \varnothing \Rightarrow \overline{E \cap V_n} \notin U).$$

Then $U^{\text{loc}} = K(\mathbf{T}) \setminus M_1^p$ would be Borel as well and so, by VI.1.3 of [6], U would admit a Borel basis, contradicting the result of Debs-Saint Raymond [1] (see also [6]).

One of the most interesting implications of the preceding result is that a proof of the conjecture would establish that U_1, U_1^* are not $\boldsymbol{\Pi}_1^1$. It is already known from work of Solovay, Kaufman that U_1, U_1^* are not $\boldsymbol{\Sigma}_1^1$ (see [3], [11], [6]). Thus these classes would be neither $\boldsymbol{\Sigma}_1^1$ nor $\boldsymbol{\Pi}_1^1$. On the other hand it can be seen that in terms of upper bounds U_1 belongs to the class of complements of $\mathscr{A}\boldsymbol{\Pi}_1^1$ sets, where $\mathscr{A} = \mathscr{A}^\omega$ is the classical operation A ([9], p. 68) and U_1^* belongs to the larger class $\boldsymbol{\Sigma}_1^1$-IND (see [9]), which is properly contained in $\boldsymbol{\Delta}_2^1 = \boldsymbol{\Sigma}_2^1 \cap \boldsymbol{\Pi}_2^1$. Thus if the conjecture holds one would have the first natural examples of sets in analysis lying strictly between levels of the projective hierarchy, a rather striking phenomenon. In fact in this case it would be reasonable to conjecture that U_1^* is of complexity exactly $\boldsymbol{\Sigma}_1^1$-IND (see [5] for results relating $\boldsymbol{\Sigma}_1^1$-IND with σ-ideals of closed sets with $\boldsymbol{\Sigma}_1^1$ bases) and perhaps similarly for U_1, in its corresponding class, i.e., the dual of $\mathscr{A}\boldsymbol{\Pi}_1^1$. A result of Dougherty and Kechris states that if X is compact metrizable, $I \subseteq K(X)$ a σ-ideal of closed subsets of X which is not $\boldsymbol{\Pi}_1^1$ then I is not $\boldsymbol{\Sigma}_1^1 \vee \boldsymbol{\Pi}_1^1$ (i.e., the union of a $\boldsymbol{\Sigma}_1^1$ and a $\boldsymbol{\Pi}_1^1$ set). Thus if the conjecture holds U_1^* cannot be in $\boldsymbol{\Sigma}_1^1 \vee \boldsymbol{\Pi}_1^1$ either.

As we mentioned earlier the first open case of the conjecture is that

$$U_1^* \subsetneqq U_{1,0}^*$$

(this problem was raised in Lyons [8]). This is equivalent to saying that $M_1^p \subsetneqq M_{1,0}^p$. In terms of $\mathscr{S}_0$ this is again equivalent to the assertion that there is a synthesizable pseudofunction which is not a *strong limit* of measures on its support. In this formulation the Piatetski-Shapiro Theorem that $U_1^* \subsetneqq U_0$ (or $M_0^p \supsetneqq M_1^p$) asserts that there is a synthesizable function which is not a *strong limit* of *Rajchman* measures, (i.e., measures in *PF*) while Lyons' stronger result $U_{1,0}^* \subsetneqq U_0$ (or $M_0^p \subsetneqq M_{1,0}^p$) amounts to saying that there is a pseudofunction which is a strong limit of measures but not *Rajchman* measures on its support. (The strict inclusion $U_1^* \subsetneqq U_{1,0}^*$ has now been established by Kaufman; see the addendum at the end of the introduction).

6. On the structure of $U_{1,\alpha}$, $U_{1,\alpha}^*$

We will establish some results about $U_{1,\alpha}$, $U_{1,\alpha}^*$ analogous to those established by Lyons for [8]. The methods are sometimes similar and we will only provide details when there are new twists.

We will first define a transfinite sequence $K_\alpha(E)$, $E \in K(\mathbf{T})$, of convex compact (in the weak*-topology) subsets of $B_1(PM)$. First for such E let

$$M^{(0)}(E) = M(E),$$
$$M^{(\alpha+1)}(E) = \left(M^{(\alpha)}(E)\right)^{(1)},$$
$$M^{(\lambda)}(E) = \bigcup_{\alpha<\lambda} M^{(\alpha)}(E), \quad \lambda \text{ limit},$$

as usual so that

$$M^{[\alpha]}(E) = \overline{M^{(\alpha)}(E)}.$$

Now let

$$K_0(E) = \mathrm{PROB}(E) = \text{the class of probability measures on } E,$$
$$K_{\alpha+1}(E) = \overline{M^{(\alpha)}(E) \cap B_1(PM)}^{w^*}$$
$$K_\lambda(E) = \left(M^{(\lambda)}(E)\right)_{(1)} \cap B_1(PM)$$

where for each $Z \subseteq PM$ we define

$$Z_{(1)} = \{S \in PM: \exists\{S_n\}(S_n \in Z \wedge S_n \xrightarrow{w^*} S \wedge R(S_n) \to 0)\},$$

with $R(S) = \overline{\lim}|S(n)|$. In [6], p. 171, $Z_{(1)}$ has been defined by taking only those $S \in Z$ that satisfy the above definition. This coincides with the above definition if Z is weak*-closed, which was the case of interest in [6]. However here the Z's we are studying are not weak*-closed.

Let us first note the following fact.

Proposition 1. *For any subspace $Z \subseteq PM$, $Z_{(1)}$ is weak*-closed. Also $PF \cap Z_{(1)} = PF \cap \overline{Z}$.*

Proof. Fix $\varepsilon > 0$ and let $C_\varepsilon = \{S \in Z: R(S) < \varepsilon\}$. If $S \in Z_{(1)}$, then $S \in \overline{C}_\varepsilon^{w^*}$ so by Lemma VIII.4.9 of [6] there are $S_n \in C_\varepsilon$, $S_n \to^{w^*} S$, $\|S_n\|_{PM} < \|S\|_{PM} + \varepsilon$. So if $S \in Z_{(1)}$, there is

$$S_n \in Z, S_n \xrightarrow{w^*} S, \quad R(S_n) \to 0, \quad \|S_n\|_{PM} < \|S\|_{PM} + 1.$$

If follows easily that $Z_{(1)}$ is weak*-sequentially closed, so by Banach's Theorem (see [6, V.2.2]) $Z_{(1)}$ is weak*-closed.

It is obvious that $PF \cap \bar{Z} \subseteq PF \cap Z_{(1)}$. Conversely assume $S \in PF \cap Z_{(1)}$. Fix $\varepsilon > 0$. Than as above, for each n, we can find $T \in C_\varepsilon$ with

$$|S(i) - T(i)| < \varepsilon \quad \text{for } |i| \leq n,$$

and

$$\|T\|_{PM} \leq \|S\|_{PM} + 1.$$

So define $n_0 < n_1 < n_2 < \ldots$ and $T_0, T_1, \ldots \in C_\varepsilon$ with $\|T_m\|_{PM} \leq \|S\|_{PM} + 1$ such that

$$|S(i) - T_m(i)| < \varepsilon \text{ if } |i| \leq n_m \text{ or } |i| \geq n_{m+1}.$$

So if N is large enough,

$$\left\| S - \frac{T_1 + \ldots + T_N}{N} \right\|_{PM} < \varepsilon \quad \text{and} \quad \frac{T_1 + \ldots + T_N}{N} = T \in Z. \quad \square$$

The main facts about K_α (that follow from their definition and the above proposition) are

(1) For $\alpha = 0$ or successor, span $(K_\alpha(E)) = M^{(\alpha)}(E)$.
For $\alpha = \lambda$ limit, $\operatorname{span}(K_\lambda(E)) = M^{(\lambda)}(E)_{(1)}$

(2) For any α, $PF \cap \overline{\operatorname{span}(K_\alpha(E))} = PF \cap \overline{M^{(\alpha)}(E)}$.

We define now a sequence of norms on A.

$$\|f\|_{\alpha, E} = \sup\{|\langle f, S\rangle \colon S \in K_\alpha(E)\} = \|f\|_{C(K_\alpha(E))}.$$

Note that

$$\|f\|_{0, E} = \|f\|_{C(E)} \; (= \text{the sup-norm of the function } f|E)$$

$$\|f\|_{0, E} \leq \|f\|_{1, E} \leq \ldots \leq \|f\|_{\alpha+1, E} \leq \ldots \leq \|f\|_{\beta+1, E} \leq \ldots \leq \|f\|_{A(E)} \quad (\alpha \leq \beta)$$

and eventually $\|f\|_{\alpha+1, E} = \|f\|_{A(E)}$. (Recall that

$$\begin{aligned}\|f\|_{A(E)} &= \inf\{\|f - g\|_A \colon g \in I(E)\} \\ &= \sup\{|\langle f, S\rangle \colon S \in N(E) \cap B_1(PM)\}.\end{aligned}$$

Moreover, if $\|f\|_{\tilde{A}(E)}$ is the tilde-norm (see [2], p. 362) defined by

$$\|f\|_{\tilde{A}(E)} = \sup\{|\langle f, S\rangle| : S \in M(E) \cap B_1(PM)\}$$

then

$$\|f\|_{\tilde{A}(E)} = \|f\|_{1,E}.$$

For $S \in PM$, define

$$\|S\|_{\alpha,E} = \sup\{\overline{\lim}|\langle f_n, S\rangle| : f_n \in B_1(A),\ \|f_n\|_{\alpha,E} \to 0\}.$$

We now have the following key fact.

PROPOSITION 2. *For each $S \in PM$,*

$$\|S\|_{\alpha,E} = \operatorname{dist}(S, \operatorname{span}(K_\alpha(E))).$$

In particular, $S \in \overline{\operatorname{span}(K_\alpha(E))} \Leftrightarrow \|S\|_{\alpha,E} = 0$.

Proof. First let $S \in PM$ and suppose $f_n \in B_1(A)$, $\|f_n\|_{\alpha,E} \to 0$. Fix $T \in \operatorname{span}(K_\alpha(E))$. Since $\|f_n\|_{\alpha,E} \to 0$, $\langle f_n, T\rangle \to 0$. So

$$\overline{\lim}|\langle f_n, S\rangle| = \overline{\lim}|\langle f_n, S - T\rangle| \leq \|f_n\|_A \cdot \|S - T\|_{PM} \leq \|S - T\|_{PM}.$$

So $\|S\|_{\alpha,E} \leq \operatorname{dist}(S, \operatorname{span}(K_\alpha(E)))$.

Conversely, let $S \in PM$. By Hahn-Banach find

$$S^* \in B_1(PM^*) \cap K_\alpha(E)^\perp \quad \text{with } \langle S, S^*\rangle = \operatorname{dist}(S, \operatorname{span}(K_\alpha(E))).$$

Fix $\varepsilon < \langle S, S^*\rangle$. We will find $f_n \in B_1(A)$ with

$$\|f_n\|_{\alpha,E} \to 0 \quad \text{and} \quad |\langle f_n, S\rangle| > \varepsilon.$$

Put

$$V = \{S^{**} \in PM^* : \operatorname{Re}\langle S, S^{**}\rangle > \varepsilon\}.$$

This is a weak*-open nbhd of S^* in PM^*, so by Goldstine's Theorem

$$W = V \cap B_1(A) \neq \varnothing.$$

Note that W is convex. We can obviously view W as a convex subset of $C(K_\alpha(E))$. (Here $K_\alpha(E)$ has the weak*-topology).

Claim. $0 \in \overline{W}$ (= the strong closure of W in $C(K_\alpha(E))$).

Granting this there are $f_n \in W$ with $\|f_n\|_{C(K_\alpha(E))} = \|f_n\|_{\alpha, E} \to 0$. Since $f_n \in W$, $|\langle f_n, S\rangle| > \varepsilon$ and we are done.

Proof of the claim. If $0 \notin \overline{W}$, there is a measure $\mu \in M(K_\alpha(E))$ and $\delta > 0$ with $\operatorname{Re}\langle f, \mu\rangle > \delta$ for all $f \in W$. Write μ as a linear combination of probability measures. Each of these has a barycenter in $K_\alpha(E)$ and thus there is $S' \in \operatorname{span}(K_\alpha(E))$ such that $\langle f, \mu\rangle = \langle f, S'\rangle$, for $f \in W$. Now

$$S^* \in \operatorname{span}(K_\alpha(E))^\perp,$$

so

$$\{S^{**} \in PM^*: S^{**} \in V \wedge \operatorname{Re}\langle S', S^{**}\rangle < \delta\}$$

is also a weak*-nbhd of S^*, thus it contains $f \in B_1(A)$. Then $f \in W$ but

$$\operatorname{Re}\langle S', f\rangle = \operatorname{Re}\langle f, \mu\rangle < \delta,$$

a contradiction. □

DEFINITION. For $E \in K(\mathbf{T})$, let

$$Z_\alpha(E) = \{f \in A: \exists f_n \in A(f_n \xrightarrow{w^*} f \wedge \|f_n\|_{\alpha, E} \to 0)\},$$
$$\tilde{Z}_\alpha(E) = \{f \in A: \exists f_n \in B_1(A)(f_n \xrightarrow{w^*} f \wedge \|f_n\|_{\alpha, E} \to 0)\}.$$

Note that $Z_\alpha(E)$ is an ideal in A and $\tilde{Z}_\alpha(E)$ is convex compact in the weak*-topology and for $S \in PF$,

$$\|S\|_{\alpha, E} = \sup\{|\langle f, S\rangle|: f \in \tilde{Z}_\alpha(E)\} = \|S\|_{C(\tilde{Z}_\alpha(E))}.$$

Note also that if $A^*(E)$ is as in [2, p. 367], then

$$Z_0(E) = A^*(E) \cap A.$$

PROPOSITION 3. *For each* $E \in K(\mathbf{T})$,

$$PF \cap Z_\alpha(E)^\perp = PF \cap M^{[\alpha]}(E).$$

Proof. Clearly for $S \in PF$,

$$S \in Z_\alpha(E)^\perp \Leftrightarrow S \in \tilde{Z}_\alpha(E)^\perp \Leftrightarrow \|S\|_{\alpha, E} = 0$$
$$\Leftrightarrow S \in \overline{\operatorname{span}(K_\alpha(E))} \Leftrightarrow S \in \overline{M^{(\alpha)}(E)}.$$ □

We have thus the following characterization

THEOREM 4. *Let $E \in K(\mathbf{T})$. Then the following are equivalent*:
(1) $E \in U_{1,\alpha}$;
(2) $M^{[\alpha]}(E) \cap PF = \{0\}$;
(3) $Z_\alpha(E)$ *is weak*-dense in* A, *i.e.*, $\overline{Z_\alpha(E)}^{w^*} = A$;
(4) $1 \in \overline{Z_\alpha(E)}^{w^*}$.
In order to introduce the classes $U'_{1,\alpha}$ we first need a definition.

DEFINITION. For each $\varnothing \neq E \in K(\mathbf{T})$, let

$$r_\alpha(E) = \sup\{r \geq 0 \colon B_r(A) \subseteq \tilde{Z}_\alpha(E)\},$$

$$s_\alpha(E) = \inf\left\{\frac{\|S\|_{\alpha,E}}{\|S\|_{PM}} : 0 \neq S \in PF\right\},$$

$$t_\alpha(E) = \inf\left\{\frac{\|S - S^*\|_{PM}}{\|S\|_{PM}} : 0 \neq S \in PF, S^* \in \operatorname{span}(K_\alpha(E))\right\},$$

$$\eta_\alpha(E) = \inf\left\{\frac{R(S)}{\|S\|_{PM}} : 0 \neq S \in \operatorname{span}(K_\alpha(E))\right\}.$$

Then the following can be established by standard methods (see e.g. [6, V.2 and V.5.3]).

PROPOSITION 5. *For any closed set $E \neq \varnothing$,*

$$r_\alpha(E) = s_\alpha(E) = t_\alpha(E) \quad \text{and} \quad t_\alpha(E) = \frac{\eta_\alpha(E)}{1 + \eta_\alpha(E)}.$$

Moreover $\eta_\alpha(E) > 0 \Leftrightarrow Z^\alpha(E) = A$.

DEFINITION. Let $E \in K(\mathbf{T})$. Put $E \in U'_{1,\alpha} \Leftrightarrow Z^\alpha(E) = A$.

PROPOSITION 6. *Let $E \in K(\mathbf{T})$. Then the following are equivalent*:
(i) $E \in U'_{1,\alpha}$;
(ii) $Z^\alpha(E) = A$;
(iii) $1 \in Z^\alpha(E)$;
(iv) $E = \varnothing$ *or* $\eta_\alpha(E) > 0$.

We will show now that $U'_{1,\alpha}$ is a basis for $U_{1,\alpha}$ and thus $U^*_{1,\alpha}$. For that consider the Cantor-Bendixson derivation associated with $U'_{1,\alpha}$ (see [6]).

Denote by $E'_{U'_{1,\alpha}}$ the corresponding derivative. The main point is the following:

PROPOSITION 7. *For each $E \in K(\mathbf{T})$,*

$$Z_\alpha(E) \subseteq I(E'_{U'_{1,\alpha}}).$$

In particular if $E \in U_{1,\alpha}$, then $E'_{U'_{1,\alpha}} \in U_1$, so that E is a countable union of $U'_{1,\alpha}$-sets. Thus

$$(U'_{1,\alpha})_\alpha = U^*_{1,\alpha},$$

i.e., $U'_{1,\alpha}$ is a basis for the σ-ideal $U^*_{1,\alpha}$.

Proof. Fix $x \in E'_{U'_{1,\alpha}}$, $f \in Z_\alpha(E)$ in order to show that $f(x) = 0$. Since $x \in E'_{U'_{1,\alpha}}$, we can find

$$S_n \in \operatorname{span}(K_\alpha(E)) \quad \text{with } S_n \xrightarrow{w*} \delta_x \text{ and } R(S_n) \leq 1/n.$$

Let $T_n \in PF$ be such that

$$\|T_n - S_n\|_{PM} \leq 1/n.$$

Since $f \in Z_\alpha(E)$ find $f_n \in A$ with $f_n \to^{w^*} f$ and $\|f_n\|_{\alpha,E} \to 0$. Say $\|f_n\|_A \leq M$. Now

$$\begin{aligned} f(x) &= \langle f, \delta_x \rangle \\ &= \langle f, \delta_x - S_n \rangle + \langle f, S_n - T_n \rangle + \langle f - f_m, T_n \rangle \\ &\quad + \langle f_m, T_n - S_n \rangle + \langle f_m, S_n \rangle. \end{aligned}$$

Fix $\varepsilon > 0$. Find then n_0 such that

$$\begin{gathered} \langle f, \delta_x - S_{n_0} \rangle < \varepsilon/5, \\ |\langle f, S_{n_0} - T_{n_0} \rangle| < \varepsilon/5, \\ |\langle f_m, T_{n_0} - S_{n_0} \rangle| < \varepsilon/5, \quad \text{all } m. \end{gathered}$$

Then choose m_0 such that

$$\begin{gathered} |\langle f - f_{m_0}, T_{n_0} \rangle| < \varepsilon/5, \\ |\langle f_{m_0}, S_{n_0} \rangle| < \varepsilon/5, \end{gathered}$$

the last being possible as $S_{n_0} \in \operatorname{span}(K_\alpha(E))$ and $\|f_n\|_{\alpha, E} \to 0$. So $|f(x)| < \varepsilon$ and we are done. □

In the case of $U'_{1,0}$ Lyons [8] shows that $U'_{1,0}$ is an ideal. We do not know if this is true for all $U'_{1,\alpha}$. It is easy to verify that $U'_{1,\alpha}$ is closed under finite unions of pairwise disjoint (closed) sets. We can use this to show:

PROPOSITION 8. *For each $E \in K(\mathbf{T})$,*

$$J\left(E'_{U'_{1,\alpha}}\right) \subseteq Z_\alpha(E) \qquad \left(\subseteq I\left(E'_{U'_{1,\alpha}}\right) \text{ by Proposition 7}\right).$$

Proof. Let $f \in J(E'_{U'_{1,\alpha}}) \cap B_1(A)$. Then $F = \operatorname{supp}(f) \cap E$ is disjoint from $E'_{U'_{1,\alpha}}$ and totally disconnected, so F is a finite union of clopen in F $U'_{1,\alpha}$-sets. These can be clearly assumed to be disjoint, so $F \in U'_{1,\alpha}$. Thus there is $g_n \in A$, $g_n \to^{w^*} 1$, $\|g_n\|_{\alpha, F} \to 0$. Then $fg_n \to^{w^*} f$, so it is enough to check that $\|fg_n\|_{\alpha, E} \to 0$. Fix $S \in K_\alpha(E)$. Then $\langle fg_n, S\rangle = \langle g_n, f \cdot S\rangle$, so

$$\|fg_n\|_{\alpha, E} \leq \|g_n\|_{\alpha, F} \to 0$$ □

and we are done.

We can deduce from this that U_α is not a σ-ideal.

PROPOSITION 9. *For each α, $U_{1,\alpha}$ is not a σ-ideal.*

Proof. Let $E \in U'_1 \setminus U$ by Körner's Theorem (see [7], also [6]). Then $E \in U'_{1,\alpha} \setminus U$.

Let $\{x_n\}$ enumerate the endpoints of the intervals contiguous to E and, denoting by E^h the Herz transform of E (see [6, 6.3, p. 226]), let $\{y_k\}$ enumerate the points of $E^h \setminus E$. Find $E_n^{(m)}$, $F_k^{(l)}$ a discrete sequence of closed sets disjoint from E such that $E_n^{(m)}$ (all m) is in the $1/n$-nbhd of x_n, $E_n^{(m)} \to^m x_n$ and $F_k^{(l)}$ (all l) is in the $1/k$-nbhd of y_k, $F_k^{(l)} \to^l x_k$ and

$$0 < \eta_\alpha\left(E_n^{(m)}\right) \to_m 0, \quad 0 < \eta_\alpha\left(F_k^{(l)}\right) \to_l 0.$$

(Such sets can be found, as for each interval I and $\varepsilon > 0$ there is $E \subseteq I$ with $E \in U'_1 \subseteq U'_{1,\alpha}$ and

$$0 < \eta_\alpha(E) \leq \eta_0(E) \overset{\text{def}}{=} \inf\{R(\mu) \colon \mu \text{ is a probability measure on } E\} < \varepsilon$$

(Piatetski-Shapiro [10], see also [6, VI.2.5]).) Let

$$F = E^h \cup \bigcup_{n,m} E_n^{(m)} \cup \bigcup_{k,l} F_k^{(l)}.$$

Then $F \in (U'_{1,\alpha})_\sigma$ is closed and $F'_{U'_{1,\alpha}} = E^h$. By the two preceding propositions

$$J(F'_{U'_{1,\alpha}}) \subseteq Z_\alpha(F) \subseteq I(F'_{U'_{1,\alpha}})$$

and $F'_{U'_{1,\alpha}} = E^h$ is a set of synthesis, so $\overline{Z_\alpha(F)} = I(E^h)$. But then $F \notin U_{1,\alpha}$, because otherwise $\overline{Z_\alpha(F)}^{w^*} = A$, i.e., $\overline{I(E^h)}^{w^*} = A$, so $E^h \in U_1$ thus $E^h \in U$ and $E \in U$, a contradiction. □

We have already seen in §5 that $U_{1,\alpha}$ is $\mathbf{\Pi}_1^1$ and thus so is the σ-ideal $U_{1,\alpha}^*$. We actually have:

THEOREM 10. *The class $U'_{1,\alpha}$ is Borel, so the σ-ideal $U_{1,\alpha}^* = (U'_{1,\alpha})_\sigma$ has a Borel basis.*

Proof. The proof is based on the method used by Solovay [12] in his original proof that the Piatetski-Shapiro rank on U is a $\mathbf{\Pi}_1^1$-rank.

Let $X \subseteq PM$ be a subspace.

DEFINITION. A sequence $\{x_n\} \in B_1(PM)^{\mathbf{N}}$ *approximates* X if $x_n \in X$, $\forall n$ and $\{x_n\colon n \in \mathbf{N}\}$ is dense in $X \cap B_1(PM)$ in the weak*-topology. (We are *not* assuming that X is weak*-dense).

Recall that $X^{(1)}$ is the set of weak*-limits of sequences from X.

LEMMA 11. *There is a Borel function $F\colon B_1(PM)^{\mathbf{N}} \to B_1(PM)^{\mathbf{N}}$ such that*

$$\{x_n\} \textit{ approximates } X \Rightarrow \{y_n\} = F(\{x_n\}) \textit{ approximates } X^{(1)}.$$

Proof. Let ρ be the metric on PM that gives the weak*-topology on each $B_r(PM)$, i.e.,

$$\rho(S,T) = \sum \frac{|S(n) - T(n)|}{2^n}.$$

Fix a dense sequence $\{d_n\}$ in $B_1(PM)$ (with the weak*-topology). For each $i \in \mathbf{N}$ let

$$U_i = \left\{x \in B_1(PM)\colon \rho(x, d_{(i)_0}) < \frac{(i)_1}{(i)_2}\right\}$$

where we have fixed a $1-1$ correspondence $i \mapsto ((i)_0, (i)_1, (i)_2)$ between $\mathbf{N}$ and $\mathbf{N}^3$. Thus $\{U_i\}$ is a basis for $B_1(PM)$.

Note that if $\{x_n\}$ approximates X and $x \in X^{(1)} \cap B_1(PM)$ then

$$x = \lim{}^{w^*} z_n, \qquad z_n \in X \cap B_N(PM),$$

for some large enough N so find x_{k_n} with $\rho(x_{k_n}, z_n/N) < 1/n$. Then $\rho(x_k \cdot N, z_n) < N/n$. So

$$x = \lim{}^{w^*}\left(x_{k_n} \cdot N\right).$$

For $\{x_n\} \in B_1(PM)^{\mathbf{N}}$, let

$$K_N\{x_n\} = \text{weak*-closure of } \{N \cdot x_n\} \text{ in } B_N(PM).$$

So we have seen that

$$X^{(1)} \cap B_1(PM) = U_N K_N\{x_n\} \cap B_1(PM).$$

Claim. For $i, N \in \mathbf{N}$ let

$$R_{N,i} = \left\{\{x_n\} \in B_1(PM)^{\mathbf{N}}: U_i \cap K_N\{x_n\} \neq \varnothing\right\}.$$

Then $R_{N,i} \subseteq B_1(PM)^{\mathbf{N}}$ is Borel.

Proof. For $K \in K(B_N(PM))$, the map $K \mapsto K \cap B_1(PM)$ is Borel from $B_N(PM)$ to $B_1(PM)$. Let

$$U_i^N = \left\{x \in B_N(PM): \rho\left(x, (d)_{i_0}\right) < \frac{(i)_1}{(i)_2}\right\}.$$

Then U_i^N is open in $B_N(PM)$ and

$$U_i \cap K \neq \varnothing \Leftrightarrow K \cap B_1(PM) \cap U_i^N \neq \varnothing;$$

thus $\{K: K \cap U_i \neq \varnothing\}$ is Borel in $K(B_N(PM))$. Now $\{x_n\} \mapsto K_N\{x_n\}$ is easily Borel from $B_1(PM)^{\mathbf{N}}$ into $K(B_N(PM))$ so we are done.

We will define now a sequence of Borel functions F_i: $B_1(PM)^{\mathbf{N}} \to B_1(PM)$ such that $\{x_n\}$ approximates $X \Rightarrow [F_i(\{x_n\}) \in X^{(1)}$ and $(U_i \cap X^{(1)} \neq \varnothing \Rightarrow F_i(\{x_n\}) \in U_i \cap X^{(1)})]$. Granting this, let $F(\{x_n\}) = \{y_n\}$ where $y_i = F_i(\{x_n\})$.

Clearly F is Borel and $\{x_n\}$ approximates $X \Rightarrow F(\{x_n\})$ approximates $X^{(1)}$, so we are done.

Definition of F_i. Fix i. Given $\{x_n\} \in B_1(PM)^{\mathbf{N}}$, first find the least $N = N(\{x_n\})$ such that $K_N\{x_n\} \cap U_i \neq \varnothing$, if such exists; else let $F_i(\{x_n\}) = x_0$.

Now define $\{l_k\}$ inductively such that the weak*-closure of U_{l_0} is contained in U_i, the weak*-closure of $U_{l_{k+1}}$ is contained in U_{l_k}, $(l_k)_1/(k_k)_2 < 1/k$ and

$$U_{l_k} \cap K_N\{x_n\} \neq \varnothing.$$

Finally put

$$F_i(\{x_n\}) = \lim{}^{w^*}(d)_{(l_k)_0} = \text{the weak*-limit of the centers of } \{U_{l_k}\}.$$

It is easy to check, using the fact that $R_{N,i}$ is Borel, that F_i is Borel. □

LEMMA 12. *There is a Borel function F: $K(\mathbf{T}) \to B_1(PM)^{\mathbf{N}}$ such that $F(E)$ approximates $M(E)$.*

Proof. Denote by $B_N(M(\mathbf{T}))$ the closed ball of radius N in $M(\mathbf{T})$ (with the weak*-topology). As in the preceding proof it is enough to check that for each fixed N, i the set

$$Q_{N,i} = \{E \in K(\mathbf{T}) \colon M(E) \cap B_N(M(\mathbf{T})) \cap U_i \neq \varnothing\}$$

is Borel. But note that the map

$$E \mapsto M(E) \cap B_N(M(\mathbf{T}))$$

from $K(\mathbf{T})$ into $K(B_N(M(\mathbf{T}))$ is Borel and that

$$L_{N,i} = \{\mu \in B_N(M(\mathbf{T})) \colon \mu \in U_i\}$$

is an F_σ in $B_N(M(\mathbf{T}))$, so that $Q_{N,i}$ is Borel. □

By putting together Lemmas 11, 12, a simple transfinite induction shows the following result.

LEMMA 13. *For each countable ordinal α, there is a Borel function F_α: $K(\mathbf{T}) \to B_1(PM)^{\mathbf{N}}$ such that $F_\alpha(E)$ approximates $M^{(\alpha)}(E)$.*

In particular for each α, $F_\alpha(E)$ is a sequence dense in $K_{\alpha+1}(E)$.

We need one more lemma to take care of $K_\lambda(E)$, λ limit.

LEMMA 14. *For each limit ordinal λ there is a Borel function G_λ: $K(\mathbf{T}) \to B_1(PM)^{\mathbf{N}}$ such that $G_\lambda(E)$ approximates $M^{(\lambda)}(E)_{(1)}$.*

Proof. It will be enough, by arguments similar to that of Lemma 12, to show that the map

$$E \mapsto M^{(\lambda)}(E)_{(1)} \cap B_1(PM)$$

from $K(\mathbf{T})$ into $K(B_1(PM))$ is Borel. And for that by Theorem 5.6, it is enough to show that the relation

$$S \in M^{(\lambda)}(E)_{(1)} \cap B_1(PM) \Leftrightarrow R(E,S)$$

is Borel in $K(\mathbf{T}) \times B_1(PM)$. But note that if $(U^{i,N})_{i=1}^{\infty}$ is a basis for the weak*-topology on each $B_N(PM)$ we have $R(E,S) \Leftrightarrow \exists N \forall i \forall p[S \in U_{i,N} \Rightarrow \exists \alpha < \lambda \exists M[\{T \in B_N(PM)\colon \forall |m| \geq M | T(m) \leq 1/p\} \cap K_N(F_\alpha(E)) \cap U_{i,N} \neq \varnothing]]$ so we are done. □

We complete now the proof that $U'_{1,\alpha}$ is Borel. Letting $\{g_{N,n}\}$ denote a sequence norm-dense in $B_N(A)$, we have

$$\begin{aligned} E \in U'_{1,\alpha} &\Leftrightarrow \exists \{f_n\}(f_n \in A \wedge f_n \xrightarrow{w^*} 1 \wedge \|f_n\|_{\alpha,E} \to 0) \\ &\Leftrightarrow \exists N \forall p \forall M \exists n \Bigg[|g_{N,n}(0) - 1| < \frac{1}{p} \wedge |g_{N,n}(m)| < \frac{1}{p}, \\ &\qquad\qquad 0 < |m| \leq M \wedge \|g_{N,n}\|_{\alpha,E} < \frac{1}{p} \Bigg]. \end{aligned}$$

For $\alpha = 0$, since $\|g\|_{0,E} = \|g\|_{C(E)}$ this is clearly Borel. For the successor case $\alpha + 1$, note that

$$\|g\|_{\alpha+1,E} = \|g\|_{C(K_{\alpha+1}(E))} = \sup_n\{|\langle g, y_n\rangle|\colon \{y_n\} = F_\alpha(E)\}$$

and for the limit case λ,

$$\|g\|_{\lambda,E} = \|g\|_{C(K_\lambda(E))} = \sup_n\{|\langle g, y_n\rangle\colon \{y_n\} = G_\lambda(E)\}$$

so we are done. □

Remark. The preceding argument can be used to show also that the inclusion in Proposition 7 is proper in general. Take $\alpha = 0$ in order to show that there are E with $Z_0(E) \neq I(E'_{U'_{1,0}})$ or in fact

$$Z_0(E)^{\perp} \neq N(E'_{U'_{1,0}}).$$

As Lyons shows in [8], $Z_0(E)^{\perp}$ $(= Z(E)^{\perp}$, in his notation$) = \overline{M(E)_{(1)}}$.

For each closed set E let $\{x_n\}$ enumerate the endpoints of the intervals contiguous to E and let $E_n^{(m)}$ be as in Proposition 9. Put $\tilde{E} = E \cup \bigcup_{n,m} E_n^{(m)}$. Thus $\tilde{E}'_{U'_{1,0}} = E$. Now recall from [6] that the map F: $K(\mathbf{T}) \to K(B_1(PM))$ given by $F(E) = N(E) \cap B_1(PM)$ is not Borel. Otherwise, since $G(E) = PM(E) \cap B_1(PM)$ is Borel, we would have

$$E \text{ is of synthesis } \Leftrightarrow F(E) = G(E)$$

so $S = \{E \in K(\mathbf{T})$: E is of synthesis$\}$ would be Borel. Put

$$H(E) = M(\tilde{E})_{(1)} \cap B_1(PM).$$

By the proof of the preceding theorem, choosing the $E_n^{(m)}$ canonically so that $E \mapsto \tilde{E}$ is Borel, we have that H: $K(\mathbf{T}) \to K(B_1(PM))$ is Borel. So for some E, $H(E) \neq F(E)$, i.e,

$$N(E) \neq M(\tilde{E})_{(1)},$$

so if $F = \tilde{E}$, $Z_0(F)^{\perp} = \overline{M(F)_{(1)}} \neq N(E) = N(F'_{U'_{1,0}})$.

We conclude with some open problems (for the definition of the concepts involved see [6]):

Is $U^*_{1,\alpha}$ $(0 \leq \alpha \leq \omega_1)$ calibrated? Is is locally non-Borel? Can every $\mathbf{\Sigma}_1^1$ set in $(U^*_{1,\alpha})^{\text{int}}$ be covered by countably many $U^*_{1,\alpha}$-sets?

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