TURBULENCE, AMALGAMATION, AND GENERIC AUTOMORPHISMS OF HOMOGENEOUS STRUCTURES

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Abstract. We study topological properties of conjugacy classes in Polish groups, with emphasis on automorphism groups of homogeneous countable structures. We first consider the existence of dense conjugacy classes (the topological Rokhlin property). We then characterize when an automorphism group admits a comeager conjugacy class (answering a question of Truss) and apply this to show that the homeomorphism group of the Cantor space has a comeager conjugacy class (answering a question of Akin-Hurley-Kennedy). Finally, we study Polish groups that admit comeager conjugacy classes in any dimension (in which case the groups are said to admit ample generics). We show that Polish groups with ample generics have the small index property (generalizing results of Hodges-Hodkinson-Lascar-Schelah) and arbitrary homomorphisms from such groups into separable groups are automatically continuous. Moreover, in the case of oligomorphic permutation groups, they have uncountable cofinality and the Bergman property. These results in particular apply to automorphism groups of many \(\omega\)-stable, \(\aleph_0\)-categorical structures and of the random graph. In this connection, we also show that the infinite symmetric group \(S_\infty\) has a unique non-trivial separable group topology. For several interesting groups we also establish Serre’s properties (FH) and (FA).

1. Introduction

1.1. Polish groups. We study in this paper topological properties of conjugacy classes in Polish groups. There are two questions which we are particularly interested in. First, does a Polish group \(G\) have a dense conjugacy class? This is equivalent (see, e.g., Kechris [31] (8.47)) to the following generic ergodicity property of \(G\): Every conjugacy invariant subset \(A \subseteq G\) with the Baire property (e.g., a Borel set) is either meager or comeager. There is an extensive list of Polish groups that have dense conjugacy classes, like, e.g., the automorphism group \(\text{Aut}(X, \mu)\) of a standard measure space \((X, \mu)\), i.e., a standard Borel space \(X\) with a non-atomic Borel probability measure \(\mu\) (see, e.g., Halmos [22]), the unitary group \(U(H)\) of separable infinite-dimensional Hilbert space \(H\) (see, e.g., Choksi-Nadkarni [10]) and the homeomorphism groups \(H(X)\) of various compact metric spaces \(X\) with the uniform convergence topology, including \(X = [0,1]^N\) (the Hilbert cube), \(X = 2^N\) (the Cantor space), \(X = S^{2d}\) (even dimensional spheres), etc. (see Glasner-Weiss [15] and Akin-Hurley-Kennedy [2]). In Glasner-Weiss [15] groups that have dense conjugacy classes are said to have the topological Rokhlin property, motivated by the existence of dense conjugacy classes in \(\text{Aut}(X, \mu)\), which is usually seen as a consequence of the well-known Rokhlin Lemma in ergodic theory. Recently, Akin, Glasner and Weiss [3] also found an example of a locally compact Polish group with dense conjugacy class. Any such group must be non-compact and it appears.
likely that it must also be totally disconnected. (Karl Hofmann has shown that if a locally compact group \( \neq \{1\} \) has a dense conjugacy class, then it is not pro-Lie and in particular, its quotient by the connected component of the identity cannot be compact.)

The second question we consider is whether \( G \) has a (necessarily unique) dense \( G_\delta \) conjugacy class (which is well-known to be equivalent to whether it has a dense non-meager class, see, e.g., Becker-Kechris [5]). Following Truss [47], we call any element of \( G \) whose conjugacy class is dense \( G_\delta \) a generic element of \( G \). This is a much stronger property which fails in very “big” groups such as \( \text{Aut}(X, \mu) \) or \( U(H) \) but can often occur in automorphism groups \( \text{Aut}(K) \) of countable structures \( K \). It was first studied in this context by Lascar [36] and Truss [47]. For example, \( \text{Aut}(\mathbb{N}, =) \) (i.e., the infinite symmetric group \( S_\infty \)), \( \text{Aut}(\mathbb{Q}, <) \), \( \text{Aut}(\mathbb{R}) \), where \( \mathbb{R} \) is the random graph, \( \text{Aut}(\mathbb{P}) \), where \( \mathbb{P} \) is the random poset, all have a dense \( G_\delta \) conjugacy class (see Truss [47], Kuske-Truss [35]). For more on automorphism groups having dense \( G_\delta \) classes, see also the recent paper Macpherson-Thomas [38]. But the question of whether certain groups have a dense \( G_\delta \) conjugacy class arose as well in topological dynamics, where Akin-Hurley-Kennedy [2] p. 104 posed the problem of the existence of a dense \( G_\delta \) conjugacy class in \( H(2^\omega) \), i.e., the existence of a generic homeomorphism of the Cantor space.

1.2. Model theory. Our goal here is to study these questions in the context of automorphism groups, \( \text{Aut}(K) \), of countable structures \( K \). It is well-known that these groups are (up to topological group isomorphism) exactly the closed subgroups of \( S_\infty \). However, quite often such groups can be densely embedded in other Polish groups \( G \) (i.e., there is an injective continuous homomorphism from \( \text{Aut}(K) \) into \( G \) with dense image) and therefore establishing the existence of a dense conjugacy class of \( \text{Aut}(K) \) implies the same for \( G \). So we can use this method to give simple proofs that such conjugacy classes exist in several interesting Polish groups. This can be viewed as another instance of the idea of reducing questions about the structure of certain Polish groups to those of automorphism groups \( \text{Aut}(K) \) of countable structures \( K \), where one can employ methods of model theory and combinatorics. An earlier use of this methodology is found in Kechris-Pestov-Todorcevic [34] in connection with the study of extreme amenability and its relation with Ramsey theory.

It is also well-known that to every closed subgroup \( G \leq S_\infty \) one can associate the structure \( K_G = (\mathbb{N}, \{ R_{i,n}\}) \), where \( R_{i,n} \subseteq \mathbb{N}^n \) is the \( i \)-th orbit in some fixed enumeration of the orbits of \( G \) on \( \mathbb{N}^n \), which is such that \( G = \text{Aut}(K_G) \) and moreover \( K_G \) is ultrahomogeneous, i.e., every isomorphism between finite substructures of \( K_G \) extends to an automorphism of \( K_G \). Fraissé has analyzed such structures in terms of their finite “approximations.”

To be more precise, let \( \mathcal{K} \) be a class of finite structures in a fixed (countable) signature \( L \), which has the following properties:

(i) (HP) \( \mathcal{K} \) is hereditary (i.e., \( A \leq B \in \mathcal{K} \) implies \( A \in \mathcal{K} \), where \( A \leq B \) means that \( A \) can be embedded into \( B \)),

(ii) (JEP) \( \mathcal{K} \) satisfies the joint embedding property (i.e., if \( A, B \in \mathcal{K} \), there is \( C \in \mathcal{K} \) with \( A, B \leq C \)),

(iii) (AP) \( \mathcal{K} \) satisfies the amalgamation property (i.e., if \( f : A \to B, g : A \to C \), with \( A, B, C \in \mathcal{K} \), are embeddings, there is \( D \in \mathcal{K} \) and embeddings \( r : B \to D, s : C \to D \) with \( r \circ f = s \circ g \)).
(iv) $\mathcal{K}$ contains only countably many structures, up to isomorphism, and contains structures of arbitrarily large (finite) cardinality.

We call any $\mathcal{K}$ that satisfies (i)-(iv) a Fraïssé class. Examples include the class of trivial structures ($L = 0$), graphs, linear orderings, Boolean algebras, metric spaces with rational distances, etc. For any Fraïssé class $\mathcal{K}$ one can define, following Fraïssé [16] (see also Hodges [25]), its so-called Fraïssé limit $\mathbf{K} = \text{Flim}(\mathcal{K})$, which is the unique countably infinite structure satisfying:

(a) $\mathbf{K}$ is locally finite (i.e., finitely generated substructures of $\mathbf{K}$ are finite),
(b) $\mathbf{K}$ is ultrahomogeneous (i.e., any isomorphism between finite substructures of $\mathbf{K}$ extends to an automorphism of $\mathbf{K}$),
(c) $\text{Age}(\mathbf{K}) = \mathcal{K}$, where $\text{Age}(\mathbf{K})$ is the class of all finite structures that can be embedded in $\mathbf{K}$.

A countably infinite structure $\mathbf{K}$ satisfying (a), (b) is called a Fraïssé structure. If $\mathbf{K}$ is a Fraïssé structure, then $\text{Age}(\mathbf{K})$ is a Fraïssé class, therefore the maps $\mathcal{K} \rightarrow \text{Flim}(\mathcal{K})$, $\mathbf{K} \rightarrow \text{Age}(\mathbf{K})$ provide a canonical bijection between Fraïssé classes and structures. Examples of Fraïssé structures include the trivial structure ($\mathbb{N}, =$), $\mathbf{R}$ = the random graph, $(\mathbb{Q}, <)$, $\mathbf{B}_\infty$ = the countable atomless Boolean algebra, $\mathbf{U}_0$ = the rational Urysohn space (= the Fraïssé limit of the class of finite metric spaces with rational distances), etc.

For further reference, we note that condition (b) in the definition of a Fraïssé structure can be replaced by the following equivalent condition, called the extension property: If $\mathbf{A}, \mathbf{B}$ are finite, $\mathbf{A}, \mathbf{B} \leq \mathbf{K}$, and $f : \mathbf{A} \rightarrow \mathbf{K}, g : \mathbf{A} \rightarrow \mathbf{B}$ are embeddings, then there is an embedding $h : \mathbf{B} \rightarrow \mathbf{K}$ with $h \circ g = f$.

Thus every closed subgroup $G \leq S_\infty$ is of the form $G = \text{Aut}(\mathbf{K})$ for a Fraïssé structure $\mathbf{K}$. We study in this paper the question of existence of dense or comeager conjugacy classes in the Polish groups $\text{Aut}(\mathbf{K})$, equipped with the pointwise convergence topology, for Fraïssé structures $\mathbf{K}$, in terms of properties of $\mathbf{K}$, and also derive consequences for other groups.

1.3. Dense conjugacy classes. Following Truss [47], we associate to each Fraïssé class $\mathcal{K}$ the class $\mathcal{K}_p$ of all systems, $\mathcal{S} = \langle \mathbf{A}, \psi : \mathbf{B} \rightarrow \mathbf{C} \rangle$, where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, $\mathbf{B}, \mathbf{C} \subseteq \mathbf{A}$ (i.e., $\mathbf{B}, \mathbf{C}$ are substructures of $\mathbf{A}$) and $\psi$ is an isomorphism of $\mathbf{B}$ and $\mathbf{C}$. For such systems, we can define the notion of embedding as follows: An embedding of $\mathcal{S}$ into $\mathcal{T} = \langle \mathbf{D}, \varphi : \mathbf{E} \rightarrow \mathbf{F} \rangle$ is an embedding $f : \mathbf{A} \rightarrow \mathbf{D}$ such that $f$ embeds $\mathbf{B}$ into $\mathbf{E}$, $\mathbf{C}$ into $\mathbf{F}$ and $f \circ \psi \subseteq \varphi \circ f$. Using this concept of embedding it is clear then what it means to say that $\mathcal{K}_p$ satisfies JEP or AP.

We now have:

**Theorem 1.1.** Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbf{K} = \text{Flim}(\mathcal{K})$. Then the following are equivalent:

(i) There is a dense conjugacy class in $\text{Aut}(\mathbf{K})$.
(ii) $\mathcal{K}_p$ satisfies the JEP.

For example, it is easy to verify JEP of $\mathcal{K}_p$ for the following classes $\mathcal{K}$:

(i) $\mathcal{K}$ = finite metric spaces with rational distances,
(ii) $\mathcal{K}$ = finite Boolean algebras,
(iii) $\mathcal{K}$ = finite measure Boolean algebras with rational measure,
and this immediately gives that $\text{Aut}(\mathbf{U}_0) = \text{Iso}(\mathbf{U}_0)$ (= the isometry group of the rational Urysohn space $\mathbf{U}_0$), $\text{Aut}(\mathbf{B}_\infty)$, $\text{Aut}(\mathbf{F}, \lambda)$ (the automorphism group of the Boolean algebra generated by the rational intervals of $[0, 1]$ with Lebesgue measure).
all have dense conjugacy classes. But \( \text{Aut}(U_0) \) can be densely embedded into \( \text{Iso}(U) \) = the isometry group of the Urysohn space (see Kechris-Pestov-Todorcevic \( [34] \)), \( \text{Aut}(B_\infty) \) is isomorphic to \( H(2^\mathbb{N}) \) by Stone duality, and \( \text{Aut}(F,\lambda) \) can be densely embedded in \( \text{Aut}(X,\mu) \), so we have simple proofs of the following:

**Corollary 1.2.** The following Polish groups have dense conjugacy classes:

(i) \( \text{Iso}(U) \),

(ii) (Glasner-Weiss \( [18] \), Akin-Hurley-Kennedy \( [2] \)) \( H(2^\mathbb{N}) \),

(iii) (Rokhlin) \( \text{Aut}(X,\mu) \).

Glasner and Pestov have also proved part (i). We also obtain a similar result for the (diagonal) conjugacy action of \( G \) on \( G^\mathbb{N} \) for all the above groups \( G \).

We say that a Polish group \( G \) has a cyclically dense conjugacy class if there are \( g, h \in G \) such that \( \{g^nh^{-n}\}_{n\in\mathbb{Z}} \) is dense in \( G \). In this case \( G \) is topologically 2-generated, i.e., has a dense 2-generated subgroup. For example, \( S_\infty \) has this property and so does the automorphism group of the random graph (see Macpherson \( [37] \)). Using a version of Theorem 1, we can give simple proofs that the following groups admit cyclically dense conjugacy classes and therefore are topologically 2-generated: \( H(2^\mathbb{N}) \), \( H(2^\mathbb{N},\sigma) \) (the group of the measure preserving homeomorphisms of \( 2^\mathbb{N} \) with the usual product measure), \( \text{Aut}(X,\mu) \), and \( \text{Aut}(\mathbb{N}^{<\mathbb{N}}) \) (the automorphism group of the infinitely splitting rooted tree).

**1.4. Comeager conjugacy classes.** We now turn to the existence of a dense \( G_\delta \) conjugacy class in \( \text{Aut}(K) \), \( K \) a Fraïssé structure, i.e., the existence of a generic automorphism of \( K \). Truss \( [47] \) showed that the existence of a subclass \( L \subseteq K_p \) such that \( L \) is cofinal under embeddability and satisfies the AP, a property which we will refer to as the cofinal amalgamation property (CAP), together with the JEP for \( K_p \), is sufficient for the existence of a generic automorphism. He also raised the question of whether the existence of a generic automorphism is equivalent to some combination of amalgamation and joint embedding properties for \( K_p \).

Motivated by this problem, we have realized that the question of generic automorphisms is closely related to Hjorth’s concept of turbulence (see, Hjorth \( [24] \) or Kechris \( [32] \)) for the conjugacy action of the automorphism group \( \text{Aut}(K) \), a connection that is surprising at first sight since turbulence is a phenomenon usually thought of as incompatible with actions of closed subgroups of the infinite symmetric group, like \( \text{Aut}(K) \). Once however this connection is realized, it leads naturally to the formulation of an appropriate amalgamation property for \( K_p \) which in combination with JEP is equivalent to the existence of a generic automorphism.

We say that \( K_p \) satisfies the weak amalgamation property (WAP) if for any \( S = \langle A, \psi : B \to C \rangle \in K_p \), there is \( T = \langle D, \varphi : E \to F \rangle \) and an embedding \( e : S \to T \), such that for any embeddings \( f : T \to T_0, g : T \to T_1 \), where \( T_0, T_1 \in K_p \), there is \( U \in K_p \) and embeddings \( r : T_0 \to U, s : T_1 \to U \) with \( r \circ f \circ e = s \circ g \circ e \).

We now have:

**Theorem 1.3.** Let \( K \) be a Fraïssé class and \( K = \text{Flim}(K) \) its Fraïssé limit. Then the following are equivalent:

(i) \( K \) has a generic automorphism.

(ii) \( K_p \) satisfies JEP and WAP.

After obtaining this result, we found out that Ivanov \( [29] \) had already proved a similar theorem, in response to Truss’ question, in a somewhat different context,
that of $\aleph_0$-categorical structures (he also calls WAP the *almost amalgamation property*). Our approach however, through the idea of turbulence, is different and we proceed to explain it in more detail.

Suppose a Polish group $G$ acts continuously on a Polish space $X$. Given $x \in X$, an open nbhd $U$ of $x$ and an open symmetric nbhd $V$ of the identity of $G$, the $(U,V)$-local orbit of $x$, $O(x,U,V)$, is the set of all $y \in U$ for which there is a finite sequence $g_0, g_1, \ldots, g_k \in V$ with $x_0 = x, g_i \cdot x_i = x_{i+1}, x_{k+1} = y$ and $x_i \in U, \forall i$. A point $x$ is *turbulent* if for every $U, V$ as above $\text{Int}(O(x,U,V)) \neq \emptyset$. This turns out to be equivalent to saying that $x \in \text{Int}(O(x,U,V))$, see Kechris [32]. It is easy to see that this only depends on the orbit $G \cdot x$ of $x$, so we can refer to turbulent orbits. This action is called (generically) turbulent if:

(i) Every orbit is meager.

(ii) There is $x \in X$ with dense, turbulent orbit.

(This is not quite the original definition of turbulence, as in Hjorth [24], but it is equivalent to it, see, e.g., Kechris [32].)

Examples of turbulent actions, relevant to our context, include the conjugacy actions of $U(H)$, see Kechris-Sofronidis [33], and $\text{Aut}(X,\mu)$, see Foreman-Weiss [15].

Now Hjorth [24] has shown that no closed subgroup of $S_\infty$ has a turbulent action and from this one has the following corollary, which can be also easily proved directly.

**Proposition 1.4.** Let $G$ be a closed subgroup of $S_\infty$ and suppose $G$ acts continuously on the Polish space $X$. Then the following are equivalent for any $x \in X$:

(i) The orbit $G \cdot x$ is dense $G_\delta$.

(ii) $G \cdot x$ is dense and turbulent.

Thus a dense $G_\delta$ orbit exists iff a dense turbulent orbit exists.

We can now apply this to the conjugacy action of $\text{Aut}(K)$, $K$ a Fraïssé structure, on itself by conjugacy and this leads to the formulation of the WAP and our approach to the proof of Theorem 3. In view of that result, it is interesting that, in the context of this action, the existence of dense, turbulent orbits is equivalently manifested as a combination of joint embedding and amalgamation properties.

1.5. **Homeomorphisms of the Cantor space.** We next use these ideas to answer the question of Akin-Hurley-Kennedy [2] about the existence of a generic homeomorphism of the Cantor space $2^\mathbb{N}$, i.e., the existence of a dense $G_\delta$ conjugacy class in $H(2^\mathbb{N})$. Since $H(2^\mathbb{N})$ is isomorphic (as a topological group) to $\text{Aut}(B_\infty)$, where $B_\infty$ is the countable atomless Boolean algebra (i.e., the Fraïssé limit of the class of finite Boolean algebras), this follows from the following result.

**Theorem 1.5.** Let $\mathcal{B}A$ be the class of finite Boolean algebras. Then $\mathcal{B}A_\omega$ has the CAP. So $B_\infty$ has a generic automorphism and there is a generic homeomorphism of the Cantor space.

Using well-known results, we can also see that there is a generic element of the group of order-preserving homeomorphisms of the interval $[0,1]$.

1.6. **Ample generics.** Finally, we discuss the concept of ample generics in Polish groups, which is a tool that has been used before (see, e.g., Hodges et al. [28]) in the structure theory of automorphism groups. We say that a Polish group $G$
has *ample generic* elements if for each finite \( n \) there is a comeager orbit for the (diagonal) conjugacy action of \( G \) on \( G^n : g \cdot (g_1, \ldots, g_n) = (g_1 g^{-1}, \ldots, g_n g^{-1}) \).

Obviously this is a stronger property than just having a comeager conjugacy class and for example \( \text{Aut}(\mathbb{Q}, \prec) \) has the latter, but not the former (see Kuske and Truss \[35\] for a discussion of this).

There is now an extensive list of permutation groups known to have ample generics. These include the automorphism groups of the following structures: many \( \omega \)-stable, \( \aleph_0 \)-categorical structures (see Hodges et al. \[26\]), the random graph (Hrushovski \[28\], see also Hodges et al. \[26\]), the free group on countably many generators (Bryant and Evans \[9\]) and arithmetically saturated models of true arithmetic (Schmerl \[44\]). Moreover, Herwig and Lascar \[23\] have extended the result of Hrushovski to a much larger class of structures in finite relational languages and the isometry group of the rational Urysohn space \( \mathbb{U}_0 \) is now also known to have ample generics (this follows from recent results of Solecki \[45\] and Vershik).

We will add another two groups to this list, which incidentally are automorphism groups of structures that are not \( \aleph_0 \)-categorical, namely, the group of (Haar) measure preserving homeomorphisms of the Cantor space, \( H(2^\mathbb{N}, \sigma) \), and the group of Lipschitz homeomorphisms of the Baire space. This latter group is canonically isomorphic to \( \text{Aut}(\mathbb{N}^{<\mathbb{N}}) \), where \( \mathbb{N}^{<\mathbb{N}} \) is seen as the infinitely splitting regular rooted tree.

Let us first notice that in the same manner as for the existence of a comeager conjugacy class, we are able to determine an equivalent model-theoretic condition on a Fraïssé class \( \mathcal{K} \) for when the automorphism group of its Fraïssé limit \( \mathcal{K} \) has ample generic elements. This criterion is in fact a trivial generalization of the one dimensional case. But we shall be more interested in the consequences of the existence of ample generics.

Recall that a second countable topological group \( G \) is said to have the small index property if any subgroup of index \( < 2^{\aleph_0} \) is open. Then we can show the following, which generalizes the case of automorphism groups of \( \omega \)-stable, \( \aleph_0 \)-categorical structures due to Hodges et al. \[26\].

**Theorem 1.6.** Let \( G \) be a Polish group with ample generic elements. Then \( G \) has the small index property.

In the case of \( G \) being a closed subgroup of \( S_\infty \), i.e., \( G \) having a neighborhood basis at the identity consisting of clopen subgroups, this essentially says that the topological structure of the group is completely determined by its algebraic structure.

We subsequently study the cofinality of Polish groups. Recall that the cofinality of a group \( G \) is the least cardinality of a well ordered chain of proper subgroups whose union is \( G \). Again generalizing results of Hodges et al. \[26\], we prove:

**Theorem 1.7.** Let \( G \) be a Polish group with ample generic elements. Then \( G \) is not the union of countably many non-open subgroups (or even cosets of subgroups).

Note that if \( G \), a closed subgroup of \( S_\infty \), is oligomorphic, i.e., has only finitely many orbits on each \( \mathbb{N}^n \), then, by a result of Cameron, any open subgroup of \( G \) is contained in only finitely many subgroups of \( G \), thus, if \( G \) has ample generics, it has uncountable cofinality. The same holds for connected Polish groups, and Polish groups with a finite number of topological generators.
It turns out that the existence of generic elements of a Polish group has implications for its actions on trees. So let us recall some basic notions of the theory of group actions on trees (see Serre [43]):

A group $G$ is said to act without inversion on a tree $T$ if $G$ acts on $T$ by automorphisms such that for no $g \in G$ there are two adjacent vertices $a, b \in T$ such that $g \cdot a = b$ and $g \cdot b = a$. The action is said to have a fixed point if there is an $a \in T$ such that $g \cdot a = a$, for all $g \in G$. We say that a group $G$ has property (FA) if whenever $G$ acts without inversion on a tree, there is a fixed point. When $G$ is not countable this is known to be equivalent to the conjunction of the following three properties (Serre [43]):

(i) $G$ is not a non-trivial free product with amalgamation,
(ii) $\mathbb{Z}$ is not a homomorphic image of $G$,
(iii) $G$ has uncountable cofinality.

Macpherson and Thomas [38] recently showed that (i) follows if $G$ has a comeager conjugacy class and, as (ii) trivially also holds in this case, we are left with verifying (iii).

Another way of proving property (FA) is through a slightly different study of the generation of Polish groups with ample generics. Obviously, any generating set for an uncountable group must be uncountable, but ideally we would still like to understand the structure of the group by studying a set of generators. Let us say that a group $G$ has the Bergman property if for each exhaustive sequence of subsets $W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq G$, there are $n$ and $k$ such that $W_k^n = G$. Bergman [7] showed this property for $S_\infty$ by methods very different from those employed here and we will extend his result to a fairly large class of automorphism groups:

**Theorem 1.8.** Let $G$ be a closed oligomorphic subgroup of $S_\infty$ with ample generic elements. Then $G$ has the Bergman property.

In particular, this result applies to many automorphism groups of $\omega$-stable, $\aleph_0$-categorical structures and the automorphism group of the random graph.

It is not hard to see, and has indeed been noticed independently by other authors (e.g., Cornulier [11]) that the Bergman property also implies that any action of the group by isometries on a metric space has bounded orbits. In fact, this is actually an equivalent formulation of the Bergman property. But well-known results of geometric group theory (see B. Bekka, P. de la Harpe and A. Valette [6]) state that if a group action by isometries on a real Hilbert space has a bounded orbit, then it has a fixed point. Similarly for an action by automorphisms without inversion on a tree. Thus Bergman groups automatically have property (FH) and (FA), where property (FH) is the statement that any isometric action on a real Hilbert space has a fixed point.

**Corollary 1.9.** Let $G$ be a closed oligomorphic subgroup of $S_\infty$ with ample generic elements. Then $G$ has properties (FA) and (FH).

The phenomenon of automatic continuity is well known and has been extensively studied, in particular in the context of Banach algebras. In this category morphisms of course preserve much more structure than homomorphisms of the underlying groups and therefore automatic continuity is easier to obtain. But there are also plenty of examples of this phenomenon for topological groups, provided we add some definability constraints on the homomorphisms. An example of this is the classical result of Pettis, saying that any Baire measurable homomorphism from a Polish
group into a separable group is continuous. Surprisingly though, when one assumes ample generics one can completely eliminate any definability assumption and still obtain the same result. In fact, one does not even need as much as separability for the target group, but essentially need only rule out that its topology is discrete. Let us recall that the Souslin number of a topological space is the least cardinal $\kappa$ such that there is no family of $\kappa$ many disjoint open sets. In analogy with this, if $H$ is a topological group, we let the uniform Souslin number of $H$ be the least cardinal $\kappa$ such that there is no non-$\emptyset$ open set having $\kappa$ many disjoint translates. Then we can prove the following:

**Theorem 1.10.** Suppose $G$ is a Polish group with ample generic elements and $\pi : G \to H$ is a homomorphism into a topological group with uniform Souslin number at most $2^{\aleph_0}$ (in particular, if $H$ is separable). Then $\pi$ is continuous.

This in particular shows that any action of a Polish group with ample generics by isometries on a Polish space or by homeomorphisms on a compact metric space is actually a continuous action. For such an action is essentially just a homomorphism into the isometry group, respectively into the homeomorphism group.

Moreover, one also sees that in this case there is a unique Polish group topology, and coupled with a result of Gaughan [19], one can in the case of $S_\infty$ prove the following stronger fact.

**Theorem 1.11.** There is exactly one non-trivial separable group topology on $S_\infty$.

In the literature one can find several results on automatic continuity of homomorphisms between topological groups when one puts restrictions on the target groups. For example, the small index property of a group $G$ can be seen to imply that any homomorphism from $G$ into $S_\infty$ is continuous. Also a classical theorem due to Van der Waerden (see Hofmann and Morris [27]) states that if $\pi : G \to H$ is a group homomorphism from an $n$-dimensional Lie group $G$, whose Lie algebra agrees with its commutator algebra, into a compact group $H$, then $\pi$ is continuous. The final result we should mention is due to Dudley [14], which says that any homomorphism from a complete metric group into a “normed” group with the discrete topology is automatically continuous. We shall not go into his definition of a normed group, other than saying that these include the additive group of the integers and more general free groups. The novelty of Theorem 1.10 lies in the fact that it places no restrictions on the target group (other than essentially ruling out the trivial case of the topology being discrete).

Extending the list of Polish groups with ample generics with $H(2^\mathbb{N},\sigma)$ and $\text{Aut}(\mathbb{N}^{<\mathbb{N}})$, we finally show the following, where a closed subgroup of $S_\infty$ has the strong small index property if any subgroup of index $< 2^{\aleph_0}$ is sandwiched between the pointwise and setwise stabilizer of a finite set.

**Theorem 1.12.** Let $G$ be either $H(2^\mathbb{N},\sigma)$, the group of measure preserving homeomorphisms of the Cantor space, or $\text{Aut}(\mathbb{N}^{<\mathbb{N}})$, the group of Lipschitz homeomorphisms of the Baire space. Then

(i) $G$ has ample generic elements.

(ii) $G$ has the strong small index property.

(iii) $G$ has uncountable cofinality.

(iv) $G$ has the Bergman property and thus properties (FH) and (FA).

The strong small index property for $\text{Aut}(\mathbb{N}^{<\mathbb{N}})$ was previously proved by Rögnvaldur Möller [39].
2. Automorphisms with Dense Conjugacy Classes

Let $\mathcal{K}$ be a Fraïssé class. We let $\mathcal{K}_p$ be the class of all systems of the form $\mathcal{S} = \langle A, \psi : B \to C \rangle$, where $A, B, C \in \mathcal{K}$, $B, C \subseteq A$ and $\psi$ is an isomorphism of $B$ and $C$. An embedding of one system $\mathcal{S} = \langle A, \psi : B \to C \rangle$ into another $\mathcal{T} = \langle D, \phi : E \to F \rangle$ is an embedding $f : A \to D$ such that $f$ embeds $B$ into $E, C$ into $F$ and moreover $f \circ \psi \subseteq \phi \circ f$.

So we can define JEP and AP for $\mathcal{K}_p$ as well.

Theorem 2.1. Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathcal{K}$. Then the following are equivalent:

(i) There is a dense conjugacy class in $\text{Aut}(\mathcal{K})$.

(ii) $\mathcal{K}_p$ satisfies the JEP.

Proof. (i) $\Rightarrow$ (ii): Fix some element $f \in \text{Aut}(\mathcal{K})$ having a dense conjugacy class in $\text{Aut}(\mathcal{K})$ and suppose $\mathcal{S} = \langle A, \psi : B \to C \rangle, \mathcal{T} = \langle D, \phi : E \to F \rangle$ are two systems in $\mathcal{K}_p$.

Replacing $\mathcal{S}$ and $\mathcal{T}$ by isomorphic copies we can of course assume that $A, D \subseteq \mathcal{K}$. Now by ultrahomogeneity of $\mathcal{K}$ we know that both $\psi$ and $\phi$ have extensions in $\text{Aut}(\mathcal{K})$, so, by the density of $f$’s conjugacy class, there are $g_1, g_2 \in \text{Aut}(\mathcal{K})$ such that $\psi \subseteq g_1^{-1} f g_1$ and $\phi \subseteq g_2^{-1} f g_2$. Let $\mathcal{H} = \langle g_1'' A \cup g_2'' D \rangle, \mathcal{M} = \langle g_1'' B \cup g_2'' E \rangle, \mathcal{N} = \langle g_1'' C \cup g_2'' F \rangle$ and let $\chi = f \upharpoonright \mathcal{M}$. Then it is easily seen that $g_1 \upharpoonright A$ and $g_2 \upharpoonright D$ embed $\mathcal{S}$ and $\mathcal{T}$ into $\langle \mathcal{H}, \chi : \mathcal{M} \to \mathcal{N} \rangle$.

(ii) $\Rightarrow$ (i): A basis for the open subsets of $\text{Aut}(\mathcal{K})$ consists of sets of the form:

$$[\psi] = \{ f \in \text{Aut}(\mathcal{K}) : f \supseteq \psi \},$$

where $\psi : B \to C$ is an isomorphism of finite substructures of $\mathcal{K}$. We refer to such $\psi : B \to C$ as conditions. We wish to construct an element $f \in \text{Aut}(\mathcal{K})$ such that for any condition $\psi : B \to C$ there is a $g \in \text{Aut}(\mathcal{K})$ such that $g \psi g^{-1} \subseteq f$. So let

$$D(\psi : B \to C) = \{ f \in \text{Aut}(\mathcal{K}) : \exists g \in \text{Aut}(\mathcal{K})(g \psi g^{-1} \subseteq f) \}. $$

$D(\psi : B \to C)$ is clearly open, but we shall see that it is also dense. If $[\phi : E \to F]$ is a basic open set, then by the JEP of $\mathcal{K}_p$ and the extension property of $\mathcal{K}$, some isomorphism $\chi : H \to L$ of finite substructures of $\mathcal{K}$ such that $\phi \subseteq \chi$, and some $g \in \text{Aut}(\mathcal{K})$ embedding $\psi : B \to C$ into $\chi : H \to L$. Then any $f \in \text{Aut}(\mathcal{K})$ extending $\chi$ witnesses that $D(\psi : B \to C) \cap [\phi : E \to D] \neq \emptyset$. Therefore any $f \in \text{Aut}(\mathcal{K})$ that is in the intersection of all $D(\psi : B \to C)$ has dense conjugacy class.

Remark. One can also give a more direct proof of Theorem 1.1 by using the following standard fact: If $G$ is a Polish group which acts continuously on a Polish space $X$ and $\mathcal{B}$ is a countable basis of non-$\emptyset$ open sets for the topology of $X$, then there is a dense orbit for this action iff $\forall U, V \in B(G \cdot U \cap V \neq \emptyset)$. (The direction $\Rightarrow$ is obvious. For the direction $\Leftarrow$, let for $V \in \mathcal{B}, D_V = \{ x \in X : G \cdot x \cap V \neq \emptyset \}$. This is clearly open, dense, and so $\bigcap_{V \in \mathcal{B}} D_V \neq \emptyset.$ Any $x \in \bigcap_{V \in \mathcal{B}} D_V$ has dense orbit.)

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We can simply apply this to the conjugacy action of \( \text{Aut}(K) \) on itself and the basis consisting of the sets \([\psi]\) as above.

We will proceed to some applications of Theorem 2.1.

**Theorem 2.2.** Let \( U_0 \) be the rational Urysohn space. Then \( \text{Aut}(U_0) \) has a dense conjugacy class. Thus, as \( \text{Aut}(U_0) \) can be continuously, densely embedded into \( \text{Iso}(U) \), the isometry group of the Urysohn space, \( \text{Iso}(U) \) also has a dense conjugacy class.

**Proof.** Let \( K \) be the class of finite metric spaces with rational distances; we will see that \( K_p \) has the JEP.

Given \( S = \langle A, \psi : B \rightarrow C \rangle \) and \( T = \langle D, \phi : E \rightarrow F \rangle \in K_p \), we let \( H = A \cup B \) be the disjoint union of the two metric spaces \( A \) and \( B \), where we have put some distance \( k > \text{diam}(A)+ \text{diam}(B) \) between any two points \( x \in A \) and \( y \in B \). The triangle inequality is clearly satisfied. So now we can let \( M \) and \( N \) be the corresponding unions of \( B, E \) and \( C, F \), and finally we let \( \chi = \psi \cup \varphi \). Then clearly both \( S \) and \( T \) embed into \( \langle H, \chi : M \rightarrow N \rangle \).

Glasner and Pestov informed us that they had also proved that \( \text{Iso}(U) \) has a dense conjugacy class.

We next consider the class \( K = \mathcal{MBA}_Q \) of all finite boolean algebras with an additive probability measure taking positive rational values on each nonzero element. So an element of \( K \) is on the form \( \langle A, \mu : A \rightarrow [0,1] \cap \mathbb{Q} \rangle \), but, in order to get into first order model theory, we will view \( \mu \) as a collection of unary predicates on \( A, \{M_r \}_{r \in [0,1] \cap \mathbb{Q}} \), where \( M_r(x) \Leftrightarrow \mu(x) = r \).

**Proposition 2.3.** \( K \) is a Fraïssé class.

**Proof.** Obviously \( K \) has the HP. Now as all structures in \( K \) have a common substructure (the trivial boolean algebra \( \{0,1\} \)) it is enough to verify AP, from which JEP will follow.

So suppose \( f : \langle A, \mu \rangle \rightarrow \langle B, \nu \rangle \) and \( g : \langle A, \mu \rangle \rightarrow \langle C, \rho \rangle \) are embeddings. Let \( a_1, \ldots, a_n \) be the atoms of \( A, b_1, \ldots, b_m \) the atoms of \( B, c_1, \ldots, c_k \) the atoms of \( C \) and let \( \Gamma_1 \sqcup \cdots \sqcup \Gamma_n \) partition \( \{1, \ldots, m\}, \Lambda_1 \sqcup \cdots \sqcup \Lambda_n \) partition \( \{1, \ldots, k\} \) such that \( f(a_i) = \bigvee_{j \in \Gamma_i} b_j \) and \( g(a_i) = \bigvee_{j \in \Lambda_i} c_j \). Let \( D \) be the boolean algebra with formal atoms \( b_i \otimes c_j \) for \( (i, j) \in \Gamma_1 \times \Lambda_1, l \leq n \), and let the embeddings \( e : B \rightarrow D, h : C \rightarrow D \) be defined by

\[
\begin{align*}
e(b_i) &= \bigvee_{j \in \Lambda_i} b_i \otimes c_j, \text{ where } i \in \Gamma_1, \\
h(c_j) &= \bigvee_{i \in \Gamma_i} b_i \otimes c_j, \text{ where } j \in \Lambda_i.
\end{align*}
\]

This is the usual amalgamation of boolean algebras and we now only have to check that we can define an appropriate measure \( \sigma \) on \( D \) to finish the proof.

Notice first that \( \mu(a_i) = \sum_{i \in \Gamma_1} \nu(b_i) = \sum_{j \in \Lambda_1} \rho(c_j) \), so the appropriate measure is \( \sigma(b_i \otimes c_j) = \frac{\nu(b_i) \rho(c_j)}{\mu(a_i)} \), for \( (i, j) \in \Gamma_1 \times \Lambda_1 \). Then \( e \) and \( h \) are indeed embeddings, as, e.g.,

\[
\sigma(e(b_i)) = \sigma(\bigvee_{j \in \Gamma_i} b_i \otimes c_j) = \sum_{j \in \Gamma_i} \frac{\nu(b_i) \rho(c_j)}{\mu(a_i)} = \nu(b_i).
\]
We can now identify the Fraïssé limit of the class $\mathcal{K}$. It follows from Fraïssé’s construction that the limit must be the countable atomless boolean algebra equipped with some finitely additive probability measure. Let $F$ be the boolean algebra generated by the rational intervals in the measure algebra of $([0,1], \lambda)$, where $\lambda$ is Lebesgue measure, together with the restriction of Lebesgue measure.

**Proposition 2.4.** $(F, \lambda)$ is the Fraïssé limit of $\mathcal{K}$.

**Proof.** The age of $(F, \lambda)$ is clearly $\mathcal{K}$, so we need only check that $(F, \lambda)$ has the extension property. So suppose $(A, \mu) \subseteq (B, \nu) \in \mathcal{K}$ and $f : A \to F$ is an embedding of $(A, \mu)$ into $(F, \lambda)$. Then we can extend $f$ to an embedding $\hat{f}$ of $(B, \nu)$ into $(F, \lambda)$ as follows:

An atom $a$ of $A$ corresponds by $f$ to a finite disjoint union of rational intervals in $[0,1]$ and is also the join of atoms $b_1, \ldots, b_k$ in $B$ with rational measure. So by appropriately decomposing these rational intervals into finitely many pieces we can find images for each of $b_1, \ldots, b_k$ of the same measure. \qed

If we let $\mathcal{K} = \mathcal{MA}_Q$ be the class of finite boolean algebras with a measure that takes positive dyadic rational values on each non-0 element, then one can show (though it is more complicated than the preceding proof) that $\mathcal{K}$ is a Fraïssé class with limit $(\text{cl}(2^N), \sigma)$, where $\sigma$ is the usual product measure on $2^N$. Moreover, Aut$(\text{cl}(2^N), \sigma)$ is (isomorphic to) the group of measure preserving homeomorphisms of $2^N$ (with the uniform convergence topology), $H(2^N, \sigma)$.

We now have:

**Theorem 2.5.** There is a dense conjugacy class in Aut$(F, \lambda)$.

**Proof.** We verify that $\mathcal{K}_p$ has the JEP. So suppose $(A, \mu, \psi : B \to C)$ and $(D, \nu, \phi : E \to H)$ are given, where $A \supseteq B, C$ and $D \supseteq E, H$ are finite boolean algebras with rational valued probability measures $\mu, \nu$ and $\psi$ and $\phi$ are isomorphisms preserving the measure. We amalgamate $(A, \mu)$ and $(D, \nu)$ over the trivial boolean algebra as in the proof of Proposition [2.4] so our atoms in the new algebra $J$ are $a_i \otimes d_j$, where $a_1, \ldots, a_n$ and $d_1, \ldots, d_m$ are the atoms at $A$ and $D$, respectively, and $i \leq n, j \leq m$. Find partitions

$$\Gamma_1 \sqcup \cdots \sqcup \Gamma_k = \Lambda_1 \sqcup \cdots \sqcup \Lambda_k = \{1, \ldots, n\}$$

and

$$\Delta_1 \sqcup \cdots \sqcup \Delta_l = \Theta_1 \sqcup \cdots \sqcup \Theta_l = \{1, \ldots, m\}$$

such that $\bigvee_{\Gamma_i} a_i$, $\bigvee_{\Lambda_i} a_i$, $\bigvee_{\Delta_j} d_j$, $\bigvee_{\Theta_j} d_j$ are the atoms of $B, C, E, H$ respectively and

$$\psi(\bigvee_{\Gamma_i} a_i) = \bigvee_{\Lambda_i} a_i, \quad \phi(\bigvee_{\Delta_j} d_j) = \bigvee_{\Theta_j} d_j.$$ 

Then we let $M$ be the subalgebra of $J$ generated by the atoms $\bigvee_{\Gamma_i \times \Delta_j} a_i \otimes d_j$, $N$ be the subalgebra generated by the atoms $\bigvee_{\Lambda_i \times \Theta_j} a_i \otimes d_j$ and $\chi : M \to N$ the isomorphism given by

$$\chi\left(\bigvee_{\Gamma_i \times \Delta_j} a_i \otimes d_j\right) = \bigvee_{\Lambda_i \times \Theta_j} a_i \otimes d_j.$$ 

Then $(J, \chi : M \to N)$ clearly amalgamates $(A, \psi : B \to C)$ and $(D, \phi : E \to H)$ over the trivial boolean algebra, so the only thing we need to check is that $\chi$
preserves the measure \( \sigma \) on \( J \) given by \( \sigma(a_i \otimes d_j) = \mu(a_i) \cdot \nu(d_j) \):

\[
\begin{align*}
\sigma\left( \bigvee_{i \in \Gamma_e \times \Delta_f} a_i \otimes d_j \right) &= \sum_{i \in \Gamma_e} \sum_{j \in \Delta_f} \mu(a_i) \nu(d_j) \\
&= \left( \sum_{i \in \Gamma_e} \mu(a_i) \right) \cdot \left( \sum_{j \in \Delta_f} \nu(d_j) \right) = \left( \sum_{i \in \Lambda_e} \mu(a_i) \right) \cdot \left( \sum_{j \in \Theta_f} \nu(d_j) \right) \\
&= \sum_{i \in \Lambda_e} \sum_{j \in \Theta_f} \mu(a_i) \nu(d_j) = \sigma\left( \bigvee_{\Lambda_e \times \Theta_f} a_i \otimes d_j \right)
\end{align*}
\]

By using \( M_{BA}Q_2 \) instead of \( M_{BA}Q \), we obtain the same result for \( H(2^N, \sigma) \).

Let now \( K = BA \) be the class of all finite Boolean algebras. It is well-known that \( K \) is a Fraïssé class (the argument is essentially that of Proposition 2.3, forgetting about the measures), whose Fraïssé limit is the countable atomless Boolean algebra \( B_\infty \). As in the proof of 2.5 we can easily verify the JEP for \( K_p \), so we have:

**Theorem 2.6.** There is a dense conjugacy class in \( \text{Aut}(B_\infty) \).

From this we immediately deduce:

**Corollary 2.7.** (Glasner-Weiss [15], Akin-Hurley-Kennedy [2]) In the uniform topology there is a dense conjugacy class on \( H(2^N) \) (the group of homeomorphisms of the Cantor space).

Note that Corollary 2.7 has as a consequence the known fact that the aperiodic homeomorphisms form a dense \( G_\delta \) in \( H(2^N) \). This is because for each \( n > 0 \), \( \{ h \in H(2^N) : \exists x h^n(x) = x \} \) is closed and conjugacy invariant, so it has empty interior. Thus, \( \text{APER} = \{ h \in H(2^N) : \forall x \forall n > 0 h^n(x) \neq x \} \) is dense \( G_\delta \).

Consider now the group \( \text{Aut}(I, \lambda) \) of measure preserving automorphisms of the unit interval \( I \) with Lebesgue measure \( \lambda \). By a theorem of Sikorski (see Kechris [31]) this is the same as the group of automorphisms of the measure algebra of \((I, \lambda)\).

Every element \( \varphi \in \text{Aut}(F, \lambda) \) induces a unique automorphism \( \varphi^* \) of the measure algebra, as \( F \) is dense in the latter. And therefore \( \varphi^* \) can be seen as an element of \( \text{Aut}(I, \lambda) \).

It is not hard to see that the mapping \( \varphi \mapsto \varphi^* \) is a continuous injective homomorphism from \( \text{Aut}(F, \lambda) \) into \( \text{Aut}(I, \lambda) \). Moreover, the image of \( \text{Aut}(F, \lambda) \) is dense in \( \text{Aut}(I, \lambda) \). To see this, use for example the following fact (see Halmos [22]): the set of \( f \in \text{Aut}(I, \lambda) \), such that for some \( n \), \( f \) is just shuffling the dyadic intervals at length \( 2^n \cdot \left[ \frac{1}{2^n}, \frac{2^n + 1}{2^n} \right] \) (linearly on each interval), is dense in \( \text{Aut}(I, \lambda) \). Obviously any such \( f \) is the image of some element of \( \text{Aut}(F, \lambda) \).

**Corollary 2.8.** \( \text{Aut}(I, \lambda) \) has a dense conjugacy class.

This is of course a weaker version of the conjugacy lemma of ergodic theory which asserts that the conjugacy class of any aperiodic transformation is dense in \( \text{Aut}(I, \lambda) \). But the above proof may be of some interest as it avoids the use of Rokhlin’s Lemma.

Finally, let \( K \) be the Fraïssé class of all finite linear orderings. Trivially \( K_p \) has the JEP, so the automorphism group of \((Q, <)\) has a dense conjugacy class. But \( \text{Aut}(Q, <) \) can be densely and continuously embedded into \( H_+(\mathbb{R}, [0, 1]) \), the group of
order preserving homeomorphisms of the unit interval with the uniform topology. So we have the following:

**Corollary 2.9.** (Glasner-Weiss \[18\]) There is a dense conjugacy class in $H_+(\{0,1\})$.

Obviously there cannot be any dense conjugacy class in $H(\{0,1\})$, as it has a proper clopen normal subgroup, $H_+(\{0,1\})$.

In the same manner one can easily check that for $K = \text{the class of finite graphs, finite hypergraphs, finite posets, etc.}, K_p$ has the JEP. So the automorphism group of the Fra"issé limit, e.g., the random graph, has a dense conjugacy class.

**Remark.** For an example of a Fra"issé class $K$ for which $K_p$ does not have the JEP, consider the class of finite equivalence relations with at most two equivalence classes. Its Fra"issé limit is $(\mathbb{N}, E)$, where $nEm \leftrightarrow \text{parity}(n) = \text{parity}(m)$. So any $f \in \text{Aut}(\mathbb{N}, E)$ either fixes each class setwise or permutes the two classes. Therefore the group of $f$ setwise fixing the classes is a clopen normal subgroup (of index 2) and there cannot be any dense conjugacy class in $\text{Aut}(\mathbb{N}, E)$. So the JEP does not hold for $K_p$ and the counterexample is of course two equivalence relations with two classes each and two automorphisms, one of which fixes the two classes and the other switches them.

In certain situations one can obtain more precise information concerning dense conjugacy classes, which also has further interesting consequences.

Suppose $G$ is a Polish group. We say that $G$ has cyclically dense conjugacy classes if there are $g, h \in G$ such that $\{g^n hg^{-n}\}_{n \in \mathbb{Z}}$ is dense in $G$.

Notice that the set

$$D = \{(g, h) \in G^2 : \{g^n hg^{-n}\}_{n \in \mathbb{Z}} \text{ is dense in } G\}$$

is $G_\delta$. Also, if some section $D_g = \{h \in G : \{g^n hg^{-n}\}_{n \in \mathbb{Z}} \text{ is dense in } G\}$ is non-empty, then it is dense in $G$. Moreover, as $D_{fgf^{-1}} = fD_g f^{-1}$, the set of $g \in G$ for which $D_g \neq \emptyset$ is conjugacy invariant. So if there is some $g$ with a dense conjugacy class such that $D_g \neq \emptyset$, then there is a dense set of $f \in G$ for which $D_f$ is dense and hence $D$ itself is dense and thus comeager in $G^2$.

For each Polish group $G$, we denote by $n(G)$ the smallest number of topological generators of $G$, i.e., the smallest $1 \leq n \leq \infty$ such that there is an $n$-generated dense subgroup of $G$. Thus $n(G) = 1$ iff $G$ is monothetic. It follows immediately that if $G$ admits a cyclically dense conjugacy class, then $n(G) \leq 2$.

Consider now a Fra"issé structure $K$ and suppose we can find $g \in \text{Aut}(K)$ with the following property:

(*) For any finite $A, B \subseteq K$ and any isomorphisms $\varphi : C \to D, \psi : E \to F$, where $C, D \subseteq A; E, F \subseteq B$, there are $m, n \in \mathbb{Z}$ such that $g^m \varphi g^{-m}, g^n \psi g^{-n}$ have a common extension.

Then, as in the proof of Theorem 1.1, we see that there is $h \in \text{Aut}(K)$ with $\{g^n hg^{-n}\}$ dense in $\text{Aut}(K)$, i.e., $K$ admits a cyclically dense conjugacy class, and moreover $\text{Aut}(K)$ is topologically 2-generated.

Macpherson \[37\] 3.3 has shown that the automorphism group of the random graph has cyclically dense conjugacy classes and so is topologically 2-generated. We note below that $\text{Aut}(\mathbb{B}_w)$, and thus $H(\mathbb{B}^w)$, as well as $H(\mathbb{B}^w, \sigma)$, $\text{Aut}(X, \mu)$, $\text{Aut}(\mathbb{N}^{<\mathbb{N}})$ admit cyclically dense conjugacy classes, so in particular, they are topologically 2-generated. That $\text{Aut}(X, \mu)$ is topologically 2-generated was earlier proved by different means in Grzaslewicz \[20\] and Prasad \[11\].
\begin{theorem}
Each of the groups \( H(2^\mathbb{N}) \), \( H(2^\mathbb{N}, \sigma) \), \( \text{Aut}(X, \mu) \), and \( \text{Aut}(\mathbb{N}^{<\mathbb{N}}) \) has cyclically dense conjugacy classes and is topologically 2-generated.
\end{theorem}

\begin{proof}
We will sketch the proof for \( H(2^\mathbb{N}) \) or rather its isomorphic copy \( \text{Aut} (\mathcal{B}_\infty) \). The proofs in the cases of \( H(2^\mathbb{N}, \sigma) \) and \( \text{Aut}(\mathbb{N}^{<\mathbb{N}}) \) are similar. Since \( H(2^\mathbb{N}, \sigma) \) can be densely embedded in \( \text{Aut}(X, \mu) \), the result follows for this group as well.

It is clear (see the proof of \[0.2\]) that it is enough to show the following, in order to verify property (*) for \( K = \mathcal{B}_\infty \): There is an automorphism \( g \) such that given any finite subalgebras \( A, B \), there is \( n \in \mathbb{Z} \) with \( g^n (A), B \) independent (i.e., any non-zero elements of \( g^n(A), B \) have non-zero join).

To find such a \( g \) view \( \mathcal{B}_\infty \) as the algebra of clopen subsets of \( 2^\mathbb{Z} \) and take \( g \) to be the Bernoulli shift on this space. \( \Box \)

It is not hard to show that also \( U(\ell_2) \) has a cyclically dense conjugacy class.

Solecki has recently shown that the group of isometries of the Urysohn space admits a cyclically dense conjugacy class and thus is topologically 2-generated.

\begin{remarks}
Prasad \[11\] showed that there is a comeager set of pairs \((g, h)\) in \( \text{Aut}(X, \mu)^2 \) generating a dense subgroup of \( \text{Aut}(X, \mu) \) (moreover, using the results of section \[0.6.6\] one sees easily that generically the subgroup generated is free). This also follows from our earlier remarks, since the \( g \) that witnesses the cyclically dense conjugacy class is in this case the shift, whose conjugacy class is dense (by the conjugacy lemma of ergodic theory).

We can also consider the diagonal conjugacy action of \( \text{Aut}(K) \) on \( \text{Aut}(K)^{\mathbb{N}} \), for \( n = 1, 2, \ldots, \mathbb{N} \). Note that there is a dense diagonal conjugacy class in \( \text{Aut}(K)^{\mathbb{N}} \) iff for each \( n = 1, 2, \ldots \), there is a dense diagonal conjugacy class in \( \text{Aut}(K)^n \). This follows from the fact that in the latter case, the set of elements \( (g_m) \in \text{Aut}(K)^{\mathbb{N}} \), whose diagonal conjugacy class is dense in \( \text{Aut}(K)^n \), is dense \( G_\delta \) for all \( n \), and hence we can pick a \( (g_m) \) such that it holds for all \( n \) and therefore also for \( \mathbb{N} \).

Consider a Fraïssé class \( K \) with Fraïssé limit \( K \) and let \( G = \text{Aut}(K) \). Define for each \( n \geq 1 \) the following sets (which are all \( G_\delta \)).

\[
F_n = \{(f_1, \ldots, f_n) \in G^n : \forall x \in K \ x \text{'s orbit under } \langle f_1, \ldots, f_n \rangle \text{ is finite} \}
\]
\[
D_n = \{(f_1, \ldots, f_n) \in G^n : \langle f_1, \ldots, f_n \rangle \text{ is relatively compact in } G \}
\]
\[
E_n = \{(f_1, \ldots, f_n) \in G^n : \langle f_1, \ldots, f_n \rangle \text{ is non-discrete in } G \}
\]
\[
H_n = \{(f_1, \ldots, f_n) \in G^n : \langle f_1, \ldots, f_n \rangle \text{ freely generates a free subgroup of } G \}
\]

Since the sets \( F_n, D_n, E_n \) and \( H_n \) are \( G_\delta \) sets invariant under the diagonal conjugacy action of \( G \) on \( G^n \), it follows that if \( G^n \) has a dense diagonal conjugacy class then each of them is either comeager or nowhere dense in \( G^n \). We also see that \( D_n \) is never dense unless \( G = \{1\} \), for if \( H < G \) is a proper open subgroup and \( (g_1, \ldots, g_n) \in H^n \), then \( (g_1, \ldots, g_n) \subseteq H \) and hence is not dense. So \( D_n \cap H^n = \emptyset \). On the other hand, if \( K \) satisfies the Hrushovski property (see Definition \[0.6.3\]), then \( F_n \) is automatically comeager. Moreover, unless \( G \) itself is compact, the sets \( F_n \) and \( D_n \) are of course disjoint and thus if also \( K \) has the Hrushovskii property, \( D_n \) is nowhere dense.

These results can also be combined with the comments in section \[0.6.6\] where we will study the sets \( H_n \).
Suppose $\mathcal{K}$ is a Fraïssé class, and $\mathbf{K}$ its Fraïssé limit. We would like to characterize as before when Aut$(\mathbf{K})^n$ has a dense diagonal conjugacy class. For this we introduce the class of $n$-systems $\mathcal{K}^n$ for each $n \geq 1$. An $n$-system $\mathcal{S} = \langle A, \psi_1 : B_1 \rightarrow C_1, \ldots, \psi_n : B_n \rightarrow C_n \rangle$ consists of finite structures $A, B_1, C_1 \in \mathcal{K}$ with $B_1, C_1 \subseteq A$ and $\psi_i$ an isomorphism of $B_i$ and $C_i$. As before, an embedding of one $n$-system $\mathcal{S} = \langle A, \psi_1 : B_1 \rightarrow C_1, \ldots, \psi_n : B_n \rightarrow C_n \rangle$ into another $T = \langle D, \phi_1 : E_1 \rightarrow F_1, \ldots, \phi_n : E_n \rightarrow F_n \rangle$ is a function $f : A \rightarrow D$ embedding $A$ into $D$, $B_i$ into $E_i$ and $C_i$ into $F_i$ such that $f \circ \psi_i \subseteq \phi_i \circ f, i = 1, \ldots, n$. So we can talk about JEP, WAP, etc., for $n$-systems as well.

We can prove exactly as before:

**Theorem 2.11.** Let $\mathcal{K}$ be a Fraïssé class and $\mathbf{K}$ its Fraïssé limit. Then the following are equivalent:

(i) There is a dense diagonal conjugacy class in Aut$(\mathbf{K})^n$,

(ii) $\mathcal{K}^n$ has the JEP.

From this we easily have the following:

**Theorem 2.12.** There is a dense diagonal conjugacy class in each of $H\langle 2^n \rangle, H\langle 2^n, \sigma^n \rangle, \text{Aut}(X, \mu)|^n, \text{Aut}(\mathbb{N}^{<\omega})|^n, \text{Aut}(\mathbb{U}_0)^n, \text{Iso}(\mathbb{U})^n$.

**Definition 2.13.** Let $\mathcal{K}$ be a Fraïssé class and let $\mathcal{K}_p$ be the corresponding class of systems. We say that $\mathcal{K}_p$ satisfies the cofinal joint embedding property (CJEP) if for each $A \in \mathcal{K}$ there is $A \leq B \in \mathcal{K}$ such that for $J_B = \langle B, id : B \rightarrow B \rangle$, any systems $T_0, T_1$ and embeddings $e : J_B \rightarrow T_0, f : J_B \rightarrow T_1$, there is a system $R$ and embeddings $g : T_0 \rightarrow R, h : T_1 \rightarrow R$ such that $g \circ e = h \circ f$.

This property is enough to ensure that if $\mathbf{K}$ is the Fraïssé limit of $\mathcal{K}$, then Aut$(\mathbf{K})$ has a neighborhood basis at the identity consisting of clopen subgroups with a dense conjugacy class.

**Theorem 2.14.** Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbf{K}$ and suppose that $\mathcal{K}_p$ satisfies the CJEP. Then for any finite substructure $A \subseteq \mathcal{K}$, there is a finite substructure $A \subseteq B \subseteq \mathbf{K}$ such that Aut$(\mathbf{K})_{[B]} = \{ g \in \text{Aut}(\mathbf{K}) : g | [B] = id_B \}$ has a dense conjugacy class.

**Proof.** Let $A \subseteq \mathbf{K}$ be given and find by the CJEP some $A \leq B \in \mathcal{K}$ satisfying the conditions of the definition. By ultrahomogeneity of $\mathcal{K}$, we can suppose that $A \subseteq B \subseteq \mathbf{K}$. Let $\vec{b} = (b_1, \ldots, b_n)$ be new names for the elements of $B$ and let $(\mathcal{K}, \vec{b})$ be the corresponding expansion of $\mathbf{K}$. Notice that Aut$(\mathcal{K}, \vec{b}) = \text{Aut}(\mathbf{K})_{[B]}$.

Similarly, we let $(\mathcal{K}, \vec{b})$ be the class of expanded structures $(D, \vec{b})$, where $B \leq D \in \mathcal{K}$ and $b_1, \ldots, b_n$ are the names for some fixed copy of $B$ in $D$. We claim that (i) $(\mathcal{K}, \vec{b})$ is a Fraïssé class and (ii) $(\mathcal{K}, \vec{b})_p$ has the JEP.

First (i): The HP for $(\mathcal{K}, \vec{b})$ follows at once from the HP of $\mathcal{K}$. Now, if $(C, \vec{b}), (D, \vec{b}), (E, \vec{b}) \in (\mathcal{K}, \vec{b})$ and $e, f$ are embeddings of $(C, \vec{b})$ into $(D, \vec{b})$ and $(C, \vec{b})$ into $(E, \vec{b})$ resp., then $e, f$ embed $C$ into $D$ and $C$ into $E$ resp. So by the AP for $\mathcal{K}$ there is a structure $F \in \mathcal{K}$ and embeddings $g : D \rightarrow F, h : E \rightarrow F$ such that $g \circ e = h \circ f$.

In particular, $g \circ e (b_i^F) = h \circ f (b_i^C)$ for $i = 1, \ldots, n$ and so we can expand $F$ by setting $b_i^F = g \circ e (b_i^C)$. Then $g$ and $h$ embed $(D, \vec{b})$ and $(E, \vec{b})$ into $(F, \vec{b})$ and $g \circ e = h \circ f$. So $(\mathcal{K}, \vec{b})$ has the AP. Finally, the JEP follows for $(\mathcal{K}, \vec{b})$ from the AP, since all structures have a common substructure, namely the one generated by $b_1, \ldots, b_n$.
Now for (ii): Suppose $S, T \in (\mathcal{K}, \mathcal{B})_p$. Then for some $D, E \subseteq C, G, H \subseteq F$,

\[ S = \langle (C, \psi) : (D, \mathcal{B}) \rightarrow (E, \mathcal{B}) \rangle \]

and

\[ T = \langle (F, \phi) : (G, \mathcal{B}) \rightarrow (H, \mathcal{B}) \rangle. \]

But as $\psi$ and $\phi$ both preserve $b_1, \ldots, b_n$, this means that $\mathcal{J}_B$ embeds into both $S = \langle C, \psi : D \rightarrow E \rangle$ and $T = \langle F, \phi : G \rightarrow H \rangle$. Therefore there is an amalgamation of $S$ and $T$, which can be expanded to a common super-system of $S$ and $T$.

We now only need to show that the Fraïssé limit of $(\mathcal{K}, \mathcal{B})$ is $(\mathcal{K}, \mathcal{B})$. But for this it is enough to notice that $(\mathcal{K}, \mathcal{B})$ is still ultrahomogeneous as $\mathcal{K}$ is, and $\text{Age}(\mathcal{K}, \mathcal{B}) = (\mathcal{K}, \mathcal{B})$ as $\text{Age}(\mathcal{K}) = \mathcal{K}$ and $\mathcal{K}$ is ultrahomogeneous. So by Fraïssé’s theorem on the uniqueness of the Fraïssé limit we have the result. For now we can just apply Theorem 2.1 to $(\mathcal{K}, \mathcal{B})$ with limit $(\mathcal{K}, \mathcal{B})$. $\square$

In any natural instance it is certainly easy to verify the CJEP, for example, it is true in any of the cases considered in this paper. However, we have no general theorem saying that it should follow from the existence of a comeager conjugacy class, and probably it does not.

### 3. Generic automorphisms

We will now consider the question of when $\text{Aut}(\mathcal{K})$ has a comeager conjugacy class, when $\mathcal{K}$ is the Fraïssé limit of some Fraïssé class $\mathcal{K}$.

Now it is known, by a theorem due to Effros, Marker and Sami (see, e.g., Becker-Kechris [5]), that any non-meager orbit of a Polish group acting continuously on a Polish space is in fact $G_\delta$. So we are actually looking for criteria for when there is a dense $G_\delta$ orbit.

Recall now some basic facts about Hjorth’s notion of turbulence (see Hjorth [24] or Kechris [32]).

Suppose a Polish group $G$ acts continuously on a Polish space $X$. A point $x \in X$ is said to be turbulent if for every open neighborhood $U$ of $x$ and every symmetric open neighborhood $V$ of the identity $e \in G$, the local orbit $O(x, U, V)$, is somewhere dense, i.e.,

\[ \text{Int}(O(x, U, V)) \neq \emptyset. \]

Here

\[ O(x, U, V) = \{ y \in X : \exists g_0, \ldots, g_k \subseteq V \forall i \leq k : (g_k \cdots g_1 \cdot x \subseteq U \text{ and } g_k g_{k-1} \cdots g_0 \cdot x = y) \}. \]

Notice that the property of being turbulent is $G$-invariant (see Kechris [32] 8.3), so we can talk about turbulent orbits. We also have the following fact (see Kechris [32] 8.4):

**Proposition 3.1.** Let a Polish group $G$ act continuously on a Polish space $X$ and $x \in X$. The following are equivalent:

(i) $x$ is turbulent.

(ii) For every open neighborhood $U$ of $x$ and every symmetric open neighborhood $V$ of $e \in G$, $x \in \text{Int}(O(x, U, V))$. 

Notice that if $V$ is an open subgroup of $G$, then $O(x, U, V) = U \cap V \cdot x$. So if $G$ is a closed subgroup of $S_\infty$, then, as $G$ has an open neighborhood basis at the identity consisting of open subgroups, we see that $x \in X$ is turbulent iff $x \in \text{Int}(V \cdot x)$ for all open subgroups $V \leq G$. Therefore, the following provide equivalent conditions for turbulence:

**Proposition 3.2.** Let $G$ be a closed subgroup of $S_\infty$ acting continuously on a Polish space $X$ and let $x \in X$. Then the following are equivalent:

(i) The orbit $G \cdot x$ is non-meager.

(ii) For each open subgroup $V \leq G$, $V \cdot x$ is non-meager.

(iii) For each open subgroup $V \leq G$, $V \cdot x$ is somewhere dense.

(iv) For each open subgroup $V \leq G$, $x \in \text{Int}(V \cdot x)$.

(v) The point $x$ is turbulent.

**Proof.** (i) $\Rightarrow$ (ii): Suppose $G \cdot x$ is non-meager and $V \leq G$ is an open subgroup of $G$. Then we can find $g_n \in G$ such that $G = \bigcup g_n V$, so some $g_n V \cdot x$ is non-meager and therefore $V \cdot x$ is non-meager.

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (iv): Suppose $V$ is an open subgroup of $G$ such that $V \cdot x$ is dense in some open set $U \neq \emptyset$. Take $g \in V$ such that $g \cdot x \in U$. Then $g^{-1} V \cdot x = V \cdot x$ is dense in $g^{-1} U \ni x$ and $x \in \text{Int}(V \cdot x)$.

(iv) $\Rightarrow$ (i): Suppose $F_n \subseteq X$ are closed nowhere dense such that $G \cdot x \subseteq \bigcup_n F_n$. Then $K_n = \{ g \in G : g \cdot x \in F_n \}$ are closed and, as they cover $G$, some $K_n$ has non-$\emptyset$ interior. So suppose $gV \subseteq K_n$ for some open subgroup $V \leq G$. Then $gV \cdot x \subseteq F_n$ and both $gV \cdot x$ and $V \cdot x$ are nowhere dense.

The equivalence of (iv), (v) is clear using the remarks following 3.3.1. \hfill $\square$

It was asked in Truss 17 whether the existence of a generic automorphism of the limit of a Fraïssé class $\mathcal{K}$ could be expressed in terms of AP and JEP for $\mathcal{K}_p$. He gave a partial answer to this showing that the existence of a generic automorphism implied JEP for $\mathcal{K}_p$, and, moreover, if $\mathcal{K}_p$ satisfies JEP and the so-called cofinal AP (CAP), then there is indeed a generic automorphism. Here we say that $\mathcal{K}_p$ satisfies the CAP if it has a cofinal subclass, under embeddability, that has the AP.

This is clearly equivalent to saying that there is a subclass $\mathcal{L} \subseteq \mathcal{K}_p$, cofinal under embeddability, such that for any $S \in \mathcal{L}, T, R \in \mathcal{K}_p$ and embeddings $f : S \rightarrow T, g : S \rightarrow R$, there is $H \in \mathcal{K}_p$ and embeddings $h : T \rightarrow H, i : R \rightarrow H$ such that $h \circ f = i \circ g$. We mention that for any Fraïssé class $\mathcal{K}$, the class $\mathcal{L}$ of $\mathcal{S} = \langle A, \psi : B \rightarrow C \rangle \in \mathcal{K}_p$, such that $A$ is generated by $B$ and $C$, is cofinal under embeddability. This follows from the ultrahomogeneity of the Fraïssé limit of $\mathcal{K}$.

**Definition 3.3.** A class $\mathcal{K}$ of finite structures satisfies the weak amalgamation property (WAP) if for every $A \in \mathcal{K}$ there is a $B \in \mathcal{K}$ and an embedding $e : A \rightarrow B$ such that for all $C, D \in \mathcal{K}$ and embeddings $i : B \rightarrow C, j : B \rightarrow D$ there is an $E \in \mathcal{K}$ and embeddings $l : C \rightarrow E, k : D \rightarrow E$ amalgamating $C$ and $D$ over $A$, i.e., such that $l \circ i \circ e = k \circ j \circ e$.

A similar definition applies to the class $\mathcal{K}_p$.

We can now use turbulence concepts to provide the following answer to Truss’ question. We have recently found out that Ivanov 29 had also proved a similar result, by other techniques, in a somewhat different context, namely that of $\aleph_0$-categorical structures.
Theorem 3.4. Let $K$ be a Fraïssé class and $K$ its Fraïssé limit. Then the following are equivalent:

(i) $K$ has a generic automorphism.

(ii) $\mathcal{K}_p$ satisfies the WAP and the JEP.

Proof. (i) $\Rightarrow$ (ii): We know that if $K$ has a generic automorphism, then some $f \in \text{Aut}(K)$ is turbulent and has dense conjugacy class. So by Theorem 1.1, $\mathcal{K}_p$ has the JEP.

Given now $S = \langle A, \psi : B \to C \rangle \in \mathcal{K}_p$, we can assume that $A \subseteq K$. Moreover, as $[\psi : B \to C]$ is open nonempty and $f$ has dense conjugacy class, we can suppose that $\psi \subseteq f$. Let $V_A = [id : A \to A]$, which is a clopen subgroup of $\text{Aut}(K)$. By the turbulence of $f$, we know that $c(V_A, f) = \{g^{-1}fg : g \in V_A\}$ is dense in some open neighborhood $[\phi : E \to F]$ of $f$.

Now, since $f \supseteq \psi$, we can suppose that $\psi \subseteq \phi$. Let $D$ be the substructure of $K$ generated by $A, E$ and $F$, and put $T = \langle D, \phi : E \to F \rangle$. So $S$ is a subsystem of $T$ and denote by $e$ the inclusion mapping of $S$ into $T$.

Assume now that $i : T \to F = \langle H, \chi : M \to N \rangle$ and $j : T \to G = \langle P, \xi : Q \to R \rangle$ are embeddings. Then we wish to amalgamate $F$ and $G$ over $S$.

By using the extension property of $K$ we can actually assume that $H$ and $P$ are substructures of $K$ and $D \subseteq H, P$, with $\chi$ and $\xi$ extending $\phi$, and $i, j$ the inclusion mappings. By the density of $c(V_A, f)$ in $[\phi : E \to F]$ there are $g, k \in V_A = [id : A \to A]$ such that $g^{-1}fg \supseteq \chi$ and $h^{-1}fh \supseteq \xi$. Let $S$ be the substructure of $K$ generated by $g''H$ and $h''P$, $T$ the substructure generated by $g''M$ and $h''Q$ and $U$ the substructure generated by $g''N$ and $h''R$. Finally put $\theta = f \mid T : T \to U$ and $\mathcal{E} = \langle S, \theta : T \to U \rangle$.

As $f$ restricts to an isomorphism of $g''M$ with $g''N$ and an isomorphism of $h''Q$ with $h''R$, it also restricts to an isomorphism of $T$ with $U$. So $\theta$ is well defined. Moreover, $g$ and $h$ obviously embed $F$ and $G$ into $\mathcal{E}$, as $g^{-1}fg \supseteq \chi$ and $h^{-1}fh \supseteq \xi$. And finally, as $g, h \in [id : A \to A]$, we have that $g \circ i \circ e = h \circ j \circ e = id_A$. So $\mathcal{E}$ is indeed an amalgam over $S$. Therefore $\mathcal{K}_p$ satisfies WAP.

(ii) $\Rightarrow$ (i): Now suppose $\mathcal{K}_p$ satisfies WAP and JEP. We will construct an $f \in \text{Aut}(K)$ with turbulent and dense conjugacy class. Notice that this will be enough to insure that the conjugacy class of $f$ is comeager. For as the conjugacy action is continuous, if the orbit of $f$ is non-meager, then it will be comeager in its closure, i.e., comeager in the whole group.

As before for $\psi : B \to C$, an isomorphism between finite substructures of $K$, we let

$$D(\psi : B \to C) = \{f \in \text{Aut}(K) : \exists g \in \text{Aut}(K)(g^{-1}fg \supseteq \psi)\}. $$

For $E$ a finite substructure of $K$, we let $V_E = [id : E \to E]$, which is a clopen subgroup of $\text{Aut}(K)$.

We have seen in the proof of Theorem 2.1 that the JEP for $\mathcal{K}_p$ implies that each $D(\psi : B \to C)$ is open dense in $\text{Aut}(K)$. And moreover, any element in their intersection has dense conjugacy class.

Now for $\psi : B \to C$ let $A$ be the substructure generated by $B, C$ and $S = \langle A, \psi : B \to C \rangle$. Then by WAP for $\mathcal{K}_p$ and the extension property of $K$, there is an extension $\hat{S} = \langle A, \hat{\psi} : B \to C \rangle$ of $S$ such that any two extensions of $\hat{S}$ can be amalgamated over $S$. By extending $\hat{S}$, we can actually assume that $A$ is generated by $B, C$. Enumerate all such $\hat{\psi}$’s as $\hat{\psi}_1, \hat{\psi}_2, \ldots$. Moreover, list all the possible finite
extensions $\theta : M \to N$ of $\psi_m : B \to C$ as $\hat{\theta}_m : \hat{\psi}_m$, $\hat{\theta}_2 : \hat{\psi}_m$, . . . . Define

$$E(\psi : B \to C) = \{ f \in \text{Aut}(K) : f \supseteq \hat{\psi}_m \}$$

(for some $\phi \supseteq \psi$ and some $m$ we have $f \supseteq \hat{\phi}_m$)

and

$$F_{m,n}(\psi : B \to C) = \{ f \in \text{Aut}(K) : f \supseteq \hat{\psi}_m \Rightarrow (c(V_B, f) \cap [\hat{\theta}_n] \neq \emptyset) \}$$

Now obviously both $E(\psi : B \to C)$ and $F_{m,n}(\psi : B \to C)$ are open and $E(\psi : B \to C)$ is dense.

**Lemma 3.5.** Suppose $f \in \text{Aut}(K)$ is in $E(\psi : B \to C)$ and $F_{m,n}(\psi : B \to C)$ for all $\psi : B \to C$ and $m, n \in N$. Then $f$ is turbulent.

**Proof.** Let a clopen subgroup $V_E \leq \text{Aut}(K)$ be given. We shall show that $c(V_E, f)$ is somewhere dense.

As $f \in E(f \mid E) = f''E$ there are some $\phi : B \to C, E \subseteq B$ and $m$ such that $f \supseteq \hat{\phi}_m$. We claim that $c(V_B, f)$ and $c(V_E, f)$ is dense in $[\hat{\phi}_m : B \to C]$. This is because any basic open subset of $[\hat{\phi}_m : B \to C]$ is of the form $[\hat{\theta}_n]$ for some $n \in N$ and we know that $c(V_B, f) \cap [\hat{\theta}_n] \neq \emptyset$.

So now we only need to show that each $F_{m,n}(\psi : B \to C)$ is dense, as any element in the intersection of the $D, E$ and $F_{m,n}$'s will do.

Given a basic open set $[\phi : A \to D]$, where we can suppose that $\phi \supseteq \hat{\psi}_m$, we need to show that for some $f \supseteq \phi$, $c(V_B, f) \cap [\hat{\theta}_n] \neq \emptyset$. Now $\theta^m$ and $\phi$ are both extensions of $\psi_m$, so by WAP for $K_p$, they can be amalgamated over $\psi$. It follows by the extension property of $K$ that there is some $g \in \text{Aut}(K)$ fixing $B$, such that $g^{-1} \phi g$ and $\theta^m$ are compatible in $\text{Aut}(K)$, i.e., for some $f \supseteq \phi, g^{-1}fg \in [\theta^m]$.

This finishes the proof. □

**Remark.** It is easy to verify that the intersection of the $D, E$, and $F_{m,n}$'s is conjugacy invariant, so actually this intersection is exactly equal to the set of generic automorphisms.

As obviously the CAP implies the WAP, we have:

**Corollary 3.6.** (Truss [47]) If $K_p$ has the cofinal amalgamation property, then there is a generic automorphism of $K$.

**Remark.** In Hodges [25] the Fraïssé theory is developed in the context of a class $K$ of finitely generated (but not necessarily finite) structures satisfying (HP), (JEP), and (AP). It is not hard to see that all the arguments and results of [22, 23] carry over without difficulty to this more general context.

Truss [47] calls an automorphism $f \in \text{Aut}(K), K = \text{Fraïssé limit of } K, K$ a Fraïssé class, locally generic if its conjugacy class is non-meager.

Given a Fraïssé class $K$, let us say that $K_p$ satisfies the local WAP if there is $S = (A, \psi : B \to C) \in K_p$ such that WAP holds for the subclass $L_p$ of $K_p$ consisting of all $T \in K_p$ into which $S$ embeds.

Using again Proposition [62] we have:
Theorem 3.7. Let $K$ be a Fraïssé class and $K$ its Fraïssé limit. Then the following are equivalent:

(i) $K$ admits a locally generic automorphism.
(ii) $K_p$ satisfies the local WAP.

Proof. (i) $\Rightarrow$ (ii): Suppose $K$ admits a locally generic automorphism $f$. Then by Proposition 3.2, the conjugacy class of $f$ is turbulent and of course dense in some open set $U \subseteq \text{Aut}(K)$. So there is some $S = \langle A, \psi : B \to C \rangle \in K_p$ such that $U \supseteq \langle \psi : B \to C \rangle$. Then, as in the proof of (i) $\Rightarrow$ (ii) in Theorem 3.4, we see that WAP holds for all $T \in K_p$ into which $S$ can be embedded.

(ii) $\Rightarrow$ (i): Suppose $S = \langle A, \psi : B \to C \rangle \in K_p$ witnesses that $K_p$ has the local WAP. Then we repeat the construction of (ii) $\Rightarrow$ (i) in the proof of Theorem 3.4, by taking as our first approximation $\psi$ (and without of course using the sets $D(\phi : D \to E)$ whose density was ensured by JEP).

Similarly one can see that the existence of a conjugacy class that is somewhere dense is equivalent to a local form of JEP defined in an analogous way.

We can also characterize WAP in terms of local genericity.

Theorem 3.8. Let $K$ be a Fraïssé class and $K$ its Fraïssé limit. Then the following are equivalent:

(i) $K$ admits a dense set of locally generic automorphisms.
(ii) $K$ admits a comeager set of locally generic automorphisms.
(iii) $K_p$ satisfies the WAP.

Proof. (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are proved as in Theorems 3.4 and 3.7, beginning the construction of a turbulent point from any given $\psi : B \to C$.

(ii) $\Rightarrow$ (i) is of course trivial and (i) $\Rightarrow$ (ii) follows from the fact that if an orbit is non-meager then it is comeager in its closure.

Remark. As is easily checked, all the structures considered in this paper that are shown to have a comeager conjugacy class actually also satisfy the above theorem. Let us just mention that for the class of $\omega$-stable, $\aleph_0$-categorical structures one just needs to notice that they stay $\omega$-stable, $\aleph_0$-categorical after having been expanded by a finite number of constants (see Hodges [25]).
4. Normal Form for Isomorphisms of Finite Subalgebras of clop(2\(^N\))

In this section we will develop some facts needed in the proof of the existence of generic homeomorphisms of 2\(^N\), which will be given in the next section.

Recall that the Fraïssé limit of the class of finite boolean algebras is the countable atomless boolean algebra \(B_\infty\), which can be concretely realized as clop(2\(^N\)), the boolean algebra of clopen subsets of Cantor space 2\(^N\).

We will eventually prove that the class \(K_p\), where \(K\) is the class of finite boolean algebras, has the CAP. Thus, we will first have to describe the cofinal class \(\mathcal{L}\) of \(K_p\) over which we can amalgamate. First of all it is clear that if \(\phi: B_\infty \to B_\infty\) is such that for some \(a \in B_\infty\), \(\phi(a) < a\), then the orbit of \(a\) under \(\phi\) is infinite. This shows that there are partial automorphisms of finite boolean algebras that cannot be extended to full automorphisms of bigger finite boolean algebras. This fact makes the situation a lot messier than in the case of measured boolean algebras and, in order to prove the amalgamation property, requires us to be able to describe the local structure of a partial automorphism of a finite boolean algebra.

**Definition 4.1.** Let \(B, C\) be finite subalgebras of clop(2\(^N\)) and \(\psi : B \to C\) an isomorphism. A refinement of \(\psi : B \to C\) consists of finite superalgebras \(B \subseteq B', C \subseteq C'\) and an isomorphism \(\psi' : B' \to C'\) such that \(\psi' |_B = \psi\).

Fix an isomorphism \(\psi : B \to C\) of finite subalgebras of clop(2\(^N\)), and let \(B \lor C\) be the algebra generated by \(B\) and \(C\).

**Definition 4.2.** A cyclic chain is a sequence \(a_1,\ldots,a_n\) of distinct atoms of \(B \lor C\) belonging to both \(B\) and \(C\) such that \(\psi(a_1) = a_2,\ldots,\psi(a_{n-1}) = a_n,\psi(a_n) = a_1:\)

\[
\begin{array}{c}
B & a_1 & a_2 & a_{n-1} & a_n \\
\psi & \downarrow & \downarrow & \ldots & \downarrow \\
C & a_2 & a_3 & a_n & a_1
\end{array}
\]  

(7)

**Definition 4.3.** A stable chain is a sequence \(a_0,\ldots,a_n\) of distinct atoms of \(B \lor C\) plus an element \(c\), which we call its end, such that one of the two following situations occur:

- **(I)**
  1. \(a_0,\ldots,a_n\) belong to \(B\),
  2. \(a_1,\ldots,a_n, c\) belong to \(C\),
  3. \(\psi(a_0) = a_1,\ldots,\psi(a_{n-1}) = a_n,\)
  4. \(c = \psi(a_n) = a_0 \lor b_1 \lor \cdots \lor b_k = a_0 \lor x\), where \(b_1,\ldots,b_k \neq a_0\) are atoms of \(B \lor C\) and \(k \geq 1\).

Diagrammatically,

\[
\begin{array}{c}
B & a_0 & a_1 & a_{n-1} & a_n \\
\psi & \downarrow & \downarrow & \ldots & \downarrow \\
C & a_1 & a_2 & a_n & a_0 \lor x
\end{array}
\]  

(8)

where \(x = b_1 \lor \cdots \lor b_k \neq 0\).

- **(II)**
  1. \(a_0,\ldots,a_n\) belong to \(C\),
  2. \(a_1,\ldots,a_n, c\) belong to \(B\),
  3. \(\psi^{-1}(a_0) = a_1,\ldots,\psi^{-1}(a_{n-1}) = a_n,\)
  4. \(c = \psi^{-1}(a_n) = a_0 \lor b_1 \lor \cdots \lor b_k = a_0 \lor x\), where \(b_1,\ldots,b_k \neq a_0\) are atoms of \(B \lor C\) and \(k \geq 1\).
Diagramatically,

\[
\begin{array}{cccc}
B & a_1 & a_2 & a_n & a_0 \lor x \\
\psi & \downarrow & \downarrow & \ldots & \downarrow \\
C & a_0 & a_1 & a_{n-1} & a_n \\
\end{array}
\]

where \( x = b_1 \lor \cdots \lor b_k \neq 0 \).

For a stable chain as above we say that an atom \( b \) in \( B \lor C \) such that \( b \neq a_0 \) and \( b < \psi(a_n) \) (respectively, \( b < \psi^{-1}(a_n) \)) is free. These are the elements \( b_1, \ldots, b_k \) above. Moreover, the atom \( a_0 \) is said to be the beginning of the stable chain.

**Definition 4.4.** A linking chain is a sequence \( a_1, \ldots, a_n \) of distinct atoms of \( B \lor C \), such that

1. \( a_1, \ldots, a_{n-1} \) belong to \( B \),
2. \( a_2, \ldots, a_n \) belong to \( C \),
3. \( \psi(a_1) = a_2, \ldots, \psi(a_{n-1}) = a_n \),
4. \( a_1, a_n \) are free in some stable chains.

\[
\begin{array}{cccc}
B & a_1 & a_2 & a_{n-1} & a_n \lor y \\
\psi & \downarrow & \downarrow & \ldots & \downarrow \\
C & a_1 \lor z & a_2 & a_3 & a_n \\
\end{array}
\]

**Definition 4.5.** An isomorphism \( \psi : B \rightarrow C \) is said to be normal iff

1. every atom of \( B \) or of \( C \) that is not an atom in \( B \lor C \) is the end of a stable chain,
2. every atom of \( B \lor C \) is a term in either a stable, linking or cyclic chain, or is free in some stable chain.

For any finite subalgebras \( B \) and \( C \) of a common algebra, we let \( n(B, C) \) be the number of atoms in \( B \) that are not atoms in \( B \lor C \) plus the number of atoms in \( C \) that are not atoms in \( B \lor C \).

**Lemma 4.6.** For any isomorphism \( \psi : B \rightarrow C \) between finite subalgebras \( B, C \subseteq \text{clop}(2^N) \), there is a refinement \( \psi' : B' \rightarrow C' \) satisfying condition (i) of normality.

*Proof.* The proof is by induction on \( n(B, C) \). For the basis of the induction, if \( n(B, C) = 0 \), then every atom of \( B \) and every atom of \( C \) is an atom of \( B \lor C \) and there is noting to prove. In this case, we see that the structure of \( \psi \) is particularly simple, since it is then just an automorphism of \( B = C = B \lor C \) and thus one can easily split \( B \lor C \) into cyclic chains, namely the \( \psi \)-orbits of atoms.

Now for the induction step: Suppose \( x \in B \) is some atom of \( B \) that is not an atom of \( B \lor C \) (the case when \( x \) is an atom of \( C \) that is not an atom of \( B \lor C \) is of course symmetric), and trace the longest chain of atoms of \( B \lor C \), \( a_i \in B, b_i \in C \), such that

\[
x \mapsto \psi b_0 = a_0 \mapsto \psi b_1 = a_1 \mapsto \psi b_2 = a_2 \mapsto \ldots
\]

If for some \( i < j \), \( a_i = a_j \), then also \( b_i = b_j \), and by the injectivity of \( \psi \), \( a_{i-1} = a_{j-1} \), etc. So in that case, by induction, we have \( b_0 = b_{j-i} \), whence \( a_{j-i-1} = x \), contradicting that \( x \) was not an atom in \( B \lor C \).

So as \( B \lor C \) is finite, the chain has to stop either (i) on some \( a_n \) or (ii) on some \( b_n \) (if it stops with \( x \) we let \( x = a_{n-1} \) and treat it as in case (i)).

Case (i): Let \( y = \psi(a_n) \), which is an atom in \( C \) but not in \( B \lor C \). Now, as \( \bigvee_{i=1}^n a_i = \bigvee_{i=1}^n b_i \), both \( x \) and \( y \) are disjoint from \( \bigvee_{i=0}^n a_i \) and let \( x = c_1 \lor \cdots \lor c_p, y = \)
$d_1 \lor \cdots \lor d_q$ be the decompositions into atoms of $B \lor C$. Suppose $p \leq q$ (the case when $q \leq p$ is similar). Split $a_i = b_i$ into non-zero elements $a_i^1, \ldots, a_i^p$ of $\text{clop}(2^n)$ for each $i \leq n$. Then we let $B'$ be the smallest algebra containing $B$ and new atoms $a_i^l$, for $l \leq p, i \leq n$, and $c_1, \ldots, c_p$. Also let $C'$ be the smallest algebra containing $C$ and new elements $a_i^l$ for $l \leq p, i \leq n$, and $d_1, d_2, \ldots, d_{p-1}, d_p \lor \cdots \lor d_q$. Finally, let $\psi$ be the unique extension of $\psi$ satisfying $\psi'(c_i) = a_0$, $\psi'(a_i^l) = a_{i+1}^l$, for $i < n$, $l \leq p$, and $\psi'(a_i^p) = d_p \lor \cdots \lor d_q$. We remark that $n(B', C') < n(B, C)$.

Case (ii): Notice that $x \lor \bigvee_{i=1}^{n-1} b_i = x \lor \bigvee_{i=1}^{n-1} a_i = 0$, and, as $b_n$ is an atom of $B \lor C$, either $b_n < x$ or $b_n \lor x = 0$. If $b_n < x$, then $x$ is the end of a stable chain. If $b_n \lor x = 0$, then we proceed as follows:

Suppose $x = c_1 \lor \cdots \lor c_p$ is the decomposition into atoms of $B \lor C$ and split $b_i$ ($i \leq n$) into $b_i^1, \ldots, b_i^p$ in $\text{clop}(2^n)$. Now let $B'$ be the algebra generated by $B$ and $b_i^1, \ldots, b_i^p$, for $i < n$ and $c_1, \ldots, c_p$. Let $C'$ be the algebra generated by $C$ and $b_i^1, \ldots, b_i^p$ for $i \leq n$, and let $\psi$ be the unique extension of $\psi$ satisfying $\psi'(c_i) = b_i^0$ ($l \leq p$), $\psi'(b_i^l) = b_{i+1}^l$ ($l \leq p$). Again we remark that $n(B', C') < n(B, C)$. So this concludes the induction step. 

**Proposition 4.7.** Any isomorphism $\psi : B \rightarrow C$ between finite subalgebras $B, C \subseteq \text{clop}(2^n)$ has a finite normal refinement.

**Proof.** By the lemma we can suppose that $\psi : B \rightarrow C$ satisfies condition (i) of normality. We will then see that it actually satisfies condition (ii) too.

So consider an atom $a_0$ of $B \lor C$. Find an atom $c$ of $B$ such that $a_0 \leq c$. Now if $a_0 < c$, then we know from condition (i) that $c$ is the end of a stable chain and $a_0$ is therefore either the beginning of a stable chain or is free. Otherwise if $a_0 = c \in B$, then we find the maximal chain of atoms of $B \lor C$, $a_i \in B$, $b_i \in C$ such that

$$\cdots \rightarrow b_{i-1} \rightarrow b_i \rightarrow b_{i+1} \rightarrow a_{i+1} \rightarrow b_{i+2} \rightarrow a_{i+2} \rightarrow \cdots$$

where the indices run over an interval of $Z$ containing 0. We have now various cases:

Case (i): $a_i = a_j$ for some $i < j$. Then obviously $b_{i+1} = b_{j+1}$ and $b_{i-1} = b_{j-1}$ by the injectivity of $\psi$ (notice also that these elements are indeed defined, i.e., are atoms in $B \lor C$ and not only in $C$). But then also $a_{i+1} = a_{j+1}$ and $a_{i-1} = a_{j-1}$, etc. So the chain is bi-infinite and periodic and we can write it as

$$B \xrightarrow{\psi} C \xrightarrow{\psi} \cdots \xrightarrow{\psi} a_0 \xrightarrow{\psi} a_1 \xrightarrow{\psi} \cdots \xrightarrow{\psi} a_n$$

So $a_0$ is term in a cyclic chain.

Case (ii): The chain ends with some $a_k$. Then $c = \psi(a_k)$ is an atom of $C$ that is not an atom of $B \lor C$ and must therefore be the end of some stable chain. So using the bijectivity of $\psi$ one sees that all of the above chain are terms in a stable chain and in particular $a_0$ is a term in a stable chain.

Case (iii): The chain begins with some $b_k$. Let $c = \psi^{-1}(b_k)$. Then $c$ must be some atom of $B$ that is not an atom of $B \lor C$, so it must be the end of some stable chain. Now looking at $\psi^{-1}$ instead of $\psi$ and switching the roles of $B$ and $C$, we see as before that $a_0$ is a term in some stable chain.

Case (iv): The chain begins with some $a_n$ and ends with some $b_k$. Then $a_n \neq b_k$ and there must be atoms $c$ of $C$ and $d$ of $B$ that are ends of stable chains such that
\(a_n \land c \neq 0\) and \(b_k \land d \neq 0\). But, as \(a_n\) and \(b_k\) are atoms, this means that \(a_n < c\) and \(b_k < d\), so the chain is linking. \(\square\)

5. Generic homeomorphisms of Cantor space

A generic homeomorphism of the Cantor space \(2^\mathbb{N}\) is a homeomorphism whose conjugacy class in \(H(2^\mathbb{N})\) is dense \(G_\delta\). As \(H(2^\mathbb{N})\) (with the uniform topology) is isomorphic as a topological group to \(\text{Aut}(\mathcal{B}_\infty)\), where \(\mathcal{B}_\infty\) is the countable atomless boolean algebra, the existence of a generic homeomorphism of \(2^\mathbb{N}\) is equivalent to the existence of a generic automorphism of \(\mathcal{B}_\infty\).

**Theorem 5.1.** Let \(K\) be the class of finite boolean algebras. Then \(K_p\) has the CAP, and therefore there is a generic automorphism of the countable atomless boolean algebra and a generic homeomorphism of the Cantor space.

Before we embark on the proof, let us first mention that one can construct counter-examples to \(K_p\) having the AP, and that therefore the added complications are necessary.

**Proof.** We will show that the class \(L\) of \(S = \langle B \lor C, \psi : B \to C \rangle\), where \(\psi\) is normal, has the AP. Notice first that \(L\) is cofinal in \(K_p\) by Proposition 4.7.

Suppose now \(S = \langle A, \psi : B \to C \rangle \in L\). Let \(S^l = \langle A^l, \psi^l : B^l \to C^l \rangle\) and \(S^r = \langle A^r, \psi^r : B^r \to C^r \rangle\) be two refinements of \(S\), which we do not necessarily demand belong to \(L\).

We claim that they can be amalgamated over \(S\). We remark first that it is trivially enough to amalgamate any two refinements of \(S^l\) and \(S^r\) over \(S\). List the atoms of \(A\) as \(a_1, \ldots, a_n\). Then by refining \(S^l, S^r\) we can suppose they have atoms

\[
\begin{align*}
& a_1^l(1), \ldots, a_1^l(k), \ldots, a_n^l(1), \ldots, a_n^l(k) \\
& a_1^r(1), \ldots, a_1^r(k), \ldots, a_n^r(1), \ldots, a_n^r(k)
\end{align*}
\]

respectively, where \(a_v^l(1), \ldots, a_v^l(k)\) \((v = l, r\) and \(t = 1, \ldots, n)\) is a splitting of \(a_t\). So \(a_t \mapsto \bigvee_{j=1}^k a_t^l(j)\) and \(a_t \mapsto \bigvee_{j=1}^k a_t^r(j)\) are canonical embeddings of \(A\) into \(A^l\) and \(A^r\) and we can furthermore suppose that both \(B^l, C^l\) and \(B^r, C^r\) contain the image of \(A\) by these embeddings. This can be done by further refining \(S^l\) and \(S^r\).

This means that any atom of \(B^v\) and \(C^v\) \((v = l, r)\) will be on the form \(\bigvee_{i \in \Gamma} a_v^l(i)\), where \(\Gamma \subseteq \{1, \ldots, k\}\) and \(1 \leq t \leq n\).

Take new formal atoms \(a_m^l(i) \otimes a_m^r(j)\) for \(m \leq n\) and \(i, j \leq k\). Our amalgam \(S^a = \langle A^a, \psi^a : B^a \to C^a \rangle\) will be such that the atoms in \(A^a\) will be a subset \(E\) of these new formal atoms and the embeddings \(l : A^l \to A^a\) and \(r : A^r \to A^a\) will be defined by

\[
\begin{align*}
& l(a_i^l(i)) = \bigvee\{a_i^l(i) \otimes a_i^r(j) \in E\}, \\
& r(a_i^r(i)) = \bigvee\{a_i^l(j) \otimes a_i^r(i) \in E\}
\end{align*}
\]

The atoms of \(B^a\) will be the following

\[
E_{i \in \Delta} = \bigvee\{a_i^l(i) \otimes a_i^r(j) \in E : (i, j) \in \Gamma \times \Delta\},
\]
where ∨_{i \in \Gamma} a^l(i) is an atom in B^l and ∨_{j \in \Delta} a^r(j) is an atom in B^r. Similarly for C^a.

Now obviously it is enough to define ψ^a between the atoms of B^a and C^a.

(1) Suppose Σ, Δ, Λ ⊆ \{1, \ldots, k\} are such that

\begin{equation}
\forall i \in \Gamma, a^l(i), a^r(i), a^l_m(i), a^r_m(i)
\end{equation}

are atoms in B^l, B^r, C^l, C^r, respectively, such that

\begin{equation}
\psi^l(\bigvee_{\Gamma} a^l(i)) = \bigvee_{\Theta} a^l_m(i) \quad \text{and} \quad \psi^r(\bigvee_{\Delta} a^r(i)) = \bigvee_{\Lambda} a^r_m(i).
\end{equation}

Then we let ψ^a(E^Γ,Δ) = E^θ,Λ,

(2) Now suppose that ∨_{Γ} a^l(i), ∨_{Δ} a^r(i), ∨_{Θ} a^l_m(i), ∨_{Λ} a^r_m(i), m ≠ s, are atoms in B^l, B^r, C^l, C^r, respectively, such that

\begin{equation}
\psi^l(\bigvee_{\Gamma} a^l(i)) = \bigvee_{\Theta} a^l_m(i) \quad \text{and} \quad \psi^r(\bigvee_{\Delta} a^r(i)) = \bigvee_{\Lambda} a^r_m(i).
\end{equation}

We notice first that, since S = (A, ψ : B → C) ∈ L is normal with A = B ▽ C, the only time ψ sends an atom of A to a non-atom of A, or inversely sends a non-atom of A to an atom of A, is in the last steps of a stable chain corresponding to a_n ↦ a_0 ▽ x in diagram \(\mathbb{3}\), or a_0 ▽ x ↦ a_0 in diagram \(\mathbb{4}\). Moreover, whenever x is an atom of B, then either x or ψ(x) is an atom of A.

Using this, it follows from equations \(\mathbb{21}\) and m ≠ s that, since ψ^l and ψ^r are refinements of ψ, a^l must be an atom of B, while a^r_m must be below the end of a stable chain in S.

Now, in this situation we cannot have a^l(i) ▽ a^r(j) ∈ E, for any (i, j) ∈ Γ × Δ, or, in other words, we must have E^Γ,Δ = 0. This is because we have ψ^l(∨_{Γ} a^l(i)) ≤ a_m, while ψ^r(∨_{Δ} a^r(i)) ≤ a_s, which would force

\begin{equation}
\psi^a(E^Γ,Δ) \leq \bigvee \{a^l_m(i) ▽ a^r_m(j) \in E\}
\end{equation}

and similarly

\begin{equation}
\psi^a(E^Γ,Δ) \leq \bigvee \{a^l_s(i) ▽ a^r_s(j) \in E\},
\end{equation}

which leaves only the possibility ψ^a(E^Γ,Δ) = 0. Or said in another way, it would force

\begin{equation}
\psi^a(E^Γ,Δ) = \psi^a(\bigvee \{a^l(i) ▽ a^r(j) \in E : (i, j) ∈ Γ × Δ\})
\end{equation}

to be the join of elements of the form a^l_m(p) ▽ a^r_s(q). But we do not include any such elements in our amalgam.

There is also a dual version of this problem, namely when

\begin{equation}
\psi^l(\bigvee_{\Theta} a^l(i)) = \bigvee_{i \in \Theta} a^l(i) \quad \text{and} \quad \psi^r(\bigvee_{\Lambda} a^r(i)) = \bigvee_{i \in \Lambda} a^r(i)
\end{equation}

for distinct s, m.

With (1) and (2) in mind, we can now formulate the necessary and sufficient conditions on E for this procedure to work out:

(a) For l to be well-defined as an embedding from A^l to A^a:

∀t ≤ n \ \∀i \leq k \ \exists j \leq k \ a^l(i) ▽ a^r(j) ∈ E
(b) For \( r \) to be well-defined as an embedding from \( \mathbf{A}^r \) to \( \Lambda^e \):

\[
\forall t \leq n \, \forall j \leq k \, \exists i \leq k \; a'_t(i) \otimes a'_r(j) \in E
\]

(c) If \( \Gamma, \Delta, \Theta, \Lambda \) and \( t, m \) are as in (1), then \( E_{t_m}^{\Gamma, \Delta} \neq 0 \iff E_{m_t}^{\Theta, \Lambda} \neq 0 \).

(d) If \( \Gamma, \Delta, \Theta, \Lambda \) and \( t, m, s \) are as in (2), then \( E_{t_{m_s}}^{\Gamma, \Delta} = 0 \).

We will define \( E \) separately for terms of stable, linking and cyclic chains.

Suppose we are given a cyclic chain

\[
\begin{array}{ccc}
\mathbf{B} & a_{t_1} & a_{t_2} & \ldots & a_{t_w} \\
\psi & \downarrow & \downarrow & \ldots & \downarrow \\
\mathbf{C} & a_{t_2} & a_{t_3} & \ldots & a_{t_1}
\end{array}
\]

Then we let \( a_{t_v}^l(i) \otimes a_{t_v}^r(j) \in E \) for all \( i, j \leq k \) and \( 1 \leq v \leq w \).

Given a linking chain

\[
\begin{array}{ccc}
\mathbf{B} & a_{t_1} & a_{t_2} & \ldots & a_{t_{w-1}} & a_{t_w} & \downarrow & y \\
\psi & \downarrow & \downarrow & \ldots & \downarrow \\
\mathbf{C} & a_{t_2} & a_{t_3} & \ldots & a_{t_1} & \downarrow & a_{t_{w+1}} & \ldots & a_{t_{w+q}}
\end{array}
\]

we let \( a_{t_v}^l(i) \otimes a_{t_v}^r(j) \in E \) for all \( i, j \leq k \) and \( 1 \leq v \leq w \).

So, since we have included all the new formal atoms in these two cases, we easily see that (a), (b) and (c) are verified. Moreover, condition (d) is only relevant for the stable chains as it only pertains to (2).

Finally, suppose we have a stable chain

\[
\begin{array}{ccc}
\mathbf{B} & a_{t_1} & a_{t_2} & \ldots & a_{t_{w-1}} & a_{t_w} \\
\psi & \downarrow & \downarrow & \ldots & \downarrow \\
\mathbf{C} & a_{t_2} & a_{t_3} & \ldots & a_{t_w} & a_{t_1} & \downarrow & a_{t_{w+1}} & \ldots & a_{t_{w+q}}
\end{array}
\]

The case when \( a_{t_1} \in \mathbf{C} \) and \( a_{t_1} \lor a_{t_{w+1}} \lor \cdots \lor a_{t_{w+q}} \in \mathbf{B} \) is symmetric to this one and is taken care of in the same manner.

By choosing \( k \) big enough and refining the partitions of \( a_{t_1}, \ldots, a_{t_w} \) in \( \mathbf{B}^l \) and \( \mathbf{B}^r \), the partitions of \( a_{t_2}, \ldots, a_{t_w}, a_{t_1} \lor a_{t_{w+1}}, \cdots \lor a_{t_{w+q}} \) in \( \mathbf{C}^l \), \( \mathbf{C}^r \) and subsequently extending \( \psi^l, \psi^r \) to these refined partitions, we can suppose that there are partitions of \( \{1, \ldots, k\} \) as follows:

\[
\{1, \ldots, k\} = \Gamma^l(\gamma, 1) \sqcup \cdots \sqcup \Gamma^l(\gamma, p) \sqcup \Delta^l(\gamma, 1, 1) \sqcup \cdots \\
\sqcup \Delta^l(\gamma, 1, p) \sqcup \cdots \sqcup \Delta^l(\gamma, q, 1) \sqcup \cdots \sqcup \Delta^l(\gamma, q, p)
\]

for \( \gamma = 1, \ldots, w \), and

\[
\Lambda^l(1) \sqcup \cdots \sqcup \Lambda^l(p) = \{1, \ldots, k\}
\]

for \( \beta = 1, \ldots, q \), such that

- for \( e = 1, \ldots, p; \gamma = 1, \ldots, w; \beta = 1, \ldots, q \):
  \( \Gamma^{l(e, \gamma)} a_{t_{e_\gamma}}^l(i) \), \( \Delta^{l(e, \gamma, \beta)} a_{t_{e_\gamma}}^l(i) \) are atoms in \( \mathbf{B}^l \);

- for \( e = 1, \ldots, p; \gamma = 2, \ldots, w; \beta = 1, \ldots, q \):
  \( \Gamma^{l(e, \gamma)} a_{t_{e_\gamma}}^l(i) \), \( \Delta^{l(e, \gamma, \beta)} a_{t_{e_\gamma}}^l(i) \) are atoms in \( \mathbf{C}^l \);

- for \( e = 1, \ldots, p \):
  \( \Delta^{l(e)} a_{t_{e_\gamma}}^l(i) \) is an atom in \( \mathbf{C}^l \);

- for \( e = 1, \ldots, p; \beta = 1, \ldots, q \):
  \( \Theta^{l(e, \beta)} a_{t_{e_\gamma}}^l(i) \) is an atom in \( \mathbf{C}^l \).
• for $e = 1, \ldots, p; \gamma = 1, \ldots, w - 1$:

\[
\psi^d \left( \bigvee_{\Gamma^i(\gamma,e)} a^t_{\Gamma^i(\gamma+1,e)}(i) \right) = \bigvee_{\Gamma^i(\gamma+1,e)} a^t_{\Gamma^i(\gamma+1,e)}(i),
\]

• for $e = 1, \ldots, p; \gamma = 1, \ldots, w - 1; \beta = 1, \ldots, q$:

\[
\psi^d \left( \bigvee_{\Delta^i(\gamma,\beta,e)} a^t_{\Gamma^i(\gamma+1,\beta,e)}(i) \right) = \bigvee_{\Delta^i(\gamma+1,\beta,e)} a^t_{\Gamma^i(\gamma+1,\beta,e)}(i),
\]

• for $e = 1, \ldots, p$:

\[
\psi^d \left( \bigvee_{\Gamma^i(w,e)} a^t_{\Gamma^i(w+1,e)}(i) \right) = \bigvee_{\Lambda^i(e)} a^t_{\Lambda^i(e)}(i),
\]

• for $e = 1, \ldots, p; \beta = 1, \ldots, q$:

\[
\psi^d \left( \bigvee_{\Delta^i(w,\beta,e)} a^t_{\Gamma^i(w+1,\beta,e)}(i) \right) = \bigvee_{\Theta^i(\beta,e)} a^t_{\Theta^i(\beta,e)}(i).
\]

And the same for $r$, i.e., with the same constants $k, p$, though different partitions.

Let us see how this can be obtained. Suppose $\psi^d : B^i \rightarrow C^i$ is given. First refine the partition of $a_{t_2}$ in $B^i$ to be finer than the partition of $a_{t_3}$ in $C^i$. Then refine the partition of $a_{t_3}$ in $C^i$ so that we can extend $\psi^d$ to be an isomorphism of the refined partitions of $a_{t_2}$ in $B^i$ and $a_{t_3}$ in $C^i$, respectively. Again we refine the partition of $a_{t_3}$ in $B^i$ to be finer than the partition of $a_{t_3}$ in $C^i$. And refine the partition of $a_{t_4}$ in $C^i$ so that we can extend $\psi^d$ to be an isomorphism of the refined partitions of $a_{t_3}$ in $B^i$ and $a_{t_4}$ in $C^i$ respectively. We continue like this until we have refined the partition of $a_{t_w}$ in $B^i$. Extend now the partition of $a_{t_1} \lor a_{t_{w+1}} \lor \cdots \lor a_{t_{w+q}}$ in $C^i$, so we can let $\psi^d$ be an isomorphism of the refined partition of $a_{t_w}$ in $B^i$ with the refined partition of $a_{t_1} \lor a_{t_{w+1}} \lor \cdots \lor a_{t_{w+q}}$ in $C^i$. We will now refine the elements of the chain in the other direction. We recall first that the partition of $a_{t_1} \lor a_{t_{w+1}} \lor \cdots \lor a_{t_{w+q}}$ in $C^i$ was supposed to be fine enough to contain the elements $a_{t_1}, a_{t_{w+1}}, \ldots, a_{t_{w+q}}$.

To begin, refine the new partition of $a_{t_w}$ in $C^i$ to be the same as the new partition of $a_{t_w}$ in $B^i$. Refine now the partition of $a_{t_{w-1}}$ in $B^i$ so that $\psi^d$ extends to an isomorphism of the refined partitions of $a_{t_{w-1}}$ in $B^i$ and $a_{t_w}$ in $C^i$. Again refine the partition of $a_{t_{w-1}}$ in $C^i$ to be the same as the partition of $a_{t_{w-1}}$ in $B^i$. And refine the partition of $a_{t_{w-2}}$ in $B^i$ so that $\psi^d$ extends to an isomorphism of the refined partitions of $a_{t_{w-2}}$ in $B^i$ and $a_{t_{w-1}}$ in $C^i$. Continue in this fashion until $a_{t_1}$ has been refined in $B^i$.

This shows that the above form can be obtained for $\psi^d : B^i \rightarrow C^i$ individually. Now we can of course also obtain it for $\psi^r : B^r \rightarrow C^r$. So the only thing lacking is to obtain the same constants $k, p$ for both $\psi^d : B^i \rightarrow C^i$ and $\psi^r : B^r \rightarrow C^r$. But this is easily done by some further splitting (this last bit is only to reduce complexity of notation).

We now define a derivation $D$ on $\mathcal{P}(\{1, \ldots, k\}^2)$ (i.e., a mapping satisfying $D(X) \subseteq X$ for all $X \subseteq \{1, \ldots, k\}^2$) as follows:

\[
D(Y) = \{(i, j) \in Y : (i, j) \in \Gamma^i(e) \times \Gamma^r(d) \Rightarrow (\Lambda^l(e) \times \Lambda^r(d)) \cap Y \neq \emptyset\}.
\]
Let
\[(37) \quad X_0 = \bigcup_{\alpha, \beta} \Gamma^\alpha(1, e) \times \Gamma^\beta(1, d) \cup \bigcup_{\beta \in 1, \alpha} \Delta^\alpha(1, \beta, e) \times \Delta^\beta(1, \beta, d).\]

So \(D_{k' + 1}^\infty(X_0) = D_{k'}^\infty(X_0)\) for some minimal \(k' \in \mathbb{N}\), and we notice first that
\[(38) \quad \bigcup_{\beta \in 1, \alpha} \Delta^\alpha(1, \beta, e) \times \Delta^\beta(1, \beta, d) \subseteq D_{k'}^\infty(X_0).\]

**Claim.** Suppose that for some \(e, d, \sigma\) and \((i, j) \in \Gamma^1(1, e) \times \Gamma^\sigma(1, d)\), we have \((i, j) \not\in D_{\sigma + 1}^\infty(X_0)\). Then \((\Lambda^\sigma(1, e) \times \Lambda^\sigma(d)) \cap D_{\sigma}^\infty(X_0) = \emptyset\).

**Proof.** By definition of \(X_0\), there is a \(\tau \geq 0\) such that \((i, j) \in \Gamma^\tau(1, e) \times \Gamma^{\tau + 1}(1, d)\) (so \(\tau \leq \sigma\)). And by definition of \(D\), we must have \((\Lambda^\tau(1, e) \times \Lambda^\tau(d)) \cap D^{\tau}(X_0) = \emptyset\), whence also \((\Lambda^\tau(1, e) \times \Lambda^\tau(d)) \cap D_{\tau}^\infty(X_0) = \emptyset\).

**Lemma 5.2.** For all \(i \in \{1, \ldots, k\}\) we have \(\{i\} \times \{1, \ldots, k\} \cap D_{k'}^\infty(X_0) \neq \emptyset\) and \(\{1, \ldots, k\} \times \{i\} \cap D_{k'}^\infty(X_0) \neq \emptyset\).

**Proof.** Otherwise we can take \(\tau\) minimal such that for some \(i\) we have, e.g., \(\{i\} \times \{1, \ldots, k\} \cap D^{\tau}(X_0) = \emptyset\). Clearly \(\tau > 0\), so, as \(\bigcup_{\beta \in 1, \alpha} \Delta^\alpha(1, \beta, e) \times \Delta^\beta(1, \beta, d) \subseteq D_{k'}^\infty(X_0)\), we must have \(i \in \Gamma^\tau(1, e)\) for some \(e\). Therefore, by the claim, letting \(\sigma + 1 = \tau\), we have \(\Lambda^\tau(1, e) \times \{1, \ldots, k\} \cap D_{k'}^\infty(X_0) = \emptyset\). So, in particular, for any \(j \in \Lambda^\tau(1, e)\) we have \(\{j\} \times \{1, \ldots, k\} \cap D_{\tau}(X_0) = \emptyset\), contradicting the minimality of \(\tau\).

Now put \(a_{\tau}^\tau(i) \otimes a_{\tau}^\tau(j) \in E\) for all \((i, j) \in D_{k'}^\infty(X_0)\). And put \(a_{\tau}^\tau(i) \otimes a_{\tau}^\tau(j) \in E\) for all \((i, j) \in \Gamma^\tau(1, e) \times \Gamma^\tau(1, d)\) such that \((\Gamma^\tau(1, e) \times \Gamma^\tau(1, d)) \cap D_{k'}^\infty(X_0) \neq \emptyset\) and \((i, j) \in \Delta^\tau(1, \gamma, e) \times \Delta^\tau(1, \gamma, d)\) \((\gamma = 2, \ldots, w; \beta = 1, \ldots, q)\).

By Lemma 5.2 (a) and (b) are satisfied for \(a_{\tau^1}\) and therefore also by construction for \(a_{\tau^2}, \ldots, a_{\tau^{w}}\). Moreover, (c) is satisfied between \(a_{\tau^1}, a_{\tau^2}, \ldots, a_{\tau^3}, \ldots, \) between \(a_{\tau^{w-1}}, a_{\tau^{w}}\). Furthermore, by definition of \(X_0\), condition (d) is satisfied in the only relevant place, namely between \(a_{\tau^w}\) and \(a_{\tau^1} \lor a_{\tau^{w+1}} \lor \cdots \lor a_{\tau^{w+q}}\). So we only have to check condition (c) between \(a_{\tau^w}\) and \(a_{\tau^1} \lor a_{\tau^{w+1}} \lor \cdots \lor a_{\tau^{w+q}}\).

Now obviously, since \(a_{\tau^w}^\tau(i) \otimes a_{\tau^w}^\tau(j) \in E\), for \((i, j) \in \Delta^\tau(1, \gamma, e) \times \Delta^\tau(1, \gamma, d)\) \((\beta = 1, \ldots, q)\), we only have to check the condition for the products \(\Gamma^\tau(1, e) \times \Gamma^\tau(1, d)\). So suppose \(a_{\tau^w}^\tau(i) \otimes a_{\tau^w}^\tau(j) \in E\) for some \((i, j) \in \Gamma^\tau(1, e) \times \Gamma^\tau(1, d)\). Then we know that \((\Gamma^\tau(1, e) \times \Gamma^\tau(1, d)) \cap D_{k'}^\infty(X_0) \neq \emptyset\), therefore, as \(D_{k'}^\infty(X_0)\) is \(D\)-stable, we must have \((\Lambda^\tau(1, e) \times \Lambda^\tau(1, d)) \cap D_{k'}^\infty(X_0) \neq \emptyset\). This means that, if
\[(39) \quad \bigvee\{a_{\tau^w}^\tau(i) \otimes a_{\tau^w}^\tau(j) \in E : (i, j) \in \Gamma^\tau(1, e) \times \Gamma^\tau(1, d)\} \neq 0,\]
then also
\[(40) \quad \bigvee\{a_{\tau^1}^\tau(i) \otimes a_{\tau^1}^\tau(j) \in E : (i, j) \in \Lambda^\tau(1, e) \times \Lambda^\tau(1, d)\} \neq 0,\]
confirming (c) in one direction. And, conversely, if
\[(41) \quad \bigvee\{a_{\tau^1}^\tau(i) \otimes a_{\tau^1}^\tau(j) \in E : (i, j) \in \Lambda^\tau(1, e) \times \Lambda^\tau(1, d)\} \neq 0,\]
then, in particular, \((A^1(e) \times A^3(d)) \cap D^{k_{\infty}}(X_0) \neq \emptyset\). So, by the claim, also \((G^1(1, e) \times \Gamma^1(1, d)) \cap D^{k_{\infty}}(X_0) \neq \emptyset\), and therefore

\[
\forall \{a_{i_w}^r(i) \otimes a_{i_w}^r(j) \in E : (i, j) \in \Gamma^1(w, e) \otimes \Gamma^1(w, d)\} \neq 0.
\]

\[\square\]

One can also use well-known results (see e.g. Hjorth [24]) to see that there is a generic increasing homeomorphism of \([0, 1]\).

**Theorem 5.3.** The group \(H_+(0, 1]\) of increasing homeomorphisms of the unit interval has a comeager conjugacy class.

**Proof.** Using the notation of Hjorth it is easy to verify that the set of \(\pi \in H_+(0, 1]\), for which \(\langle \mathcal{M}(\pi) \rangle\) is isomorphic to \(\mathbb{Q}\), \(P_{\mathcal{M}(\pi)} = \emptyset\) and \(P^{\mathcal{M}(\pi)}_{+}\) are dense and unbounded in both directions in \(\langle \mathcal{M}(\pi) \rangle\), is comeager and forms a single conjugacy class. \(\square\)

Notice that this group is connected, so is not a topological subgroup of \(S_\infty\) (In fact, it has been recently proved in Rosendal–Solecki [42] that it is not even an abstract subgroup of \(S_\infty\) either.)

## 6. Ample Generic Automorphisms

### 6.1. The general concept.

Suppose a Polish group \(G\) acts continuously on a Polish space \(X\). Then there is a natural induced (diagonal) action of \(G\) on \(X^n\), for any \(n = 1, 2, \ldots\), defined by \(g \cdot (x_1, x_2, \ldots, x_n) = (g \cdot x_1, g \cdot x_2, \ldots, g \cdot x_n)\). Now, by Kuratowski–Ulam, if there is a comeager \(G\)-orbit in \(X^n\), and \(k \leq n\), then there is also a comeager orbit in \(X^k\). However this is far from being true in the other direction. Let us reformulate the property for \(n = 2\).

Recall that \(\forall^* x R(x)\) means that \(\{x : R(x)\}\) is comeager, where \(x\) varies over elements of a topological space \(X\).

**Proposition 6.1.** Let a Polish group \(G\) act continuously on a Polish space \(X\) and suppose \(X\) has a comeager orbit \(O\). Then the following are equivalent:

(i) There is a comeager orbit in \(X^2\),

(ii) \(\forall x \in O \forall^* y \in \mathcal{O}(G_x \cdot y = \text{comeager in } X)\), where \(G_x = \{g \in G : g \cdot x = x\}\)

is the stabilizer of \(x\),

(iii) \(\forall x \in O \forall^* y \in \mathcal{O}(G_xG_y = \text{comeager in } G)\)

(iv) \(\forall x \in O \forall^* h \in \mathcal{G}(G_xhG_x = \text{comeager in } G)\),

(v) \(\exists x, y \in O(G_x \cdot y = \text{comeager})\).

**Proof.** (i) \(\Rightarrow\) (ii) Let \(C\) be a comeager orbit in \(X^2\). Then by the Kuratowski–Ulam Theorem \(\forall^* x \forall^* y(x, y) \in C\). Thus \(\{x : \forall^* y(x, y) \in C\}\) is comeager and clearly \(G\)-invariant, so for every \(x \in O, \forall^* y(x, y) \in C\). It follows that \(\forall^* y \in \mathcal{O}(G_x \cdot y = \text{comeager})\).

(ii) \(\Rightarrow\) (iii) Fix \(x \in O \) and \(y \in O\) such that \(G_x \cdot y = \text{comeager}\). Since \(O = G_\delta\) in \(X\), by Effros’ Theorem (see, e.g., Becker–Kechris [3]) the map \(\pi : G \to O\) given by \(\pi(g) = g \cdot y\) is continuous and open and therefore \(\pi^{-1}_{\text{comeager}}(G_x \cdot y) = \{g : g \cdot y \in G_x \cdot y = \{g : \exists h \in G_x(h \cdot y \in G_x)\} = G_xG_y\text{ is comeager in } G\).

(iii) \(\Rightarrow\) (iv) We have for any \(x \in O, \forall^* y \in \mathcal{O}(G_xG_y = \text{comeager in } G)\), so, by Effros’ Theorem again, applied this time to \(\sigma(h) = h \cdot x\), we have \(\forall^* h(G_xhG_x = \text{comeager in } G)\). But \(G_{h \cdot x} = hG_xh^{-1}\), so \(\forall^* h(G_xhG_x = \text{comeager})\).
Remark. Given a Polish space \( X \) action of \( S_\sigma \) is invariant by permutation of the coordinates, i.e., invariant under the action by \( (46) \).

\( \sigma \) is continuous and open from \( G \) to \( O \), it follows that \( \pi(\sigma y) = \pi y \) is isomorphism.

(iv) \( \Rightarrow \) (v) Fix \( x, y \in O \) with \( G \cdot y \) comeager. If \( z = g \cdot x \), then \( G \cdot y = G \cdot g \cdot y = G \cdot g^{-1} \cdot g \cdot y = G \cdot y \) is comeager. But \( \{ z \} \times G \cdot y = \{ g \cdot x \} \times G \cdot y \subseteq G \cdot (x, y) \), so \( \forall z \in O \forall u \in X(z, u) \in G \cdot (x, y) \), thus by the Kuratowski–Ulam Theorem, \( G \cdot (x, y) \) is a comeager orbit in \( X^2 \).

In particular, if \( G \) is abelian, there cannot be a comeager orbit in \( X^2 \), unless \( X \) is a singleton.

Remark. Given a Polish space \( X \) and \( n = 1, 2, \ldots \), there is a natural continuous action of \( S_n \) on \( X^n \) given by

\[
\sigma \cdot (x_1, x_2, \ldots, x_n) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}).
\]

If there is a comeager orbit \( A \) in \( X^n \) under the action of \( G \), this orbit will be invariant by permutation of the coordinates, i.e., invariant under the action by \( S_n \). Because for each \( \sigma \in S_n \), \( \sigma \cdot A \) will be comeager, so \( \sigma \cdot A \cap A \neq \emptyset \). Take some \( (y_1, y_2, \ldots, y_n) \in \sigma \cdot A \cap A \) and \( h \in G \) such that \( (y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)}) = (h \cdot y_1, \ldots, h \cdot y_n) \). Then for any \( (x_1, x_2, \ldots, x_n) = (k \cdot y_1, \ldots, k \cdot y_n) \in A \),

we have

\[
(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = (k \cdot y_{\sigma(1)}, \ldots, k \cdot y_{\sigma(n)}) = (k \cdot y_1, \ldots, k \cdot y_n) \in A.
\]

Remark. Naturally one would like to know if there is any Polish group \( G \) acting continuously on a Polish space \( X \), with \( \text{card}(X) > 1 \), for which there is a comeager orbit in \( X^N \). This however turns out not to be the case. For suppose \( \text{card}(X) > 1 \) and let \( \emptyset \neq V \subseteq X \) be open, but not dense. Let \( C \subseteq X^N \) be a comeager orbit, towards a contradiction. Then \( A = \{(x_n) \in X^N : \exists n \in \mathbb{N} (x_n \in V) \} \) is dense and \( G_\delta \), so fix \( (x_n) \in A \cap C \). Then \( \Omega = \{ n \in \mathbb{N} : x_n \in V \} \) is infinite. Put \( B = \{(y_n) \in X^N : \{y_n\}_{n \in \Omega} \text{ is dense in } X \} \), which is a dense \( G_\delta \). So fix \( (y_n) \in B \cap C \). Then there is \( h \in G \) such that \( h \cdot (x_n) = (y_n) \), so that \( \{h \cdot x_n\}_{n \in \Omega} \subseteq h \cdot V \) is dense, a contradiction.

If \( G \) acts continuously on \( X \), we call \( (x_1, \ldots, x_n) \in X^n \) generic if its orbit is comeager. We say that the action has ample generics if for each \( n \), there is a generic element in \( X^n \). In particular, we say that a Polish group \( G \) has ample generic elements if there is a comeager orbit in \( G^n \) (with the conjugacy action of \( G \) on \( G^n \)), for each finite \( n \). If \( \bar{g} \in G^n \) has a comeager orbit, we will refer to it as a generic element of \( G^n \). This is an abuse of terminology as \( G^n \) is itself a Polish group and so it makes sense to talk about a generic element of \( G^n \) viewed as a group. Note that a generic element in the sense we use here is also generic in the group \( G^n \), so our current notion is stronger, and it will be the only one we will use in the rest of the paper. Note that \( G^n \) as a group has a generic element iff \( G \) has a generic element.
Suppose $\mathcal{K}$ is a Fraïssé class, and $\mathbf{K}$ its Fraïssé limit. We would like to characterize as before when $\mathbf{K}$ has ample generic automorphisms.

We know that Aut($\mathbf{K}$)$^n$ has a comeager diagonal conjugacy class iff some $\bar{f} = (f_1, \ldots, f_n) \in$Aut($\mathbf{K}$)$^n$ is turbulent and has dense diagonal conjugacy class. Therefore we can prove exactly as before:

**Theorem 6.2.** Let $\mathcal{K}$ be a Fraïssé class and $\mathbf{K}$ its Fraïssé limit. Then the following are equivalent:

(i) There is a comeager diagonal conjugacy class in Aut($\mathbf{K}$)$^n$,

(ii) $\mathcal{K}_p^n$ has the JEP and the WAP.

So $\mathbf{K}$ has ample generic automorphisms iff for every $n$, $\mathcal{K}_p^n$ has the JEP and the WAP.

We recall the results of Hodges et al. [26] stating that many $\omega$-stable, $\mathcal{L}_0$-categorical structures and the random graph have ample generic automorphisms, but this also holds for many other structures.

**6.2. The Hrushovski property.**

**Definition 6.3.** We say that a Fraïssé class $\mathcal{K}$ satisfies the Hrushovski property if any system $S = \langle A, \psi_1 : B_1 \rightarrow C_1, \ldots, \psi_n : B_n \rightarrow C_n \rangle$ in $\mathcal{K}_p^n$ can be extended to some $T = \langle D, \varphi_1 : D \rightarrow D, \ldots, \varphi_n : D \rightarrow D \rangle$ in $\mathcal{K}_p^n$, i.e., to a sequence of automorphisms of the same finite structure.

Let us first reformulate the Hrushovski property in topological terms as a condition on the automorphism group.

**Proposition 6.4.** Let $\mathcal{K}$ be a Fraïssé class (of finite structures), $\mathbf{K}$ its Fraïssé limit, and Aut($\mathbf{K}$) the automorphism group of $\mathbf{K}$. Then $\mathcal{K}$ has the Hrushovski property if and only if there is a countable chain $C_0 \leq C_1 \leq C_2 \leq \ldots \leq \text{Aut}(\mathbf{K})$ of compact subgroups whose union is dense in Aut($\mathbf{K}$).

**Proof.** Suppose that $\mathcal{K}$ has the Hrushovski property and $G = \text{Aut}(\mathbf{K})$. Then each of the sets

$$F_n = \{ (f_1, \ldots, f_n) \in G^n : \exists x \in K, x's \text{ orbit under } (f_1, \ldots, f_n) \text{ is finite} \}$$

$$= \{ (f_1, \ldots, f_n) \in G^n : (f_1, \ldots, f_n) \text{ is relatively compact in } G \},$$

is comeager, and hence the generic infinite sequence $(f_i)$ in $G$ will generate a dense subgroup all of whose finitely generated subgroups are relatively compact. In particular, $G$ is the closure of the union of a countable chain of compact subgroups.

Conversely, suppose $C_0 \leq C_1 \leq C_2 \leq \ldots \leq G$ is a chain of compact subgroups whose union is dense in $G$. Let $A \subseteq K$ be a finite substructure of $\mathbf{K}$ and let $p_1, \ldots, p_n$ be a sequence of partial automorphisms of $A$. By density, we can find some $m$ such that there are $f_1, \ldots, f_n \in C_m$ with $f_i \supseteq p_i$ for each $i \leq n$. But then $B = C_m \cdot A$ is a finite set and if we let $B$ be the finite substructure of $\mathbf{K}$ generated by $B$ then, as $C_m$ is a group, one can check that $B$ is invariant under elements of $C_m$. In particular, $B$ is closed under the automorphisms $f_i$, whence $(B, f_1[B], \ldots, f_n[B])$ is the extension of the system $(A, p_1, \ldots, p_n)$ needed. □

Hrushovski [28] originally proved the Hrushovski property for the class $\mathcal{K}$ of finite graphs and this was used in Hodges et al. [26] to show that the automorphism group of the random graph has ample generic automorphisms. In fact, the Hrushovski property simply means that the class of all $\mathcal{T}$ as above is cofinal and this is often
enough to show that $\mathcal{K}_p^n$ has the CAP for all $n$. This, combined with the JEP for $\mathcal{K}_p^n$, which is usually not hard to verify, implies the existence of ample generics.

For example, this easily works for the class of finite graphs. Another case is the class of finite metric spaces with rational distances satisfies the Hrushovski property. From this it easily follows that $\mathcal{K}_p^n$ has the CAP for each $n$. Take for notational simplicity $n = 1$.

Let $\mathcal{L} \subseteq \mathcal{K}_p$ be the class of systems $\mathcal{S} = \langle A, \psi : A \to A \rangle$, $A \in \mathcal{K}$. By the Hrushovski property of $\mathcal{K}$ this class is cofinal under embeddability in $\mathcal{K}_p$. We claim that $\mathcal{L}$ has the AP. Let $\psi : A \to A, \phi : B \to B, \chi : C \to C$ be in $\mathcal{K}_p$ with $A \subseteq B, A \subseteq C$ and $\psi \subseteq \phi, \psi \subseteq \chi$. Let $\theta$ be the metric on $B$ and $\rho$ the metric on $C$ and suppose without loss of generality that $B \cap C = A$.

We define the following metric $d$ on $B \cup C$:

- for $x, y \in B$, let $d(x, y) = \theta(x, y)$
- for $x, y \in C$, let $d(x, y) = \rho(x, y)$
- for $x \in B, y \in C$ let $d(x, y) = \min(\theta(x, z) + \rho(z, y) : z \in A)$

One easily checks that this satisfies the triangle inequality. So now we only need to see that $\theta = \phi \cup \chi$ is actually an automorphism of $B \cup C$. Let us first notice that it is well defined as $\phi$ and $\chi$ agree on their common domain $A$. Moreover, trivially $d(x, y) = d(\theta(x), \theta(y))$, whenever both $x, y \in B$ or both $x, y \in C$. So let $x \in B, y \in C$ and find $z \in A$ such that $d(x, y) = d(z, z)$. Then

$$d(\theta(x), \theta(y)) \leq d(\theta(x), \theta(z)) + d(\theta(z), \theta(y)) = d(x, z) + d(z, y) = d(x, y),$$

as $\theta(z) \in A$. And reasoning with $\theta^{-1}$ we get $d(x, y) \leq d(\theta(x), \theta(y))$, so $\theta$ is indeed an isometry of $B \cup C$ with the metric $d$. This can therefore be taken to be our amalgam.

Since the argument in the proof of [22] also shows that $\mathcal{K}_p^n$ has the JEP, for all $n$, it follows from [6] that $U_0$ has ample generic automorphisms.

For a further example of a structure with the Hrushovski property and ample generic automorphisms take $\mathcal{K} = \text{MBA}_Q$, with Fraïssé limit $(F, \lambda)$. If

$$\mathcal{S} = \langle A, \psi_1 : B_1 \to C_1, \ldots, \psi_n : B_n \to C_n \rangle$$

is in $\mathcal{K}_p^n$, then it can be extended to some

$$\mathcal{T} = \langle D, \phi_1 : D \to D, \ldots, \phi_n : D \to D \rangle,$$

i.e., to a sequence of automorphisms of the same structure. Moreover, $D$ can be found such that all the atoms of $D$ have the same measure.

To see this, notice that, as the measure on $A$ only takes rational values, we can refine $A$ to some $B$ such that all its atoms have the same measure. But then, as $\psi_1$ preserves the measure, for any $b \in B_1, b$ is composed of the same number of $D$ atoms as $\psi_1(b)$. So $\psi_1$ can easily be extended to an automorphism $\phi_1$ of $D$.

Let $\mathcal{L}^n$ be the subclass of $\mathcal{K}_p^n$ consisting of systems of the same form as $\mathcal{T}$. We claim that $\mathcal{L}^n$ has the AP. For suppose that $\mathcal{S} = \langle A, \psi_1, \ldots, \psi_n \rangle, \mathcal{T} = \langle B, \phi_1, \ldots, \phi_n \rangle, \mathcal{R} = \langle C, \chi_1, \ldots, \chi_n \rangle$ are in $\mathcal{L}^n$ with $\mathcal{S}$ being a subsystem of $\mathcal{T}$ and $\mathcal{R}$. List the atoms of $A$ as $a_1, \ldots, a_p$, the atoms of $B$ as

$$b_1(1), \ldots, b_1(k_1), \ldots, b_p(1), \ldots, b_p(k_p),$$

and the atoms of $C$ as

$$c_1(1), \ldots, c_1(l_1), \ldots, c_p(1), \ldots, c_p(l_p),$$
where
\[ a_i = b_i(1) \lor \cdots \lor b_i(k_i) = c_i(1) \lor \cdots \lor c_i(l_i) \]

Then we can amalgamate \( T \) and \( R \) over \( S \) by taking atoms \( b_i(e) \otimes c_i(d) \) and sending \( b_i(e) \) to \( \bigvee_{d \leq l_i} b_i(e) \otimes c_i(d) \), \( c_i(d) \) to \( \bigvee_{c \leq k_i} b_i(e) \otimes c_i(d) \) and letting
\[
\nu(b_i(e) \otimes c_i(d)) = \frac{\delta(b_i(e)) \gamma(c_i(d))}{\mu(a_i)}
\]
where \( \mu, \delta \) and \( \gamma \) are the measures on \( A, B \) and \( C \) respectively. Moreover, let
\[
\theta_j(b_i(e) \otimes c_i(d)) = \phi_j(b_i(e)) \otimes \chi_j(c_i(d)).
\]

One easily checks that this is indeed an amalgam of \( T \) and \( R \) over \( S \). As, in this case, all \( n \)-systems have a common subsystem in \( L^n \), AP for \( L^n \) also implies JEP for \( \mathbb{K}^n \), and hence:

**Theorem 6.5.** Let \( (F, \lambda) \) be the Fraïssé limit of \( \mathcal{M} \mathcal{S} \mathcal{A}_Q \). Then \( (F, \lambda) \) has ample generic automorphisms.

It is easy to check that the above works for \( \mathcal{M} \mathcal{S} \mathcal{A}_{\mathbb{Q}_2} \) as well, so the group \( \text{Aut}(\text{clop}(2^{N}), \sigma) \) has ample generic elements and so does the group of measure preserving homeomorphisms of \( 2^N \).

Another property that has been studied in the context of automorphism groups is the existence of a dense locally finite subgroup. Bhattacharjee and Macpherson [8] showed that such a group exists in the automorphism group of the random graph and it is not difficult to see that also \( H(2^{N}, \sigma) \) has one. Let us just mention that if \( \text{Aut}(M) \), for \( M \) a (locally finite) Fraïssé structure, has a dense locally finite subgroup \( H \), then \( M \) has the Hrushovski property. This follows directly from Proposition 6.3.

In particular, there is no dense locally finite subgroup of \( H(2^{N}) \), since \( B_\infty \) does not even have the Hrushovski property. Vershik [18] poses the question of whether \( \text{Aut}(U_0) \) has a locally finite dense subgroup.

### 6.3. Two lemmas
We will now prove two technical lemmas, which generalize and extend some results in Hodges et al. [26]. The second will be repeatedly used later on.

**Lemma 6.6.** Let \( G \) be a Polish group acting continuously on a Polish space \( X \) with ample generics. Let \( A, B \subseteq X \) be such that \( A \) is not meager and \( B \) is not meager in any non-\( \emptyset \) open set. Then if \( \bar{x} \in X^n \) is generic and \( V \) is an open nbhd of the identity of \( G \), there are \( y_0 \in A, y_1 \in B, h \in V \) such that \( (\bar{x}, y_0), (\bar{x}, y_1) \in X^{n+1} \) are generic and \( h \cdot (\bar{x}, y_0) = (\bar{x}, y_1) \).

**Proof.** Let \( C \subseteq X^{n+1} \) be a comeager orbit. Then, by Kuratowski-Ulam, \( \{ \bar{z} \in X^n : \forall^* y(\bar{z}, y) \in C \} \) is comeager and clearly \( G \)-invariant, so it contains \( \bar{x} \) and thus \( \forall^* y(\bar{x}, y) \in C \). If \( y \in C_{\bar{x}} = \{ z : (\bar{x}, z) \in C \} \), then \( C_{\bar{x}} = G_{\bar{x}} \cdot y \), where \( G_{\bar{x}} \) is the stabilizer of \( \bar{x} \) for the action of \( G \) on \( X^n \). Thus for any \( y \in C_{\bar{x}}, G_{\bar{x}} \cdot y \) is comeager. Fix \( y_0 \in A \cap C_{\bar{x}} \). Consider now the action of \( G_{\bar{x}} \) on \( X \). Since \( G_{\bar{x}} \cdot y_0 \) is comeager, it is \( G_{\bar{x}} \), so by Effros’ Theorem, the map
\[
\pi : G_{\bar{x}} \to G_{\bar{x}} \cdot y_0
\]
\[
g \mapsto g \cdot y_0
\]
is continuous and open. Thus \( \pi(G_{\bar{x}} \cap V) = (G_{\bar{x}} \cap V) \cdot y_0 \) is open in \( G_{\bar{x}} \cdot y_0 \), so \( (G_{\bar{x}} \cap V) \cdot y_0 \cap B \neq \emptyset \). Fix then \( y_1 \in (G_{\bar{x}} \cap V) \cdot y_0 \cap B \). Then for some \( h \in G_{\bar{x}} \cap V, h \cdot y_0 = y_1 \) and clearly \( h \cdot (\bar{x}, y_0) = h \cdot (\bar{x}, y_1) \). \( \square \)
One can also avoid the use of Effros’ Theorem in the above proof and instead give an elementary proof along the lines of Proposition 3.2.

Lemma 6.7. Let $G$ be a Polish group acting continuously on a Polish space $X$ with ample generics. Let $A_n, B_n \subseteq X$ be such that, for each $n$, $A_n$ is not meager and $B_n$ is not meager in any non-$\emptyset$ open set. Then there is a continuous map $a \mapsto h_a$ from $2^\mathbb{N}$ into $G$, such that if $a|n = b|n, a(n) = 0, b(n) = 1$, we have $h_a \cdot A_n \cap h_b \cdot B_n \neq \emptyset$.

Proof. Fix a complete metric $d$ on $G$. For $s \in 2^{<\mathbb{N}}$ define $f_s \in G, x_s, x_s^0, x_s^1 \in X$ such that if $x_s = (x_s(0), x_s(2), \ldots, x_s)$, $h_s = f_s|1 \cdots f_s$ ($s \neq 0$), we have

1. $\bar{x}_s$ is generic,
2. $x_s^0 \in A|s|$, $x_s^1 \in B|s|$,
3. $f_s|0 = 1_G$,
4. $d(h_s, h_s f_s|1) < 2^{-|s|}$,
5. $f_s|1 \cdot \bar{x}_s = \bar{x}_s^0$.

We begin by using (47) to find $x_0, x_1, f_0, f_1$ (taking $h_0 = 1$).

Suppose $f_s, x_s$ are given. By (47) again, we can find $x_s^0 \in A|s|$, so that $\bar{x}_s^0, \bar{x}_s^1$ are generic and $g_s \in V = \{f : d(h_s, h_s f) < 2^{-|s|}\}^{-1}$ with $g_s \cdot \bar{x}_s^0 = \bar{x}_s^1$. Let $f_s|1 = g_s|1$.

By (3) and (4), for any $a \in 2^\mathbb{N}$ the sequence $(h_a|n)$ is Cauchy, so converges to some $h_a \in G$. Now consider some $a \in 2^\mathbb{N}$ and $m \prec n$. If $a(n - 1) = 0$, then by (3), $f_a|n = 1_G$ and hence

$$f_a|n \cdot x_a|n - m = x_a|n - m.$$ 

On the other hand, if $a(n - 1) = 1$, then by (5) we have

$$f_a|n \cdot \bar{x}(a|n - 1)^{\cdot 1} = f(a|n - 1)^{\cdot 1} \cdot \bar{x}(a|n - 1) = \bar{x}(a|n - 1)^{\cdot 0};$$

whence, in particular,

$$f_a|n \cdot \bar{x}_a|n - 1 = \bar{x}_a|n - 1.$$

But since $m \prec n$, $x_a|n - m$ is a term in $\bar{x}_a|n - 1$ and thus also

$$f_a|n \cdot x_a|n = x_a|n.$$

Fix now $a \in 2^\mathbb{N}$. Then, as $h_a|n \to a$, we have for any $n$, $h_a|n \cdot x_a|n \to a \cdot x_a|n$. Thus, for any $m$,

$$h_a \cdot x_a|n = \lim_n h_a|n \cdot x_a|n$$

$$= \lim_n f_a|1 \cdots f_a|m \cdot x_a|n$$

$$= f_a|1 \cdots f_a|m \cdot x_a|n$$

$$= h_a|n \cdot x_a|n.$$

So if $a|n = b|n = s, a(n) = 0, b(n) = 1$, we have

$$h_a \cdot x_s^0 = h_s^0 \cdot x_s^0 \cdot x_s^0 = h_a \cdot x_s^0,$$

as $f_s|0 = 1_G$, and

$$h_a \cdot x_s^1 = h_s^1 \cdot x_s^1 = h_s^1 \cdot x_s^1 = h_a \cdot x_s^1,$$

so $h_a \cdot x_s^0 = h_b \cdot x_s^1$, and, since $x_s^0 \in A_n, x_s^1 \in B_n$, we have $h_a \cdot A_n \cap h_b \cdot B_n \neq \emptyset$. \hfill $\square$
6.4. The small index property. We will now discuss the connection of ample generics with the so-called small index property of a Polish group $G$, which asserts that any subgroup of index $< 2^{|G|}$ is open.

Lemma 6.8. (Hodges et al. [26]) Let $G$ be a Polish group. Then any meager subgroup has index $2^{|G|}$ in $G$.

Proof. (Solecki) Notice that $G$ is perfect, i.e., has no isolated points, as otherwise it would be discrete. Let $E = \{(g, h) \in G^2 : gh^{-1} \in H\}$, where $H \leq G$ is a meager subgroup. Then as $(g, h) \mapsto gh^{-1}$ is continuous and open, $E$ must be a meager equivalence relation and therefore by Mycielski’s Theorem (see Kechris [31] (19.1)) have $2^{|G|}$ classes. □

Theorem 6.9. Let $G$ be a Polish group with ample generics. Then $G$ has the small index property.

Proof. Suppose $H \leq G$ has index $< 2^{|G|}$ but is not open. Then $H$ is not meager by [26, 31]. Also $G \setminus H$ is not meager in any non-$\emptyset$ open set, since otherwise $H$ would be comeager in some non-$\emptyset$ open set and thus, by Pettis’ Theorem (see Kechris [31] (9.9)), $H$ would be open. We apply now Lemma 6.7 to the action of $G$ on itself by conjugation and $A_1 = H$, $B_1 = G \setminus H$. Then, if $a \neq b \in 2^{|G|}$, with say $a|n = b|n, a(n) = 0, b(n) = 1, h_aHh_a^{-1} \cap h_b(G \setminus H)h_b^{-1} \neq \emptyset$, so $(h_a^{-1}h_b)H(h_b^{-1}h_a) \cap (G \setminus H) \neq \emptyset$, therefore $h_b^{-1}h_a \notin H$, thus $h_a, h_b$ belong to different cosets of $H$, a contradiction. □

We will now apply this to the group of measure preserving homeomorphisms of the Cantor space.

Lemma 6.10. Suppose $A, B$ are two finite subalgebras of $\text{clop}(2^{|G|})$ and $G = \text{Aut}(\text{clop}(2^{|G|}), \sigma)$. Then $\langle G(A), G(B) \rangle = G(A \cap B)$, where $G(A)$ is the pointwise stabilizer of $A$.

Proof. Since $G(A)$ is open, so is $\langle G(A), G(B) \rangle$, and it is therefore also closed. Moreover, it is trivially contained in $G(A \cap B)$, so it is enough to show that $\langle G(A) \cup G(B) \rangle$ is dense in $G(A \cap B)$ Suppose $D$ is a finite subalgebra of $\text{clop}(2^{|G|})$ with a set of atoms $X$ all of which have the same measure. Suppose moreover, that $A, B \subseteq D$. Let $\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_m\}, \{c_1, \ldots, c_k\}$ be the partitions of $X$ given by the atoms of the subalgebras $A, B, A \cap B = C$ of $D = \mathcal{P}(X)$. Since all of the elements of $X$ have the same measure, it is enough to show that any permutation $\rho$ of $X$, pointwise fixing $C$, i.e., $\rho^n c_i = c_i, i = 1, \ldots, k$, is in $\langle G(A) \cup G(B) \rangle$. In fact it is enough to show this for any transposition $\rho$. Let $\sigma_{x, y}$ denote the transposition switching $x$ and $y$ (we allow here $x = y$). Fix $x_0 \in X$ and let $V_{x_0} \subseteq X$ be the set of atoms $y \in X$ such that $\sigma_{x_0, y} \in \langle G(A) \cup G(B) \rangle$. Then $V_{x_0} \subseteq C$. For if, e.g., $a_i \cap V_{x_0} \neq \emptyset$, then there is some $y \in a_i \cap V_{x_0}$. So for any $z \in a_i$ we have $\sigma_{x_0, y} \circ \sigma_{y, z} = \sigma_{x_0, z} \in \langle G(A) \cup G(B) \rangle$, and hence $\sigma_{x, z} = \sigma_{x_0, y} \circ \sigma_{y, z} = \sigma_{x, y} \in \langle G(A) \cup G(B) \rangle$. This shows that $V_{x_0} \subseteq A$ and a similar argument shows that $V_{x_0} \subseteq B$. Therefore, if $\sigma_{x, y}$ is a transposition in $\text{Sym}(X)$ pointwise fixing $C$, then $x, y$ belong to the same atom of $C$ and we have $y \in V_x$, showing that $\sigma_{x, y} \in \langle G(A) \cup G(B) \rangle$. □

Let $G$ be a closed subgroup of $S_\infty$. $G$ is said to have the strong small index property if whenever $H \leq G$ and $[G : H] < 2^{|G|}$ there is a finite set $X \subseteq \mathbb{N}$ such that $G(X) \leq H \leq G(X)$, where $G(X)$ is the setwise stabilizer of $X$. 

Theorem 6.11. Let \( G = \text{Aut}(\text{clop}(2^\mathbb{N}), \sigma) \cong H(2^\mathbb{N}, \sigma) \). Then

(i) \( G \) has ample generics,

(ii) \( G \) has the strong small index property.

Proof. We know that \( G \) has ample generics, so has the small index property.

Suppose \([G : H] < 2^{\mathfrak{c}}\). Then \( H \) is open and therefore for some finite \( Y \subseteq \text{clop}(2^\mathbb{N}) \), we have \( G(Y) \leq H \). If \( A \) is the subalgebra generated by \( Y \), then obviously \( G(A) = G(Y) \leq H \). We notice that if \( h \in H \) then \( G_{(h^nA)} = hG(A)h^{-1} \leq H \) and by the lemma \( G_{(A \setminus h^nA)} = G(A) \cup G_{(h^nA)} \leq H \). So, as \( A \) is finite, we have, for \( B = \bigcap_{h \in H} h^nA \), that \( G(B) \leq H \). But trivially also \( H \leq G(B) \). So \( G \) has the strong small index property.

6.5. Uncountable cofinality. In the same manner as in Hodges et al. [26], we can also obtain the following result (analogous to their Theorem 6.1).

Theorem 6.12. Let \( G \) be a Polish group with ample generic elements. Then \( G \) is not the union of a countable chain of non-open subgroups.

Proof. Let \( G = \bigcup_n A_n \), where \( A_0 \subseteq A_1 \subseteq \ldots \) are non-open subgroups. We can then assume that each \( A_n \) is non-meager. Also, as in the proof of (ii) of 6.5, each \( G \setminus A_n \) is non-meager in every non-\( \emptyset \) open set. Apply then Lemma 6.7 to \( A_n \) and \( B_n = G \setminus A_n \) to find \( h_n, a \in 2^\mathbb{N} \), in \( G \), so that if \( a|n = b|n, a(n) = 0, b(n) = 1 \), then

\[
(h_n A_n h_n^{-1}) \cap (h_b (G \setminus A_n) h_b^{-1}) \neq \emptyset,
\]

whence \( h_a \notin A_n \) or \( h_b \notin A_n \). Find uncountable \( C \subseteq 2^\mathbb{N} \) and \( m \) so that \( h_a \in A_m \) for all \( a \in C \). Then find \( a, b \in C \) such that \( h_a, h_b \in A_m \) and \( a|n = b|n, a(n) = 0, b(n) = 1 \) for some \( n > m \). As \( h_a, h_b \in A_n \), we have a contradiction.

Let us also mention here the following result of Cameron (see Hodges et al. [26]): If \( G \) is an oligomorphic closed subgroup of \( S_\mathbb{N} \), i.e., if \( G \) is the automorphism group of some \( K_\mathbb{N} \)-categorical structure or equivalently \( G \) has finitely many orbits on each \( \mathbb{N}^n \), then any open subgroup of \( G \) is only contained in finitely many subgroups of \( G \). Recall also that any open subgroup of a Polish group is actually clopen. So we have that if \( G \) is either connected or an oligomorphic closed subgroup of \( S_\mathbb{N} \), then \( G \) is not the union of a chain of proper subgroups. And therefore if \( G \) has ample generic elements it must have uncountable cofinality, i.e., \( G \) is not the union of a countable chain of proper subgroups.

Notice also that if a Polish group \( G \) is topologically finitely generated, then it cannot be the union of a countable chain of proper open subgroups, so, if it also has ample generics, then it has uncountable cofinality.

To summarize: If \( G \) is a Polish group with ample generics and one of the following holds:

(i) \( G \) is an oligomorphic closed subgroup of \( S_\mathbb{N} \),

(ii) \( G \) is connected,

(iii) \( G \) is topologically finitely generated,

then \( G \) has uncountable cofinality.

6.6. Coverings by cosets. One can also use the main lemma to prove an analog of the following well-known group theoretical result due to B.H. Neumann [40]: If a group \( G \) is covered by finitely many cosets \( \{g_iH_i\}_{i \leq n} \), then for some \( i \), \( H_i \) has finite index.
Before we consider the main result in this setting, let us briefly look at a simple fact.

**Proposition 6.13.** Suppose $G$ is a Polish group with a comeagre conjugacy class. Then the smallest number of cosets of proper subgroups needed to cover $G$ is $\aleph_0$.

**Proof.** Assume $G = g_1H_1 \cup \ldots \cup g_nH_n$. Then by Neumann’s lemma, some $H_i$ is of finite index in $G$. But then $\langle H_i \rangle$ must contain some subgroup $N$ which is normal and of finite index in $G$. As a subgroup of finite index cannot be meagre, $N$ must intersect and therefore contain the comeagre conjugacy class, and thus be equal to $G$. So $G = N \subseteq H_i$. \qed

In the same manner one can show that if $G$ is partitioned into finitely many pieces, one of these pieces generates $G$. We should mention that this result does not generalize to partitions of a generating set. For example, Abért \cite{a} has shown that $S_\infty$ is generated by two abelian subgroups, but evidently $S_\infty$ is not abelian itself.

**Theorem 6.14.** Let $G$ be a Polish group with ample generics. Then for any countable covering of $G$ by cosets $\{g_iH_i\}_{i \in \mathbb{N}}$, there is an $i$ such that $H_i$ is open and thus has countable index.

The theorem clearly follows from the following more general lemma.

**Lemma 6.15.** Let $G$ be a Polish group with ample generics. If $\{k_iA_i\}_{i \in \mathbb{N}}$ is any covering of $G$, with $k_i \in G$ and $A_i \subseteq G$, then for some $i$

$$A_i^{-1}A_iA_i^{-1}A_i^{-1}A_iA_i^{-1}A_i,$$

contains an open nbhd of the identity.

**Proof.** First enumerate in a list $\{B_n\}_{n \in \mathbb{N}}$ all non-meager $A_i$’s, so that each one of them appears infinitely often. Then $\bigcup_{l,n \in \mathbb{N}} k_lB_n = B$ is clearly comeagre. Notice now that if for some $n$ the set $B_n^{-1}B_nB_n^{-1}B_n$ is comeagre in some non-$\emptyset$ open set, then, by Pettis’ Theorem, we are done, so we can assume that $C_n = G \setminus (B_n^{-1}B_nB_n^{-1}B_n)$ is not meager in any non-$\emptyset$ open set, so, by Lemma \cite{6.7} there is a continuous map $\alpha \mapsto h_\alpha$ from $2^{\mathbb{N}}$ into $G$ such that if $a|n = b|n, a(n) = 0, b(n) = 1$, then $h_\alpha B_n h_\alpha^{-1} \cap h_\beta C_n h_\beta^{-1} \neq \emptyset$. Since $B_n \cap C_n = \emptyset$ it also follows that $a \mapsto h_\alpha$ is injective, so $K = \{h_\alpha : a \in 2^{\mathbb{N}}\}$ is homeomorphic to $2^{\mathbb{N}}$.

The map $(g, h) \in G \times K \mapsto g^{-1}h \in G$ is continuous and open so, since

$$\bigcup_{l,n \in \mathbb{N}} k_lB_n = B$$

is comeagre in $G$, we have, by Kuratowski–Ulam,

$$\forall^* g \in G \forall^* h \in K (g^{-1}h \in B),$$

so fix $g \in G$ with $\forall^* h \in K(h \in gB)$. Then fix $l, n \in \mathbb{N}$ such that, letting $g_0 = gk_l$, the set

$$\{a \in 2^{\mathbb{N}} : h_\alpha \in g_0B_n\}$$

is non-meager, so dense in some

$$N_t = \{a \in 2^{\mathbb{N}} : t \subseteq a\},$$
Suppose $G$ is a Polish group with ample generics and that $G$ acts on a set $X$. Then the following are equivalent:

(i) All orbits are of size $|X|$. 
(ii) All orbits are uncountable. 
(iii) For every countable set $A \subseteq X$ there is a $g \in X$ such that $g \cdot A \cap A = \emptyset$.

Proof. (i)$$\implies$$ (ii) is trivial.

(ii)$$\implies$$ (iii): Suppose $A = \{a_0, a_1, \ldots\}$ and that (iii) fails for $A$. Then clearly, $G = \bigcup_{i,j \in \mathbb{N}} G_{a_i,a_j}$, where $G_{a_i,a_j} = \{g \in G \mid g \cdot a_i = a_j\}$. Since each $G_{a_i,a_j}$ is a coset of the subgroup $G_{a_i}$, this implies by Theorem 6.14 that for some $i$, $|G : G_{a_i}| \leq \aleph_0$, and hence $G \cdot a_i$ is countable.

(iii)$$\implies$$ (i): Suppose some orbit $\mathcal{O}$ has cardinality strictly smaller than the continuum. Then for any $x \in \mathcal{O}$, $|G : G_x| = |\mathcal{O}| < 2^{\aleph_0}$, so by the small index property of $G$, $G_x$ is open and hence of countable index in $G$. Thus $|\mathcal{O}| \leq \aleph_0$, and hence $A = \mathcal{O}$ contradicts (iii). \qed

6.7. Finite generation of groups. We will now investigate a strengthening of uncountable cofinality that concerns the finite generation of permutation groups, a subject recently originated in Bergman [7] and also studied in Droste-Göbel [12], Droste-Holland [13].

Definition 6.17. A group $G$ is said to have the Bergman property iff for each exhaustive sequence of subsets $W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq G$, there are $n$ and $k$ such that $W_n^k = G$.

If $G$ has the stronger property that for some $k$ and each exhaustive sequence of subsets $W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq G$, there is $n$ such that $W_n^k = G$, we say that $G$ is $k$-Bergman.

Bergman [7] proved the $k$-Bergman property for $S_\infty$ and subsequently Droste and Göbel [12] found a sufficient condition for certain permutation groups to have this property.

We will see that ample generics also provide an approach to this problem.

Proposition 6.18. Let $G$ be a Polish group with ample generic elements and suppose $A_0 \subseteq A_1 \subseteq \cdots \subseteq G$ is an exhaustive sequence of subsets of $G$. Then there is an $i$ such that $1 \in \text{Int}(A_i^{10})$.

Proof. Notice first that also $A_0 \cap A_0^{-1} \subseteq A_1 \cap A_1^{-1} \subseteq \cdots$ exhausts $G$. For given $g \in G$ find $m$ such that $g, g^{-1} \in A_m$; then $g \in A_m \cap A_m^{-1}$. So we can suppose that each $A_n$ is symmetric. As $\{A_n\}_{n \in \mathbb{N}}$ is a covering of $G$, there is, by Lemma 6.7 some $i$ such that

$$A_i^{10} = A_i^{-1} A_i^{-1} A_i^{-1} A_i^{-1} A_i^{-1} A_i^{-1} A_i^{-1} A_i^{-1} A_i^{-1} A_i$$
contains an open neighborhood of the identity. □

In the case of oligomorphic groups we have the following, whose proof generalizes some ideas of Cameron:

**Theorem 6.19.** Suppose $G$ is a closed oligomorphic subgroup of $S_\infty$ with ample generic elements. Then $G$ is 21-Bergman.

**Proof.** Suppose that $W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq G$ is an exhaustive chain of subsets of $G$. Then by Proposition 6.18 there is an $n$ such that $W_n^{10}$ contains an open neighborhood of the identity. Find some finite sequence $\mathcal{P} \in N^{<N}$, $|\mathcal{P}| = m$, such that $G(\mathcal{P}) \subseteq W_n^{10}$. Then as $G$ is oligomorphic there are only finitely many distinct orbits of $G(\mathcal{P})$ on $N^m$. Choose representatives $\mathcal{P}_1, \ldots, \mathcal{P}_k$ for each of the $G(\mathcal{P})$ orbits on $N^m$ that intersect $G \cdot \mathcal{P}$ and find $h_1, \ldots, h_k \in G$ such that $h_i \cdot \mathcal{P} = \mathcal{P}_i$. In other words, for each $f \in G$ there are $i \leq k$ and $g \in G(\mathcal{P})$ such that $f \cdot \mathcal{P} = g \cdot \mathcal{P}_i = gh_i \cdot \mathcal{P}$. Now find $l \geq n$ sufficiently big such that $h_1, \ldots, h_k \in W_l$. We claim that $W_l^{21} = G$. Let $f$ be any element of $G$ and find $i \leq k$ and $g \in G(\mathcal{P})$ with $f \cdot \mathcal{P} = g \cdot h_i \cdot \mathcal{P}$. Then $h_i^{-1}f^{-1}g \in G(\mathcal{P})$ and thus $f \in G(\mathcal{P})W_lG(\mathcal{P}) \subseteq W_l^{21}$, i.e., $W_l^{21} = G$. □

Let us now make the following trivial but useful remarks: Suppose $M$ is some countable structure and $G = \text{Aut}(M)$ has ample generics. If furthermore for any finitely generated substructure $A \subseteq M$ there is a $g \in G$ such that $G = \langle G(A), G(g''A) \rangle$, then $G$ has uncountable cofinality. This follows easily from Theorem 6.12 as $G(g''A) = gG(A)g^{-1}$.

If moreover there is a finite $n$ such that $G = \langle G(A), G(g''A) \rangle^n$, then $G$ has the Bergman property. For suppose $W_0 \subseteq W_1 \subseteq W_2 \subseteq \ldots \subseteq G$ is an exhaustive chain of subsets of $G$. Then by Proposition 6.18 there is an $m$ such that $W_m^{10}$ contains an open neighborhood of the identity. Say it contains $G(A)$ for some finitely generated substructure $A \subseteq M$. Find $g \in G$ and $n$ as above. Then for $k \geq m$ big enough, $g, g^{-1} \in W_k$ and

$$G = \langle G(A), G(g''A) \rangle^n = \langle G(A), gG(A)g^{-1} \rangle^n \subseteq (W_m^{10}W_kW_m^{10}W_k)^n \subseteq W_k^{22n}. \quad (50)$$

This situation is less rare than one might think. In fact:

**Theorem 6.20.** The group $H(2^N, \sigma)$ of measure preserving homeomorphisms of the Cantor space is 32-Bergman.

We will, as always, identify $H(2^N, \sigma)$ and $\text{Aut}(\text{clop}(2^N), \sigma)$.

**Proof.** Recall that $2^Z$ is measure preserving homeomorphic to $2^N$, so we will temporarily work on the former space. Let $g$ be the Bernoulli shift on $2^Z$ seen as an element of $\text{Aut}(\text{clop}(2^Z), \sigma)$.

Suppose $A$ is a finite subalgebra of $\text{clop}(2^Z)$. By refining $A$, we can suppose that $A$ has atoms $N_s = \{x \in 2^Z : x \upharpoonright [-k,k] = s\}$ for $s \in 2^{2k+1}$. Then $(g^{2k+1})''A$ is independent of $A$. Let $h = g^{2k+1}$. We wish to show that $G(A)G((h''A))G(A) = G$. Since $G(A)$ is an open neighborhood of the identity, it is enough to show that $G(A)G((h''A))G(A)$ is dense in $G$, and for this it suffices to show that whenever $D$ is a finite subalgebra and $f_0 \in G(D)$, the setwise stabilizer of $D$, then there is some $f_1 \in G(A)G((h''A))G(A)$ agreeing with $f_0$ on $D$, i.e., $f_0|_D = f_1|_D$. We notice first that by refining $D$, we can suppose that $D$ is the subalgebra of $\text{clop}(2^Z)$ generated by $A$ and some finite subalgebra $B \supseteq h''A$ that is independent of $A$ and all of
whose atoms have the same measure. For concreteness, we can suppose that $\mathbf{B}$ has atoms $N_{l,r} = \{x \in 2^\omega : x \mid_{-l,-k} = t$ and $x \mid_{k,l} = r\}$, for $t, r \in 2^{l-k}$, where $l$ is some number $> k$. List the atoms of $\mathbf{A}$ as $a_1, \ldots, a_n$ and the atoms of $\mathbf{B}$ as $b_1, \ldots, b_m$. Then the atoms of $\mathbf{D}$, namely $a_i \cap b_j$, all have the same measure and can be identified with formal elements $a_i \otimes b_j$, $i \leq n, j \leq m$. So an element of $G_{\{\mathbf{D}\}}$ gives rise to an element of $S = \text{Sym}(\{a_i \otimes b_j : i \leq n, j \leq m\})$, whereas elements of $G_{\{\mathbf{A}\}} \cap G_{\{\mathbf{D}\}}$ and $G_{\{\mathbf{B}\}} \cap G_{\{\mathbf{D}\}}$ give rise to elements of $S$ that preserve respectively the first and the second coordinates.

We now need the following well-known lemma, see, for example, Abért [1] for a proof:

**Lemma 6.21.** Let $\Omega = \{a_i \otimes b_j : i \leq n, j \leq m\}$ and let $F \leq \text{Sym}(\Omega)$ be the subgroup that preserves the first coordinates and $H \leq \text{Sym}(\Omega)$ be the subgroup that preserves the second coordinates. Then $\text{Sym}(\Omega) = FHF$.

This finishes the proof. $\square$

Some of the results of this section are related to independent research by A. Ivanov [30]. His setup is slightly different from ours as he formulates his results in terms of amalgamation bases.

### 6.8. Actions on trees

Macpherson and Thomas [38] have recently found a relationship between the existence of a comeager conjugacy class in a Polish group and actions of the group on trees. This is further connected with Serre’s property (FA) that we will verify for the group of (measure preserving) homeomorphisms of $2^\mathbb{N}$.

A *tree* is a graph $T = (V, E)$ that is uniquely path connected, i.e., $E$ is a symmetric irreflexive relation on the set of vertices $V$ such that any two vertices are connected by a unique path.

A group $G$ is said to *act without inversions* on a tree $T$ if there is an action of $G$ by automorphisms on $T$ such that for any $g \in G$ there are no two adjacent vertices $a, b$ on $T$ such that $g \cdot a = b$ and $g \cdot b = a$. Now we can state Serre’s property (FA):

A group $G$ is said to have *property* (FA) if whenever $G$ acts without inversions on a tree $T = (V, E)$, there is a vertex $a \in V$ such that for all $g \in G, g \cdot a = a$.

We say that a free product with amalgamation $G = G_1 \star_A G_2$ is *trivial* in case one of the $G_i$ is equal to $G$.

For $G$’s that are not countable we have the following characterization of property (FA), from Serre [33]: $G$ has property (FA) iff

(i) $G$ is not a non-trivial product with amalgamation,

(ii) $\mathbb{Z}$ is not a homomorphic image of $G$,

(iii) $G$ is not the union of a countable chain of proper subgroups.

**Theorem 6.22.** (Macpherson-Thomas [38]) Let $G$ be a Polish group with a comeager conjugacy class. Then $G$ cannot be written as a free product with amalgamation.

Moreover, it is trivial to see that if a Polish group has a comeager conjugacy class then also (ii) holds. In fact every element of $G$ is a commutator. For suppose $C$ is the comeager conjugacy class and $g$ is an arbitrary element of $G$. Then both $C$ and $gC$ are comeager, so $C \cap gC \neq \emptyset$. Take some $h, f \in C$ such that $h = gf$. Then for some $k \in G$ we have $gf = h = kf^{-1}$ and $g = kf^{-1}f^{-1}$. So $G$ has only one abelian quotient, namely $\{e\}$. For if for some $H \leq G, G/H$ is abelian, then $G = [G, G] \leq H$. 

So to verify whether a Polish group with a comeager conjugacy class has property (FA), we only need to show it has uncountable cofinality.

Another way of approaching property (FA) is through the Bergman property, which one can see is actually very strong. For example, one can show that it implies that any action of the group by isometries on a metric space has bounded orbits. But in the case of isometric actions on real Hilbert spaces or automorphisms of trees, having a bounded orbit is enough to ensure that there is a fixed point. So groups with the Bergman property automatically have property (FA) and property (FH) (the latter says that any action by isometries on a real Hilbert space has a fixed point). Thus in particular, \( H(2^\mathbb{N}, \sigma) \) has both properties (FA) and (FH).

6.9. Generic freeness of subgroups. Let us next mention another application of the existence of dense diagonal conjugacy classes in each \( G^n \). Suppose \( G \) is Polish and has a dense diagonal conjugacy class in \( G^n \), for each \( n \in \mathbb{N} \). Then there is a dense \( G_\delta \) subset \( C \subseteq 2^\mathbb{N} \) such that any two sequences \( (f_n) \) and \( (g_n) \in C \) generate isomorphic groups, i.e., the mapping \( f_n \mapsto g_n \) extends to an isomorphism of \( \langle f_n \rangle \) and \( \langle g_n \rangle \).

To see this, suppose \( w(X_1, \ldots, X_n) \) is a reduced word. Then either \( w(g_1, g_2, \ldots, g_n) = e \), for all \( g_1, \ldots, g_n \), or \( w(g_1, g_2, \ldots, g_n) \neq e \) for an open dense set of \( (g_1, \ldots, g_n) \in G^n \). (This is trivial as \( w(g_1, g_2, \ldots, g_n) = e \iff w(kg_1k^{-1}, kg_2k^{-1}, \ldots, kg_nk^{-1}) = e \), for all \( k \in G \) and \( (g_1, \ldots, g_n) \in G^n \).) In any case, taking the intersection over all reduced words, we have that there is a dense \( G_\delta \) set \( C \subseteq 2^\mathbb{N} \) such that any two sequences \( (f_n) \) and \( (g_n) \) in \( C \) satisfy the same equations \( w(X_1, \ldots, X_n) = e \), i.e., \( f_n \mapsto g_n \) extends to an isomorphism.

In particular, if for any nontrivial reduced word \( w(X_1, \ldots, X_n) \) there are \( g_1, \ldots, g_n \in G \) such that \( w(g_1, \ldots, g_n) \neq e \), then any sequence in \( C \) freely generates a free group and, using the Kuratowski–Mycielski Theorem (see Kechris [31] (19.1)) the generic compact subset of \( G \) also freely generates a free group.

Macpherson [37] shows that any oligomorphic closed subgroup of \( S_\infty \) contains a free subgroup of infinite rank. It is not hard to see that it also holds for the groups \( \text{Aut}([0,1], \lambda), H(2^\mathbb{N}, \sigma), \text{Aut}(U_0) \) and \( \text{Iso}(U) \). Moreover, \( H(2^\mathbb{N}) \) and these latter groups also have dense conjugacy classes in each dimension (for \( \text{Aut}(U_0) \) and \( H(2^\mathbb{N}) \)) it is enough to use the multidimensional version of Theorem 2.11 analogous to Theorem 4.2.

This property has been studied by a number of authors (see e.g. Gartside and Knight [17]) and it seems to be a fairly common phenomenon in bigger Polish groups.

6.10. Automatic continuity of homomorphisms. We now come to the study of automatic continuity of homomorphisms from Polish groups with ample generics. Notice that the small index property can be seen as a phenomenon of automatic continuity. In fact, if a topological group \( G \) has the small index property, then any homomorphism of \( G \) into \( S_\infty \) will be continuous. For the inverse image of a basic open neighborhood of \( 1_{S_\infty} \) will be a subgroup of \( G \) with countable index and therefore open. But of course this puts a strong condition on the target group, namely that it should have a neighborhood basis at the identity consisting of open subgroups. We would like to have some less restrictive condition on the target group that still ensures automatic continuity of any homomorphism from a Polish
group with ample generics. Of course, some restriction is necessary, for the identity function from a Polish group into itself, equipped with the discrete topology, is never continuous unless the group is countable.

Lemma 6.23. Let $H$ be a topological group and $\kappa$ a cardinal number. Then the following are equivalent:

(i) For each open neighborhood $V$ of $1_H$, $H$ can be covered by $< \kappa$ many right translates of $V$.

(ii) For each open neighborhood $V$ of $1_H$, there are not $\kappa$ many disjoint right translates of $V$.

Proof. $(i) \Rightarrow (ii)$: Suppose $(ii)$ fails and $\{Vf_\xi\}_{\xi<\kappa}$ is a family of $\kappa$ many disjoint right translates of some open neighborhood $V$ of $1_H$. By choosing $V$ smaller, we can suppose that $V$ is actually symmetric. Then if $\{Vg_\nu\}_{\nu<\lambda}$ covers $H$ for some $\lambda < \kappa$, there are $f_\xi, f_\zeta$ and $g_\nu$, $\xi \neq \zeta$, such that $f_\xi, f_\zeta \in Vg_\nu$, and hence $g_\nu \in Vf_\xi \cap Vf_\zeta$, contradicting that $Vf_\xi \cap Vf_\zeta = \emptyset$. So $(i)$ fails.

$(ii) \Rightarrow (i)$: Suppose that $(ii)$ holds and $V$ is some open neighborhood of $1_H$. Find some open neighborhood $U \subseteq V$ of $1_H$ such that $U^{-1}U \subseteq V$ and choose a maximal family of disjoint right translates $\{Uf_\xi\}_{\xi<\lambda}$ of $U$. By $(ii)$, $\lambda < \kappa$. Suppose that $g \in H$. Then there is a $\xi < \lambda$ such that $Ug \cap Uf_\xi \neq \emptyset$, whereby $g \in U^{-1}Uf_\xi \subseteq Vf_\xi$. So $\{Vf_\xi\}_{\xi<\lambda}$ covers $H$ and $(i)$ holds.

Recall that the Souslin number of a topological space is the least cardinal $\kappa$ such that there is no family of $\kappa$ many disjoint open subsets of the space. By analogy, for a topological group $H$, let the uniform Souslin number be the least cardinal $\kappa$ satisfying the equivalent conditions of the above lemma.

Notice that the uniform Souslin number is at most the Souslin number, which is again at most density$(H)^+$, where the density of a topological space is the smallest cardinality of a dense subset. In particular, if $H$ is a separable topological group, then its (uniform) Souslin number is at most $\aleph_1 \leq 2^{\aleph_0}$. We should mention that groups with uniform Souslin number at most $\aleph_1$ have been studied extensively under the name $\aleph_0$-bounded groups (see the survey article by Tkachenko [18]).

A well-known result (see, e.g., Guran [21]) states that a Hausdorff topological group is $\aleph_0$-bounded if and only if it (topologically) embeds as a subgroup into a direct product of second countable groups. $\aleph_0$-bounded groups are easily seen to contain the $\sigma$-compact groups. Moreover, the uniform Souslin number is productive in contradistinction to separability, i.e., any direct product of groups with uniform Souslin number at most $\kappa$ has uniform Souslin number at most $\kappa$ (for any infinite $\kappa$).

Theorem 6.24. Suppose $G$ is a Polish group with ample generic elements and $\pi : G \to H$ is a homomorphism into a topological group with uniform Souslin number at most $2^{\aleph_0}$ (in particular, if $H$ is separable). Then $\pi$ is continuous.

Proof. It is enough to show that $\pi$ is continuous at $1_G$. So let $W$ be an open neighborhood of $1_H$. We need to show that $\pi^{-1}(W)$ contains an open neighborhood of $1_G$. Pick a symmetric open neighborhood $V$ of $1_H$ such that $V^{20} \subseteq W$ and put $A = \pi^{-1}(V^{-1}V) = \pi^{-1}(V^2)$.

Claim 1. $A$ is non-meager.

Proof. Otherwise, as $(g,h) \mapsto gh^{-1}$ is open and continuous from $G^2$ to $G$, there is by the Mycielski theorem a Cantor set $C \subseteq G$, such that for any $g \neq h$ in $C,$
So this means that there are continuum many disjoint translates of \( V \), contradicting that the uniform Souslin number of \( H \) is at most \( 2^{\aleph_0} \).

Claim 2. A covers \( G \) by \( < 2^{\aleph_0} \) many right translates.

Proof. By the condition on the uniform Souslin number we can find a covering \( \{V_{f_\xi}\}_{\xi<\lambda} \) of \( H \) by \( \lambda < 2^{\aleph_0} \) many right translates of \( V \). So for each \( V_{f_\xi} \) intersecting \( \pi(G) \) take some \( g_\xi \in G \) with \( \pi(g_\xi) \in V_{f_\xi} \). Then \( V_{f_\xi} \subseteq VV^{-1}\pi(g_\xi) = V^2\pi(g_\xi) \), so the latter cover \( \pi(G) \). Now, if \( g \in G \) find \( \xi < \lambda \) such that \( \pi(g) \in V^2\pi(g_\xi) \), whence \( \pi(gg_\xi^{-1}) \in V^2 \), i.e., \( gg_\xi^{-1} \in A \) and \( g \in Ag_\xi \). So the \( Ag_\xi \) cover \( G \).

Claim 3. \( A^5 \) is comeager in some non-\( \emptyset \) open set.

Proof. Otherwise, by Lemma 6.24 we can find \( h_a, a \in 2^{\aleph_1} \), in \( G \) so that if \( a|n = b|n, a(n) = 0, b(n) = 1 \),

\[
h_aAh_a^{-1} \cap h_b(G \setminus A^5)h_b^{-1} \neq \emptyset
\]
or equivalently

\[
h_b^{-1}h_aAh_a^{-1}h_b \cap (G \setminus A^5) \neq \emptyset.
\]

Since \( A \) covers \( G \) by \( < 2^{\aleph_0} \) right translates, and thus left translates (as \( A \) is symmetric), there is uncountable \( B \subseteq 2^{\aleph_1} \) and \( g \in G \) such that for \( a \in B, h_a \in gA \). If \( a, b \in B, a|n = b|n, a(n) = 0, b(n) = 1 \), then let \( g_a, g_b \in A \) be such that \( h_a = gg_a, h_b = gg_b \). Then

\[
h_b^{-1}h_aAh_a^{-1}h_b = g_b^{-1}g_aAg_a^{-1}g_a^{-1}gg_b = g_b^{-1}g_aAg_a^{-1}g_b \subseteq A^5,
\]
a contradiction.

So, by Pettis’ theorem, \( A^{10} \subseteq \pi^{-1}(W) \) contains an open neighborhood of \( 1_G \) and \( \pi \) is continuous.

Corollary 6.25. Suppose \( G \) is a Polish group with ample generic elements. Then \( G \) has a unique Polish group topology.

Proof. Suppose that \( \tau \) and \( \sigma \) are two Polish group topologies on \( G \) such that \( G \) has ample generics with respect to \( \tau \). Then the identity mapping from \( G, \tau \) to \( G, \sigma \) is continuous, i.e., \( \sigma \subseteq \tau \). But then \( \tau \) must be included in the Borel algebra generated by \( \sigma \) and in particular the identity mapping is Baire measurable from \( G, \sigma \) to \( G, \tau \), so continuous.

We see from the above proof, that the only thing we need is that the two topologies be inter-definable, which is exactly what Theorem 6.24 gives us.

As one can easily show that \( S_\infty \) has ample generics, our result applies in particular to this group. So this implies that, e.g., any unitary representation of \( S_\infty \) on separable Hilbert space is actually a continuous unitary representation. Moreover, whenever \( S_\infty \) acts by homeomorphisms on some locally compact Polish space or by isometries on some Polish metric space, then it does so continuously. For these results it is enough to notice that the actions in question correspond to homomorphisms into the unitary group, resp. the homeomorphism and the isometry group, which are Polish when the spaces are separable, resp. locally compact.
In a beautiful paper Gaughan [19] proves that any Hausdorff group topology on $S_\infty$ must extend its usual Polish topology. So coupled with the above result this gives us the following rigidity result for $S_\infty$ (we would like to thank V. Pestov for suggesting how to get rid of a Hausdorff condition in a previous version of the result):

**Theorem 6.26.** $S_\infty$ has exactly two separable group topologies, namely the trivial one and the usual Polish topology.

**Proof.** Suppose that $\tau$ is a separable group topology on $S_\infty$ and define

$$N = \bigcap \{ U : \text{open } \& 1 \in U \}$$

We easily see that $N$ is conjugacy invariant, $N = N^{-1}$ and that $N$ is closed under products. For if $x, y \in N$ and $W$ is any open neighborhood of $xy$, then by the continuity of the group operations, we can find open sets $x \in U$ and $y \in V$ such that $UV \subseteq W$. But then as 1 cannot be separated from $x$ by an open set, $x$ cannot be separated from 1 either. Thus 1 is in $U$ and similarly 1 is in $V$, whence 1 is in $W$. So $xy$ cannot be separated from 1 by an open set and hence 1 cannot be separated from $xy$ either. Thus $xy \in N$.

So $N$ is a normal subgroup of $S_\infty$ and hence equal to either $\{1\}$, $\text{Alt}$, $\text{Fin}$ or $S_\infty$ itself. In the first case, we see that $\tau$ is Hausdorff and thus that it extends the Polish topology and in the last case that $\tau = \{\emptyset, S_\infty\}$. Moreover, by the separability of $\tau$, we know that $\tau$ is weaker than the Polish topology on $S_\infty$ and thus in the first case we know that it must be exactly equal to the Polish topology. We are therefore left with the two middle cases that we claim cannot occur. The subgroups $\text{Alt}$ and $\text{Fin}$ are dense in the Polish topology and thus also dense in $\tau$. If $x \notin N$, then we can find some open neighborhood $V$ of $x$ not containing 1. But then $V \cap N = \emptyset$, because any element of $V$ can be separated from 1 and thus does not belong to $N$. This shows that $N$ is closed and contains a dense subgroup, so $N = S_\infty$, a contradiction. \hfill $\square$

We should also mention the following result that allows us to see the small index property as a special case of automatic continuity.

**Proposition 6.27.** Suppose $G$ is a Polish group such that any homomorphism from $G$ into a group with uniform Souslin number $\leq 2^{\aleph_0}$ is continuous. Then $G$ has the small index property.

**Proof.** Suppose $H$ is a subgroup of $G$ of small index. Then $\text{Sym}(G/H)$, where $G/H$ is the set of left cosets of $H$, has uniform Souslin number $\leq 2^{\aleph_0}$ and clearly the action of left translation of $G$ on $G/H$ gives rise to a homomorphism of $G$ into $\text{Sym}(G/H)$. By automatic continuity, this shows that the pointwise stabiliser of the coset $H$ is open in $G$, i.e., $H$ is open in $G$. \hfill $\square$

### 6.11. Automorphisms of trees

We will finally investigate the structure of the group of Lipschitz homeomorphisms of the Baire space $N$. This group is of course canonically isomorphic to $\text{Aut}(N^{<\omega})$, where $N^{<\omega}$ is seen as a copy of the uniformly countably splitting rooted tree. Though $\text{Age}(N^{<\omega})$ is not a Fraïssé class in its usual relational language, we can still see $N^{<\omega}$ as being the generic limit of the class of finite rooted trees and we will see that the theory goes through in this context. Alternatively, one can change the language by replacing the tree relation by a unary
function symbol that assigns to each node its predecessor in the tree ordering. In this way, \( \text{Age}(\mathbb{N}^{<\mathbb{N}}) \) outright becomes a Fraïssé class.

In the following, \( \mathbb{N}^{<\mathbb{N}} \) is considered a tree with root the empty string, \( \emptyset \), such that the children of a vertex \( s \in \mathbb{N}^{<\mathbb{N}} \) are \( s \cdot n \), for all \( n \in \mathbb{N} \). A subtree of \( \mathbb{N}^{<\mathbb{N}} \) is a subset \( T \subseteq \mathbb{N}^{<\mathbb{N}} \) closed under initial segments, i.e., if \( \langle n_0, n_1, \ldots, n_k \rangle \in T \), then so are \( \emptyset, \langle n_0 \rangle, \ldots, \langle n_0, n_1, \ldots, n_{k-1} \rangle \). By \( m^{<m} \) we denote the tree of sequences \( \langle n_0, \ldots, n_k \rangle \) such that \( n_i < m \) and \( k < m \). Moreover, if \( s, t \in \mathbb{N}^{<\mathbb{N}} \), we write \( s \subseteq t \) to denote that \( t \) extends \( s \) as a sequence.

**Lemma 6.28.** Suppose \( \phi : T \to S \) is an isomorphism between finite subtrees of \( \mathbb{N}^{<\mathbb{N}} \), say \( T, S \subseteq m^{\leq m} \). Then there is an automorphism \( \psi \) of \( m^{\leq m} \) extending \( \phi \).

**Proof.** Notice first that \( \phi \) restricts to a bijection between two subsets of \( m^1 = \{(0), (1), \ldots, (m-1)\} \), so can be extended to a permutation \( \phi_1 \) of \( m^1 \). Then \( \phi_1 \) is an isomorphism of \( T_1 = T \cup m^1 \) and \( S_1 = S \cup m^1 \). Now suppose \( \phi_1((0)) = (j) \). Then \( \phi_1 \) restricts to a bijection between two subsets of \( \{(0), (0, 1), \ldots, (0, m-1)\} \) and \( \{(j, 0), (j, 1), \ldots, (j, m-1)\} \) and can be extended as before to some isomorphism of

\[
T_2 = T_1 \cup \{(0, 0), (0, 1), \ldots, (0, m-1)\}
\]

and

\[
S_2 = S_1 \cup \{(j, 0), (j, 1), \ldots, (j, m-1)\}
\]

Now continue with \( \phi_1((1)) = (\ell) \), etc. Eventually, we will obtain an automorphism \( \psi \) of \( m^{\leq m} \) that extends \( \phi \). \( \square \)

**Lemma 6.29.** Suppose \( m \leq n \) and \( (\psi_1, \ldots, \psi_k), (\phi_1, \ldots, \phi_k) \) are sequences of automorphisms of \( n^{<n} \) extending automorphisms \( (\chi_1, \ldots, \chi_k) \) of \( m^{\leq m} \), i.e., \( \chi_i \subseteq \psi_i, \phi_i \). Then there is an automorphism \( \xi \) of \( \ell^{\leq \ell} \), for \( \ell = 2n \), such that \( (\psi_1, \ldots, \psi_k) \) and \( (\xi \circ \phi_1 \circ \xi^{-1}, \ldots, \xi \circ \phi_k \circ \xi^{-1}) \) can be extended to some common sequence of automorphisms of \( \ell^{\leq \ell} \). Moreover, \( \xi \) fixes \( m^{\leq m} \) pointwise.

**Proof.** Let \( \xi \) be an automorphism of \( \ell^{\leq \ell} \) pointwise fixing \( m^{\leq m} \) and such that \( \xi(n^{<n}) \cap n^{<n} = m^{<m} \). Then for each \( i \leq k \), \( \psi_i \) and \( \xi \circ \phi_i \circ \xi^{-1} \) agree on their common domain, and the same for \( \psi_i^{-1} \) and \( (\xi \circ \phi_i \circ \xi^{-1})^{-1} \). So \( \psi_i \cup \xi \circ \phi_i \circ \xi^{-1} \) is an isomorphism of finite subtrees of \( \ell^{\leq \ell} \) and the result follows from Lemma 6.28. \( \square \)

Though \( \mathbb{N}^{<\mathbb{N}} \) is not ultrahomogeneous in its relational language, it is in the functional language, and the preceding results show that its age in the functional language satisfies the conditions of Theorem 6.24. Therefore \( \text{Aut}(\mathbb{N}^{<\mathbb{N}}) \) has ample generics.

**Lemma 6.30.** Suppose \( S \) and \( T \) are finite subtrees of \( \mathbb{N}^{<\mathbb{N}} \) and \( G = \text{Aut}(\mathbb{N}^{<\mathbb{N}}) \). Then \( G_{(S \cap T)} = G_{(T)}G_{(S)}G_{(T)} \).

**Proof.** Since \( G_{(T)}G_{(S)}G_{(T)} = G_{(T)}G_{(S)}G_{(T)} \) and \( G_{(T)} \) is an open neighborhood of the identity, it is enough to show that \( G_{(T)}G_{(S)}G_{(T)} \) is dense in \( G_{(S \cap T)} \). By Lemma 6.28, it is enough to show that if \( \phi \) is an automorphism of \( m^{\leq m} \) for some \( m \) such that \( S, T \subseteq m^{\leq m} \) and \( \phi \) pointwise fixes \( S \cap T \), then there is a \( g \in G_{(T)}G_{(S)}G_{(T)} \) with \( g \supseteq \phi \).
For \( s \in S \cap T \), let \( A_s = \{ n \in \mathbb{N} : s^n \notin S \cap T \text{ and } s^n \in S \} \) and fix a permutation \( \sigma_s \) of \( \mathbb{N} \) pointwise fixing \( \{0, 1, 2, \ldots, m - 1\} \setminus A_s \), but such that \( \sigma_s(A_s) \cap A_s = \emptyset \). Define \( f \in G(T) \) as follows: If \( u \in S \cap T \), let \( f(u) = u \). Otherwise if
\[
(54) \quad u = s^n t, s \in S \cap T, s^n \notin S \cap T, t \in \mathbb{N}^{<\infty},
\]
let \( f(u) = s^n \sigma_s(n)^t \).

Now let \( g \supseteq \phi \) be defined by: If \( u = s^n t \), where \( s \) is the maximal initial segment such that \( s \in m^{\leq m} \), let \( g(u) = \phi(s)^t \). Then \( f^{-1} g f \in G(S) \). For suppose \( u = s^n t \in S, s^n \notin T, s \in S \cap T \). Then
\[
(55) \quad f^{-1} g \circ f(u) = f^{-1} g(s^n \sigma_s(n)^t) = f^{-1}(\phi(s)^n \sigma_s(n)^t) = f^{-1}(s^n \sigma_s(n)^t) = u,
\]
since \( \phi \) fixes \( S \cap T \) pointwise. And if \( u \in S \cap T \), then \( f^{-1} g \circ f(u) = u \), as both \( f, g \in G(S \cap T) \). So
\[
(56) \quad g = f f^{-1} g f^{-1} \in G(T) G(S) G(T).
\]

\[\square\]

**Theorem 6.31.** Let \( G = \text{Aut}(\mathbb{N}^{\leq \infty}) \). Then:

(i) \( G \) has ample generic elements.

(ii) [R. Möller \([39]\)] \( G \) has the strong small index property.

(iii) \( G \) is 32-Bergman.

(iv) \( G \) has a locally finite dense subgroup.

**Proof.** (i) has been verified and so \( G \) has the small index property. Suppose \( G(S) \leq H \leq G \) for some finite subtree \( S \subseteq \mathbb{N}^{<\infty} \) and open subgroup \( H \). Then, by Lemma \[\text{[63]}\]
we have, for any \( h \in H \), that \( G(S \cup h) = G(S) G(h \cup S) G(S) \leq H \). So, as \( S \) is finite, we have, if \( T = \bigcap_{h \in H} h \cup S \), that \( G(T) \leq H \) and also \( H \leq G(T) \). So \( G \) has the strong small index property.

Now suppose \( W_0 \subseteq W_1 \subseteq \ldots \subseteq G \) is an exhaustive sequence of subsets. Then, by Proposition \[\text{[61]}\]
there is an \( n \in \mathbb{N} \) such that \( W_n^{\infty} \) contains an open subgroup \( G(S) \), where \( S \subseteq \mathbb{N}^{<\infty} \) is some finite tree. Take \( g \in G(S) \) such that \( g^{\prime} S \cap S = \{0\} \). Then for \( m \geq n \) sufficiently big we have \( g, g^{-1} \in W_m \) and \( G = G(S) G(S) G(S) G(S) \subseteq G(S) G(S) G(S) G(S) \subseteq W_m^{32} \). This verifies the Bergman property.

Finally, the following is easily seen to be a dense locally finite subgroup of \( G : K = \{ g \in G : \exists m \exists \phi \text{ automorphism of } m^{\leq m} \text{ such that if } u = s^n t \in \mathbb{N}^{<\infty}, \text{ where } s \text{ is the maximal initial segment of } u \text{ such that } s \in m^{\leq m}, \text{ then } g(u) = \phi(s)^t \} \).

We denote by \( \mathbf{T} \) the \( \aleph_0 \)-regular tree on \( \mathbb{N} \), i.e., the tree in which each vertex has valency \( \aleph_0 \). Notice that \( \text{Aut}(\mathbb{N}^{<\infty}) \cong \text{Aut}(\mathbf{T}, a_0) = \text{Aut}(\mathbf{T})_{a_0} \), for any \( a_0 \in \mathbf{T} \). Hence, \( \text{Aut}(\mathbb{N}^{<\infty}) \) is isomorphic to a clopen subgroup of \( \text{Aut}(\mathbf{T}) \).

**Corollary 6.32.** \( \text{Aut}(\mathbf{T}) \) satisfies automatic continuity and has the small index property.

This follows from the general fact that automatic continuity passes from an open subgroup to the whole group.

For the next couple of results, we need some more detailed information about the structure of group actions on trees. If \( g \) is an automorphism of a tree \( S \) that acts without inversion, then \( g \) either has a fixed point, in which case \( g \) is said to be elliptic, or \( g \) acts by translation on some line in the tree, in which case \( g \) is said to be hyperbolic. Serre’s book \[\text{[33]}\] is a good reference for more information on these concepts.
Lemma 6.33. Suppose \( \ell_g = (a_i \mid i \in \mathbb{Z}) \subseteq T \) is a line and \( g \) a hyperbolic element of \( \text{Aut}(T) \) acting by translation on \( \ell_g \) with amplitude 1, i.e. \( g \cdot a_i = a_{i+1}, \forall i \in \mathbb{Z}. \) Then \( \text{Aut}(T) = \langle \text{Aut}(T, a_0) \cup \{g\} \rangle. \)

Proof. Suppose \( t_0 \) is any element of \( T. \) Then there is an \( h \in \text{Aut}(T, a_0) \) such that \( t_0 \in h \cdot \ell_g \) and so \( h g^{d(t_0, a_0)} \cdot a_0 = t_0. \) Put \( k = h g^{d(t_0, a_0)}, \) then

\[
(57) \quad \text{Aut}(T, t_0) = k \text{Aut}(T, a_0) k^{-1} \subseteq \langle \text{Aut}(T, a_0) \cup \{g\} \rangle.
\]

This shows that \( \langle \text{Aut}(T, a_0) \cup \{g\} \rangle \) contains all elliptic elements of \( \text{Aut}(T). \)

Now, suppose \( k \) is any other hyperbolic element of \( \text{Aut}(T) \) acting by translation on a line \( \ell_k \) with amplitude \( m = \|k\|. \) Let \( \alpha = (b_n, \ldots, b_0) \) be the geodesic from \( \ell_k \) to \( \ell_g \) and \( a_i = b_0 \) be its endpoint.

There are two cases: Either \( \ell_k \cap \ell_g \neq \emptyset, \) in which case it is easy to find some \( h \in \text{Aut}(T, a_i) \) such that \( k = hg^m h^{-1} \in \langle \text{Aut}(T, a_0) \cup \{g\} \rangle. \) Otherwise, take some \( f \in \text{Aut}(T, a_i) \) such that \( f \cdot a_{i+j} = b_j \) for \( j = 0, \ldots, n. \) So

\[
(58) \quad b_n = f \cdot a_{i+n} \in (f \cdot \ell_g) \cap \ell_k = \ell_{fgf^{-1}} \cap \ell_k.
\]

Replacing \( g \) by \( f gf^{-1} \) we can repeat the argument above to find an \( h \in \text{Aut}(T, b_n) \) such that \( k = h g g^{-1} h^{-1} \in \langle \text{Aut}(T, a_0) \cup \{g\} \rangle. \) Therefore, \( \langle \text{Aut}(T, a_0) \cup \{g\} \rangle \) contains all hyperbolic elements of \( \text{Aut}(T). \)

We now only have the inversions left. So suppose \( k \) inverts an edge \( (a, b) \) of \( T. \) Find some hyperbolic \( f \) with amplitude \( \|f\| = 1 \) such that its characteristic subtree \( \ell_f \) passes through \( a \) and \( b \) with \( f \cdot a = b. \) Take also some elliptic \( h \in \text{Aut}(T, b) \) such that \( h \cdot (f \cdot b) = a, \) then clearly

\[
(59) \quad h fk \in \text{Aut}(T, a) \subseteq \langle \text{Aut}(T, a_0) \cup \{g\} \rangle.
\]

Therefore, \( \text{Aut}(T) = \langle \text{Aut}(T, a_0) \cup \{g\} \rangle. \)

\( \square \)

Theorem 6.34. \( \text{Aut}(T) \) is of uncountable cofinality.

Proof. Suppose \( H_0 \leq H_1 \leq \ldots \leq \text{Aut}(T) \) exhausts \( \text{Aut}(T). \) Then evidently, \( H_0 \cap \text{Aut}(T, t_0) \leq H_1 \cap \text{Aut}(T, t_0) \leq \ldots \leq \text{Aut}(T, t_0) \) also exhausts \( \text{Aut}(T, t_0) \) and thus by the uncountable cofinality of the latter, there is some \( n \) such that \( \text{Aut}(T, t_0) \leq H_n. \) Now, fix a line \( \ell_g = (a_i \mid i \in \mathbb{Z}) \subseteq T \) and a hyperbolic element \( g \in \text{Aut}(T) \) acting by translation on \( \ell_g \) with amplitude 1, i.e. \( g \cdot a_i = a_{i+1}, \forall i \in \mathbb{Z} \) such that \( t_0 = a_0. \) We know by Lemma 6.33 that \( \text{Aut}(T) = \langle \text{Aut}(T, t_0) \cup \{g\} \rangle, \) whence if \( m \geq n \) is such that \( g \in H_m, \) also \( \text{Aut}(T) = H_m. \)

\( \square \)

Remark 6.35. It is easy to see that \( \text{Aut}(T) \) fails property (FA). For we can just add a vertex to every edge of \( T \) and extend the action of \( \text{Aut}(T) \) to the tree obtained. We now see that \( \text{Aut}(T) \) acts without inversion, but does not fix a vertex. An exercise in Serre’s book \([13]\) p. 34 implies that \( \text{Aut}(T) \) is actually a non-trivial free product with amalgamation.

We shall now see that \( \text{Aut}(T) \) also satisfies the analogue of Neumann’s Lemma. This is a special case of the following general fact.

Proposition 6.36. Suppose \( G \) is a Polish group having an open subgroup \( K \) with ample generics. Then \( G \) satisfies Theorem 6.14 and Corollary 6.12.
Proof. Suppose towards a contradiction that \( \{g_i H_i\}_{i \in \mathbb{N}} \) covers \( G \) such that no \( H_i \) is open. Then as \( G \) has the small index property, every \( H_i \) has index \( 2^{\aleph_0} \) in \( G \) and hence when \( G \) acts by left translation on
\[
X = G/H_0 \sqcup G/H_1 \sqcup \ldots
\]
any orbit is of size \( 2^{\aleph_0} \). Since \( K \) is open in \( G \) there are \( f_i \in G \) such that \( \{f_i K\}_{i \in \mathbb{N}} \) cover \( G \). So for any \( x \in X \), \( G \cdot x = \bigcup f_i K \cdot x \), whence every \( K \)-orbit is of size \( 2^{\aleph_0} \). Applying Theorem 6.16 to \( K \), we find some \( k \in K \) such that \( k \cdot A \cap A = \emptyset \), where \( A = \{H_i, g_i H_i\}_{i \in \mathbb{N}} \). Thus for each \( i \), \( k H_i \neq g_i H_i \), contradicting that \( \{g_i H_i\}_{i \in \mathbb{N}} \) covers \( G \). Therefore, some \( H_i \) is open proving Theorem 6.14 for \( G \). Theorem 6.16 for \( G \) now follows from the small index property and Theorem 6.14 for \( G \). \( \square \)

6.12. Concluding remarks. It is interesting to see that there seem to be two different paths to the study of automorphism groups of homogeneous structures. On the one hand, there are the methods of moieties dating at least back to Anderson [4] which have been developed and used by a great number of authors, and, on the other hand, the use of genericity.

The main tenet of this last section is that although ample genericity can sometimes be quite non-trivial to verify, it is nevertheless a sufficiently powerful tool for it to be worth the effort looking for. In particular, it provides a uniform approach to proving the small index property, uncountable cofinality, property (FA), the Bergman property and automatic continuity.

6.13. Some questions.

(1) Are there any examples of Polish groups that are not isomorphic to closed subgroups of \( S_\infty \) but have ample generic elements? That have the small index property? **Addendum.** It has now been verified by S. Solecki and the second author that the homeomorphism group of the reals indeed has the small index property, but of course is not a topological subgroup of \( S_\infty \).

(2) Does \( \text{Aut}(B_\infty) \) have ample generic elements?

(3) Can a Polish locally compact group have a comeager conjugacy class?

(4) Characterize the generic elements of \( \text{Aut}(B_\infty) \) and \( \text{Aut}(F, \lambda) \), where \( (F, \lambda) \) is as in Proposition 2.4. **Added in proof.** We have recently received a preprint of Akin, Glasner, and Weiss [3] in which they give another proof of the existence of a comeager conjugacy class in the homeomorphism group of the Cantor space, a proof that also gives an explicit characterization of the elements of this class.

(5) Is the conjugacy action of \( \text{Iso}(U) \) turbulent? (Vershik) Does \( \text{Iso}(U_0) \) have a dense locally finite subgroup?

(6) Suppose a Polish group has ample generics and acts by homeomorphisms on a Polish space. Is the action necessarily continuous?

(7) Is property (T) somehow related to the existence of dense or comeager conjugacy classes? Concretely, if \( G \) is a Polish group, which is not the union of a countable chain of proper open subgroups and such that there is a diagonally dense conjugacy class in \( G^{\mathbb{N}} \), does \( G \) have property (T)?

References


[42] C. Rosendal and S. Solecki, Automatic continuity of group homomorphisms and discrete groups with the fixed point on metric compacta property, to appear in *Israel J. Math*.


