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# COUNTABLE SECTIONS FOR LOCALLY COMPACT GROUP ACTIONS. II

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ABSTRACT. In this paper we study the structure of the orbit equivalence relation induced by a Borel action of a second countable locally compact group on a standard Borel space.

Let $G$ be a second countable locally compact group and $(g, x) \mapsto g.x$ a Borel action of $G$ on a standard Borel space $X$. (A *standard Borel space* is a Polish space with the associated Borel structure, and an action is Borel if the function $(g, x) \mapsto g.x$ is Borel from $G \times X$ into $X$.) We denote by $E_G$ the equivalence relation induced by this action, i.e., $xE_Gy \Leftrightarrow \exists g \in G(g.x = y)$. It is well known that $E_G$ is a Borel equivalence relation on $X$.

A Borel equivalence relation $E$ on a standard Borel space $X$ is called *countable* if every equivalence class $[x]_E$ of $E$ is countable. Given two Borel equivalence relations $E, E'$ on $X, X'$ resp., we say that $E$ is (*Borel*) *reducible* to $E'$, in symbols $E \leq E'$, if there is a Borel map $f: X \to X'$ such that $xEy \Leftrightarrow f(x)E'f(y)$, and we say that $E, E'$ are *bireducible*, in symbols $E \approx^* E'$, if $E \leq E'$ and $E' \leq E$. It has been shown in [DJK] that bireducibility for countable Borel equivalence relations is also equivalent to the following notion: We say that $E, E'$ are *stably isomorphic*, in symbols $E \cong^s E'$, if there are Borel sets $A \subseteq X$, $B \subseteq X'$ which meet every $E$- (resp. $E'$-) equivalence class (such sets are called *complete sections* or *full*) and $E|A \cong E'|B$, where $E|A$ denotes the restriction of $E$ to $A$ and $\cong$ is (Borel) isomorphism. (Two Borel equivalence relations $F, F'$ on standard Borel spaces $Y, Y'$ resp. are (*Borel*) *isomorphic* if there is a Borel bijection $f: Y \to Y'$ with $xFy \Leftrightarrow f(x)F'f(y)$.) Denoting by $I_S$ the transitive equivalence relation $S \times S$ on $S$, it turns out that we have, for countable $E, E'$,

$$E \approx E' \Leftrightarrow E \cong^s E' \Leftrightarrow E \times I_{\mathbb{N}} \cong E' \times I_{\mathbb{N}}$$

where as usual the product of two equivalence relations $E, F$ is the equivalence relation

$$(x, y)(E \times F)(x', y') \Leftrightarrow xEx' \,\&\, yFy'.$$

We now state our main results in this paper.

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**Theorem 1.** *Let $G$ be a second countable locally compact group acting in a Borel way on a standard Borel space $X$. Then there is a unique decomposition $X = C \cup U$ into invariant Borel sets such that $E_G|C$ is countable and $E_G|U \cong F \times I_{\mathbb{R}}$, with $F = E_G|Z$ for $Z$ a complete Borel section of $E_G|U$ such that $E_G|Z$ is countable.*

We call $E|C$ the *countable part* of $E_G$ and $E_G|U$ the *uncountable* (or *continuous*) *part* of $E_G$. For the latter we have the following additional information.

**Theorem 2.** *The map $F \mapsto F \times I_{\mathbb{R}}$ gives a one-to-one correspondence between:* (1) *countable Borel equivalence relations up to stable isomorphism, and* (2) *Borel equivalence relations of the form $E_G$, $G$ second countable locally compact, with all equivalence classes uncountable, up to isomorphism.*

These results provide pure Borel theoretic versions of analogous results of Feldman-Hahn-Moore [FHM, §§4–6], who study nonsingular measurable actions of second countable locally compact groups on standard measure spaces. The proofs of Theorems 1 and 2 use, among other things, a result in [K], which provides a purely Borel theoretic extension of the main result of [FHM], concerning the existence of countable complete Borel sections in Borel actions of such groups.

In the case of $\mathbb{R}$-actions, i.e., flows, one can obtain further information using [W] (see also [K]) and one of the results in [DJK]. Note that if $E_{\mathbb{R}}$ is the orbit equivalence relation of a flow, then every equivalence class is either a singleton or uncountable (this is because the stabilizer of a point under the action is a closed subgroup of $\mathbb{R}$; see [V]). Recall that a Borel equivalence relation $E$ on a standard Borel space $X$ is called *smooth* if there is a standard Borel space $Y$ and a Borel map $f: X \to Y$ with $xEy \Leftrightarrow f(x) = f(y)$. For $E$ of the form $E_G$, this is equivalent (as it follows from [B], see also [K]) to saying that there is a Borel set $A \subseteq X$ picking exactly one element out of each $E$-equivalence class. It is easy to classify $E_{\mathbb{R}}$ up to Borel isomorphism when it is smooth. For nonsmooth ones we have

**Theorem 3.** *Let $E^1_{\mathbb{R}}$, $E^2_{\mathbb{R}}$ be two nonsmooth Borel equivalence relations, induced by Borel actions of $\mathbb{R}$. Let $c_i =$ the cardinality of the set of singleton equivalence classes for $E^i_{\mathbb{R}}$. Then $E^1_{\mathbb{R}} \cong E^2_{\mathbb{R}} \Leftrightarrow c_1 = c_2$. In particular, any two nonsmooth equivalence relations induced by Borel flows with no fixed points are Borel isomorphic.*

It should also be true that any two nonsmooth $E_G$, with $G$ amenable, which have all equivalence classes uncountable are Borel isomorphic, but this is an open problem.

*Added in proof.* This has now been verified by Jackson, Kechris, and Louveau for compactly generated $G$ of polynomial growth, extending a result of B. Weiss.

## 1. Some lemmas

The following lemmas are needed for the proofs of the main results.

**Lemma 1.1.** *Let $Z$, $X$ be standard Borel spaces, $H \subseteq Z \times X$ a Borel set with $H_z = \{x : (z, x) \in H\} \neq \varnothing$, for every $z \in Z$. Assume to each $z \in Z$ we have assigned a $\sigma$-ideal $I_z$ of subsets of $H_z$ (i.e., a collection containing $\varnothing$ and* *closed under subsets and countable unions) such that:*

(1) $H_z \notin I_z$, $\{x\} \in I_z$ *for every* $x \in H_z$;

(2) *For each Borel* $P \subseteq H$, *the set* $P_I = \{z \in Z : P_z \in I_z\}$ *is Borel.*

*Then there is a Borel map* $\varphi: Z \times \mathbb{R} \to X$ *such that for each* $z \in Z$, $\varphi_z: \mathbb{R} \to X$ *given by* $\varphi_z(r) = \varphi(z, r)$ *is a Borel isomorphism of* $\mathbb{R}$ *with* $H_z$.

*Proof.* Since $H$ is Borel, we can find a family $\{H^s\}$, where $s$ varies over the set $\mathbb{N}^{<\mathbb{N}}$ of finite sequences from $\mathbb{N}$, such that:

(i) $H^s$ is Borel;

(ii) $H^{\varnothing} = H$, $H^{s\hat{}n} \cap H^{s\hat{}m} = \varnothing$, if $n \neq m$, $H^s = \bigcup_n H^{s\hat{}n}$;

(iii) If $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $H^{\alpha|n} \neq \varnothing$ for all $n$, then $H^\alpha := \bigcap_n H^{\alpha|n}$ is a singleton $\{w\}$ and if $w_n \in H^{\alpha|n}$, then $w_n \to w$.

(We view here $Z$, $X$ and thus $Z \times X$ as Polish spaces, so $w_n \to w$ means convergence in $Z \times X$.)

Put now $H_z^s = \{x : (z, x) \in H^s\}$. Then it is easy to see that $\{H_z^s\}$ has properties (i)–(iii) for $H_z$.

For each $z$ now define a tree $T_z$ on $\mathbb{N}$ (i.e., a set of finite sequences from $\mathbb{N}$ closed under initial segments) as follows: $s \in T_z \Leftrightarrow H_z^s \notin I_z$. Since every singleton is in $I_z$, it follows that

$$\forall s \in T_z \exists s', s'' \in T_z[s \subseteq s', s \subseteq s'', \text{ and } s', s'' \text{ are incompatible}].$$

(Otherwise for some $s \in T_z$ there is $\alpha \in \mathbb{N}^{\mathbb{N}}$, $s \subseteq \alpha$, such that $H_z^{\alpha|n} \notin I_z$ for all $n$, but if $t \supseteq s$, $t \in \mathbb{N}^{\mathbb{N}}$, and $t \neq \alpha|n$, then $H_z^t \in I_z$. Then $H_z^s = H_z^\alpha \cup \{H_z^t : s \subseteq t,\ t \in \mathbb{N}^n,\ t \neq \alpha|n\}$, where $H_z^\alpha = \bigcap_n H_z^{\alpha|n}$. Thus $H_z^\alpha \notin I_z$, which is impossible since $H_z^\alpha$ is a singleton.)

Define now inductively on the length of $u \in 2^{<\mathbb{N}}$ a sequence $s_u \in T_z$, so that $s_\varnothing = \varnothing$ and $s_{u\hat{}0}, s_{u\hat{}1}$ are the first two (in some enumeration of $\mathbb{N}^{<\mathbb{N}}$) incompatible extensions of $s_u$ in $T_z$. Finally define for each $\alpha \in 2^{\mathbb{N}}$

$$\tilde{\varphi}_z(\alpha) = \text{ the unique element of } \bigcap_n H_z^{s_{\alpha|n}}.$$

In view of property (2) of the assignment $z \mapsto I_z$, it is easy to check that the map $\tilde{\varphi}(z, \alpha) = \tilde{\varphi}_z(\alpha)$ from $Z \times 2^{\mathbb{N}}$ into $X$ is Borel. Moreover, it is clear that $\tilde{\varphi}_z: 2^{\mathbb{N}} \to H_z$ is a Borel injection. Let also $\tilde{\psi}: H \to 2^{\mathbb{N}}$ be a Borel injection, so that in particular $\tilde{\psi}_z: H_z \to 2^{\mathbb{N}}$ given by $\tilde{\psi}_z(x) = \tilde{\psi}(z, x)$ is a Borel injection. By the usual Schroeder-Bernstein argument applied to $\tilde{\varphi}_z$, $\tilde{\psi}_z$ we can finally find a Borel map $\varphi: Z \times 2^{\mathbb{N}} \to X$ such that $\varphi_z$ is a bijection of $2^{\mathbb{N}}$ with $H_z$. Since $2^{\mathbb{N}}$, $\mathbb{R}$ are Borel isomorphic, this completes the proof. □

**Lemma 1.2.** *Let* $E$ *be a Borel equivalence relation on a standard Borel space* $X$. *Assume to each* $E$ *equivalence class* $C$ *we have assigned a* $\sigma$*-ideal* $J_C$ *of subsets of* $C$ *such that:*

(1) $C \notin J_C$, $\{x\} \in J_C$ *for all* $x \in C$;

(2) *If* $P \subseteq E$ *is Borel, then the set* $P_J = \{x : P_x \in J_{[x]_E}\}$ *is Borel.*

*Let also* $F$ *be a countable Borel equivalence relation on a standard Borel space* $Y$ *and* $f: X \to Y$ *a Borel map with* $xEy \Leftrightarrow f(x)Ff(y)$.

*Then there is a Borel set* $Z \subseteq f[X]$, *such that* $Z$ *and* $f[X]$ *meet the same* $F$*-equivalence classes and such that* $E \cong (F|Z) \times I_{\mathbb{R}}$.

*Proof*. Note that if $C$ is an $E$-equivalence class, then $f[C]$ is countable; thus, for some $y \in f[C]$, $f^{-1}(\{y\}) \subseteq C$ is not in $J_C$. So define

$$R(y, x) \Leftrightarrow f(x) = y \,\&\, f^{-1}(\{y\}) \notin J_{[x]_E}$$
$$\Leftrightarrow f(x) = y \,\&\, \{x' : f(x) = f(x')\} \notin J_{[x]_E}$$

so that $R$ is Borel.

*Claim*. There is a Borel uniformization $R^* \subseteq R$, i.e., a Borel set $R^* \subseteq R$ such that $\{y : \exists x(y, x) \in R^*\} = \{y : \exists x(y, x) \in R\}$ and $(y, x) \in R^* \,\&\, (y, x') \in R^* \Rightarrow x = x'$.

Granting this claim it follows immediately that $Z = \{y : \exists x(y, x) \in R\}$ is Borel and the function $r: Z \to X$ given by $r(z) = x \Leftrightarrow R^*(z, x)$ is Borel too. Note that $Z$ and $f[X]$ meet the same $F$-equivalence classes.

If

$$X^* = \{x : \exists y(x, y) \in R\} = \{x : \{x' : f(x) = f(x')\} \notin J_{[x]_E}\}$$

then $X^*$ is also a Borel subset of $X$ and $f$ is a surjection of $X^*$ onto $Z$. Also $\forall x \in X \exists z \in Z(f(x)Fz)$, so since every $F$-equivalence class is countable, it follows that $\forall x \in X$ (there are at most countably many $z \in Z$ with $f(x)Fz$); so by a standard uniformization theorem, let $g: X \to Z$ be Borel with $f(x)Fg(x)$, $g(x) \in Z$. Let $h: X \to Z$ be the Borel surjection defined by $h(x) = f(x)$ if $x \in X^*$, and $h(x) = g(x)$ if $x \notin X^*$. Then $h(x)Ff(x)$, so $xEy \Leftrightarrow h(x)Fh(y)$ and, for each $z \in Z$, $h^{-1}[\{z\}] \notin J_{[x]_E}$ if $h(x) = z$. Put $H(z, x) \Leftrightarrow h(x) = z$. To each $z \in Z$ assign the following $\sigma$-ideal $I_z$ of subsets of $H_z$:

$$I_z = \{A \subseteq H_z : A \in J_{[x]_E}\}$$

where $h(x) = z$. Then all the conditions of Lemma 1.1 are satisfied. (To verify condition (2) note that if $P \subseteq H$ then

$$z \in P_I \Leftrightarrow P_z \in I_z$$
$$\Leftrightarrow \exists x[h(x) = z \,\&\, \{x' : (h(x), x') \in P\} \in J_{[x]_E}]$$
$$\Leftrightarrow \forall x[h(x) = z \Rightarrow \{x' : (h(x), x') \in P\} \in J_{[x]_E}]$$

so $P_I$ is Borel.) Then there is Borel $\varphi: Z \times \mathbb{R} \to X$ such that for each $z$, $\varphi_z$ is a bijection of $\mathbb{R}$ into $H_z$. But then clearly $\varphi$ is a Borel isomorphism of $(F|Z) \times I_{\mathbb{R}}$ with $E$.

So it remains to give the

*Proof of the claim*. The proof is similar to that of 2.4 in [K], so we only give a sketch. Let $\{R^s\}_{s \in \mathbb{N}^{<\mathbb{N}}}$ be a family of Borel sets having properties (i)–(iii) (for $R$) as in the beginning of the proof of Lemma 1.1. For each $y$ with $\exists x(y, x) \in R$, i.e., $R_y \neq \varnothing$, so that actually $R_y \notin J_{[x]_E}$, where $f(x) = y$, define $\alpha_y: \mathbb{N} \to \mathbb{N}$ inductively so that $\alpha_y(n)$ is the least number $m$ for which $R_y^{(\alpha_y|n)\hat{}m} = \{x' : (y, x') \in R^{(\alpha_y|n)\hat{}m}\} \notin J_{[x]_E}$. Let $x_y$ be the unique element of $R_y^\alpha = \bigcap_n R_y^{\alpha|n}$. Put $R^*(y, x) \Leftrightarrow x = x_y$. $\square$

## 2. The main results

We have first

**Theorem 2.1.** *Let $G$ be a second countable locally compact group and $(g, x) \mapsto g.x$ a Borel action of $G$ on a standard Borel space $X$. Let $E_G$ be the induced*

*Borel equivalence relation*

$$xE_Gy \Leftrightarrow \exists g \in G(g.x = y).$$

*Then if* $C = \{x : [x]_{E_G}$ *is countable*$\}$ *and* $U = X \setminus C$, *the sets* $C$, $U$ *are Borel invariant and there is a Borel set* $Z \subseteq U$, *which meets every* $E_G|U$*-equivalence class in a countable non-*$\varnothing$ *set and* $E_G|U \cong (E_G|Z) \times I_{\mathbb{R}}$. (*Notice that* $E_G|Z$ *is a countable Borel equivalence relation.*)

*Proof*. For each $E_G$-equivalence class $D$ define the $\sigma$-ideal $J_D$ of a subset of $D$ as

$$A \in J_D \Leftrightarrow \{g \in G : g.x \in A\} \text{ is meager in } G$$

where $x \in D$ (this is easily seen to be independent of the choice of $x$). Note (see, e.g., [K]) that if $P \subseteq E_G$ is Borel, so is $P_J = \{x : P_x \in J_{[x]_E}\}$. Note now that

$$[x]_{E_G} \text{ is countable } \Leftrightarrow G_x = \{g \in G : g.x = x\} \text{ has countable index in } G$$
$$\Leftrightarrow G_x \text{ is not meager in } G$$

(the last equivalence following from the Baire Category Theorem). So

$$x \in C \Leftrightarrow \{x\} \notin J_{[x]_{E_G}},$$

thus $C$ (and therefore $U$) is Borel.

By the main Theorem 1.2 of [K], there is a Borel set $Y \subseteq U$ which meets every $E_G|U$-equivalence class in a countable non-$\varnothing$ set. Since the set

$$R(x, z) \Leftrightarrow x \in U \,\&\, y \in Y \,\&\, xE_Gy$$

is Borel and has countable sections, while $\forall x \in U \exists y \in Y R(x, y)$, it follows by a standard uniformization theorem that there is a Borel function $f$ such that $\forall x \in U R(x, f(x))$; thus, in particular, $xE_Gy \Leftrightarrow f(x)Ff(y)$, where $F = E_G \restriction Y$. All the conditions of Lemma 1.2 are now satisfied, so there is a Borel set $Z \subseteq Y$ meeting every $E_G \restriction U$-equivalence class in a non-$\varnothing$ countable set and $E_G|U \cong (E_G|Z) \times I_{\mathbb{R}}$. □

If $E$, $E'$ are Borel equivalence relations on disjoint standard Borel spaces $X$, $X'$, we denote by $E \oplus E'$ their *direct sum* (i.e., the union of $E$, $E'$ which is a Borel equivalence relation on $X \cup X'$). It follows from the preceding that every $E_G$, where $G$ is second countable locally compact, can be uniquely written as $E \oplus E'$, where $E$ is a countable Borel equivalence relation and $E' = F \times \mathbb{I}_{\mathbb{R}}$ for some countable Borel equivalence relation $F$.

Since by a result of Feldman-Moore [FM] every countable Borel equivalence relation $E$ is of the form $E = E_G$ for some *countable* group $G$, and so, in particular, $E = E_{F_\omega}$, where $F_\omega$ is the free group on a countably infinite set of generators, it follows that every $E_G$, with $G$ second countable locally compact, is Borel isomorphic to some $E_{F_\omega \times \mathbb{R}}$.

For two Borel equivalence relations $E$, $E'$ on $X$, $X'$ respectively, we write

$$E \sqsubseteq^i E' \Leftrightarrow \text{ there is a Borel invariant } Y \subseteq X' \text{ with } E \cong E' \restriction Y.$$

Since it has been shown in [V] that, for every second countable locally compact group $G$, there is a universal Borel $G$-action, it follows in particular that there is some $E_G^0$ such that for every $E_G$, $E_G \sqsubseteq^i E_G^0$. Thus, in view of the preceding remarks, $E_G \sqsubseteq^i E^0_{F_\omega \times \mathbb{R}}$ for any $E_G$, $G$ second countable locally compact.

Let us note now some simple facts concerning equivalence relations of the form $F \times I_{\mathbb{R}}$, $F$ a countable Borel equivalence relation.

**Proposition 2.2.** *Let $F$, $F'$ be countable Borel equivalence relations. If $F \leq F'$ then $F \times I_{\mathbb{R}} \sqsubseteq^i F' \times I_{\mathbb{R}}$. In particular, $F \approx^* F' \Leftrightarrow F \times I_{\mathbb{R}} \cong F' \times I_{\mathbb{R}}$.*

*If $G$, $G'$ are second countable locally compact groups and $E_G$, $E_{G'}$ have uncountable equivalence classes, then $E_G \leq E_{G'} \Leftrightarrow E_G \sqsubseteq^i E_{G'}$ and thus $E_G \approx^* E_{G'} \Leftrightarrow E_G \cong E_{G'}$.*

*Proof*. Let $X$, $X'$ be the underlying spaces of $F$, $F'$, and let $f: X \to X'$ be Borel such that $xFy \Leftrightarrow f(x)F'f(y)$. Since $f$ is countable-to-1, $f[X] = Z$ is Borel and so is $W = [Z]_{F'}$. Also clearly, $F \approx^* F'|W$. Then $F \times I_{\mathbb{N}} \cong (F'|W) \times I_{\mathbb{N}}$, so $F \times I_{\mathbb{R}} \cong F \times I_{\mathbb{N}} \times I_{\mathbb{R}} \cong (F'|W) \times I_{\mathbb{N}} \times I_{\mathbb{R}} \cong (F'|W) \times I_{\mathbb{R}} = (F' \times I_{\mathbb{R}})|(W \times \mathbb{R})$. But $W \times \mathbb{R}$ is invariant Borel in $F' \times I_{\mathbb{R}}$, so $F \times I_{\mathbb{R}} \sqsubseteq^i F' \times I_{\mathbb{R}}$. The second assertion follows from the usual Schroeder-Bernstein argument.

If now $E_G$, $E_{G'}$ are as in the statement of the proposition, then by Theorem 2.1 $E_G \cong F \times I_{\mathbb{R}}$, $E_{G'} \cong F' \times I_{\mathbb{R}}$, for some countable Borel $F$, $F'$. As $F \leq F \times I_{\mathbb{R}}$ and $F' \times I_{\mathbb{R}} \leq F'$, we have that $E_G \leq E_{G'}$ implies $F \leq F'$, and so $E_G \cong F \times I_{\mathbb{R}} \sqsubseteq^i F' \times I_{\mathbb{R}} \cong E_{G'}$. □

**Corollary 2.3.** *The map $F \mapsto F \times I_{\mathbb{R}}$ gives a* 1-1 *correspondence between countable Borel equivalence relations up to stable isomorphism* (*or equivalently, bireducibility*) *and Borel equivalence relations $E_G$, $G$ a second countable locally compact group, with uncountable equivalence classes, up to isomorphism* (*or equivalently, bireducibility*).

Finally, we consider $\mathbb{R}$-actions, i.e., flows. It follows from a result of Wagh [W] (see also [K]) that, for every Borel action of $\mathbb{R}$ on a standard Borel space $X$, there is $Y \subseteq X$ such that $E_{\mathbb{R}}|Y$ is hyperfinite, where a Borel equivalence relation $E$ is *hyperfinite* if it can be written as $E = \bigcup_n E_n$, with $E_0 \subseteq E_1 \subseteq \cdots$ Borel equivalence relations with finite equivalence classes (thus $E$ is countable).

It is shown in [DJK] that any two hyperfinite nonsmooth Borel equivalence relations are stably isomorphic. It thus follows that we have the following classification.

**Theorem 2.4.** *Let $E^0_{\mathbb{R}}$, $E^1_{\mathbb{R}}$ be two Borel equivalence relations induced by Borel $\mathbb{R}$-actions. Let*

$$c_i = \operatorname{card}(\{[x]_{E^i_{\mathbb{R}}} : [x]_{E^i_{\mathbb{R}}} \textit{ is a singleton}\}).$$

*If $E^0_{\mathbb{R}}$, $E^1_{\mathbb{R}}$ are not smooth, then $E^0_{\mathbb{R}} \cong E^1_{\mathbb{R}} \Leftrightarrow c_0 = c_1$. In particular, all nonsmooth $E_{\mathbb{R}}$ with uncountable equivalence classes are Borel isomorphic.*

*Proof*. Let $X^0$, $X^1$ be the spaces of $E^0_{\mathbb{R}}$, $E^1_{\mathbb{R}}$. Put

$$\widetilde{X}^i = \{x \in X^i : [x]_{E^i_{\mathbb{R}}} \text{ is a singleton}\}.$$

Then $\widetilde{X}^i$ are Borel, and if $c_0 = c_1$, $E^0_{\mathbb{R}}|\widetilde{X}^0 = E^1_{\mathbb{R}}|\widetilde{X}^1$. Let $Y^i = X^i \backslash \widetilde{X}^i$. As $E^i_{\mathbb{R}}|\widetilde{X}^i$ is smooth, $E^i_{\mathbb{R}}|Y^i$ is not smooth. Moreover, each $E^i_{\mathbb{R}}$-equivalence class in $Y^i$ is uncountable, since the cardinality of $[x]_{E^i_{\mathbb{R}}}$ is the same as the index of the stabilizer of $x$ in the action inducing $E^i_{\mathbb{R}}$, which is a closed subgroup of $\mathbb{R}$ (see [V]). So $E^i_{\mathbb{R}}|Y^i \cong F_i \times I_{\mathbb{R}}$ with $F_i = E^i_{\mathbb{R}}|Z_i$ hyperfinite and nonsmooth. So $F_0 \cong^s F_1$; thus, $E^0_{\mathbb{R}}|Y^0 \cong E^0_{\mathbb{R}}|Y^1$. □

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