THE JOURNAL OF SYMBOLIC LOGIC
Volume 43, Number 3, Sept. 1978

MINIMAL UPPER BOUNDS FOR SEQUENCES OF Δ^1_{2n}-DEGREES

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It is proved here, assuming Projective Determinacy, that every ascending sequence of Δ^1_{2n}-degrees has a minimal strict upper bound but no least strict upper bound. This generalizes a result of Friedman for $n = 1$.

Our general notation and terminology will be that of [Ke1] and [Mo1]. Letters $i, j, k, \ldots$ denote members of ω and $\alpha, \beta, \gamma, \ldots$ members of ω^ω i.e. *reals*. Projective Determinacy (PD) is the hypothesis that every projective set of reals in determined, while in general for a collection of sets of reals Γ, Determinacy (Γ) abbreviates the statement that every set in Γ is determined.

§1. Δ^1_m-degrees. For each $m \geq 1$ and $\alpha, \beta \in \omega^\omega$ let $\alpha \leq_m \beta \Leftrightarrow \alpha \in \Delta^1_m(\beta)$, $\alpha <_m \beta \Leftrightarrow \alpha \leq_m \beta \wedge \beta \not\leq_m \alpha$, and $\alpha \equiv_m \beta \Leftrightarrow \alpha \leq_m \beta \wedge \beta \leq_m \alpha$. Clearly $\equiv_m$ is an equivalence relation on ω^ω. The $\equiv_m$-equivalence class of $\alpha \in \omega^\omega$ is called its Δ^1_m-*degree*, in symbols

$$[\alpha]_m = \{\beta : \beta \equiv_m \alpha\}.$$

If $\mathbf{d} = [\alpha]_m$, $\mathbf{e} = [\beta]_m$ then we define

$$\mathbf{d} \leq \mathbf{e} \Leftrightarrow \alpha \leq_m \beta, \qquad \mathbf{d} < \mathbf{e} \Leftrightarrow \alpha <_m \beta.$$

Clearly $\leq$ is a partial ordering on the set of Δ^1_m-degrees with least element $\mathbf{0} = [\lambda t . 0]_m$. If D is a set of Δ^1_m-degrees a *minimal strict upper bound* of D is a Δ^1_m-degree $\mathbf{b}$ such that

$$\forall \mathbf{d} \in D\,(\mathbf{d} < \mathbf{b}) \wedge \forall \mathbf{e}\,(\mathbf{e} \leq \mathbf{b} \wedge \forall \mathbf{d} \in D\,(\mathbf{d} < \mathbf{e}) \Rightarrow \mathbf{b} = \mathbf{e}).$$

Our main result in this paper is

THEOREM (PD). *For $n \geq 1$, every sequence $\mathbf{d}_0 \leq \mathbf{d}_1 \leq \cdots$ of Δ^1_{2n}-degrees has a minimal strict upper bound.*

The proof will combine the methods of Friedman [Fr], Sacks [Sa1], [Sa2] with the techniques developed in [Ke1], [Ke2], [Ke3].

§2. Proof of the main theorem.

2.1. As with Friedman's work on Δ^1_2-degrees the key ingredient will be the use of Δ^1_{2n}-pointed perfect sets. In general, below, T, with various embellishments, will be used to denote perfect nonempty binary splitting trees on ω. For

Received April 15, 1977.

[1] Research and preparation for this paper were partially supported by NSF Grants MPS75-07562 and MCS76-17254 respectively.

each such tree, $[T]$ denotes the set of all (infinite) paths through T and $h_T: [T] \to 2^\omega$ is the cononical homeomorphism.

We call a tree T as above Δ^1_m*-pointed* if $\forall \alpha \in [T]\ (T \leq_m \alpha)$. Clearly we think of T here as coded by a real in the obvious fashion.

The following two lemmas stated in even more general form in [Sa2] are very useful in the following.

LEMMA 1 (SACKS). *If T is Δ^1_m-pointed and $T^* \subseteq T$ is Δ^1_m in T then $T^* \equiv_m T$ and T^* is Δ^1_m-pointed.*

PROOF. To see that T is Δ^1_m in T^* notice that $T \leq_m \alpha \leq_m T^*$, where α is the leftmost path of T^*. ⊣

LEMMA 2 (SACKS). *If T is Δ^1_m-pointed and $T \leq_m \alpha$, there is $T^* \subseteq T$, T^* Δ^1_m-pointed such that $T^* \equiv_m \alpha$.*

PROOF. Let $h_T: [T] \to 2^\omega$ be the canonical homeomorphism. Let $P = \{\beta \in 2^\omega : \forall n\ (\beta(2n) = \alpha(n))\}$, where without loss of generality we can assume $\alpha \in 2^\omega$. Let now $T^* \subseteq T$ be such that $[T^*] = h_T^{-1}[P]$. ⊣

Our next lemma lifts to this context a basic fact used in Spector's construction of a minimal Turing degree. For $m = 1$ it was proved by Gandy and Sacks (see [Ga-Sa]) and used—without the pointed part—in their construction of a minimal hyperdegree.

LEMMA 3 (PD). *If T is Δ^1_m-pointed and $F: [T] \to \omega^\omega$ is Δ^1_m in T, there is $T^* \subseteq T$, T^* Δ^1_m-pointed, $T^* \equiv_m T$ such that $F \restriction [T^*]$ is either constant (with value Δ^1_m in T of course) or 1-1.*

PROOF. By the relativized form of Theorem 5.5.1. in [Ke2] one can get $T^* \leq_m T$, $T^* \subseteq T$ such that $F \restriction [T^*]$ is either constant or 1-1. By Lemma 1 $T^* \equiv_m T$ and T^* is Δ^1_m-pointed. ⊣

2.2. Let now $\mathscr{S}(k, \alpha)$ be a universal Σ^1_{2n} set such that for each $\alpha \in \omega^\omega$, $\mathscr{S}^\alpha = \{k : \mathscr{S}(k, \alpha)\}$ is a complete $\Sigma^1_{2n}(\alpha)$ subset of ω. Say $\mathscr{S}(k, \alpha) \Leftrightarrow \exists \beta \mathscr{P}(k, \alpha, \beta)$, where $\mathscr{P} \in \Pi^1_{2n-1}$ and (by [Mo2]) uniformize $\mathscr{P}$ on β by $\mathscr{P}^* \in \Pi^1_{2n-1}$. We use PD here and below of course. Let σ be a Π^1_{2n-1}-norm on $\mathscr{P}^*$ and define the norm τ on $\mathscr{S}$ by

$$\tau(k, \alpha) = \sigma(k, \alpha, \alpha^*), \quad \text{where } \mathscr{P}^*(k, \alpha, \alpha^*).$$

Clearly τ is a Σ^1_{2n}-norm on $\mathscr{S}$. For each real α put now

$$\mu^\alpha_{2n} = \mu^\alpha = \sup\{\tau(k, \alpha) : \mathscr{S}(k, \alpha)\}.$$

Let also $\mu_{2n} = \mu = \mu^{\lambda t.0}$.

REMARK. Although it seems that μ^α_{2n} depends on the particular $\mathscr{S}, \mathscr{P}, \mathscr{P}^*, \tau$ we picked, it can be shown that $\mu^\alpha_{2n} = \sup\{\xi : \xi$ is the length of a $\Delta^1_{2n-1}(\beta)$ prewellordering of ω^ω, for some $\beta \leq_{2n} \alpha\}$, so that μ^α_{2n} is actually intrinsically defined. Note here also that

$$\mu^\alpha_{2n} = \sup\{\tau(k, \beta) : (k, \beta) \in \mathscr{S} \wedge \beta \leq_{2n} \alpha\}.$$

For each real $\varepsilon \in \omega^\omega$, for each $k \in \omega$ such that $(k, \varepsilon) \in \mathscr{S}$ and for each $e \in \omega$ define now $F^\varepsilon_{k,e}: \omega^\omega \to \omega^\omega$ as follows:

Let ε^* be such that $\mathscr{P}^*(k, \varepsilon, \varepsilon^*)$. Clearly $\varepsilon^* \leq_{2n} \varepsilon$. Put

$$F^{\varepsilon}_{k,e}(\alpha)(m) = \begin{cases} \text{least } t \text{ such that } \exists\beta\,[(e,\langle m,t,\alpha\rangle,\beta) \leq^*_\sigma (k,\varepsilon,\varepsilon^*)], \\ \hfill \text{if such exists} \\ 0, \quad \text{otherwise.} \end{cases}$$

Here $x \leq^*_\sigma y \Leftrightarrow x \in \mathscr{P}^* \wedge (y \notin \mathscr{P}^* \vee \sigma(x) \leq \sigma(y))$. It is now routine to check that $F^{\varepsilon}_{k,e}$ is $\Delta^1_{2n}(\varepsilon)$. So by Lemma 3 we have

LEMMA 4 (PD). *If T is Δ^1_{2n}-pointed, $\varepsilon \leq_{2n} T$, $(k,\varepsilon) \in \mathscr{S}$ and $e \in \omega$, then there is $T^* \subseteq T$, T^* Δ^1_{2n}-pointed such that $T^* \equiv_{2n} T$ and $F^{\varepsilon}_{k,e} \restriction [T^*]$ is either* 1-1 *or constant.*

The following plays here the role of Lemma 10 in [Fr].

LEMMA 5 (PD). *Fix $k \in \omega$. If T is Δ^1_{2n}-pointed and $\exists\alpha \in [T]$ $(\alpha \notin \mathscr{C}_{2n}(T) \wedge (k,\alpha) \in \mathscr{S})$, then there is $T^* \subseteq T$, $T^* \equiv_{2n} T$, T^* Δ^1_{2n}-pointed and there are $\beta \leq_{2n} T$, $l \in \omega$ such that $\mathscr{S}(l,\beta)$ and for all $\alpha \in [T^*]$, $\mathscr{S}(k,\alpha) \wedge \tau(k,\alpha) = \tau(l,\beta)$.*

PROOF. Here $\mathscr{C}_{2n}(x)$ is the largest countable $\Sigma^1_{2n}(x)$ set of reals. Find $\alpha_0 \in [T]$, $\alpha_0 \notin \mathscr{C}_{2n}(T)$ such that $(k,\alpha_0) \in \mathscr{S}$. Find α^*_0 such that $(k,\alpha_0,\alpha^*_0) \in \mathscr{P}^*$. Clearly $(k,\alpha_0,\alpha^*_0) \notin \mathscr{C}_{2n-1}(T) =$ largest countable $\Pi^1_{2n-1}(T)$ set of reals. Then by the proof of Lemma 12 in [Ke3] we can find some $(l,\beta,\beta^*) \in \mathscr{P}^*$ such that

$$\sigma(l,\beta,\beta^*) = \sigma(k,\alpha_0,\alpha^*_0) \wedge \langle k,\alpha_0,\alpha^*_0\rangle \not\leq_{2n-1} \langle l,\beta,\beta^*,T\rangle.$$

[We repeat the general argument here: We are given a Π^1_{2n-1} set $P \subseteq \omega^\omega$, a Π^1_{2n-1}-norm ρ on P and a real $x \in P - \mathscr{C}_{2n-1}$ and we want to conclude that there is $y \in P$ with $\rho(x) = \rho(y)$ and $x \not\leq_{2n-1} y$. (By direct relativization we immediately obtain what we stated above.) If this conclusion fails then we have

$$\forall y \in P(\rho(x) = \rho(y) \Rightarrow x \leq_{2n-1} y).$$

So if $R = \{z \in P : \forall y \in P\ [\rho(y) = \rho(z) \Rightarrow z \leq_{2n-1} y]\}$ then clearly $R \in \Pi^1_{2n-1}$ and $x \in R$. But also $R \subseteq \bigcup_\xi \{z \in P : \rho(z) = \xi \wedge \forall y \in P[\rho(y) = \xi \Rightarrow z \leq_{2n-1} y]\} = \bigcup_\xi A_\xi$, where clearly each A_ξ is $\boldsymbol{\Pi}^1_{2n-1}$ and countable, so by Theorem (1A-1) of [Ke1] $\bigcup_\xi A_\xi$ and therefore R is thin, thus $R \subseteq \mathscr{C}_{2n-1}$ and in particular $x \in \mathscr{C}_{2n-1}$, a contradiction.]

The rest of the argument parallels that of the proof of Lemma 10 in [Fr]. Indeed, by the above, we have

$$\exists l,\beta,\beta^*[\mathscr{P}^*(l,\beta,\beta^*) \wedge \exists\alpha,\alpha^*(\mathscr{P}^*(k,\alpha,\alpha^*) \wedge \alpha \in [T] \wedge \sigma(k,\alpha,\alpha^*)$$
$$= \sigma(l,\beta,\beta^*) \wedge \langle k,\alpha,\alpha^*\rangle \not\leq_{2n-1} \langle l,\beta,\beta^*,T\rangle)].$$

By the Basis Theorem of [Mo2] we can find such l,β,β^* for which $\beta,\beta^* \leq_{2n} T$. Then the set of all $\langle\alpha,\alpha^*\rangle$ with the property

$$\alpha \in [T] \wedge \mathscr{P}^*(k,\alpha,\alpha^*) \wedge \sigma(k,\alpha,\alpha^*) = \sigma(l,\beta,\beta^*)$$

is $\Delta^1_{2n-1}(\beta,\beta^*,T)$ and contains a member which is not in $\Delta^1_{2n-1}(\beta,\beta^*,T)$ so by the Perfect Set Theorem (see [Mo1]) it contains a perfect set $[T']$, where $T' \leq_{2n} T$. Project now $[T']$ to its first coordinate to obtain a $\Delta^1_{2n}(T)$ tree T^* such that $T^* \subseteq T$ and for all $\alpha \in [T^*]$ there is (unique) α^* with $\mathscr{P}^*(k,\alpha,\alpha^*)$ and $\sigma(k,\alpha,\alpha^*) = \sigma(l,\beta,\beta^*)$. Then for all $\alpha \in [T^*]$, $\mathscr{S}(k,\alpha)$ holds and $\tau(k,\alpha) = \sigma(k,\alpha,\alpha^*) = \sigma(l,\beta,\beta^*) = \tau(l,\beta)$, so we are done. ⊣

2.3. Assume now that a sequence $\mathbf{d}_0 \le \mathbf{d}_1 \le \mathbf{d}_2 \le \cdots$ of Δ^1_{2n}-degrees is given, where without loss of generality $\mathbf{d}_0 = \mathbf{0}$. Pick for each i, $\alpha_i \in \mathbf{d}_i$ and then $n_i \in \omega$ such that $(n_i, \alpha_i) \in \mathcal{S}$, $\tau(n_i, \alpha_i) < \tau(n_{i+1}, \alpha_{i+1})$ and

$$\sup\{\tau(n_i, \alpha_i) \colon i \in \omega\} = \sup\{\mu^{\alpha_i} \colon i \in \omega\}.$$

Let also $\{e_i\}_{i\in\omega}$ be an enumeration of ω in which each $e \in \omega$ appears infinitely often. Using our previous lemmas we shall construct, following a procedure analogous to that in [Fr], a binary system of Δ^1_{2n}-pointed trees T_s, indexed by the binary finite sequences $s \in 2^{<\omega}$, as follows:

$$T_\varnothing = \text{full binary tree } 2^{<\omega}.$$

Assume now T_s has been defined for all s of length $\le i$ so that the following hold.

(i) If $\mathrm{lh}(s) = j \le i$, then $T_s \equiv_{2n} \alpha_j$ and

(ii) T_s is Δ^1_{2n}-pointed.

For each s of length i we shall now define $T_{s^\frown 0}, T_{s^\frown 1} \subseteq T_s$ so that at least properties (i), (ii) are preserved. First split T_s into two disjoint trees T'_s and T''_s both Δ^1_{2n}-pointed and of the same Δ^1_{2n}-degree as T_s, using Lemma 1. Then find subtrees T^*_s, T^{**}_s of T'_s, T''_s respectively both Δ^1_{2n}-pointed and both of the same Δ^1_{2n}-degree as α_{i+1}, using Lemma 2. At the next step thin down T^*_s, T^{**}_s to T^0_s, T^1_s respectively, which are still Δ^1_{2n}-pointed, of the same Δ^1_{2n}-degree as α_{i+1} and on each one of them $F^{\alpha_{i+1}}_{n_{i+1}, e_{i+1}}$ is either 1-1 or constant, using Lemma 4. Finally, using Lemma 5 define $T_{s^\frown 0}$ to be a Δ^1_{2n}-pointed subtree of T^0_s of the same Δ^1_{2n}-degree as α_{i+1} and such that for some $\beta \le_{2n} \alpha_{i+1}$, $l \in \omega$ we have $\mathcal{S}(l, \beta)$ and for all $\alpha \in [T_{s^\frown 0}]$, $\mathcal{S}(i, \alpha)$ holds and $\tau(i, \alpha) = \tau(l, \beta)$, provided such a tree exists. Otherwise let $T_{s^\frown 0} = T^0_s$. Similarly define $T_{s^\frown 1}$ (relative to T^1_s). Clearly $T_{s^\frown 0} \cup T_{s^\frown 1} \subseteq T_s$ and $T_{s^\frown 0} \cap T_{s^\frown 1} = \varnothing$. Put $[T] = \bigcup_{f\in 2^\omega} \bigcap_n [T_{f\restriction n}]$. By the usual fusion argument T is a perfect binary tree. We shall prove that if $\alpha \in [T] - \bigcup_i \mathscr{C}_{2n}(\alpha_i)$ then $[\alpha]_{2n}$ is a minimal strict upper bound for $\{\mathbf{d}_i\}_{i\in\omega}$. Since obviously there are many such α's ($\bigcup_i \mathscr{C}_{2n}(\alpha_i)$ being countable) this will complete our proof.

Fix such an α and let $f \in 2^\omega$ be such that $\forall n$, $\alpha \in [T_{f\restriction n}]$. Since each T_s is Δ^1_{2n}-pointed and $T_{f\restriction i} \equiv_{2n} \alpha_i$, clearly $\alpha_i \le_{2n} \alpha$ so that $[\alpha]_{2n}$ is an upper bound for the $\mathbf{d}_i$'s. Since $\alpha \notin \bigcup_i \mathscr{C}_{2n}(\alpha_i)$, $[\alpha]_{2n}$ is a strict upper bound.

We prove next that $\mu^\alpha = \sup\{\mu^{\alpha_i} \colon i \in \omega\}$. Since $\mu^{\alpha_i} \le \mu^\alpha$ it is enough to show that $\mu^\alpha \le \sup\{\mu^{\alpha_i} \colon i \in \omega\}$. Fix $(k, \alpha) \in \mathcal{S}$. Then look at $T^{f(k)}_{f\restriction k}$. Since $\alpha \in [T_{f\restriction(k+1)}] \subseteq [T^{f(k)}_{f\restriction k}]$ and $\alpha \notin \mathscr{C}_{2n}(T^{f(k)}_{f\restriction k})$ we have by Lemma 5 and the construction of the T_s's that for some $l \in \omega$ and some $\beta \le_{2n} \alpha_{k+1}$, $(l, \beta) \in \mathcal{S}$ and $\forall \alpha' \in [T_{f\restriction(k+1)}](\tau(k, \alpha') = \tau(l, \beta))$. In particular, $\tau(k, \alpha) = \tau(l, \beta) < \mu^\beta \le \mu^{\alpha_{k+1}}$. So $\mu^\alpha = \sup\{\tau(k, \alpha) \colon (k, \alpha) \in \mathcal{S}\} \le \sup\{\mu^{\alpha_i} \colon i \in \omega\}$ and we are done.

Finally assume that $\beta \le_{2n} \alpha$ and $[\beta]_{2n}$ is a strict upper bound for the $\mathbf{d}_i$'s. We want to show that $\alpha \le_{2n} \beta$. For that first find $e \in \omega$ and $(t, \alpha) \in \mathcal{S}$ such that

$$\begin{aligned} \beta(k) = l &\Leftrightarrow (e, \langle k, l, \alpha\rangle) \in \mathcal{S} \\ &\Leftrightarrow (e, \langle k, l, \alpha\rangle) \le^*_\tau (t, \alpha), \end{aligned}$$

by boundedness. Find $i+1$ large enough so that $e_{i+1}=e$ and $\tau(t,\alpha)\leq \tau(n_{i+1},\alpha_{i+1})$. We are using here of course the facts that $\mu^{\alpha}=\sup\{\mu^{\alpha_i}: i\in\omega\}$ and $\sup\{\tau(n_i,\alpha_i): i\in\omega\}=\sup\{\mu^{\alpha_i}: i\in\omega\}$. Then clearly

$$\begin{aligned}\beta(k)=l &\Leftrightarrow (e,\langle k,l,\alpha\rangle)\in\mathscr{S}\\ &\Leftrightarrow (e,\langle k,l,\alpha\rangle)\leq^*_{\tau}(n_{i+1},\alpha_{i+1})\\ &\Leftrightarrow \exists\gamma\,[(e,\langle k,l,\alpha\rangle,\gamma)\leq^*_{\sigma}(n_{i+1},\alpha_{i+1},\alpha^*_{i+1})],\end{aligned}$$

so that

$$\beta = F^{\alpha_{i+1}}_{n_{i+1},e_{i+1}}(\alpha).$$

Now $F^{\alpha_{i+1}}_{n_{i+1},e_{i+1}}\restriction[T_{fl(i+1)}]$ is either 1-1 or constant. In the first case, since $\alpha\in[T_{fl(i+1)}]$ we have $\alpha\leq_{2n}\langle\beta,\alpha_{i+1},T_{fl(i+1)}\rangle\leq_{2n}\beta$ so $\alpha\equiv_{2n}\beta$ and we are done. In the second case, $\{\beta\}=F^{\alpha_{i+1}}_{n_{i+1},e_{i+1}}[T_{fl(i+1)}]$ so that $\beta\leq_{2n}\langle\alpha_{i+1},T_{fl(i+1)}\rangle\leq_{2n}\alpha_{i+1}$, a contradiction and the proof is complete.

§3. Further results. By a closer inspection of the above proof one can actually extract the following more precise version of the main result and also settle the question of the existence of least strict upper bounds, where $\mathbf{d}$ is a *least strict upper bound* of $\mathbf{d}_0\leq\mathbf{d}_1\leq\mathbf{d}_2\leq\cdots$ if it is a strict upper bound and for every other such $\mathbf{e}$, $\mathbf{d}\leq\mathbf{e}$.

THEOREM. *Assume $n\geq 1$ and Determinacy $(\mathbf{\Delta}^1_{2n-2})$. Let $\mathbf{d}_0\leq\mathbf{d}_1\leq\cdots$ be an ascending sequence of Δ^1_{2n}-degrees amd let $\alpha_i\in\mathbf{d}_i$, $\forall i\in\omega$. Then*

(i) *There is a perfect tree $T\in\mathscr{C}_{2n}(\langle\alpha_i\rangle_{i\in\omega})$ such that for every $\alpha\in[T]$, $[\alpha]_{2n}$ is an upper bound of $\{\mathbf{d}_i\}_{i\in\omega}$, while if $\alpha\in[T]-\bigcup_i\mathscr{C}_{2n}(\alpha_i)$ then $[\alpha]_{2n}$ is a minimal strict upper bound of $\{\mathbf{d}_i\}_{i\in\omega}$.*

(ii) *There exists a minimal strict upper bound of $\{\mathbf{d}_i\}_{i\in\omega}$.*

(iii) *There is a least strict upper bound of $\{\mathbf{d}_i\}_{i\in\omega}$ iff for some $i_0\in\omega$, $\omega^{\omega}\subseteq\mathscr{C}_{2n}(\alpha_{i_0})$.*

In particular if $\forall\alpha\exists\beta\ (\beta\notin\mathscr{C}_{2n}(\alpha))$, for example if Determinacy $(\mathbf{\Sigma}^1_{2n-1})$ holds, no ascending sequence of Δ^1_{2n}-degrees has a least strict upper bound.

PROOF. (i) follows immediately from the arguments in §2.

(ii) Let T be as in (i). Let $\alpha\in[T]$. If $\alpha\in[T]-\bigcup_i\mathscr{C}_{2n}(\alpha_i)$ we are done. Otherwise say $\alpha\in\mathscr{C}_{2n}(\alpha_{i_0})$. Then $\alpha_i\in\mathscr{C}_{2n}(\alpha_{i_0})$, for all i. Let $\leq$ be a $\Delta^1_{2n}(\alpha_{i_0})$-good wellordering of $\mathscr{C}_{2n}(\alpha_{i_0})$ (by [Ke1]) and let $\beta\in\mathscr{C}_{2n}(\alpha_{i_0})$ be the $\leq$-least real in $\mathscr{C}_{2n}(\alpha_{i_0})$ which is $\leq$-bigger than all $\gamma\in\mathscr{C}_{2n}(\alpha_{i_0})$ with $\gamma\leq_{2n}\alpha_i$ for some i. Such β exists since $\alpha_i\leq_{2n}\alpha$ for all i and $\gamma\leq\delta\Rightarrow\gamma\leq_{2n}\delta$. We can define now a sequence $\{\beta_i\}_{i\in\omega}\in\mathscr{C}_{2n}(\alpha_{i_0})$ such that $\beta_0\leq_{2n}\beta_1\leq\cdots$ and $\forall i\,\exists j(\alpha_i\leq\beta_j)\wedge\forall j\,\exists i(\beta_j\leq\alpha_i)$. So without loss of generality we can assume $\{\alpha_i\}_{i\in\omega}\in\mathscr{C}_{2n}(\alpha_{i_0})$, therefore $T\in\mathscr{C}_{2n}(\alpha_{i_0})$. Then $[T]\subseteq\mathscr{C}_{2n}(\alpha_{i_0})$ (otherwise we are done) so $\omega^{\omega}\subseteq\mathscr{C}_{2n}(\alpha_{i_0})$ and $\{\mathbf{d}_i\}_{i\in\omega}$ has a least strict upper bound, namely $[\beta]_{2n}$, where β is as above. This completes the proof of (ii).

(iii) As in (ii), noticing that a least strict upper bound is also the unique minimal strict upper bound. ⊣

Of course for $n=1$ (where clearly one does not need any hypotheses) the preceding result is due in its entirety to Friedman; see [Fr]. Friedman uses this to show also that if ω_1 is inaccessible in L then every stable countable ordinal is of the form $\delta^1_2(\alpha) = \mu^\alpha_2$ for some α. We do not know any such characterization of the ordinals μ^α_{2n} for $n > 1$.

At the suggestion of the referee we comment briefly on the problem of strict minimal upper bounds for sequences of Δ^1_{2n+1}-degrees: To start with, for $n = 0$ i.e. on the context of hyperdegrees Sacks [Sa2] has shown that most "reasonable" sequences of hyperdegrees have strict minimal upper bounds. It is still open however if *all* sequences of hyperdegrees have strict minimal upper bounds. For $n > 0$ much less is known. In a forthcoming paper entitled *Forcing in analysis,* II we will show (from PD) that for every Δ^1_{2n+1}-degree d there are continuum many minimal covers of d i.e. Δ^1_{2n+1}-degrees e such that $d < e$ but there is no e' such that $d < e' < e$. Our method also shows that if a sequence of Δ^1_{2n+1}-degrees is not "far spread out" then it has strict minimal upper bounds. As opposed to Δ^1_{2n}-degrees, some sequences of Δ^1_{2n+1}-degrees *have* least strict upper bounds. This situation is investigated in more detail in [Sa2] and in our forthcoming paper mentioned above.

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