Π^1_1 Borel Sets
Author(s): Alexander S. Kechris, David Marker and Ramez L. Sami
Source: *The Journal of Symbolic Logic*, Vol. 54, No. 3 (Sep., 1989), pp. 915-920
Published by: Association for Symbolic Logic
Stable URL: http://www.jstor.org/stable/2274751
Accessed: 20/05/2013 12:56

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at
http://www.jstor.org/page/info/about/policies/terms.jsp

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Association for Symbolic Logic is collaborating with JSTOR to digitize, preserve and extend access to *The Journal of Symbolic Logic*.

http://www.jstor.org

This content downloaded from 131.215.71.79 on Mon, 20 May 2013 12:56:39 PM
All use subject to JSTOR Terms and Conditions

THE JOURNAL OF SYMBOLIC LOGIC
Volume 54, Number 3, Sept. 1989

Π_1^1 BOREL SETS

ALEXANDER S. KECHRIS, DAVID MARKER, AND RAMEZ L. SAMI

§0. Introduction. The results in this paper were motivated by the following question of Sacks. Suppose T is a recursive theory with countably many countable models. What can you say about the least ordinal α such that all models of T have Scott rank below α? If Martin's conjecture is true for T then $\alpha \leq \omega \cdot 2$.

Our goal was to look at this problem in a more abstract setting. Let E be a Σ_1^1 equivalence relation on ${}^\omega\omega$ with countably many classes each of which is Borel. What can you say about the least α such that each equivalence class is $\boldsymbol{\Pi}_\alpha^0$? This problem is closely related to the following question. Suppose $X \subseteq {}^\omega\omega$ is Π_1^1 and Borel. What can you say about the least α such that X is $\boldsymbol{\Pi}_\alpha^0$?

In §1 we answer these questions in ZFC. In §2 we give more informative answers under the added assumptions $V = L$ or Π_1^1-determinacy. The final section contains related results on the separation of Π_{2n+1}^1 sets by Borel sets.

Our notation is standard. The reader may consult Moschovakis [5] for undefined terms.

Some of these results were proved first by Sami and rediscovered by Kechris and Marker.

§1. Borel Π_1^1-sets.

DEFINITION 1.1. If $X \subseteq \omega_1$, let $\hat{X} = \{\omega \in \mathrm{WO}: |w| \in X\}$. If $\hat{X}$ is Π_n^1 we say that X is Π_n^1 in the codes. An ordinal α is a *basis* for subsets of ω_1 which are Π_n^1 in the codes iff whenever $X \subseteq \omega_1$, $X \neq \varnothing$ and $\hat{X}$ is Π_n^1, there is $\beta \in X$ such that $\beta < \alpha$. We let γ_n^1 be the least such ordinal.

In [1] Kechris showed that, assuming PD, $\gamma_{2n+1}^1 = \delta_{2n+1}^1$ for $n \geq 1$.

LEMMA 1.2. *If $X \subseteq {}^\omega\omega$ is Π_1^1 and Borel, then X is $\boldsymbol{\Pi}_\beta^0$ for some $\beta < \gamma_2^1$.*

PROOF. Let F be a recursive function such that $x \in X$ if and only if $f(x) \in \mathrm{WO}$. For $\eta < \omega_1$ let $X_\eta = \{x \in X: |f(x)| < \eta\}$. Since X is Borel, $f(X)$ is a $\boldsymbol{\Sigma}_1^1$ subset of WO. Thus there is $\eta < \omega_1$ such that $X = X_\eta$. Let $Z = \{w \in \mathrm{WO}: \forall x \in X |f(x)| < |w|\}$. Then Z is Π_2^1, so there is $w \in Z$ such that $\eta = |w| < \gamma_2^1$. Thus X is Wadge reducible to WO_η, but by Stern [9] WO_η is $\boldsymbol{\Sigma}_{2\eta}^0$. Hence since γ_2^1 is closed under ordinal addition X is $\boldsymbol{\Pi}_\beta^0$ for some $\beta < \gamma_2^1$. □

Received January 28, 1988; revised May 9, 1988.
The first two authors were partially supported by the National Science Foundation.

© 1989, Association for Symbolic Logic
0022-4812/89/5403-0019/$01.60

This content downloaded from 131.215.71.79 on Mon, 20 May 2013 12:56:39 PM
All use subject to JSTOR Terms and Conditions

We will next see that γ_2^1 is the least such ordinal. Suppose X is a nonempty bounded initial segment of ω_1 and $\hat{X}$ is Σ_2^1. There is a tree $\mathscr{T}$ on $\omega \times \omega_1$ such that $\hat{X} = \{x \in {}^\omega\omega : \exists f : \omega \to \omega_1 \langle x, f\rangle \in [\mathscr{T}]\}$. For $\alpha < \omega_1$ and $x \in {}^\omega\omega$, let $\mathscr{T}_x^\alpha = \{\tau : n \to \alpha : n \in \omega$ and $\langle x|n, \tau\rangle \in \mathscr{T}\}$. Then there is a recursive function f such that if $\omega \in \mathrm{WO}$ and $x \in {}^\omega\omega$, then $f(\omega, x)$ is a code for $\mathscr{T}_x^{|w|}$ and $x \in \hat{X}$ iff $\exists w \in \mathrm{WO}\ f(w,x) \notin \mathrm{WF}$. Let $R \subseteq \omega_1 \times \omega_1 \times {}^\omega\omega$ be defined by $R(\alpha, \beta, x)$ if and only if $|\mathscr{T}_x^\alpha| \nless \beta \wedge \forall \sigma \in \alpha^{<\omega} |(\mathscr{T}_x^\alpha)_\sigma| \neq \beta$. Then $x \in \hat{X} \Leftrightarrow \exists \alpha, \beta < \omega_1\, R(\alpha, \beta, x)$. Let $R^* \in \Pi_1^1$ and $R^{**} \in \Sigma_1^1$ be such that for $u, v \in \mathrm{WO}$ and $x \in {}^\omega\omega$

$$R^*(u, v, x) \Leftrightarrow R^{**}(u, v, x) \Leftrightarrow R(|u|, |v|, x).$$

Let

$$A(v, w', w) \Leftrightarrow w, w', v \in \mathrm{WO} \wedge |w| = |w'| \wedge \exists n_0, n_1\, R^*(v|n_0, v|n_1, w') \\ \wedge \forall v'(|v'| < |v| \to \forall w^*(|w^*| = |w| \to \forall n_0, n_1 \neg R^{**}(v'|n_0, v'|n_1, w^*))).$$

Then A is Π_1^1.

For all $\alpha \in X$ we can find $\beta_\alpha = \mu\beta\, \exists w \in \mathrm{WO}|w| = \alpha \wedge \mathscr{T}_w^\beta \notin \mathrm{WF}$. Let w_α be chosen such that $\mathscr{T}_{w_\alpha}^{\beta_\alpha} \notin \mathrm{WF}$. Let $\gamma_\alpha = \mu\gamma R(\beta_\alpha, \gamma, w_\alpha)$ and let $\delta_\alpha = \sup\{\beta_\alpha + 1, \gamma_\alpha + 1\}$. Then

$$A = \bigcup_{\alpha \in X} \{(v, w', w) : |w| = \alpha \wedge |v| = \delta_\alpha \wedge |w| = |w'| \wedge R(\beta_\alpha, \gamma_\alpha, w')\}.$$

Since X is bounded, A is a countable union of Borel set and hence Borel.

For all $\alpha \in X$ there are V and w' such that $\{w : A(v, w', w)\} = \{w \in \mathrm{WO} : |w| = \alpha\}$. Thus, for all $\alpha \in X$, A has Borel rank greater than or equal to α.

LEMMA 1.3 (STERN [9]). *Suppose $\alpha = \omega^\beta$. Then $\{x : x \in \mathrm{WO}, |x < \alpha\}$ is $\boldsymbol{\Sigma}_{2\cdot\beta}^0$ and $\{x \in \mathrm{WO} : |x| \leq \alpha\}$ is not $\boldsymbol{\Sigma}_{2\cdot\beta+1}^0$. In particular, $\{x \in \mathrm{WO} : |x| = \alpha\}$ is not $\boldsymbol{\Pi}_{2\cdot\beta}^0$.*

THEOREM 1.4. *$\gamma_2^1 = \sup\{\alpha : \exists X \subseteq {}^\omega\omega\ X$ is Π_1^1, Borel and α is the least ordinal such that X is $\boldsymbol{\Pi}_\alpha^0\}$.*

PROOF. In view of Lemma 1.2 we need only show that if $\delta < \gamma_2^1$, there is a Π_1^1 Borel set A which is not $\boldsymbol{\Pi}_\delta^0$.

Let $X \subset \omega_1$ be nonempty, bounded and Σ_2^1 in the codes such that, for all $\alpha \leq \delta$, $\alpha \in X$. Let $X^* = \{\beta : \exists \alpha \in X\, \forall \gamma \leq \alpha\, \gamma \in X \wedge \beta \leq \omega^\alpha\}$. There is a recursive f such that if $x \in \mathrm{WO}$, then $f(x) \in \mathrm{WO}$ and $|f(x)| = \omega^{|x|}$. Thus $\hat{X}^* = \{w \in \mathrm{WO} : \exists v \in X\ \forall n \in w\ v|n \in X \wedge |w| \leq |f(v)|\}$. Then X^* is a proper initial segment of ω_1 containing ω^δ which is Σ_2^1 in the codes. By the above construction we can find a Π_1^1 Borel set A which has $\{w \in \mathrm{WO} : |w| = \omega^\delta\}$ as a section. By Lemma 1.3, A is not $\boldsymbol{\Pi}_\delta^0$. □

COROLLARY 1.5. *For all $\alpha < \gamma_2^1$ there is a Σ_1^1 equivalence relation with countably many classes such that each class is Borel but at least one class is not $\boldsymbol{\Pi}_\alpha^0$.*

PROOF. Let A be Π_1^1 and Borel but not $\boldsymbol{\Pi}_{\beta+2}^0$. Let $\Psi : A \to \omega_1$ be a Π_1^1-norm. Since A is Borel, there is $\delta < \omega_1$ such that $\forall x \in A\ \Psi(x) < \delta$. Define an equivalence relation

$$xEy \Leftrightarrow (x \notin A \wedge y \notin A) \vee \Psi(x) = \Psi(y).$$

If each E class is $\boldsymbol{\Pi}_\beta^0$, then A would be $\boldsymbol{\Sigma}_{\beta+1}^0$, a contradiction. □

If E is a Σ_1^1 equivalence relation with countably many equivalence classes each of which is Borel, then there is $\alpha < \gamma_2^1$ such that all E-classes are $\boldsymbol{\Pi}_\alpha^0$. In fact the following stronger theorem is true.

This content downloaded from 131.215.71.79 on Mon, 20 May 2013 12:56:39 PM
All use subject to JSTOR Terms and Conditions

THEOREM 1.6 [6]. *If E is a Σ_1^1 equivalence relation with Borel equivalence classes and there is a bound on the ranks of the classes, then there is $\alpha < \gamma_2^1$ such that every E class is $\boldsymbol{\Pi}_\alpha^0$.*

PROOF. Let f be a recursive function such that $xEy \Leftrightarrow f(x,y) \notin \mathrm{WO}$. For $\eta < \omega_1$, say

$$xE_\eta y \Leftrightarrow \neg(f(x,y) \in \mathrm{WO} \wedge f(x,y) \leq \eta).$$

Then $E = \bigcap_{\eta<\omega_1} E_\eta$. For any x, since $\{y: yEx\}$ is Borel, by boundedness we can find a $\gamma_x < \omega_1$ such that $\forall y\, x\not{E} y \rightarrow x\not{E}_{\gamma_x} y$, so $xEy \Leftrightarrow xE_{\gamma_x} y$. If each E class is $\boldsymbol{\Pi}_\alpha^0$, then for each β and x we can separate $\{y: xEy\}$ from $\{y: x\not{E}_\beta y\}$ by a $\boldsymbol{\Pi}_\alpha^0$ set. On the other hand if for all β we can separate $\{y: xEy\}$ from $\{y: x\not{E}_\beta y\}$ by a $\boldsymbol{\Pi}_\alpha^0$ set, then since eventually $\{y: xEy\} = \{y: xE_\beta y\}$, $\{y: xEy\}$ is $\boldsymbol{\Pi}_\alpha^0$.

Suppose $v, w \in \mathrm{WO}$. Since $\{y: yEx\}$ is $\Sigma_1^1(x)$ and $\{y: y\not{E}_{|w|} x\}$ is $\Delta_1^1(x,w)$, if they can be separated by a $\boldsymbol{\Pi}_{|v|}^0$ set, by Louveau's separation theorem [3] they can be separated by a $\boldsymbol{\Pi}_{|v|}^0$ set with code hyperarithmetic in $\langle v, w, x\rangle$.

Thus if $Z = \{w \in \mathrm{WO}$: every E class is $\boldsymbol{\Pi}_{|w|}^0\}$, then

$$w \in Z \Leftrightarrow w \in \mathrm{WO} \wedge \forall x, v(v \in \mathrm{WO} \rightarrow \exists z \leq_{\mathrm{hyp}} \langle x, v, w\rangle\ z \text{ is a } \boldsymbol{\Pi}_{|w|}^0\text{-code}$$
$$\wedge\ \forall y((xEy \rightarrow y \in B(z)) \wedge (x\not{E}_{|v|} y \rightarrow y \notin B(z)))),$$

where $B(z)$ is the Borel set coded by z. Since the quantifier $\exists z \leq_{\mathrm{hyp}} \langle x, v, w\rangle$ is really universal, Z is Π_2^1. Thus there is $w \in Z$ with $|w| < \gamma_2^1$.

Question. Suppose G is a Polish group acting on ${}^\omega\omega$ with countably many orbits. What can we say about the least α such that every orbit is $\boldsymbol{\Pi}_\alpha^0$? By results of Sami [7], $\alpha < \delta_2^1$.

§2. Bounds on γ_2^1.

LEMMA 2.1. $\delta_2^1 < \gamma_2^1$.

PROOF. If $X = \{\alpha: \alpha < \delta_2^1\}$, then $\hat{X} = \{y \in \mathrm{WO}: \exists x \in \Delta_2^1\ x \in \mathrm{WO} \wedge |x| = |y|\}$. Since $\hat{X}$ is Σ_2^1, there is $\delta \in \omega_1 - X$ such that $\delta_2^1 \leq \delta < \gamma_2^1$.

THEOREM 2.2 [5]. *If $V = L$, then $\gamma_2^1 = \delta_3^1$.*

PROOF. If $V = L$, then Δ_3^1 is a basis for Σ_3^1. Thus every nonempty Π_2^1 set contains a Δ_3^1 member. So $\gamma_2^1 \leq \delta_3^1$.

Suppose $y \in \mathrm{WO}$ is Δ_3^1. Say $y(n) = m \Leftrightarrow \exists r\ A(r, \langle n,m\rangle)$ and $y(n) \neq m \Leftrightarrow \exists r\ B(r, \langle n,m\rangle)$, where A and B are Π_2^1. Then

$$x = y \Leftrightarrow \exists r \forall n, m((x(n) = m \rightarrow A(r_n, \langle n,m\rangle)) \wedge (x(n) \neq m \rightarrow B(r_m, \langle n,m\rangle))).$$

Thus $x = y \Leftrightarrow \exists r\ C(r,x)$, where C is Π_2^1 and x is recursive in every element of C.

Let $Z = \{z \in \mathrm{WO}: L_{|z|} \models \mathrm{KP} \wedge \exists r, x \in L_{|z|}(r,x) \in C\}$. Since $V = L$, Z is nonempty. Thus

$$Z' = \{z \in \mathrm{WO}: \forall x \in L_{|z|}(x \in \mathrm{WO} \rightarrow |x| < |z|) \wedge \exists r, x \in L_{|z|}((r,x) \in C)\} \supseteq Z \neq \varnothing.$$

But "$x \in L_{|z|}$" is Δ_1^1 and if $r, x \in L_{|z|}$, then $r, x \leq_{\mathrm{hyp}} z$, so

$$z \in Z' \Leftrightarrow \forall x((x \in L_{|z|} \wedge x \in \mathrm{WO}) \rightarrow |x| < |z|) \wedge \exists r, x \leq_{\mathrm{hyp}} z\ (r,x) \in C.$$

So Z' is Π_2^1. Thus there is $z \in Z'$ such that $|z| < \gamma_2^1$ and $y \in L_{|z|}$. Thus $|y| < |z| < \gamma_2^1$. □

This content downloaded from 131.215.71.79 on Mon, 20 May 2013 12:56:39 PM
All use subject to JSTOR Terms and Conditions

We will see that under the assumption of Π^1_1-determinacy γ^1_2 is quite large in L but much smaller than δ^1_3.

THEOREM 2.3. *Suppose, for all* $\xi < \gamma^1_2, \aleph^L_\xi < \aleph_1$. *Then, for all* $\xi < \gamma^1_2, \aleph^L_\xi < \gamma^1_2$.

PROOF. Pick $\xi < \gamma^1_2$. Let $X \subseteq w_1$ be a bounded initial segment of ω_1 containing ξ which is Σ^1_2 in the codes.

Suppose (ω, E) is a transitive, well founded model of KP + $V = L$. If $w \in \mathrm{WO}$ and $f: \mathrm{dom}(w) \to \omega$ we say that f is a *w-chain* in (ω, E) if and only if for all $n \in \mathrm{dom}(w)$ if $f(n) = m$, then $(w, E) \models$ "$m = \aleph_{|w|n|}$."

Claim. For all $\alpha < \gamma^1_2, \aleph^L_\alpha = \sup\{\aleph^{(\omega,E)}_\alpha$: (ω, E) *is a transitive, well founded model of* KP + $V = L$, $\alpha < \mathrm{On}(\omega, E)$ *and* $(\omega, E) \models \aleph_\alpha$ *exists*$\}$.

($\geq$) Clear since, for some β, $(\omega, E) \simeq (L_{\beta,\varepsilon})$.

($\leq$) Let $(\omega, E) \simeq (L_{\omega^L_{\alpha+1}}, \varepsilon)$; then $\aleph^{(\omega,E)}_\alpha = \aleph^L_\alpha$. This is possible since $\alpha + 1 < \gamma^1_2$, so $\omega^L_{\alpha+1} < \aleph_1$.

Let $Z = \{v \in \mathrm{WO}: \exists E, f, w, g(\omega, E)$ is a transitive well founded model of KP + $V = L$, $w \in \hat{X}$, $f: \mathrm{dom}(w) \to \omega$ is a w-chain for (ω, E) and $g: \mathrm{dom}(v) \to \{m \in \omega: \exists n \in \mathrm{dom}(w)(\omega, E) \models$ "m is an ordinal and $m \leq f(n)$"$\}$ is order preserving$\}$. Then Z is Σ^1_2 and $Z = \{v \in \mathrm{WO}: \exists\delta \in X\ |v| \leq \aleph^L_\delta\}$. Let $z \in \mathrm{WO} - Z$ be such that $|z| < \gamma^1_2$. Then $\aleph^L_\xi < \gamma^1_2$. □

COROLLARY 2.4. $(\forall x \aleph^{L[x]}_1 < \aleph_1)\aleph^L_{\gamma^1_2} = \gamma^1_2$.

PROOF. $\aleph_1$ is inaccessible in L. Thus $\aleph_1 = \aleph^L_{\aleph_1}$. So $\aleph^L_{\gamma^1_2} < \aleph_1$. □

On the other hand, γ^1_2 will always behave reasonably well in L.

THEOREM 2.4. γ^1_2 *is definable in L and* $\mathrm{cf}^L(\gamma^1_2) = \omega$.

PROOF. In [2] Kechris and Moschovakis show that every subset of ω_1 which is Π^1_2 in the codes is constructible. In fact if $U \subseteq \omega \times R$ is an ω-universal Π^1_2 set and $\bar{U} = \{(n, x): x \in \mathrm{WO} \wedge \forall y \in \mathrm{WO}\ |y| = |x| \to (n, y) \in U\}$, then $Y = \{(n, \alpha) \in \omega \times \omega_1: \exists x \in \mathrm{WO}\ \alpha = |x| \wedge (n, x) \in \bar{U}\}$ is constructible. Thus in L we can define $\langle \alpha_n: n \in \omega \rangle \in L$, where

$$\alpha_n = \begin{cases} 0 & \forall\alpha < \omega_1 (n, \alpha) \notin Y, \\ \text{least } \alpha\ (n, \alpha) \in Y & \text{otherwise.} \end{cases}$$

Then $\gamma^1_2 = \sup_n \alpha_n$. [We thank the referee for pointing out this simple argument.] □

COROLLARY 2.5 (Π^1_1-AD). γ^1_2 *is less than the first Silver indescernible, so* $\gamma^1_2 < \delta^1_3$.

§3. A separation theorem for Π^1_{2n+1}-sets. We assume projective determinacy. Let U be an ω-universal Π^1_{2n+1} set. Let $\varphi: U \to \boldsymbol{\delta}^1_{2n+1}$ be a Π^1_{2n+1}-norm. Let A and B be disjoint Π^1_{2n+1} sets, and let $e_0, e_1 \in \omega$ be such that $A = \{x: (e_0, x) \in U\}$ and $B = \{x: (e_1, x) \in U\}$. For $\eta < \boldsymbol{\delta}^1_{2n+1}$, let $A_\eta = \{x: \varphi(e_0, x) < \eta\}$ and $B_\eta = \{x: \varphi(e_1, x) < \eta\}$.

The following is a generalization of a weak version of a theorem of Stern [10].

THEOREM 3.1. *Suppose, for all* $\eta < \boldsymbol{\delta}^1_{2n+1}$, A_η *and* B_η *are* $\boldsymbol{\Pi}^0_\xi$*-separable. Then A and B are* $\boldsymbol{\Pi}^0_\xi$*-separable.*

The proof uses the analysis of certain Wadge-like games from [8]. If $X_0 \cap X_1 = Y_0 \cap Y_1 = \varnothing$, consider the game $G(\langle X_0, X_1\rangle, \langle Y_0, Y_1\rangle)$ where if I plays $\alpha \in {}^\omega\omega$ and II plays $\beta \in {}^\omega\omega$, then II wins if and only if $\alpha \in X_0 \to \beta \in Y_0$ and $\alpha \in X_1 \to \beta \in Y_1$. We write $\langle X_0, X_1\rangle \leq \langle Y_0, Y_1\rangle$ if and only if II has a winning strategy.

This content downloaded from 131.215.71.79 on Mon, 20 May 2013 12:56:39 PM
All use subject to JSTOR Terms and Conditions

LEMMA 3.2. *Assume all X_i and Y_i are projective. If $\langle X_0, X_1\rangle \not\leq \langle Y_0, Y_1\rangle$, then $\langle Y_0, Y_1\rangle \leq \langle X_1, X_0\rangle$.*

PROOF. By PD, I has a winning strategy σ in $G(\langle X_0, X_1\rangle, \langle Y_0, Y_1\rangle)$. Suppose II plays $G(\langle Y_0, Y_1\rangle, \langle X_1, X_0\rangle)$ using σ (and ignoring I's last move). If I plays α and II plays β, then I wins $G(\langle X_0, X_1\rangle, \langle Y_0, Y_1\rangle)$ on the play β, α. So either $\beta \in X_0$ and $\alpha \notin Y_0$ or $\beta \in X_1$ and $\alpha \notin Y_1$. Thus $\alpha \in Y_0 \to \beta \in X_1$ and $\alpha \in Y_1 \to \beta \in X_0$. So this is a winning strategy for II in $G(\langle Y_0, Y_1\rangle, \langle X_1, X_0\rangle)$. □

LEMMA 3.3. *Let $\mathscr{C}$ be any Wadge class. If $\langle X_0, X_1\rangle \leq \langle Y_0, Y_1\rangle$ and Y_0, Y_1 can be separated by some $D \in \mathscr{C}$, then X_0 and X_1 can be separated by some $\hat{D} \in \mathscr{C}$.*

PROOF. Let $\hat{D}$ be the inverse image of D under the winning strategy. □

LEMMA 3.4. *If X_0 and X_1 are projective and C is complete $\boldsymbol{\Pi}^0_\alpha$, then $\langle X_0, X_1\rangle \leq \langle C, \neg C\rangle$ if and only if X_0 and X_1 are $\boldsymbol{\Pi}^0_\alpha$-separable.*

PROOF. ($\Rightarrow$) Clear by 3.3.

($\Leftarrow$) Let $D \in \boldsymbol{\Pi}^0_\alpha$ separate X_0 and X_1. Then D is Lipschitz reducible to C. II can win $G(\langle X_0, X_1\rangle, \langle C, \neg C\rangle)$ by playing the winning strategy in the Lipschitz game. □

PROOF OF 3.1. Suppose A and B are not $\boldsymbol{\Pi}^0_\xi$-separable. Then $\langle A, B\rangle \not\leq \langle C, \neg C\rangle$, where C is complete $\boldsymbol{\Pi}^0_\xi$. Thus, by 3.2, $\langle \neg C, C\rangle \leq \langle A, B\rangle$. Let σ be II's strategy in $G(\langle \neg C, C\rangle, \langle A, B\rangle)$ and let f_σ be the continuous function it determines. Then $f(\neg C) \subseteq A$ and $f(C) \subseteq B$. Since $f(\neg C)$ and $f(C)$ are $\boldsymbol{\Sigma}^1_1$ sets, by boundedness there is $\eta < \boldsymbol{\delta}^1_{2n+1}$ such that $f(\neg C) \subseteq A_\eta$ and $f(C) \subseteq B_\eta$. Thus, using σ, II also wins $G(\langle \neg C, C\rangle, \langle A_\eta, B_\eta\rangle)$. So $\langle \neg C, C\rangle \leq \langle A_\eta, B_\eta\rangle$. Since A_η and B_η are $\boldsymbol{\Pi}^0_\xi$-separable, $\langle A_\eta, B_\eta\rangle \leq \langle C, \neg C\rangle$. But then $\langle \neg C, C\rangle \leq \langle C, \neg C\rangle$. But then $\neg C$ is Lipschitz reducible to C, a contradiction. □

This proof also works if we replace $\boldsymbol{\Pi}^0_\alpha$ by a Wadge class of Δ^1_{2n+1} sets containing a complete set.

Question. Suppose X and Y are disjoint Σ^1_{2n+2} sets, A and B are Π^1_{2n+1} sets such that $\pi(A) = X$ and $\pi(B) = Y$ and, for $\eta < \boldsymbol{\delta}^1_{2n+1}$, $X_\eta = \pi(A_\eta)$ and $Y_\eta = \pi(B_\eta)$. Suppose, for all η, X_η and Y_η can be separated by a $\boldsymbol{\Pi}^0_\alpha$ set. Can X and Y be separated by a $\boldsymbol{\Pi}^0_\alpha$-set? Stern [10] showed the answer is yes if $n = 0$ (using the weaker assumption $\forall x \aleph_1^{L[x]} < \aleph_1$).

COROLLARY 3.5. *If A and B are Borel separable $\boldsymbol{\Pi}^1_{2n+1}$ sets, then A and B can be separated by a Π^0_α set for some $\alpha < \gamma^1_{2n+2}$.*

PROOF. Let $Z = \{w \in \mathrm{WO}\colon A \text{ and } B \text{ are } \boldsymbol{\Pi}^0_{|w|}\text{-separable}\}$. Then

$$w \in Z \Leftrightarrow \forall \eta < \boldsymbol{\delta}^1_{2n+1}\, (\exists z (z \text{ is a } \boldsymbol{\Pi}^0_{|w|}\text{-code} \wedge \forall x (x \in A_\eta \to x \in B(z)) \wedge \forall y (y \in B_\eta \to y \notin B(z)))),$$

where $B(z)$ denotes the Borel set coded by z. By Louveau and Saint Raymond ([3] and [4]), A_η and B_η are $\boldsymbol{\Pi}^0_{|w|}$-separable if and only if for any $(e, s) \in U$ with $\varphi(e, s) = \eta$, there is a $\boldsymbol{\Pi}^0_{|w|}$ set separating A_η and B_η with code in $\Delta^1_{2n+1}(w, s)$. Thus

$$w \in Z \Leftrightarrow \forall e, s((e, s) \in U \to \exists z \in \Delta^1_{2n+1}(w, s)(z \text{ is a } \boldsymbol{\Pi}^0_{|w|}\text{-code} \wedge \forall x (\varphi(e_0, x) < \varphi(e, s) \to x \in B(z)) \wedge \forall y (\varphi(e_1, y) < \varphi(e, s)) \to y \notin B(z))).$$

Thus Z is Π^1_{2n+2}. □

The next result shows this is best possible.

This content downloaded from 131.215.71.79 on Mon, 20 May 2013 12:56:39 PM
All use subject to JSTOR Terms and Conditions

PROPOSITION 3.6. *For each* $\alpha < \gamma^1_{2n+2}$ *there are* Π^1_{2n+1} *sets A and B which are Borel separable but not separable by a* Π^0_α *set.*

PROOF. As in 1.4 we can find a bounded initial segment X of $\aleph_1$ containing ω^α such that $\hat{X}$ is Σ^1_{2n+2} and if U is a universal Π^1_{2n+1} set and $\Psi: U \to \boldsymbol{\delta}^1_{2n+1}$ is a Π^1_{2n+1} norm, we can find a Π^1_{2n+1} $A \subseteq U \times \hat{X} \times \hat{X}$ such that the following conditions hold:

(i) If $A(v, w', w)$, then $|w| = |w'|$.

(ii) For all $w \in \hat{X}$ there is a unique v such that for some w', w''

$$|w'| = |w''| = |w| \wedge A(v, w', w'').$$

(iii) If $A(v, w', w)$ and $|w''| = |w|$, then $A(v, w', w'')$.

Let $B(v, w', w) \Leftrightarrow A(v, w', w') \wedge |w| \neq |w'|$. Then B is Π^1_{2n+1} and $A \cap B = \varnothing$. Let $C = \{(v, w', w) : |w'| = |w| \in X\}$. Then since X is bounded, C is Borel. Clearly $A \subseteq C$ and $B \cap C = \varnothing$, so A and B are Borel separable. But if D separates A and B, then $\{z \in \mathrm{WO} : |z| = \omega^\alpha\}$ is a section of D. Thus D is not $\boldsymbol{\Pi}^0_\alpha$. □

REFERENCES

[1] A. S. KECHRIS, *Countable ordinals and the analytical hierarchy*. I, ***Pacific Journal of Mathematics***, vol. 60 (1975), pp. 223–227.

[2] A. S. KECHRIS and Y. MOSCHOVAKIS, *Two theorems about projective sets*, ***Israel Journal of Mathematics***, vol. 12 (1972), pp. 391–399.

[3] A. LOUVEAU, *A separation theorem for* Σ^1_1 *sets*, ***Transactions of the American Mathematical Society***, vol. 260 (1980), pp. 363–378.

[4] A. LOUVEAU and J. SAINT RAYMOND, *Borel classes and closed games: Wadge-type and Hurewicz-type results*, ***Transactions of the American Mathematical Society***, vol. 304 (1987), pp. 431–467.

[5] Y. MOSCHOVAKIS, ***Descriptive set theory***, North-Holland, Amsterdam, 1980.

[6] R. SAMI, *On* Σ^1_1 *equivalence relations with Borel classes of bounded rank*, this JOURNAL, vol. 49 (1984), pp. 1273–1283.

[7] ———, *Polish group actions and the Vaught conjecture* (preprint).

[8] J. STEEL, *Determinateness and the separation property*, this JOURNAL, vol. 46 (1981), pp. 41–44.

[9] J. STERN, *Evaluation du rang de Borel de certains ensembles* (preprint).

[10] ———, *On Lusin's restricted continuum problem* (preprint).

DEPARTMENT OF MATHEMATICS
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA 91125

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT CHICAGO
CHICAGO, ILLINOIS 60680

DEPARTMENT OF MATHEMATICS
CAIRO UNIVERSITY
CAIRO, EGYPT

UER DE MATHÉMATIQUE
UNIVERSITÉ PARIS-VII
75251 PARIS, FRANCE

This content downloaded from 131.215.71.79 on Mon, 20 May 2013 12:56:39 PM
All use subject to JSTOR Terms and Conditions