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ON PROJECTIVE ORDINALS[1]

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We study in this paper the projective ordinals $\boldsymbol{\delta}^1_n$, where $\boldsymbol{\delta}^1_n = \sup\{\xi : \xi$ *is the length of a* $\boldsymbol{\Delta}^1_n$ *prewellordering of the continuum*$\}$. These ordinals were introduced by Moschovakis in [8] to serve as a measure of the "definable length" of the continuum. We prove first in §2 that projective determinacy implies $\boldsymbol{\delta}^1_n < \boldsymbol{\delta}^1_{n+1}$, for all even $n > 0$ (the same result for odd n is due to Moschovakis). Next, in the context of full determinacy, we partly generalize (in §3) the classical fact that $\boldsymbol{\delta}^1_1 = \aleph_1$ and the result of Martin that $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$ by proving that $\boldsymbol{\delta}^1_{2n+1} = \lambda^+_{2n+1}$, where λ_{2n+1} is a cardinal of cofinality ω. Finally we discuss in §4 the connection between the projective ordinals and Solovay's uniform indiscernibles. We prove among other things that $\forall\alpha$ ($\alpha^\#$ exists) implies that every $\boldsymbol{\delta}^1_n$ with $n \geq 3$ is a fixed point of the increasing enumeration of the uniform indiscernibles.

§1. Preliminaries.

1A. Let $\omega = \{0, 1, 2, \dots\}$ be the set of natural numbers and $\mathscr{R} = {}^\omega\omega$ the set of all functions from ω into ω or (for simplicity) *reals*. Letters $i, j, k, l, m, \dots$ will denote elements of ω and $\alpha, \beta, \gamma, \delta, \dots$ elements of $\mathscr{R}$. We study subsets of the *product spaces*

$$\mathscr{X} = X_1 \times \cdots \times X_k,$$

where X_i is ω or $\mathscr{R}$. We call such subsets *pointsets*. Sometimes we think of them as relations and write interchangeably

$$x \in A \Leftrightarrow A(x).$$

A *pointclass* is a class of pointsets, usually in all the product spaces. Most of the time we shall be working here with the *analytical pointclasses* Σ^1_m, Π^1_m, Δ^1_m and their corresponding *projective pointclasses* $\boldsymbol{\Sigma}^1_m$, $\boldsymbol{\Pi}^1_m$, $\boldsymbol{\Delta}^1_m$. We use $\Sigma^1_m(\alpha)$, $\Pi^1_m(\alpha)$, $\Delta^1_m(\alpha)$ for the relativized (to any $\alpha \in \mathscr{R}$) analytical pointclasses.

Various determinacy hypotheses occur frequently as assumptions in the statements of theorems in this paper. Nevertheless we never make direct use of them. We simply draw conclusions from some of their known consequences. The reader can consult [10], [8] or the recent survey article [2] for the basic facts concerning

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games, determinacy, etc. In general we write Determinacy(Γ), where Γ is a pointclass, to indicate that every set of reals in Γ is determined. Furthermore we put

$$\text{Projective Determinacy(PD)} \Leftrightarrow \text{every projective set of reals is determined,}$$
$$\text{Full Determinacy(AD)} \Leftrightarrow \text{every set of reals is determined.}$$

1B. A *prewellordering* on a set X is a relation $\preccurlyeq \subseteq X \times X$ which satisfies the following conditions:

(a) $x \preccurlyeq x, \forall x \in X$;

(b) $x \preccurlyeq y \mathbin{\&} y \preccurlyeq z \Rightarrow x \preccurlyeq z$;

(c) $x \preccurlyeq y$ or $y \preccurlyeq x$;

(d) if $Y \subseteq X$ then there exists $y \in Y$ such that for all $y' \in Y$, $y \preccurlyeq y'$.

Let A be a set. A *norm* on A is a map $\sigma: A \twoheadrightarrow \lambda$ from A *onto* an ordinal λ, the *length* of σ. With each such norm we associate the prewellordering $\leq^\sigma$ on A defined by

$$x \leq^\sigma y \Leftrightarrow \sigma(x) \leq \sigma(y).$$

Conversely, each prewellordering $\preccurlyeq$ on a set A gives rise to a unique norm $\sigma: A \twoheadrightarrow \lambda$ such that $\preccurlyeq \; = \; \leq^\sigma$; we call λ the length of the prewellordering $\preccurlyeq$.

If Γ is a pointclass and σ a norm on a pointset A, we say that σ is a Γ-*norm* if there exist relations $\leq^\sigma_\Gamma$, $\leq^\sigma_{\check\Gamma}$ in Γ, $\check\Gamma = \{\mathscr{X} - B : B \subseteq \mathscr{X}, B \in \Gamma\}$ respectively, so that

$$y \in A \Rightarrow \forall x\{[x \in A \mathbin{\&} \sigma(x) \leq \sigma(y)] \Leftrightarrow x \leq^\sigma_\Gamma y \Leftrightarrow x \leq^\sigma_{\check\Gamma} y\}.$$

We write Prewellordering(Γ) if every set in Γ admits a Γ-norm. The prewellordering property was formulated (in a more complicated form, equivalent to the above for most interesting Γ) by Moschovakis; see [8] for details. Martin [6] and (independently) Moschovakis [1] proved that

$$\text{Determinacy}(\Delta^1_{2n}) \Rightarrow \text{Prewellordering}(\Pi^1_{2n+1}) \mathbin{\&} \text{Prewellordering}(\Sigma^1_{2n+2})$$

(thus also Prewellordering($\boldsymbol{\Pi}^1_{2n+1}$) & Prewellordering($\boldsymbol{\Sigma}^1_{2n+2}$)).

A *scale* on a pointset A is a sequence $\{\sigma_n\}_{n\in\omega}$ of norms on A with the following *limit property*:

> If $x_i \in A$, for all i, if $\lim_{i\to\infty} x_i = x$ and if, for each n and all large enough i, $\sigma_n(x_i) = \lambda_n$, then $x \in A$ and, for each n, $\sigma_n(x) \leq \lambda_n$.

(Following Solovay, we call the condition "$\sigma_n(x) \leq \lambda_n$" the *semicontinuity* property of scales.)

If Γ is a pointclass and $\{\sigma_n\}_{n\in\omega}$ is a scale on A we say that $\{\sigma_n\}_{n\in\omega}$ is a Γ-*scale* if there exist relations S_Γ, $S_{\check\Gamma}$ in Γ, $\check\Gamma$ respectively so that

$$y \in A \Rightarrow \forall x\{[x \in A \mathbin{\&} \sigma_n(x) \leq \sigma_n(y)] \Leftrightarrow S_\Gamma(n, x, y) \Leftrightarrow S_{\check\Gamma}(n, x, y)\}.$$

We write Scale(Γ) if every set in Γ admits a Γ-scale. The notion of a scale was formulated by Moschovakis in [9], where the scale property is called property $\mathscr{S}$. One of the basic results of [9] is that

$$\text{Determinacy}(\Delta^1_{2n}) \Rightarrow \text{Scale}(\Pi^1_{2n+1}) \mathbin{\&} \text{Scale}(\Sigma^1_{2n+2})$$

(similarly for the boldface classes).

Finally if $\{\sigma_n\}_{n\in\omega}$ is a scale on a pointset A we call $\{\sigma_n\}_{n\in\omega}$ a λ-*scale*, where λ is an ordinal, if every σ_n has length $\leq \lambda$ (or equivalently if each σ_n maps A onto λ).

REMARK. It happens very often in practice that one defines a sequence $\{\sigma_n\}_{n\in\omega}$ of maps from a pointset A into the ordinals, that has all the properties of a scale except possibly that some σ_n is not a norm, i.e., it is not *onto* an ordinal. Then one can associate to $\{\sigma_n\}_{n\in\omega}$ a unique scale $\{\bar{\sigma}_n\}_{n\in\omega}$ so that $\leq^{\bar{\sigma}_n} = \leq^{\sigma_n}$, where $\leq^{\sigma_n}$ is the prewellordering

$$x \leq^{\sigma_n} y \Leftrightarrow \sigma_n(x) \leq \sigma_n(y).$$

It is convenient to abuse language here and refer to $\{\sigma_n\}_{n\in\omega}$ itself as a scale, although what we have in mind is $\{\bar{\sigma}_n\}_{n\in\omega}$.

1C. We will have to deal very often in this paper with wellfounded relations and trees. If X is a set, a *wellfounded relation* on X is a relation $\prec \subseteq X \times X$ such that for no sequence $x_0, x_1, \ldots$ of elements of X we have $\cdots \prec x_2 \prec x_1 \prec x_0$. The set

$$\text{Field}(\prec) = \{x : \exists y(x \prec y) \text{ or } \exists y(y \prec x)\}$$

is called the *field* of $\prec$. For $x \in \text{Field}(\prec)$, we define the *length* of x by the $\prec$-induction

$$|x|_\prec = \sup\{|y|_\prec + 1 : y \prec x\},$$

where we assume $\sup(\varnothing) = 0$. The *length* of $\prec$ itself is given by

$$|\prec| = \sup\{|x|_\prec + 1 : x \in \text{Field}(\prec)\}.$$

Notice here the following minimality property of the function $|x|_\prec$: If

$$f\colon \text{Field}(\prec) \to \text{ordinals} \quad \text{and} \quad x \prec y \Rightarrow f(x) < f(y)$$

then for every $x \in \text{Field}(\prec)$ we have $|x|_\prec \leq f(x)$.

Now let X be a set. A *tree* on X is a set T of finite sequences from X closed under subsequences, i.e.,

$$(x_1, \ldots, x_n) \in T \;\&\; k \leq n \Rightarrow (x_1, \ldots, x_k) \in T.$$

The empty sequence ( ) is always a member of a nonempty tree. A *branch* of T is a sequence $f \in {}^\omega X$ such that, for every n,

$$f \upharpoonright n = (f(0), \ldots, f(n-1)) \in T.$$

We denote the set of branches of T by $[T]$, following Mansfield. A tree T is *wellfounded* if it has no branches (i.e., $[T] = \varnothing$) or equivalently if $\prec \cap\, T \times T$ is wellfounded, where $\prec$ is the usual (*proper*) *extension* relation between finite sequences

$$(x_1, \ldots, x_n) \prec (y_1, \ldots, y_m) \Leftrightarrow n > m \;\&\; x_i = y_i \quad \text{for } i \leq m.$$

Thus if T is a wellfounded tree we can put, for each $u \in T$,

$$|u|_T = \sup\{|v|_T + 1 : v \in T, v \prec u\} = \sup\{|u^\frown(x)|_T + 1 : u^\frown(x) \in T\}$$

(where $u^\frown v$ denotes concatenation) and we can define the *length* of T by $|T| = |(\ )|_T$. Finally for $u \in T$, let $T_u = \{v : u^\frown v \in T\}$. Then $|u|_T = |T_u|$.

We will be usually working with trees of pairs of integers and ordinals, i.e., trees on sets $X = \omega \times \lambda$, where λ is an ordinal. They contain elements of the form $((k_0, \xi_0), \ldots, (k_n, \xi_n))$, where $k_i \in \omega$ and $\xi_i < \lambda$, for all i. A branch of such a tree is a sequence $g \in {}^\omega(\omega \times \lambda)$, but for convenience it will be represented by the unique pair $(\alpha, f) \in {}^\omega\omega \times {}^\omega\lambda$, such that $g(n) = (\alpha(n), f(n))$, for all n. For each $\alpha \in \mathscr{R}$ the tree $T(\alpha)$ on λ is defined by

$$T(\alpha) = \{(\xi_0, \ldots, \xi_n) : ((\alpha(0), \xi_0), \ldots, (\alpha(n), \xi_n)) \in T\}.$$

Notice that

$$(\xi_0, \ldots, \xi_n) \in T(\alpha) \;\&\; \bar{\alpha}(n+1) = \bar{\beta}(n+1) \Rightarrow (\xi_0, \ldots, \xi_n) \in T(\beta).$$

From this it follows immediately that the sets

$$A_{(\xi_0, \ldots, \xi_n)} = \{\alpha : (\xi_0, \ldots, \xi_n) \in T(\alpha)\}$$

are all clopen.

1D. We work in this paper entirely in Zermelo-Fraenkel set theory with *dependent choices* (ZF + DC) where

(DC) $$\forall u \in x \exists v (u, v) \in r \Rightarrow \exists f \forall n (f(n), f(n+1)) \in r.$$

We state all other hypotheses explicitly.

§2. Relations between projective ordinals.

2A. The projective ordinals $\boldsymbol{\delta}^1_n$ are defined by

$$\boldsymbol{\delta}^1_n = \sup\{\xi : \xi \text{ is the length of a } \boldsymbol{\Delta}^1_n \text{ prewellordering of } \mathscr{R}\}.$$

They have been introduced by Moschovakis in [8] and several results about them were proved there.

It is clear that $\boldsymbol{\delta}^1_0 = \boldsymbol{\delta}^1_1 \le \boldsymbol{\delta}^1_2 \le \cdots \le \boldsymbol{\delta}^1_n \le \boldsymbol{\delta}^1_{n+1} \le \cdots$, but is it possible that, for some $n > 0$, $\boldsymbol{\delta}^1_n = \boldsymbol{\delta}^1_{n+1}$? Moschovakis proved in [8] that

$$\text{Determinacy}(\boldsymbol{\Delta}^1_{2n}) \Rightarrow \boldsymbol{\delta}^1_{2n+1} < \boldsymbol{\delta}^1_{2n+2}.$$

This is a consequence of the following basic fact.

THEOREM (2A-1) (MOSCHOVAKIS [8]). *Assume Determinacy*($\boldsymbol{\Delta}^1_{2n}$). *Let σ be a $\boldsymbol{\Pi}^1_{2n+1}$-norm on a complete $\boldsymbol{\Pi}^1_{2n+1}$ set. Then the length of σ is precisely $\boldsymbol{\delta}^1_{2n+1}$.*

(A $\boldsymbol{\Pi}^1_{2n+1}$ set A is *complete* if for any $B \in \boldsymbol{\Pi}^1_{2n+1}$ there is a continuous f so that $x \in B \Leftrightarrow f(x) \in A$.)

Nevertheless the problem of the relationship between $\boldsymbol{\delta}^1_{2n}$ and $\boldsymbol{\delta}^1_{2n+1}$ (for $n > 0$) was left open. We prove below that, for $n > 0$, $\boldsymbol{\delta}^1_{2n} < \boldsymbol{\delta}^1_{2n+1}$ (assuming PD).

2B. The observation that lies behind the proof of this fact is that one can work much better with wellfounded relations than directly with prewellorderings. We therefore find it convenient to introduce here another kind of projective ordinals. Let

$$\boldsymbol{\sigma}^1_n = \sup\{\xi : \xi \text{ is the length of a } \boldsymbol{\Sigma}^1_n \text{ wellfounded relation on reals}\}.$$

The following can be proved using a simple variation of the proof of Lemma 10 in [8].

THEOREM (2B-1) (MOSCHOVAKIS). *For any n*, *Determinacy*($\boldsymbol{\Delta}^1_{2n}$) $\Rightarrow \boldsymbol{\sigma}^1_{2n+1} = \boldsymbol{\delta}^1_{2n+1}$.

Using (2B-1) we now show

THEOREM (2B-2). *Assume $n \geq 1$. Then Determinacy$(\mathbf{\Delta}^1_{2n}) \Rightarrow \boldsymbol{\delta}^1_{2n} < \boldsymbol{\delta}^1_{2n+1}$.*

PROOF. Since $\boldsymbol{\delta}^1_{2n} \leq \boldsymbol{\sigma}^1_{2n}$ and $\boldsymbol{\sigma}^1_{2n+1} = \boldsymbol{\delta}^1_{2n+1}$, it is enough to prove that $\boldsymbol{\sigma}^1_{2n} < \boldsymbol{\sigma}^1_{2n+1}$. The idea is to "put together" all $\mathbf{\Sigma}^1_{2n}$ wellfounded relations to create a longer $\mathbf{\Sigma}^1_{2n+1}$ wellfounded relation. This can be done as follows:

Let $S \subseteq \mathscr{R}^3$ be a Σ^1_{2n} set which is universal for $\mathbf{\Sigma}^1_{2n}$ subsets of $\mathscr{R}^2$. This means that every $\mathbf{\Sigma}^1_{2n}$ subset of $\mathscr{R}^2$, A, has the form

$$A = S_\alpha = \{(\beta, \gamma) : (\alpha, \beta, \gamma) \in S\},$$

for some real α, a *code* of A. Let $W = \{\alpha : S_\alpha \textit{ is wellfounded}\}$ be the set of codes of $\mathbf{\Sigma}^1_{2n}$ wellfounded relations. Then our "big" wellfounded relation is on $\mathscr{R}^2$ and is given by

$$(\alpha, \beta) \prec (\gamma, \delta) \Leftrightarrow \alpha = \gamma \in W \;\&\; (\beta, \delta) \in S_\alpha.$$

It is trivial to verify that $\prec$ is wellfounded and it is not harder to observe that, if $R = S_{\alpha_0}$ is wellfounded, then R is isomorphic to the restriction of $\prec$ to pairs of the form (α_0, β). Thus $|R| \leq |\prec|$ and therefore $\boldsymbol{\sigma}^1_{2n} \leq |\prec|$.

The proof will be complete once we show that $\prec \in \mathbf{\Sigma}^1_{2n+1}$. But this is immediate since

$$\alpha \in W \Leftrightarrow \neg\exists\beta\forall n[((\beta)_{n+1}, (\beta)_n) \in S_\alpha]. \quad \square$$

REMARK 1. Let $\boldsymbol{\pi}^1_n = \sup\{\xi : \xi$ *is the length of a* $\mathbf{\Pi}^1_n$ *wellfounded relation on reals*$\}$. Then we have

PROPOSITION. *For each n, $\boldsymbol{\pi}^1_n = \boldsymbol{\sigma}^1_{n+1}$.*

PROOF. It is enough to show $\boldsymbol{\pi}^1_n \geq \boldsymbol{\sigma}^1_{n+1}$. Let $\prec$ be a $\mathbf{\Sigma}^1_{n+1}$ wellfounded relation. We shall define for each real α a wellfounded tree T_α on $\mathscr{R}^3$ such that $T_\alpha \in \mathbf{\Pi}^1_n$ and $\alpha \prec \beta \Rightarrow |T_\alpha| < |T_\beta|$. This implies that for any $\alpha \in \text{Field}(\prec)$ we have $|\alpha|_\prec \leq |T_\alpha|$ and since any $\mathbf{\Pi}^1_n$ tree has length $< \boldsymbol{\pi}^1_n$ we get $|\prec| \leq \sup\{|T_\alpha| + 1 : \alpha \in \mathscr{R}\} \leq \boldsymbol{\pi}^1_n$. Thus $\boldsymbol{\sigma}^1_{n+1} \leq \boldsymbol{\pi}^1_n$.

To define T_α let $\alpha \succ \beta \Leftrightarrow \exists\gamma((\alpha, \beta, \gamma) \in P)$, where $P \in \mathbf{\Pi}^1_n$. Applying the "unfolding trick" put for each α,

$$\begin{aligned} T_\alpha = \{((\alpha, \beta_0, \gamma_0), (\beta_0, \beta_1, \gamma_1), \ldots, (\beta_{k-1}, \beta_k, \gamma_k)) :& \\ (\alpha, \beta_0, \gamma_0) \in P \;\&\; (\beta_0, \beta_1, \gamma_1) \in P \;\&\cdots\&\; (\beta_{k-1}, \beta_k, \gamma_k) \in P\}.& \end{aligned}$$

Clearly $T_\alpha \in \mathbf{\Pi}^1_n$ and T_α is wellfounded. If $\alpha \prec \beta$, pick a γ_0 such that $(\beta, \alpha, \gamma_0) \in P$. Then (in the notation of 1C) $T_\alpha = (T_\beta)_{\langle(\beta,\alpha,\gamma_0)\rangle}$ which implies $|T_\alpha| < |T_\beta|$. $\square$

Notice also that the proof of (2B-2) establishes (in ZF + DC only) that $\boldsymbol{\sigma}^1_n < \boldsymbol{\sigma}^1_{n+1}$ $(n > 0)$.

REMARK 2. Kunen and Martin have independently shown that

$$\text{Determinacy}(\mathbf{\Delta}^1_{2n}) \Rightarrow \boldsymbol{\sigma}^1_{2n+2} = \boldsymbol{\delta}^1_{2n+2} \quad \text{(see [7])}.$$

Thus assuming projective determinacy we have the following picture:

$$(\boldsymbol{\delta}^1_0 =)\, \boldsymbol{\pi}^1_0 = \boldsymbol{\delta}^1_1 = \boldsymbol{\sigma}^1_1 < \boldsymbol{\pi}^1_1 = \boldsymbol{\delta}^1_2 = \boldsymbol{\sigma}^1_2 < \boldsymbol{\pi}^1_2 = \boldsymbol{\delta}^1_3 = \boldsymbol{\sigma}^1_3 < \boldsymbol{\pi}^1_3 = \cdots.$$

§3. Projective ordinals in the completely playful universe.

3A. Assuming the (full) Axiom of Determinacy (AD) Moschovakis has shown in [8] that every $\boldsymbol{\delta}^1_n$ is a cardinal. It is classically known that $\boldsymbol{\delta}^1_1 = \aleph_1$ and Martin proved in 1968 that AD $\Rightarrow \boldsymbol{\delta}^1_2 = \aleph_2$ (see [7]). For some time it seemed likely that, with AD, $\boldsymbol{\delta}^1_n = \aleph_n$ would hold for every $n \geq 1$. Thus it came as a surprise when Martin proved in 1970 that AD $\Rightarrow \boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$ (see [7]). Our main result in this section partly generalizes the classical result $\boldsymbol{\delta}^1_1 = \aleph_1$ and Martin's theorem. We prove

$$\mathrm{AD} \Rightarrow \boldsymbol{\delta}^1_{2n+1} = (\lambda_{2n+1})^+, \quad \text{where } \lambda_{2n+1} \text{ is a cardinal of cofinality } \omega.$$

(Here λ^+ = least cardinal bigger than λ.) This gives also lower bounds for the $\boldsymbol{\delta}^1_n$'s.

3B. Before we proceed to prove this fact we have to set up some of the machinery concerning κ-Souslin and κ-Borel pointsets. (Further details can be found in [7] or [4]). It was Martin who first applied in a nontrivial way these methods to the study of the projective sets beyond the second level of the hierarchy. In fact our proof below parallels the arguments Martin used to prove $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$, but in addition uses the basic theorem of Moschovakis on the existence of scales on projective sets.

DEFINITION (3B-1). A pointset $A \subseteq \mathscr{X}$ is called *κ-Souslin*, where κ is a cardinal, if it can be written as

$$A = \bigcup_{f \in {}^\omega\kappa} \bigcap_{n \in \omega} A_{f \upharpoonright n},$$

where for each sequence $(\xi_0, \ldots, \xi_n)$ from κ, $A_{\langle \xi_0, \ldots, \xi_n \rangle}$ is a clopen pointset. We denote by $\mathscr{S}_\kappa$ the pointclass of κ-Souslin pointsets.

It is easy to see that a set $A \subseteq \mathscr{R}$ is κ-Souslin if and only if there exists a tree T on $\omega \times \kappa$ so that $\alpha \in A \Leftrightarrow T(\alpha)$ is not wellfounded ($\Leftrightarrow \exists f((\alpha, f) \in [T])$). Similarly for subsets of the product spaces.

DEFINITION (3B-2). A pointset $A \subseteq \mathscr{X}$ is called κ-Borel, where κ is a cardinal, if it belongs to the smallest pointclass which contains all open pointsets and is closed under complements and unions of length $< \kappa$. This pointclass is denoted by $\mathscr{B}_\kappa$.

It is a standard fact that $\mathscr{S}_{\aleph_0} = \boldsymbol{\Sigma}^1_1$, and the classical Souslin theorem asserts that $(\mathscr{B}_{\aleph_1} =) \mathscr{B}_{\boldsymbol{\delta}^1_1} = \boldsymbol{\Delta}^1_1$. This last result has been generalized to all odd levels usingAD. Martin proved first that AD $\Rightarrow \mathscr{B}_{\boldsymbol{\delta}^1_3} = \boldsymbol{\Delta}^1_3$ (see [7]) and also

THEOREM (3B-3) (MARTIN [7]). *For any n,*

$$\mathrm{AD} \Rightarrow \mathscr{B}_{\boldsymbol{\delta}^1_{2n+1}} \subseteq \boldsymbol{\Delta}^1_{2n+1}.$$

Then Moschovakis [9] showed the other inclusion of (3B-3), i.e.,

$$\mathrm{AD} \Rightarrow \boldsymbol{\Delta}^1_{2n+1} \subseteq \mathscr{B}_{\boldsymbol{\delta}^1_{2n+1}}.$$

(In fact he needs only PD here.)

We shall see later how $\mathscr{S}_{\aleph_0} = \boldsymbol{\Sigma}^1_1$ generalizes.

The next fact connects the notions of κ-Souslin and κ-Borel. It was proved by Sierpiński for $\kappa = \omega$ (see [5, p. 32]), but his proof works as well for any κ.

THEOREM (3B-4) (SIERPIŃSKI). *If $A \subseteq \mathscr{X}$ is κ-Souslin, where κ is a cardinal, then A is the intersection of κ^+ sets in $\mathscr{B}_{\kappa^+}$. Thus $A \in \mathscr{B}_{\kappa^{++}}$.*

PROOF. Assume for simplicity $A \subseteq \mathscr{R}$ and let T be a tree on $\omega \times \kappa$ such that $\alpha \in A \Leftrightarrow T(\alpha)$ is not wellfounded. For each $0 \leq \xi < \kappa^+$ and any finite sequence u from κ put

$$A^\xi_u = \{\alpha : |T(\alpha)_u| < \xi\},$$

where, for a tree J, $|J| < \xi$ abbreviates both that J is wellfounded and that $|J| < \xi$. We agree that $|J_u| = -1$ if $u \notin J$ and $-1 < \xi$ for any ordinal ξ. It is now easy to check that for length$(u) = n$ we have

$$A_u^0 = \{\alpha : ((\alpha(0), u_0), \ldots, (\alpha(n-1), u_{n-1})) \notin J\},$$
$$A_u^{\xi+1} = A_u^\xi \cup \bigcap_{\eta<\kappa} A^\xi_{u^\frown\langle\eta\rangle},$$
$$A_u^\lambda = \bigcup_{\xi<\lambda} A_u^\xi, \quad \text{if } \lambda = \bigcup \lambda > 0.$$

Thus $A_u^\xi \in \mathscr{B}_{\kappa^+}$ for any ξ and u. But $A = \bigcap_{\xi<\kappa^+} (\mathscr{R} - A^\xi_{\langle\rangle})$, which completes the proof. $\square$

One can do much better if cofinality$(\kappa) > \omega$.

THEOREM (3B-5) (MARTIN [7]). *If $A \subseteq \mathscr{X}$ is κ-Souslin, where κ is a cardinal and cofinality$(\kappa) > \omega$, then $A \in \mathscr{B}_{\kappa^+}$.*

PROOF. Assume again $A \subseteq \mathscr{R}$ is κ-Souslin and let T be a tree on $\omega \times \kappa$ such that $\alpha \in A \Leftrightarrow T(\alpha)$ is not wellfounded. Since cofinality$(\kappa) > \omega$ we have

$$\alpha \in A \Leftrightarrow \exists \xi < \kappa(T^\xi(\alpha) \text{ is not wellfounded}),$$

where T^ξ is the restriction of T to ordinals $< \xi$. Apply now (3B-4). $\square$

The hypothesis "cofinality$(\kappa) > \omega$" in the statement of (3B-5) is necessary as the example $\kappa = \omega$ shows. Nevertheless if *both* A and $\mathscr{X} - A$ are κ-Souslin we have again that $A \in \mathscr{B}_{\kappa^+}$ (without restrictions on cofinality(κ)); see [7].

And we conclude this preliminary discussion with the following basic fact.

THEOREM (3B-6) (FOLCLORE-TYPE RESULT). *Assume $A \subseteq \mathscr{X}$ is a pointset which admits a λ-scale. Then A is $|\lambda|$-Souslin, where $|\lambda|$ = cardinality of λ.*

PROOF. Put

$$T = \{((\alpha(0), \sigma_0(\alpha)), \ldots, (\alpha(n), \sigma_n(\alpha))) : \alpha \in A\},$$

where $\{\sigma_n\}_{n\in\omega}$ is a λ-scale on A. We check that $\alpha \in A \Leftrightarrow T(\alpha)$ is not wellfounded.

If $\alpha \in A$, then $\vec{\sigma}(\alpha) = (\sigma_0(\alpha), \sigma_1(\alpha), \ldots, \sigma_n(\alpha), \ldots)$ is a branch of $T(\alpha)$. Conversely, if $(\xi_0, \xi_1, \ldots, \xi_n, \ldots)$ is a branch of $T(\alpha)$, there exist reals $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ all in A, such that, for every n,

$$((\alpha_n(0), \sigma_0(\alpha_n)), \ldots, (\alpha_n(n), \sigma_n(\alpha_n))) = ((\alpha(0), \xi_0), \ldots, (\alpha(n), \xi_n)).$$

Then $\alpha_n \to \alpha$ and $\sigma_m(\alpha_n) = \xi_m$ for all $n \geq m$. So $\alpha \in A$.

Now T is a tree on $\omega \times \lambda$ and it can be easily replaced by an isomorphic one on $\omega \times |\lambda|$, without changing its integer part. Thus $A \in \mathscr{S}_{|\lambda|}$. $\square$

We are now ready to prove

THEOREM (3B-7). *For any n, AD $\Rightarrow \boldsymbol{\delta}^1_{2n+1} = (\lambda_{2n+1})^+$, where λ_{2n+1} is a cardinal of cofinality ω.*

PROOF. Let $S \subseteq \mathscr{R}$ be a set which is $\boldsymbol{\Sigma}^1_{2n+1}$ but not $\boldsymbol{\Pi}^1_{2n+1}$. Say $\alpha \in S \Leftrightarrow \exists\beta Q(\alpha, \beta)$, where $Q \in \boldsymbol{\Pi}^1_{2n}$. Let $\{\sigma_m\}_{m\in\omega}$ be a $\boldsymbol{\Pi}^1_{2n+1}$-scale on Q. Since $Q \in \boldsymbol{\Pi}^1_{2n}$, the prewellorderings $\leq^{\sigma_m}$ are actually $\boldsymbol{\Delta}^1_{2n+1}$; thus

$$\text{length}(\leq^{\sigma_m}) < \boldsymbol{\delta}^1_{2n+1}, \quad \text{for all } m.$$

It is easy to see that cofinality$(\boldsymbol{\delta}^1_{2n+1}) > \omega$, so that $\{\sigma_m\}_{m\in\omega}$ is a λ-scale for some $\lambda < \boldsymbol{\delta}^1_{2n+1}$. Put $\lambda_{2n+1} = |\lambda|$. We proceed to show that

$$(\lambda_{2n+1})^+ = \boldsymbol{\delta}^1_{2n+1} \quad \text{and} \quad \text{cofinality}(\lambda_{2n+1}) = \omega.$$

Since Q admits a λ-scale it is λ_{2n+1}-Souslin by (3B-6) and thus, as a simple argument shows, S is λ_{2n+1}-Souslin. Then, by (3B-4), $S \in \mathscr{B}_{(\lambda_{2n+1})^{++}}$. If $(\lambda_{2n+1})^+$ was less than $\boldsymbol{\delta}^1_{2n+1}$, $(\lambda_{2n+1})^{++}$ would be at most $\boldsymbol{\delta}^1_{2n+1}$ (recall that each $\boldsymbol{\delta}^1_k$ is a cardinal), therefore $S \in \mathscr{B}_{\boldsymbol{\delta}^1_{2n+1}} \subseteq \boldsymbol{\Delta}^1_{2n+1}$ (by (3B-3)), which is a contradiction. Thus $(\lambda_{2n+1})^+ = \boldsymbol{\delta}^1_{2n+1}$.

If cofinality$(\lambda_{2n+1}) > \omega$, then (by (3B-5))

$$S \in \mathscr{B}_{(\lambda_{2n+1})^+} = \mathscr{B}_{\boldsymbol{\delta}^1_{2n+1}} \subseteq \boldsymbol{\Delta}^1_{2n+1},$$

again a contradiction. Thus cofinality$(\lambda_{2n+1}) = \omega$. $\square$

COROLLARY (3B-8). *For any n,*

$$\mathrm{AD} \Rightarrow \boldsymbol{\delta}^1_{2n+1} \geq \aleph_{\omega\cdot n+1}, \qquad \boldsymbol{\delta}^1_{2n+2} \geq \aleph_{\omega\cdot n+2}.$$

REMARK 1. One can easily check by examining the proof of (3B-7) that λ_{2n+1} = smallest cardinal λ such that $\boldsymbol{\Sigma}^1_{2n+1} \subseteq \mathscr{S}_\lambda$. If we put $\lambda_{2n} = \boldsymbol{\delta}^1_{2n-1}$ $(n \geq 1)$, then as Martin already observed, in [7], $\mathrm{AD} \Rightarrow \boldsymbol{\Sigma}^1_n = \mathscr{S}_{\lambda_n}$ $(n \geq 1)$, which generalizes the fact that $\boldsymbol{\Sigma}^1_1 = \mathscr{S}_{\aleph_0}$. In fact λ_n is the least such cardinal. Solovay (unpublished) proved that

$$\mathrm{AD} \Rightarrow \mathscr{S}_{\lambda_n} = \mathscr{S}_\kappa \quad \text{for } \lambda_n \leq \kappa < \lambda_{n+1}$$

and this gives a complete description of the growth of the pointclasses $\mathscr{S}_\kappa$ for $\kappa < \boldsymbol{\delta}^1_\infty = \sup_n \boldsymbol{\delta}^1_n$, under AD. How the classes $\mathscr{B}_\kappa$ grow remains open.

REMARK 2.[2] It may be interesting at this point (although irrelevant to the problem of the $\boldsymbol{\delta}^1_n$'s) to see to what extent there is a converse to Theorem (3B-6). In other words we would like to see for what cardinals κ we have

$$A \text{ is } \kappa\text{-Souslin} \Rightarrow A \text{ admits a } \kappa\text{-scale}.$$

We prove first that this is true if cofinality$(\kappa) > \omega$.

PROPOSITION. *Let κ be a cardinal and assume cofinality$(\kappa) > \omega$. Then, for any $A \subseteq \mathscr{X}$, A is κ-Souslin $\Rightarrow$ A admits a κ-scale.*

PROOF. Let T be a tree on $\omega \times \kappa$ and assume $\alpha \in A \Leftrightarrow T(\alpha)$ is not wellfounded. The first attempt for defining a scale on A is to put, for $\alpha \in A$,

$$\sigma'_n(\alpha) = \langle h_{T(\alpha)}(0), \ldots, h_{T(\alpha)}(n)\rangle,$$

where for any nonwellfounded tree J on an ordinal λ we denote by h_J its *leftmost branch*, defined as follows by induction:

$$h_J(0) = \text{least } \xi \text{ such that } J_{(\xi)} \text{ is not wellfounded},$$

$$h_J(n+1) = \text{least } \xi \text{ such that } J_{h_J\restriction(n+1)^\frown(\xi)} \text{ is not wellfounded}.$$

We use $\langle \xi_1, \ldots, \xi_n\rangle$, where $\xi_i < \kappa$, to denote the ordinal of the n-tuple $(\xi_1, \ldots, \xi_n)$ in the lexicographical wellordering of ${}^n\kappa$. One can easily check now that $\{\sigma'_n\}_{n\in\omega}$ is a scale. (Recall here the remark in 1B.) In fact it is a κ^ω-scale (κ^ω denotes ordinal exponentiation), but not necessarily a κ-scale.

[2] The results in this remark (which extends to the end of 3B) will not be used in the rest of the paper.

To avoid this problem we use the hypothesis cofinality$(\kappa) > \omega$ to write $\alpha \in A \Leftrightarrow \exists \xi < \kappa(T^{\xi}(\alpha)$ is not wellfounded), where of course T^{ξ} is the restriction of T to ordinals $< \xi$. Then we put, for $\alpha \in A$,

$$\sigma_{-1}(\alpha) = \text{least } \xi \text{ such that } T^{\xi}(\alpha) \text{ is not wellfounded},$$
$$\sigma_n(\alpha) = \langle \sigma_{-1}(\alpha), h_{\alpha}^{\sigma_{-1}(\alpha)}(0), \ldots, h_{\alpha}^{\sigma_{-1}(\alpha)}(n) \rangle,$$

where we abbreviate h_{α}^{ξ} = the leftmost branch of $T^{\xi}(\alpha)$. One can now easily check that $\{\sigma_n\}_{n\in\omega}$ is indeed a κ-scale. □

What if cofinality$(\kappa) = \omega$? Contrary to the previous fact we prove that there exist ω-Souslin (i.e., $\boldsymbol{\Sigma}_1^1$) sets which do not admit ω-scales. This follows trivially from the next result which gives also a new characterization of the Borel sets.

PROPOSITION. *For any $A \subseteq \mathscr{X}$, A admits an ω-scale $\Leftrightarrow$ A is $\boldsymbol{\Delta}_1^1$.*

PROOF. If $A \subseteq \mathscr{R}$ is $\boldsymbol{\Delta}_1^1$, then for some $B \in \boldsymbol{\Pi}_1^0$ (i.e., B closed) we have

$$\alpha \in A \Leftrightarrow \exists \beta B(\alpha, \beta) \Leftrightarrow \exists\,!\,\beta B(\alpha, \beta).$$

Put, for $\alpha \in A$, $\sigma_n(\alpha) = \beta(n)$, where $B(\alpha, \beta)$. It is easy to check that $\{\sigma_n\}_{n\in\omega}$ is an ω-scale on A.

Conversely suppose that A admits an ω-scale $\{\sigma_n\}_{n\in\omega}$. Let T be the tree on $\omega \times \omega$ coming from this scale as in (3B-6). Then $\alpha \in A \Leftrightarrow \exists \beta((\alpha, \beta) \in [T])$. Put

$$Q(\alpha, \beta) \Leftrightarrow (\alpha, \beta) \in [T] \;\&\; \forall \gamma \leq^* \beta((\alpha, \gamma) \in [T] \Rightarrow \gamma = \beta),$$

where $\gamma \leq^* \beta \Leftrightarrow \forall n(\gamma(n) \leq \beta(n))$. Then

$$\alpha \in A \Leftrightarrow \exists \beta Q(\alpha, \beta) \Leftrightarrow \exists\,!\,\beta Q(\alpha, \beta).$$

Because if $\alpha \in A$, take $\beta_0 = \vec{\sigma}(\alpha) = (\sigma_0(\alpha), \sigma_1(\alpha), \ldots)$. Then $Q(\alpha, \beta_0)$ by the semicontinuity property of scales and the proof of (3B-6). If also $Q(\alpha, \beta)$ holds, we have $\beta_0 \leq^* \beta$ so that $\beta_0 = \beta$.

The proof will be complete if we can show that Q is arithmetical in T. But for $(\alpha, \beta) \in [T]$ we have

$$\neg \forall \gamma \leq^* \beta((\alpha, \gamma) \in [T] \Rightarrow \gamma = \beta) \Leftrightarrow \exists \gamma \leq^* \beta((\alpha, \gamma) \in [T] \;\&\; \gamma \neq \beta)$$
$$\Leftrightarrow (\exists s)(s \in T(\alpha) \;\&\; s \text{ precedes } \bar{\beta}(\mathrm{lh}(s)) \text{ lexicographically}$$
$$\& \; \{t : t \in T(\alpha) \;\&\; t \text{ extends } s \;\&\; \forall i < \mathrm{lh}(t)((t)_i \leq \beta(i))\} \text{ is infinite}).$$

The last equivalence follows from the Brouwer-König infinity lemma (see [12, p. 187]) and proves what we want. □

It would seem now probable that the converse of (3B-6) fails for cofinality$(\kappa) = \omega$. Nevertheless Busch, Martin and Solovay (unpublished) proved that if $\kappa > \omega$ and if cofinality$(\kappa) = \omega$ then again A is κ-Souslin $\Rightarrow$ A admits a κ-scale. Moreover Busch (unpublished) proved that every $\boldsymbol{\Sigma}_1^1$ set admits an $(\omega + 1)$-scale.

To summarize:

For $\kappa > \omega$, A is κ-Souslin iff A admits a κ-scale.

For $\kappa = \omega$, A is ω-Souslin iff $A \in \boldsymbol{\Sigma}_1^1$ iff A has an $(\omega + 1)$-scale. Also A has an ω-scale iff $A \in \boldsymbol{\Delta}_1^1$.

3C. We close this section with a few comments on the problem of computing the $\boldsymbol{\delta}_n^1$'s, assuming AD. It has been already proved by Kunen and Martin (independently) that

$$\mathrm{AD} \Rightarrow \boldsymbol{\delta}_{2n+2}^1 = (\boldsymbol{\delta}_{2n+1}^1)^+ \quad \text{(see [7])}.$$

(Thus $\boldsymbol{\delta}_n^1 = (\lambda_n)^+$, for any $n \geq 1$.) Therefore we know AD $\Rightarrow \boldsymbol{\delta}_4^1 = \aleph_{\omega+2}$ and we will know all $\boldsymbol{\delta}_n^1$'s, once we know the ones with odd $n \geq 5$. From the results already mentioned one is tempted to conjecture that AD $\Rightarrow \boldsymbol{\delta}_{2n+1}^1 = \aleph_{\omega \cdot n+1}$. Kunen (unpublished) disproved this by showing that AD $\Rightarrow \boldsymbol{\delta}_5^1 > \aleph_{\omega \cdot 2+1}$. This result and (3B-7) improve the lower bounds of (3B-8). Nevertheless the problem of the exact computation of $\boldsymbol{\delta}_n^1$ for $n \geq 5$ seems very difficult and remains still unsolved, although Kunen has made important progress.

§4. Projective ordinals and uniform indiscernibles.

4A. The aim of this last section is to establish connections between the projective ordinals and Solovay's uniform indiscernibles. Inevitably we will have to use several facts from Silver's elaborate theory of indiscernibles for the models $L[\alpha]$, for $\alpha \in \mathscr{R}$. We will also need the results of Solovay on "sharps" and uniform indiscernibles. We try to summarize what we need in 4B below. One can find details in [11], [13] and [7].

4B. Consider the theory

$$\mathrm{ZF} + V = L[\dot{\alpha}] + \dot{\alpha} \in \mathscr{R},$$

abbreviated ZFL($\dot{\alpha}$), in a language which besides ϵ contains a constant $\dot{\alpha}$. Let $v_1, v_2, v_3, \ldots$ be the variables of this language. It is well known that in ZFL($\dot{\alpha}$) one can define a formula $\chi(\dot{\alpha}, v_1, v_2)$ abbreviated $v_1 <_{\dot{\alpha}} v_2$, which gives a wellordering of the universe, so that if ξ, η are ordinals it can be proved that $\xi <_{\dot{\alpha}} \eta \Leftrightarrow \xi < \eta$ ($\Leftrightarrow \xi \in \eta$). For any formula $\varphi(v, v_1, \ldots, v_n)$ we define the term

$$\begin{aligned} t_\varphi(v_1, \ldots, v_n) &= <_{\dot{\alpha}}\text{-}\ \text{least } v \text{ such that } \varphi(v, v_1, \ldots, v_n), \text{ if such exists,} \\ &= 0, \text{ otherwise.} \end{aligned}$$

Let $\mathfrak{A} = \langle A, E, a \rangle$ be a (not necessarily wellfounded) model of ZFL($\dot{\alpha}$). An infinite subset $I \subseteq A$ is called a set of *indiscernibles* for $\mathfrak{A}$ if for any formula $\varphi(v_1, \ldots, v_n)$ and any $x_1, \ldots, x_n, y_1, \ldots, y_n$ in I we have

$$\begin{aligned} &x_1 <_a^{\mathfrak{A}} x_2 <_a^{\mathfrak{A}} \cdots <_a^{\mathfrak{A}} x_n \ \&\ y_1 <_a^{\mathfrak{A}} y_2 <_a^{\mathfrak{A}} \cdots <_a^{\mathfrak{A}} y_n \\ &\qquad \Rightarrow \mathfrak{A} \models \varphi(x_1, \ldots, x_n) \Leftrightarrow \varphi(y_1, \ldots, y_n). \end{aligned}$$

(Superscript $\mathfrak{A}$ means as usual interpretation.) A set I of indiscernibles *generates* $\mathfrak{A}$, if $\mathfrak{A}$ is the smallest elementary submodel of itself containing I. This is equivalent to saying that every element of A can be written in the form $t_\varphi^{\mathfrak{A}}(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in I$ and $x_1 <_a^{\mathfrak{A}} x_2 <_a^{\mathfrak{A}} \cdots <_a^{\mathfrak{A}} x_n$.

The *character of I in* $\mathfrak{A}$, $\Phi(\mathfrak{A}, I)$, is the set

$$\begin{aligned} \{\varphi(v_1, \ldots, v_n) : &\text{ for some } x_1, \ldots, x_n \in I, \text{ with} \\ &x_1 <_a^{\mathfrak{A}} x_2 <_a^{\mathfrak{A}} \cdots <_a^{\mathfrak{A}} x_n, \text{ we have } \varphi^{\mathfrak{A}}(x_1, \ldots, x_n)\}. \end{aligned}$$

A *character* is a character of some I in some $\mathfrak{A}$. It is a well-known result of the Ehrenfeucht-Mostowski theory that for each character Φ and each infinite ordinal ξ there exists a unique (up to isomorphism) model $\Gamma(\Phi, \xi)$ of ZFL($\dot{\alpha}$), which is generated by a set of indiscernibles of order type ξ (under $<_{\alpha^{\Gamma(\Phi,\xi)}}^{\Gamma(\Phi,\xi)}$).

Silver proved that if a Ramsey cardinal exists then for each $\alpha \in \mathscr{R}$ there exists a character Φ_α which has the following properties (where $\operatorname{Ord}(v)$ abbreviates "v is an ordinal"):

(a) "$\operatorname{Ord}(v_1)$" $\in \Phi_\alpha$.
(b) "$t_\varphi(v_1, \ldots, v_n) <_\alpha v_{n+1}$" $\in \Phi_\alpha$, all φ.
(c) "$\operatorname{Ord}(t_\varphi(v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+k}))\ \&\ t_\varphi(v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+k}) < v_{n+1} \Rightarrow t_\varphi(v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+k}) = t_\varphi(v_1, \ldots, v_n, v_{n+k+1}, \ldots, v_{n+2k})$" $\in \Phi_\alpha$, all φ.
(d) $\alpha(n) = m \Leftrightarrow$ "$\dot\alpha(\mathbf{n}) = \mathbf{m}$" $\in \Phi_\alpha$, where $\mathbf{n}$ is the nth numeral.
(e) For all ξ, $\Gamma(\Phi_\alpha, \xi)$ is wellfounded.

A character satisfying (a)–(e) is called *remarkable* (for α). If Φ_α is remarkable, then (by (e)), for each limit ordinal λ, $\Gamma(\Phi_\alpha, \lambda)$ is isomorphic to a unique $L_{\lambda_\alpha^*}[\alpha]$ and there exists a unique subset $I_\lambda^\alpha \subseteq \lambda_\alpha^*$ which is a generating set of indiscernibles for $L_{\lambda_\alpha^*}[\alpha]$ and has character Φ_α. Silver proved

(1) I_λ^α is cofinal in λ_α^*,
(2) I_λ^α is an initial segment of I_μ^α if $\lambda < \mu$ and $\lambda_\alpha^* = \lambda$th element of I_μ^α,
(3) $L_{\lambda_\alpha^*}[\alpha]$ is an elementary submodel of $L_{\mu_\alpha^*}[\alpha]$, if $\lambda < \mu$,
(4) $\kappa_\alpha^* = \kappa$, if κ is a cardinal,
(5) if $I^\alpha = \bigcup_\lambda I_\lambda^\alpha$, then I^α is a closed unbounded class of ordinals which contains all cardinals and generates $L[\alpha]$. Call I^α the class of *Silver indiscernibles* for $L[\alpha]$.

From (3) and (5) it follows that

$$\Phi_\alpha = \{\varphi(v_1, \ldots, v_n) : L_{\aleph_\omega}[\alpha] \models \varphi(\aleph_1 \cdots \aleph_n)\},$$

so that a remarkable character for α is unique. It is customary after Solovay to write $\alpha^\#$ for the real coding Φ_α (i.e., $\alpha^\# : \omega \to 2$ and $\alpha^\#(n) = 0 \Leftrightarrow n$ is the Gödel number of a formula in Φ_α).

It is important to state here that all the results about indiscernibles for $L[\alpha]$ can be deduced only from the assumption that the remarkable character for α exists, usually abbreviated "$\alpha^\#$ exists." In particular, if $\alpha^\#$ exists, the theory of indiscernibles for $L[\alpha]$ can be done in $L[\alpha^\#]$ (e.g., the class I^α is definable in $L[\alpha^\#]$).

Solovay proved that $\beta = \alpha^\#$ is a Π_2^1 relation. Thus if $\forall\alpha\exists\beta(\beta = \alpha^\#)$, i.e., if $\forall\alpha(\alpha^\#$ *exists*), then $\alpha \mapsto \alpha^\#$ is a Δ_3^1 function from $\mathscr{R}$ into $\mathscr{R}$ which can be easily seen to have a recursive inverse on its range, i.e., for some recursive $f: \mathscr{R} \to \mathscr{R}$ we have $f(\alpha^\#) = \alpha$.

Assuming $\forall\alpha(\alpha^\#$ exists), Solovay defined the class of *uniform indiscernibles* by $\mathscr{U} = \bigcap_\alpha I^\alpha$. Then clearly $\mathscr{U}$ is a closed unbounded class of ordinals containing all the cardinals. Let $\mathscr{U} = \{u_1, u_2, \ldots, u_\xi, \ldots\}$ in increasing order. Then $u_1 = \aleph_1 = \boldsymbol{\delta}_1^1$. We prove in the rest of this section that all the $\boldsymbol{\delta}_n^1$'s are uniform indiscernibles and we study their position in the above enumeration.

4C. We begin with a result on subsets of $\aleph_1$ constructible from a real. It provides a converse to the first theorem of [3] but gives also immediately that $\boldsymbol{\delta}_2^1 \geq u_2$ (which can be proved also directly; see Martin [7]).

Let, for each $\alpha \in \mathscr{R}$,

$$\leq_\alpha = \{(m, n) : \alpha(\langle m, n\rangle) = 0\},$$
$$<_\alpha = \{(m, n) : m \neq n\ \&\ m \leq_\alpha n\}.$$

Put WO $= \{\alpha : \leq_\alpha$ is a wellordering$\}$. For $\alpha \in$ WO, let $|\alpha| =$ length of $\leq_\alpha$. The set WO and the map $\alpha \mapsto |\alpha|$ provide a natural coding system for ordinals $< \aleph_1$. If $A \subseteq \aleph_1$, we define the *code set* of A by

$$\mathrm{Code}(A) = \{\alpha \in \mathrm{WO} : |\alpha| \in A\}.$$

A set $A \subseteq \aleph_1$ is called Γ *in the codes* (where Γ is a pointclass) if $\mathrm{Code}(A) \in \Gamma$. It was proved in [3] that $\mathrm{Code}(A) \in \Sigma^1_2(\alpha) \Rightarrow A \in L[\alpha]$. Thus if $\boldsymbol{L} = \bigcup_{\alpha \in \mathscr{R}} L[\alpha]$, then

$$\mathrm{Code}(A) \in \boldsymbol{\Sigma}^1_2 \Rightarrow A \in \boldsymbol{L}.$$

We prove here a converse assuming $\forall\alpha(\alpha^\#$ exists).

THEOREM (4C-1). *Assume $\forall\alpha(\alpha^\#$ exists). Then, for any $A \subseteq \aleph_1$, $A \in \boldsymbol{L} \Rightarrow A$ is $\boldsymbol{\Pi}^1_1$ in the codes.*

PROOF. Let $A \subseteq \aleph_1$ and $A \in L$ (without loss of generality since the proof is "uniform"). Then

$$A \in L_{(\aleph_1^+)^L} \subseteq L_{\iota_{\aleph_1+\omega}},$$

where, for any transitive model $\mathscr{M}$ of ZF and any $\lambda \in \mathscr{M}$, $(\lambda^+)^{\mathscr{M}} =$ least cardinal of $\mathscr{M}$ bigger than λ and $I = I^{\lambda t 0} = \{\iota_0, \iota_1, \ldots, \iota_\xi, \ldots\}$ is the increasing enumeration of the Silver indiscernibles for L. Thus

$$A = t^L_\varphi(\eta_1, \ldots, \eta_k, \aleph_1, \iota_{\aleph_1+m_1}, \ldots, \iota_{\aleph_1+m_l}),$$

where $\eta_1 < \cdots < \eta_2 < \aleph_1 < \iota_{\aleph_1+m_1} < \cdots < \iota_{\aleph_1+m_l}$. Find an α such that $\eta_1 \cdots \eta_k$ are definable in $L[\alpha]$ and $\iota_{\aleph_1+m_1}, \ldots, \iota_{\aleph_1+m_l}$ are definable in $L[\alpha]$ *from* $\aleph_1$. Then $A = t^{L[\alpha]}_\chi(\aleph_1)$, for some χ.

Let $\xi < \aleph_1$. Then $\xi = t^{L[\alpha]}_\psi(\xi_1, \ldots, \xi_n, \aleph_2, \ldots, \aleph_r)$, where $\xi_1 \cdots \xi_n \in I^\alpha$ and $\xi_1 < \cdots < \xi_n \leq \xi$. From this we get

$$\begin{aligned}
\xi \in A &\Leftrightarrow t^{L[\alpha]}_\psi(\xi_1, \ldots, \xi_n, \aleph_2, \ldots, \aleph_r) \in t^{L[\alpha]}_\chi(\aleph_1)\\
&\Leftrightarrow t^{L[\alpha]}_\psi(\xi_1, \ldots, \xi_n, \aleph_2, \ldots, \aleph_r) \in t^{L[\alpha]}_\chi(\xi'), \quad \text{for any } \xi' \in I^\alpha,\ \xi' > \xi,\\
&\Leftrightarrow \xi \in t^{L[\alpha]}_\chi(\xi')\\
&\Leftrightarrow L_{\lambda^*_\alpha}[\alpha] \models \xi \in t_\chi(\xi'), \quad \text{for any limit } \lambda > \xi'.
\end{aligned}$$

Thus taking $\lambda = \xi + \omega$ we have

$$\xi \in A \Leftrightarrow \exists\xi'(\xi' \in I^\alpha \ \&\ \xi < \xi' < (\xi + \omega)^*_\alpha \ \&\ L_{(\xi+\omega)^*_\alpha[\alpha]} \models \xi \in t_\chi(\xi')).$$

And going to the codes $\delta \in \mathrm{Code}(A) \Leftrightarrow \delta \in \mathrm{WO}$ & $\exists\beta\exists\gamma[\langle\omega, <_\beta, 0\rangle$ is a wellfounded model of ZFL($\dot{\alpha}$) *and* γ is (the characteristic function of) a generating set of indiscernibles for this model with character $\alpha^\#$ and order type $|\delta| + \omega$ *and* for some m, n we have $\pi(m) = |\delta|$, $\gamma(n) = 0$ and $\langle\omega, <_\beta, 0\rangle \models m \in n \ \&\ m \in t_\chi(n)$ (where $\pi : \langle\omega, <_\beta, 0\rangle \to \mathscr{M}$ is the transitive realization)].

This looks like a Σ^1_2 in $\alpha^\#$ expression. But writing $P(\beta, \gamma, \delta, \alpha^\#)$ for the matrix following $\exists\beta\exists\gamma$ we notice that P is Π^1_1 in $\alpha^\#$, while β, γ can be restricted to be Δ^1_1 in δ and $\alpha^\#$ (this is because a copy of $L_{(|\delta|+\omega)^*_\alpha}[\alpha]$ "with" $I^\alpha_{|\delta|+\omega}$ can be constructed in a Δ^1_1 fashion from δ and $\alpha^\#$). Thus

$$\delta \in \mathrm{Code}(A) \Leftrightarrow \delta \in \mathrm{WO} \ \&\ \exists\beta \in \Delta^1_1(\delta, \alpha^\#)\exists\gamma \in \Delta^1_1(\delta, \alpha^\#)P(\beta, \gamma, \delta, \alpha^\#),$$

which shows that $\mathrm{Code}(A) \in \Pi^1_1(\alpha^\#)$. $\square$

Corollary (4C-2). *Assume* $\forall\alpha(\alpha^{\#}$ *exists) and let* $A \subseteq \aleph_1$. *Then*

$$A \in \mathbf{L} \Leftrightarrow A \text{ is } \boldsymbol{\Pi}^1_1 \text{ in the codes}$$
$$\Leftrightarrow A \text{ is } \boldsymbol{\Sigma}^1_2 \text{ in the codes.}$$

Corollary (4C-3) (*Proved also independently by Martin* [7]). *Assume* $\forall\alpha(\alpha^{\#}$ *exists). Then* $u_2 \leq \boldsymbol{\delta}^1_2$.

Proof. Let $\xi < u_2$. By a theorem of Solovay (see [7]),

$$u_{\xi+1} = \sup\{(u_\xi^+)^{L[\alpha]} : \alpha \in \mathscr{R}\}.$$

Thus find α such that $\xi < (\aleph_1^+)^{L[\alpha]}$. In $L[\alpha]$ there exists a map $f: \aleph_1 \twoheadrightarrow \xi$, from $\aleph_1$ 1-1 and onto ξ. Let $\eta \preccurlyeq \theta \Leftrightarrow f(\eta) \leq f(\theta)$. Then $\preccurlyeq$ is a prewellordering of length ξ on $\aleph_1$ and $\preccurlyeq \in L[\alpha]$. Thus

$$\mathrm{Code}(A) = \{(\alpha, \beta) \in \mathrm{WO}^2 : (|\alpha|, |\beta|) \in \preccurlyeq\} \in \boldsymbol{\Pi}^1_1.$$

But Code(A) is a $\boldsymbol{\Pi}^1_1$ prewellordering of WO of length ξ; therefore $\xi < \boldsymbol{\delta}^1_2$. □

Martin [7] proved that $\forall\alpha(\alpha^{\#}$ *exists*) $\Rightarrow \boldsymbol{\delta}^1_2 \leq u_2$. Thus $\forall\alpha(\alpha^{\#}$ *exists*) $\Rightarrow \boldsymbol{\delta}^1_2 = u_2$ (Martin [7]).

4D. Since (assuming $\forall\alpha(\alpha^{\#}$ exists)) $u_1 = \boldsymbol{\delta}^1_1$ and $u_2 = \boldsymbol{\delta}^1_2$, one is confronted again with a tempting conjecture, i.e., $u_n = \boldsymbol{\delta}^1_n$ for all $n \geq 1$. That this is not the case is obvious to the believer in some strong form of definable determinacy, e.g., determinacy of all games in $L(\mathscr{R})$ = the smallest model of ZF containing all ordinals and $\mathscr{R}$. This is equivalent to the assertion $L(\mathscr{R}) \models$ AD. But from work of Moschovakis [8] and Kunen (unpublished) it is known that AD $\Rightarrow$ all $\boldsymbol{\delta}^1_n$ are regular. On the other hand Solovay proved that

$$\forall\alpha(\alpha^{\#} \text{ exists}) \Rightarrow \mathrm{Cofinality}(u_{\xi+1}) = \mathrm{Cofinality}(u_2),$$

for all $\xi \geq 1$; see [7]. These two facts (and of course AD $\Rightarrow \forall\alpha(\alpha^{\#}$ exists)) force $\boldsymbol{\delta}^1_n$ for $n \geq 3$ to be a fixed point of $\{u_\xi\}_{\xi\in\mathrm{Ord}}$, in $L(\mathscr{R})$. But this is absolute from $L(\mathscr{R})$ to the world. Thus $\boldsymbol{\delta}^1_n = u_{\boldsymbol{\delta}^1_n}$, $n \geq 3$. This is what we prove below, using only $\forall\alpha(\alpha^{\#}$ exists). The proof is motivated by the following (unpublished) result of Solovay.

Theorem (4D-1) (Solovay). *Assume* $\forall\alpha(\alpha^{\#}$ *exists). Then, for every* $\xi \geq 1$ *and every* $\eta < u_{\xi+1}$, *we can find a formula* φ, *a real* α *and uniform indiscernibles* $u_{\xi_1} < \cdots < u_{\xi_n} \leq u_\xi$ *such that* $\eta = t_\varphi^{L[\alpha]}(u_{\xi_1}, \ldots, u_{\xi_n})$.

We are now ready to prove

Theorem (4D-2). *Assume* $\forall\alpha(\alpha^{\#}$ *exists). Then, for any* $n \geq 3$, $\boldsymbol{\delta}^1_n = u_{\boldsymbol{\delta}^1_n}$.

Proof. Let $n \geq 3$. If $\boldsymbol{\delta}^1_n \neq u_{\boldsymbol{\delta}^1_n}$, then $u_\xi \leq \boldsymbol{\delta}^1_n \leq u_{\xi+1}$, for some $\xi < \boldsymbol{\delta}^1_n$. We derive a contradiction by producing a map $\sigma: \omega \times \mathscr{R}^2 \twoheadrightarrow u_{\xi+1}$, onto $u_{\xi+1}$, whose corresponding prewellordering $\leq^\sigma$ is $\boldsymbol{\Delta}^1_n$.

Since $\xi + 1 < \boldsymbol{\delta}^1_n$ let $\tau: \mathscr{R} \twoheadrightarrow \xi + 1$ be a $\boldsymbol{\Delta}^1_n$-norm on $\mathscr{R}$ of length $\xi + 1$ and put for simplicity $\tau(\beta) = |\beta|$. By (4D-1) every ordinal $\eta < u_{\xi+1}$ has the form

$$\eta = t_\varphi^{L[\alpha]}(u_{|(\beta)_1|}, \ldots, u_{|(\beta)_r|}),$$

for some φ and some α, β. Here $r = r(\ulcorner\varphi\urcorner)$, with $\ulcorner\varphi\urcorner$ = Gödel number of φ and $r: \omega \to \omega$ recursive. We assume also for convenience that every integer is of the form $\ulcorner\varphi\urcorner$. If $F(\dot{\alpha}, v_1) \equiv F(v_1)$ is a function definable in ZFL($\dot{\alpha}$), which maps the universe onto the ordinals keeping them fixed and sending everything else to 0 we have also $\eta = F^{L[\alpha]}(t_{\varphi}^{L[\alpha]}(u_{|(\beta)_1|}, \ldots, u_{|(\beta)_r|}))$, and the expression on the right always gives an ordinal. This suggests defining

$$\sigma(\ulcorner\varphi\urcorner, \alpha, \beta) = F^{L[\alpha]}(t_{\varphi}^{L[\alpha]}(u_{|(\beta)_1|}, \ldots, u_{|(\beta)_r|})).$$

That σ maps $\omega \times \mathscr{R}^2$ onto $u_{\xi+1}$ is obvious by the preceding remarks. That $\leq^{\sigma} \in \mathbf{\Delta}_n^1$ follows from the computation below:

$$\sigma(\ulcorner\varphi\urcorner, \alpha, \beta) \leq \sigma(\ulcorner\chi\urcorner, \gamma, \delta) \Leftrightarrow F^{L[\alpha]}(t_{\varphi}^{L[\alpha]}(u_{|(\beta)_1|}, \ldots)) \leq F^{L[\gamma]}(t_{\chi}^{L[\gamma]}(u_{|(\delta)_1|}, \ldots,))$$
$$\Leftrightarrow L[\alpha, \gamma] \models \psi(u_{|(\beta)_1|}, \ldots, u_{|(\delta)_1|}, \ldots)$$

(for a ψ obtained explicitly from φ, χ)

$$\Leftrightarrow \langle \alpha, \gamma \rangle^{\#}(g(\ulcorner\varphi\urcorner, \ulcorner\chi\urcorner, \beta, \delta)) = 0,$$

where $g : \omega^2 \times \mathscr{R}^2 \to \omega$ is $\mathbf{\Delta}_n^1$ and $\langle \alpha, \gamma \rangle = (\alpha(0), \gamma(0), \alpha(1), \gamma(1), \ldots)$. Roughly speaking g specifies the interweaving of $|(\beta)_1|, \ldots, |(\delta)_1|, \ldots$ and this can be done in a $\mathbf{\Delta}_n^1$ fashion. $\square$

REMARK. From (4D-2) and Martin's result that $\boldsymbol{\delta}_2^1 \leq u_2$ it is clear that one can prove $\boldsymbol{\delta}_2^1 < \boldsymbol{\delta}_3^1$ using only $\forall\alpha(\alpha^{\#}$ exists). This is also implicit in [7].

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