SUPPLEMENTARY INFORMATION
Topologically Protected State Transfer in a Chiral Spin Liquid

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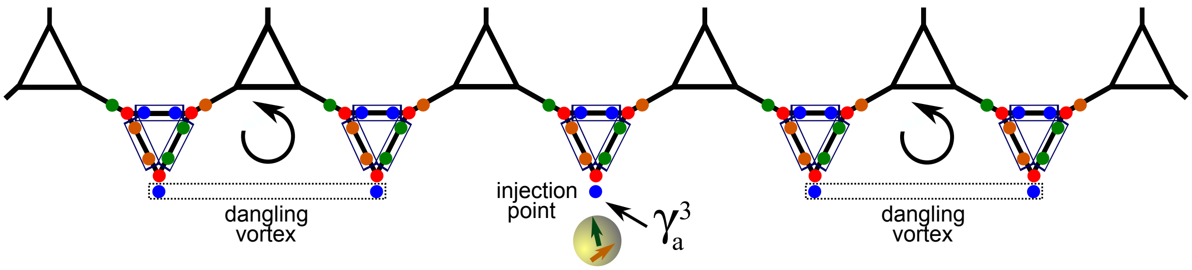
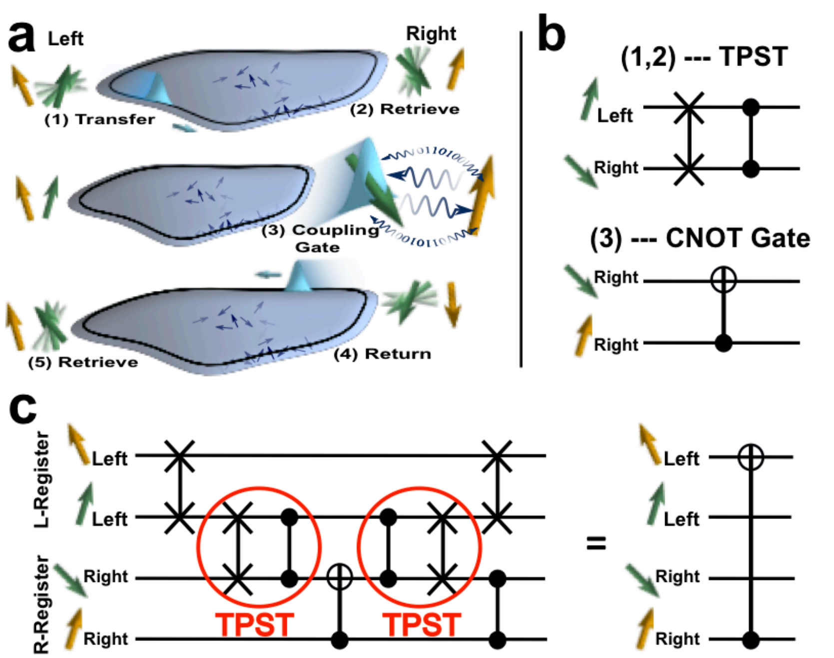
Supplementary Figures



Supplementary Fig. S1: Dangling Majoranas and Vortices. A schematic microscopic edge of the decorated honeycomb with the injection point explicitly labeled. Other pairs of dangling edge spins are paired into decoupled dangling vortices. It is these dangling vortices which lead to the large degeneracy found in the exactly solvable model.



Supplementary Fig. S2: Schematic representation of TPST and a remote CNOT gate. (a) The quantum registers L and R each contain two spins with the gold spin corresponding to the memory qubit. A single step of evolution SWAPs the information between the green spins of the left and right register. However, in addition to the SWAP gate, it also creates entanglement in the form of a controlled phase gate between these spins. To perform a remote CNOT gate between the memory qubits, we first perform an intra-register operation to SWAP the quantum information between the green and gold qubit of the left spin. (b) Next, the first step of TPST is performed (corresponding to a SWAP gate and a controlled phase gate). Afterwards, an intra-register CNOT gate between the green and gold qubit of the right register is performed. (c) The second step of TPST is then performed to return the information to the left register. Finally, intra-register gates are performed to yield a remote CNOT between the memory qubits. This enables universal computation.

Supplementary Discussion

Having demonstrated topologically protected state transfer by working in the extended Hilbert space in the main text, here, we consider the gauge projection back to the physical subspace. We illustrate the SWAP gate associated with TPST in the language of physical spin states. We consider the decorated honeycomb lattice model along with two additional spin registers as in Figure 2b of the main text. The full Hamiltonian, $H_T = H_0 + H_L + H_R + H_{int}$ is composed of

$$H_0 = \frac{1}{2} \sum_{\substack{x,x' \\ links}} \sigma_i^x \sigma_j^x + \frac{1}{2} \sum_{\substack{y,y' \\ links}} \sigma_i^y \sigma_j^y + \frac{1}{2} \sum_{\substack{z,z' \\ links}} \sigma_i^z \sigma_j^z \tag{S1}$$

$$H_{L/R} = -\frac{\Delta_S}{2}\sigma_L^z - \frac{\Delta_S}{2}\sigma_R^z, \quad H_{int} = g_L \sigma_L^x \sigma_a^x + g_R \sigma_R^x \sigma_b^x \tag{S2}$$

where we have chosen units in which $\kappa = 1$. The extension of the Hamiltonian to the Majorana Hilbert space results in a model of Majorana fermions γ_i^0 coupled to a static $\mathbb{Z}_2$ gauge field $\hat{U}_{i,j}$ residing on the lattice links:

$$H_0^\gamma = \frac{i}{4} \sum_{ij} \hat{U}_{i,j} \gamma_i^0 \gamma_j^0 \tag{S3}$$

$$H_{L/R}^\gamma = -\frac{\Delta_S}{2} i\gamma_L^3 \gamma_L^0 - \frac{\Delta_S}{2} i\gamma_R^3 \gamma_R^0 \tag{S4}$$

$$H_{int}^\gamma = -ig_L \hat{U}_{L,a} \gamma_L^0 \gamma_a^0 - ig_R \hat{U}_{R,b} \gamma_R^0 \gamma_b^0 \tag{S5}$$

where $\hat{U}_{i,j} = i\gamma_i^\alpha \gamma_j^\alpha$ and $\alpha = x, y, z$ is the link type of $\langle ij \rangle$, or $\hat{U}_{i,j} = 0$ if i and j are not connected. We extend the definition of the gauge field $\hat{U}_{i,j}$ to the paired dangling edge spins on the boundary as depicted in Supplementary Figure S1, and to $\hat{U}_{L,R} = i\gamma_L^2 \gamma_R^2$. With this choice of pairings, all $\hat{U}_{i,j}$ are conserved by the total Hamiltonian and thus time evolution may be understood in each $\hat{U}_{i,j}$ sector. We label the sectors of $\hat{U}_{i,j}$ by field configurations $\{U_{i,j} = \pm 1\}$. Due to the antisymmetry of $U_{i,j}$, there is some subtlety in correctly labeling sectors: in all our formulae, we take ij to be oriented according to the arrows in Figure 2b of the main text. Thus, $U_{i.j} = 1$ corresponds to a ground state gauge sector. Finally, we define $c_{L/R} = \frac{1}{2}(\gamma_{L/R}^0 + i\gamma_{L/R}^3)$ as in the main text so that we may think of $H_{L/R}^\gamma$ as measuring the occupation of left and right register fermions.

The $U_{i,j}$ are gauge dependent quantities as $\{D_i, \hat{U}_{i,j}\} = 0$, but the net flux around any closed loop $w(C) = \prod_{ij \in C} \hat{U}_{i,j}$ is gauge invariant; thus $w(C)$ is physical and conserved. The extended Hilbert space may be divided into conserved *gauge* sectors while the physical Hilbert space splits into conserved *flux* sectors after projection. As usual, we say that any plaquette P such that $w(\partial P) = -1$ contains a vortex; here, we additionally extend this definition of vortices to include the dangling plaquettes defined by the $U_{i,j}$ links between dangling edges, as shown in Supplementary Figure S1. These *dangling vortices* are completely decoupled from the fermions and lead to a large degeneracy of the model with open boundaries ($2^{N_e/4}$ where N_e is the number of dangling edge spins).

Let us consider the physical ground state of the system in the absence of interaction g between the registers and the decorated honeycomb. The spin registers both point up, disentangled from the rest of the system, while the lattice spins sit in their collective ground state: $|\uparrow\rangle_L |GS\rangle_0 |\uparrow\rangle_R$. We seek a reference ground state $|\Omega\rangle$ in a fixed gauge sector of the extended Hilbert space such that $|\uparrow\rangle_L |GS\rangle_0 |\uparrow\rangle_R = P|\Omega\rangle$ up to normalization. We choose $U_{i,j} = +1$ (*i.e.* the flux configuration of the ground state sector as described in the main text) and we choose $|\Omega\rangle$ to be annihilated by c_L, c_R and c_k for $k > 0$ of $H_0^\gamma(U) + H_{L/R}^\gamma$. This state is, by construction, a lowest energy eigenstate of the system, but it may not survive projection. The norm of $P|\Omega\rangle$ is $\langle\Omega| PP |\Omega\rangle = \frac{1}{2^N} \langle\Omega| 1 + \prod_i D_i |\Omega\rangle$ where we have exploited the orthogonality of states with different gauge configurations $U_{i,j}$. The product over all gauge transformations D_i measures the product of all $U_{i,j}$ and the parity of the $\gamma_1^0 \gamma_2^0 \cdots \gamma_N^0$ fermionic state. Thus, by flipping the choice of $U_{L,R}$ in $|\Omega\rangle$, we may guarantee that $P|\Omega\rangle$ survives projection. In general, in any fixed gauge sector related to our reference sector by an even (odd) number of flipped $U_{i,j}$'s, P will annihilate states with odd (even) fermionic parity.

We now construct an explicit representation of the 4 possible register states coupled to the intermediate ground state $|GS\rangle_0$ by acting with $\sigma^x_{L/R}$:

$$
\begin{aligned}
|\uparrow\rangle_L |GS\rangle_0 |\uparrow\rangle_R &= P|\Omega\rangle \\
|\downarrow\rangle_L |GS\rangle_0 |\uparrow\rangle_R &= -iPc_L^\dagger\gamma_L^1|\Omega\rangle \\
|\uparrow\rangle_L |GS\rangle_0 |\downarrow\rangle_R &= -iPc_R^\dagger\gamma_R^1|\Omega\rangle \\
|\downarrow\rangle_L |GS\rangle_0 |\downarrow\rangle_R &= -iPc_L^\dagger c_R^\dagger|\Omega\rangle
\end{aligned}
\tag{S6}
$$

where in the last line we have used $U_{L,R} = 1$ acting on $|\Omega\rangle$. More generally, degenerate and/or low energy states may be found in either the same flux sector with (pairs of) extra edge fermions, *e.g.* $Pc_p^\dagger c_{p'}^\dagger|\Omega\rangle$ or in degenerate flux sectors (containing dangling vortices), *e.g.* $Pc_k^\dagger\gamma_i^\alpha|\Omega\rangle$ where γ_i^α is a dangling edge Majorana. We note that since $H_0^\gamma(U)$ does not depend on the dangling edge $U_{i,j}$'s, neither do the fermionic eigenmodes c_k in these degenerate sectors nor the fermionic vacuum.

Let us now consider time evolution $\mathcal{U}(t)$ in the presence of the coupling H^γ_{int}. The gauge field $U_{i,j}$ remains conserved and the time evolution of the Majorana field γ_i^0 within each gauge sector is that of noninteracting fermions. The full Hamiltonian in our chosen ground state gauge sector is given by equation (5) of the main text. In general, ϵ_k, $Q_{k,a}$, and c_k depend on the gauge field but not on the dangling pieces of it, so the following analysis applies identically in each sector containing dangling vortices so long as the gauge is chosen the same way in the bulk and on $U_{L,a}$, $U_{R,b}$. Assuming that $g \ll \Delta_S$, we may use the secular approximation to eliminate the c-fermion number non-conserving terms in equation (5),

$$
\begin{aligned}
H^\gamma(U) \approx &\sum_{k>0}\epsilon_k(c_k^\dagger c_k - \frac{1}{2}) + \Delta_S(c_L^\dagger c_L - \frac{1}{2}) + \Delta_S(c_R^\dagger c_R - \frac{1}{2}) \\
&- \frac{i}{\sqrt{2}}g_L(c_L^\dagger\sum_k Q_{k,a}^* c_k + c_L\sum_k Q_{k,a}c_k^\dagger) \\
&- \frac{i}{\sqrt{2}}g_R(c_R^\dagger\sum_k Q_{k,b}^* c_k + c_R\sum_k Q_{k,b}c_k^\dagger).
\end{aligned}
\tag{S7}
$$

This Hamiltonian leaves the c-fermion vacuum $|\Omega\rangle$ invariant and evolves the modes as usual non-interacting Dirac fermions: $\mathcal{U}c_{k_1}^\dagger c_{k_2}^\dagger\cdots c_{k_m}^\dagger|\Omega\rangle = c_{k_1(t)}^\dagger c_{k_2(t)}^\dagger\cdots c_{k_m(t)}^\dagger|\Omega\rangle$, where $k_i(t)$ denotes the time evolved wavefunction of the k_i mode according to the single particle Schrödinger equation. We note that the most general Majorana evolution would mix the $c^\dagger$ and c modes and accordingly the instantaneous c-vacuum would evolve in time.

To enable state transfer, we now tune $\Delta_S = \epsilon_{\tilde{k}}$ for an edge mode $\tilde{k}$. In the dot regime, we further require $|g_L Q_{\tilde{k},a}|, |g_R Q_{\tilde{k},b}| \ll |\epsilon_{\tilde{k}} - \epsilon_{\tilde{k}\pm 1}|$. This condition enables single-mode resolution of the edge eigenmodes and state transfer proceeds by resonant fermionic tunneling in an effective three mode model (dropping constants and the uninvolved modes):

$$
H_{eff} = -\frac{i}{\sqrt{2}}g_L Q_{\tilde{k},a}^* c_L^\dagger c_{\tilde{k}} - \frac{i}{\sqrt{2}}g_R Q_{\tilde{k},b}^* c_R^\dagger c_{\tilde{k}} + h.c.
\tag{S8}
$$

Since the individual quantum registers are fully controllable, we tune g_L and g_R to ensure that the effective tunneling rate $t_{\tilde{k}} = |\frac{g_L}{\sqrt{2}}Q_{\tilde{k},a}| = |\frac{g_R}{\sqrt{2}}Q_{\tilde{k},b}|$ between the modes is equivalent. Re-expressing $-iQ_{\tilde{k},a}^* = e^{i\phi_{\tilde{k},a}}|Q_{\tilde{k},a}|$ and $-iQ_{\tilde{k},b}^* = e^{i\phi_{\tilde{k},b}}|Q_{\tilde{k},b}|$, subsequent evolution under H_{eff} for a time $\tau = \frac{\pi}{\sqrt{2}t_{\tilde{k}}}$ results in mode evolution,

$$
\begin{aligned}
c_L^\dagger &\longrightarrow -e^{-i\phi}c_R^\dagger \\
c_{\tilde{k}}^\dagger &\longrightarrow -c_{\tilde{k}}^\dagger \\
c_R^\dagger &\longrightarrow -e^{i\phi}c_L^\dagger
\end{aligned}
\tag{S9}
$$

where $\phi = \phi_{\tilde{k},a} - \phi_{\tilde{k},b}$. Using these relations to evolve the states from equation (S6), we find

$$
\begin{aligned}
|\uparrow\rangle_L |GS\rangle_0 |\uparrow\rangle_R &\longrightarrow |\uparrow\rangle_L |GS'\rangle_0 |\uparrow\rangle_R \\
|\downarrow\rangle_L |GS\rangle_0 |\uparrow\rangle_R &\longrightarrow -ie^{-i\phi}|\uparrow\rangle_L |GS'\rangle_0 |\downarrow\rangle_R \\
|\uparrow\rangle_L |GS\rangle_0 |\downarrow\rangle_R &\longrightarrow ie^{i\phi}|\downarrow\rangle_L |GS'\rangle_0 |\uparrow\rangle_R \\
|\downarrow\rangle_L |GS\rangle_0 |\downarrow\rangle_R &\longrightarrow -|\downarrow\rangle_L |GS'\rangle_0 |\downarrow\rangle_R\,.
\end{aligned}
\tag{S10}
$$

up to known dynamical phases. The time evolution presented in equation (S10) generates our desired SWAP gate in addition to a controlled phase gate between the register modes (up to single qubit rotations). Here $|GS'\rangle_0$ indicates a state which evolves from $|GS\rangle_0$ *independent* of the state of the two register qubits. As depicted in Supplementary Figure S2, in combination with intra-register manipulations, the gate described by equation (S10) enables universal computation between the memory qubits of the remote spin registers.

This schematic evolution holds identically for *any* initial state of the intermediate system $|GS\rangle_0$ containing extra fermions or dangling vortices, since such states may be represented in a gauge sector where all bulk $U_{i,j} = 1$. Furthermore, in flux sectors in which there are an *even* number of bulk vortices, it is possible to choose a gauge in which $U_{i,j} = 1$ for all links near the edge. The evolution proceeds nearly identically in this case as well. On the other hand, in flux sectors where there are an *odd* number of bulk vortices, the energy of the edge modes is shifted by $\sim \kappa/L$ implying that the spin registers are off-resonant. This can be corrected for through tomography and subsequent retuning.

Supplementary Methods

Here, we describe the shaping of the fermionic wavepacket in the droplet regime of TPST. The edge mode energies ϵ_k of a finite-sized droplet are split at order $1/\ell$. As discussed in the main text, since single mode energy resolution becomes impossible in the macroscopic limit, we encode the spin register's quantum information into a fermionic wavepacket traveling along the chiral edge of the 2D droplet. This requires the shaping of $g_L(t)$ and $g_R(t)$ in order to ensure the sending and receiving of the packet. Let us first consider the shaping of the initial wavepacket at the left register, so $g_R(t) = 0$. It is sufficient for us to consider the single particle problem since the modes evolve as usual non-interacting Dirac fermions. By tuning Δ_S to an energy in the middle of the edge dispersion and restricting $|g_L| \ll \Delta_S$, we have (assuming a plane wave description of the low energy chiral edge modes)

$$H_{wp} = \sum_k E_k |k\rangle\langle k| + \frac{g_L}{\sqrt{l}} \sum_k (|k\rangle\langle L| + |L\rangle\langle k|), \tag{S11}$$

where $|k\rangle$ is the edge mode with momentum k, we have absorbed all numerical factors into g_L and $E_k = vk$ is shifted by Δ_S (here, we have correspondingly shifted the definition of zero energy and the indexing of k to begin at the state with energy Δ_S). We choose this notation for the Hamiltonian to be consistent with the literature regarding photonic wavepacket storage and retrieval, where an analogous problem is solved; thus, in this section, c_i, rather than being fermionic operators, will represent the amplitude of the $|i\rangle$ mode. Initially, we consider a state $|\psi\rangle$ whose amplitude is fully localized on the left spin register, $|\psi\rangle = c_L|L\rangle + \sum c_k|k\rangle$, where $c_L(t=0) = 1$ and $c_k(t=0) = 0$. After making a continuum approximation in both position and momentum, we formally solve the Schrödinger equation to obtain $\dot{c}_L(t) = -\frac{1}{2v}|g_L(t)|^2 c_L(t)$, yielding $c_L(t) = e^{h(t)}$ where $h(t) = \frac{1}{2v}\int_0^t dt' |g_L(t')|^2$. Substituting this result into the formal solution of $c_k(t)$ yields

$$c_k(t) = -i \int_0^t dt' e^{-ivk(t-t')} \frac{1}{\sqrt{l}} g_L(t') e^{-h(t')}. \tag{S12}$$

Thus, the shape of the outgoing wavepacket is

$$\begin{aligned} c(x,t) &= \frac{1}{\sqrt{l}} \sum_k e^{ikx} c_k(t) \approx \frac{\sqrt{l}}{2\pi} \int dk e^{ikx} c_k(t) = \frac{-i}{2\pi} \int dk e^{ikx} \int_0^t dt' e^{-ivk(t-t')} g_L(t') e^{-h(t')} \\ &= -i \int_0^t dt' \delta(x - v(t-t')) g_L(t') e^{-h(t')} = -i \frac{1}{v} g_L(t - x/v) e^{-h(t-x/v)} \theta(t - x/v). \end{aligned} \tag{S13}$$

where θ is the Heaviside step function and we have assumed linear dispersion with group velocity v. The assumption of a linear dispersion will have corrections $\mathcal{O}(k^3)$, which can be pre-compensated during the shaping of the wavepacket. Here, we note that in converting from a k sum to an integral, we have assumed that the amplitude on both $k < 0$ and bulk modes will be negligible since $|g_L| \ll \Delta_S$. As previously discussed, this assumption is crucial to ensure that the vacuum does not undergo time evolution.

It is natural to think of the wave-packet in the time domain and evaluate $c(x,t)$ at $x = 0$. Thus, the solution to the problem of shaping any desired wavepacket, $f(t)$, simplifies to deriving the requisite $g_L(t)$ control function that satisfies $\frac{1}{v} g_L(t) e^{-h(t)} = f(t)$ where $h(t) = \frac{1}{2v}\int_0^t dt' |g_L(t')|^2$; such a solution then yields,

$$g_L(t) = \frac{\sqrt{v} f(t)}{\sqrt{\int_t^\infty dt' |f(t')|^2}}. \tag{S14}$$

The subsequent retrieval of the wave-packet at the location of the right spin register can be understood by using time-reversal; indeed, the control function $g_R(t)$ should be the time-reversed form of the control used to generate the time-reversed form of the sent wavepacket. While, for simplicity, we have considered $g_L, g_R \in \mathbb{R}$ above, generalizing to complex $g_{L/R}$ can easily be achieved, for example, by employing a Λ-configuration spin register.