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SETS OF ORDINALS CONSTRUCTIBLE FROM TREES AND THE THIRD VICTORIA DELFINO PROBLEM

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A very important part of the structure theory of Σ^1_2 sets of reals is based on their close interrelationship with the Gödel constructible universe L. The fundamental fact underlying this connection is the theorem of Shoenfield which asserts that every Σ^1_2 set of reals is Souslin over L. This means that given any Σ^1_2 subset of the reals ($= \omega^\omega$ in this paper), there is a tree T on $\omega \times \lambda$ (λ some ordinal, which can be taken to be $\aleph_1$ here) such that $T \in L$ and $A = p[T] = \{\alpha \in \omega^\omega : \exists f \in \lambda^\omega\, \forall n(\alpha{\restriction}n, f{\restriction}n) \in T\}$.

In 1971 Moschovakis [14] introduced the notion of scale and showed that under the hypothesis of Projective Determinacy (PD) the pointclasses Π^1_{2n+1}, Σ^1_{2n+2} for all $n \geq 0$ have the Scale Property. To take for instance the case of Π^1_{2n+1} this asserts that given any Π^1_{2n+1} set $A \subseteq \omega^\omega$ there is a sequence of norms $\varphi_m : A \to$ Ordinals (which we are always assuming here to be *regular*, i.e., having as range an initial segment of the ordinals) such that the following properties hold:

(i) If $\alpha_0, \alpha_1, \ldots \in A$, $\alpha_i \to \alpha$ and $\varphi_m(\alpha_i) \to \lambda_m$ (in the sense that $\varphi_m(\alpha_i) = \lambda_m$ for all large enough i), then $\alpha \in A$ and $\varphi_m(\alpha) \leq \lambda_m$.

(ii) The relations (on m, α, β): $\alpha \leq^*_{\varphi_m} \beta \Leftrightarrow \alpha \in A \wedge (\beta \notin A \vee \varphi_m(\alpha) \leq \varphi_m(\beta))$, $\alpha <^*_{\varphi_m} \beta \Leftrightarrow \alpha \in A \wedge (\beta \notin A \vee \varphi_m(\alpha) < \varphi_m(\beta))$ are Π^1_{2n+1}.

Such a sequence $\{\varphi_m\}$ is called a (regular) Π^1_{2n+1}-*scale* on A. Now *any* scale $\bar{\varphi} = \{\varphi_m\}$ on a set A (i.e. a sequence of norms satisfying (i) above) gives rise to a tree $T = T(A,\bar{\varphi})$ on $\omega \times \lambda$, where $\lambda = \sup_m\{\text{range}(\varphi_m)\}$, given by

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$$((k_0,k_1,\ldots,k_n),(\eta_0,\eta_1,\ldots,\eta_n)) \in T \Leftrightarrow \exists\alpha \in A\ \forall m \leq n[\alpha(m) = k_m \wedge \varphi_m(\alpha) = \eta_m].$$

It is then easy to check the key fact that $A = p[T]$, so that T verifies that A is Souslin. Moreover $\{\varphi_m\}$ can be canonically recovered from T, since for $\alpha \in A$, $(\varphi_0(\alpha),\varphi_1(\alpha),\ldots)$ = the leftmost branch of $T(\alpha)$, where $T(\alpha) = \{(\eta_0,\ldots,\eta_n) : (\alpha\restriction(n+1),(\eta_0,\ldots,\eta_n)) \in T \wedge n \in \omega\}$. Thus T is a canonical embodiment of the scale $\{\varphi_m\}$ as (essentially) a set of ordinals.

Now, always assuming PD, let $\mathcal{P} \subseteq \omega^\omega$ be a complete Π^1_{2n+1} set of reals and $\{\varphi_m\} = \bar{\varphi}$ a regular Π^1_{2n+1}-scale on $\mathcal{P}$. Let $T_{2n+1}(\mathcal{P},\bar{\varphi})$ be the associated tree. This is a tree on $\omega \times \underset{\sim}{\delta}^1_{2n+1}$, where $\underset{\sim}{\delta}^1_{2n+1} = \sup\{\xi : \xi$ is the length of a $\underset{\sim}{\Delta}^1_{2n+1}$ prewellordering of $\omega^\omega\}$. If we consider the model

$$L[T_{2n+1}(\mathcal{P},\bar{\varphi})].$$

of sets constructible from $T_{2n+1}(\mathcal{P},\bar{\varphi})$, then by [9], 9A - 2 every Σ^1_{2n+2} set is Souslin over this model. Thus the Σ^1_{2n+2} sets have the same relationship to this model as the Σ^1_2 sets have with respect to L, and it is natural to consider $L[T_{2n+1}(\mathcal{P},\bar{\varphi})]$ as a higher level analog of the counstructible universe.

Indeed Moschovakis (see [9], p. 42) showed (in ZF + DC) that

$$L[T_1(\mathcal{P},\bar{\varphi})] = L.$$

In particular $L[T_1(\mathcal{P},\bar{\varphi})]$ is independent of the choice of the Π^1_1-complete set $\mathcal{P}$ and the Π^1_1-scale $\bar{\varphi}$ on $\mathcal{P}$. The question whether the same invariance holds for $2n+1 > 1$ arose almost immediately after the introduction of these models and Moschovakis conjectured in 1971 that this was indeed the case. This problem of the invariance of $L[T_{2n+1}(\mathcal{P},\bar{\varphi})]$ was eventually formulated as the 3rd Victoria Delfino Problem (see [10], p. 280) for the first open case $2n + 1 = 3$. (It has been already proved in [5] that $L[T_{2n+1}(\mathcal{P},\bar{\varphi})] \cap \omega^\omega$ is invariant for all n.) We provide in this paper an affirmative solution of this problem for all $2n + 1$.

Theorem 1. Assume Projective Determinacy when $n \geq 1$. Let ρ be a complete Π^1_{2n+1} set of reals and $\bar{\varphi} = \{\varphi_m\}$ a regular Π^1_{2n+1}-scale on ρ. Then the model $L[T_{2n+1}(\rho,\bar{\varphi})]$ is independent of the choice of ρ, $\bar{\varphi}$.

Thus one can legitimately refer for each n to the model

$$L[T_{2n+1}]$$

without any embellishments.

Since the tree $T_{2n+1}(\rho,\bar{\varphi})$ is a natural set theoretic manifestation of the scale $\bar{\varphi}$ on ρ this result shows also a strong uniqueness property inherent in the concept of Π^1_{2n+1}-scale.

Actually the way the theorem is proved establishes quite a bit more. To explain the stronger statement let us go back to one of the immediate consequences of the Souslin representation of Σ^1_2 sets by trees in L, namely the fact that every Σ^1_2 subset of ω is constructible. Following work of Solovay (see [16]) it was shown in Kechris-Moschovakis [8] that every Σ^1_2 subset of $\aleph_1 = \underset{\sim}{\delta}^1_1$ is also constructible. (Apparently H. Friedman has also independently proved that theorem without publishing it--see also [4]). As usual, when we say that a set $X \subseteq \aleph_1$ is Σ^1_2 we mean that given a Π^1_1-norm $\varphi : \rho \xrightarrow{\text{onto}} \underset{\sim}{\delta}^1_1$ on a complete Π^1_1 set ρ, the set

$$X^* = \{w \in \rho : \varphi(w) \in X\}$$

is $\underset{\sim}{\Sigma}^1_2$. It is not hard to check that this notion is intrinsic, i.e., independent of the choice of ρ, φ.

Following the work of Harrington-Kechris [5] it has become possible to generalize this notion to higher levels, assuming PD. We call a subset X of $\underset{\sim}{\delta}^1_{2n+1}$ Σ^1_{2n+2} if given a Π^1_{2n+1}-norm $\varphi : \rho \xrightarrow{\text{onto}} \underset{\sim}{\delta}^1_{2n+1}$ on a complete Π^1_{2n+1} set ρ the set

$$X^* = \{w \in \rho : \varphi(w) \in X\}$$

is Σ^1_{2n+2}. (It is worth recalling here that the length of any such norm is exactly $\underset{\sim}{\delta}^1_{2n+1}$, see [12], p. 216.) From [5], p. 125 it follows that this notion is again independent of $\wp$, φ. We can now generalize the result about Σ^1_2 subsets of $\aleph_1$ and L as follows.

Theorem 2. Assume Projective Determinacy when $n \geq 1$. Let $\wp$ be a complete Π^1_{2n+1} set of reals and $\bar{\varphi} = \{\varphi_m\}$ a regular Π^1_{2n+1}-scale on $\wp$. Then every Σ^1_{2n+2} subset of $\underset{\sim}{\delta}^1_{2n+1}$ is in $L[T_{2n+1}(\wp,\bar{\varphi})]$.

Now it has been verified (see [5], p. 131 or [12], p. 164) that any $T_{2n+1}(\wp,\bar{\varphi})$ (viewed by some simple coding of tuples as a subset of $\underset{\sim}{\delta}^1_{2n+1}$) is indeed Σ^1_{2n+2}, thus Theorem 2 immediately implies Theorem 1.

Moschovakis [12] has introduced (under PD) the models H_{2n+1}, which are defined as $L[P_{2n+2}]$, where $P_{2n+2} \subseteq \omega \times \underset{\sim}{\delta}^1_{2n+1}$ is universal Σ^1_{2n+2}. It follows immediately from Theorem 2 and the remarks following it that

$$L[T_{2n+1}] = H_{2n+1}.$$

Martin showed in 1976 (see an exposition in [6], p. 67) that if the tree $T_{2n+1}(\wp,\bar{\varphi})$ has a special property, namely it is homogeneous with fully (i.e., $\underset{\sim}{\delta}^1_{2n+1}$-) additive measures, then indeed $L[T_{2n+1}(\wp,\bar{\varphi})]$ satisfies Theorem 2 and thus also $H_{2n+1} = L[T_{2n+1}(\wp,\bar{\varphi})]$ for such a $T_{2n+1}(\wp,\bar{\varphi})$. (For the definition of homogeneous tree see [6]). He also succeeded in 1982 (unpublished) in proving the existence of such trees. Thus Theorems 1 and 2 have been known to be equivalent. Martin's proof of the special case of Theorem 2 for the homogeneous trees used an idea for a game which has found several applications in the study of the models H_{2n+1}, and is also being used in our proof as well.

The models $L[T_{2n+1}(\wp,\bar{\varphi})]$ and H_{2n+1} are defined and studied in Section 8G of [12]. Other work on these models can be found in the following references: [1], [2], [3], [5], [6], [9], and [15]. These models have a very interesting and useful structure theory, and they are related to many topics in descriptive set theory.

Our demonstration of Theorem 2 is actually quite general and applies to many other pointclasses "resembling Π^1_1." In fact our main result, from which everything else is an immediate corollary (when combined with already known theorems) is a very simple and general constructibility theorem for the tree associated with a scale, which is just a theorem of ZF + DC. We formulate and prove this result in §1. In §2 we derive as corollaries Theorem 2 and its generalizations to move general pointclasses, and in §3 we discuss some analogs of Theorem 2 for the even levels of the projective hierarchy. Finally in §4 we prove the Π^1_3 analog of Kleene's theorem that Π^1_1 equals inductive on the structure $\langle\omega,<\rangle$, solving an open problem raised in Kechris-Martin [7].

§1. The Main Theorem. We will follow below basically the notation and terminology of Moschovakis [12], except for calling ω^ω the *set of reals* and denoting it by $\mathbb{R}$. We work in ZF + DC stating all extra hypotheses explicitly.

Theorem. Let Γ be an ω-parametrized pointclass containing all the recursive pointsets and closed under conjunctions and recursive substitutions. Let $\wp$ be a complete Γ set of reals, let $\bar{\varphi} = \{\varphi_m\}$ be a regular $\exists^{\mathbb{R}}\Gamma$-scale on $\wp$, and let $\varphi_0 : \wp \xrightarrow{\text{onto}} \kappa$. For any $X \subseteq \kappa$, if X is $\exists^{\mathbb{R}}\Gamma$ in the codes provided by φ_0(i.e., $\{w \in \wp : \varphi_0(w) \in X\}$ is in the pointclass $\exists^{\mathbb{R}}\Gamma$), then $X \in L[T(\wp,\bar{\varphi})]$, where $T(\wp,\bar{\varphi})$ is the tree associated with the scale $\bar{\varphi}$ on $\wp$.

Proof. Our proof relies on the absoluteness of open games. An open game can be identified with a pair $(K,\mathcal{S})$, where K is a set and $\mathcal{S}$ is a set of finite sequences from K. In the game players I and II alternately play elements of K and I wins iff after some finite number of plays the sequence produced (by both players) is in $\mathcal{S}$; II wins otherwise, that is iff II has not already lost at some finite time. This game is open for I and closed for II. By the Gale-Stewart Theorem, all open games on a well-orderable set K are determined. Furthermore, if M is a transitive model of ZFC and the open game is in M (i.e., $(K,\mathcal{S}) \in M$), then the same player who wins in V wins also in M (see [9], p. 40).

To prove the theorem, we will assign to each ordinal $\xi < \kappa$, an open game G_ξ satisfying the following two properties:

(i) $\forall\xi < \kappa[\xi \in X \Leftrightarrow$ II has a winning strategy in $G_\xi]$,

(ii) The map $\xi \mapsto G_\xi$ is in $L[T]$, where $T = T(\wp,\bar{\varphi})$. It clearly follows from this and the absoluteness of open games that $X \in L[T]$, which proves the theorem.

Fix $\xi < \kappa$. We will define the game G_ξ and prove (i). The proof of (ii) is obvious.

Definition of G_ξ.

Since $\wp = p[T]$ is a complete Γ set, for any $n \geq 1$ and any $B \subseteq \mathbb{R}^n$, $B \in \exists^{\mathbb{R}}\Gamma$, there is a tree S_B such that $B = p[S_B]$ and $S_B \in L[T]$. Moreover S_B can be constructed from (a $\exists^{\mathbb{R}}\Gamma$-code of) B in a uniform and $L[T]$-absolute manner (cf.[9], Section 9).

For $x \in \wp$, let

$$|x| = \varphi_0(x),$$

and let

$$X^* = \{x \in \wp : |x| \in X\}$$

be the code set of X. Since by hypothesis $X^* \in \exists^{\mathbb{R}}\Gamma$ there is a tree $S = S_{X^*}$ in $L[T]$ such that $p[S] = X^*$. For each $\eta < \kappa$ let

$$T^{\eta} = \{((a_0,\ldots,a_n),(\xi_0,\ldots,\xi_n)) \in T : \xi_0 \leq \eta\}.$$

Our open game G_ξ will involve the two trees T^ξ and S plus a sequence S_0, S_1, S_2, ... of other trees which we now describe.

For all $n \in \omega$, let $Q_n \subseteq \mathbb{R} \times (\omega^{n+1} \times \lambda^{n+1})$ (for an appropriate λ) be the following set:

$$\{(x,(a_0,\ldots,a_n),(\xi_0,\ldots,\xi_n)) : x \in \wp \wedge ((a_0,\ldots,a_n),(\xi_0,\ldots,\xi_n)) \in T^{|x|}\}.$$

Let Q_n^* be the code set of Q_n, where the ith coordinate is encoded via φ_i. Formally, $Q_n^* \subseteq \mathbb{R} \times \omega^{n+1} \times \mathbb{R}^{n+1}$ and

$$(x,a_0,\ldots,a_n,z_0,\ldots,z_n) \in Q_n^* \Leftrightarrow x \in \wp \wedge \forall i \leq n(z_i \in \wp) \wedge$$

$$((a_0,\ldots,a_n),(\varphi_0(z_0),\ldots,\varphi_n(z_n))) \in T^{|x|}.$$

By the definition of the tree $T = T(\wp,\bar{\varphi})$ associated with the scale $\bar{\varphi}$, we have then

$$(x,a_0,\ldots,a_n,z_0,\ldots,z_n) \in Q_n^* \Leftrightarrow x \in \wp \wedge \forall i \leq n(z_i \in \wp) \wedge$$

$$\exists y[y \in \wp \wedge \forall i \leq n(y(i) = a_i) \wedge \forall i \leq n(\varphi_i(y) = \varphi_i(z_i)) \wedge \varphi_0(y) \leq \varphi_0(x)].$$

Since Γ is closed under conjunction, $\exists^{\mathbb{R}}\Gamma$ is closed under both conjunction and existential quantification over $\mathbb{R}$. Thus, since $\bar{\varphi}$ is a $\exists^{\mathbb{R}}\Gamma$-scale the above formula shows that Q_n^* is in $\exists^{\mathbb{R}}\Gamma$ uniformly in n. Hence there is a sequence of trees S_0, S_1, ... such that

1) $\forall n[p[S_n] = Q_n^*]$.

(Although $Q_n^* \subseteq \mathbb{R} \times \omega^{n+1} \times \mathbb{R}^{n+1}$, we can naturally consider it as a subset of $\mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ identifying $k \in \omega$ with the constant real $\lambda n.k$.)

2) The map $n \mapsto S_n$ is in $L[T]$.

Now we can describe the game G_ξ. I and II will play finitely many integers and ordinals in each move. If the game lasts infinitely long (and hence II wins), they will have played elements of ω^ω and ORD^ω as shown below:

$$\begin{array}{ll} \mathrm{I} & y,\ F \\ \mathrm{II} & x,\ f,\ g,\ z_0,\ h_0,\ z_1,\ h_1,\ z_2,\ h_2,\ \ldots \end{array}$$

where

$$y,\ x,\ z_0,\ z_1,\ z_2,\ \ldots \in \omega^\omega,$$
$$F,\ f,\ g,\ h_0,\ h_1,\ \ldots \in \mathrm{ORD}^\omega.$$

Player I must play so that his moves are in the tree T^ξ (i.e., for each n, $(y\restriction n, F\restriction n) \in T^\xi$). Player II must play so that particular moves (of I and II combined) are in particular trees, as described in the table below. Finite initial segments of the tuples of infinite sequences in the left column must be in the tree listed in the center column. The right column of the table explains the meaning of being a branch through that tree.

	Sequence	Tree	Meaning
Player I	y,F	T^ξ	$\lvert y\rvert \leq \xi$
Player II	x,f	T^ξ	$\lvert x\rvert \leq \xi$
	x,g	S	$\lvert x\rvert \in X$
	$x,y(0),z_0,h_0$	S_0	$\varphi_0(z_0)=\xi_0 \wedge (y(0),\xi_0) \in T^{\lvert x\rvert}$
	$x,y(0),y(1),z_0,z_1,h_1$	S_1	$\varphi_0(z_0)=\xi_0 \wedge \varphi_1(z_1)=\xi_1 \wedge ((y(0),y(1)),(\xi_0,\xi_1)) \in T^{\lvert x\rvert}$
	$x,y(0),y(1),y(2),$ z_0,z_1,z_2,h_2	S_2	$\varphi_0(z_0)=\xi_0 \wedge \varphi_1(z_1)=\xi_1 \wedge \varphi_2(z_2)=\xi_2$ $\wedge ((y(0),y(1),y(2)),(\xi_0,\xi_1,\xi_2)) \in T^{\lvert x\rvert}$
	⋮	⋮	⋮

The first player to fail to play into his required tree loses. If neither player fails, the game lasts forever and II wins. Then the sequences in the left column give infinite branches through the trees in the center column.

The integers and ordinals are to be played in some reasonable order fixed in advance such that the following condition is satisfied:

Player II does not have to begin playing z_n or h_n until after I has played $y(0) \cdots y(n)$ and $F(0) \cdots F(n)$. For example, in the nth round ($n = 0,1,2,\ldots$) player I plays $(y(n),F(n))$ and II plays $(x(n),f(n),g(n),z_0(n),$ $h_0(n),z_1(n-1),h_1(n-1),\ldots,z_n(0),h_n(0))$.

This completes the definition of G_ξ. We now prove the main claim:

Claim. $\xi \in X \Leftrightarrow$ II has a winning strategy in G_ξ.

Proof. $\Leftarrow$: Let σ be a winning strategy for II in G_ξ. Fix $y \in \wp$ with $|y| = \xi$, and let $F \in \mathrm{ORD}^\omega$ be the sequence $F = (\varphi_0(y),\varphi_1(y),\ldots)$. Consider the run of the game G_ξ in which I plays this y, F and II plays according to σ. Since (y,F) is a path through T^ξ(since $F(0) = \varphi_0(y) = |y| = \xi$), I never fails to play into this tree, so he does not lose for that trivial reason. Since σ is a winning strategy for II, II produces in this run of the game some x, f, g, z_0, h_0, z_1, h_1, ... and these must also be in the required trees as shown in the above table. Thus (x,f) is a path through T^ξ so $x \in p[T^\xi]$, thus by the definition of T^ξ and the "lower semicontinuity" property of scales (the clause $\varphi_m(\alpha) \leq \lambda_m$ in the definition) $x \in \wp \wedge \varphi_0(x) = |x| \leq \xi$. Also (x,g) is a path through S, so $x \in p[S] = X^*$, thus $|x| \in X$. So it is enough to show $|x| = \xi$, i.e., (since we know already that $|x| \leq \xi$) that $\xi \leq |x|$ or since $|y| = \xi$, that $|y| \leq |x|$. Again by the "lower semicontinuity" property, this amounts to showing that $y \in p[T^{|x|}]$. But this is exactly what the moves z_0, h_0, z_1, h_1, ... guarantee. Since the appropriate moves are in the trees S_0, S_1, S_2, ... it follows that z_0, z_1, z_2, ... are in $\wp$ and code a sequence of ordinals $\xi_i = \varphi_i(z_i)$ such that $(y,(\xi_0,\xi_1,\xi_2,\ldots))$ is a path through $T^{|x|}$, i.e., $y \in p[T^{|x|}]$ and we are done.

$\Rightarrow$: Assume $\xi \in X$. We will show that II has a winning strategy in G_ξ. Since G_ξ is determined it is enough to prove that I does not have one. So assume, towards a contradiction, that he does have a winning strategy, say τ. We will find a run of the game in which II plays against τ and wins. Recall that for II to win, he needs only to make sure that he has not lost at any finite time, that is as long as I has not yet failed to play into T^ξ, II must play into all the required trees. We now describe a run of the game in which II does this. In this run, I follows τ.

Let x be a fixed real in $\wp$ such that $|x| = \xi$. Find then an f such that (x,f) is a path through T^ξ (say $f = (\varphi_0(x),\varphi_1(x),\ldots)$). Since $\xi \in X$, find g such that (x,g) is a path through S. In this run of G_ξ, II plays these fixed x, f, g (independently of what I does). So II does not lose the game on account of these x, f, g.

Now we describe how II plays z_0, h_0. Say I plays $y(0) = a_0$, $F(0) = \xi_0$. If $(a_0,\xi_0) \notin T^\xi$, I already lost and we are done. So assume $(a_0,\xi_0) \in T^\xi$. Then choose a code (with respect to φ_0) for ξ_0, i.e., choose $z_0 \in \wp$ with $\varphi_0(z_0) = \xi_0$.

Since $|x| = \xi$, $(a_0,\xi_0) \in T^{|x|}$. By definition of Q_0^*, $(x,a_0,z_0) \in Q_0^*$ and $Q_0^* = p[S_0]$, so find h_0 with (x,a_0,z_0,h_0) a path through S_0. II plays these z_0, h_0. (The z_0,h_0 played by II depend both on I's first move a_0, ξ_0 and on the choice of a code for ξ_0. They do not depend on subsequent moves of I.) Note that II will play into the tree S_0 as required and so will never loose on account of z_0, h_0.

Next we explain how II plays z_1, h_1. Recall that he does not have to begin playing z_1 and h_1 until after I has played $y(1)$, $F(1)$. Say I plays $y(1) = a_1$ and $F(1) = \xi_1$. Again if $((a_0,a_1), (\xi_0,\xi_1)) \notin T^{\xi}$, I has already lost, so we can assume that $((a_0,a_1), (\xi_0,\xi_1)) \in T^{\xi}$. II then chooses a code for ξ_1 with respect to φ_1 (i.e., a $z_1 \in \wp$ with $\varphi_1(z_1) = \xi_1$). As before this means that $(x,a_0,a_1,z_0,z_1) \in Q_1^*$, so pick an h_1 with (x,a_0,a_1,z_0,z_1,h_1) a branch through S_1. II plays these z_1, h_1 (which are independent of subsequent moves of I).

Player II continues playing in this manner. After I plays $y(n) = a_n$ and $F(n) = \xi_n$, we choose a code z_n for ξ_n. (We are using DC to choose codes.) Assuming I has not already lost the game, there is an h_n such that $(x,a_0,\ldots,a_n,z_0,\ldots,z_n,h_n)$ is a branch through S_n, and II plays this z_n, h_n. This way II beats I's strategy τ and our proof is complete.

§2. The models $L[T_{2n+1}]$ and the Third Victoria Delfino Problem. We will first apply the Main Theorem to pointclasses Γ which resemble Π_1^1. Recall from [12] that a pointclass Γ *resembles* Π_1^1 if

(i) Γ is a Spector pointclass with the scale property and closed under $\forall^{\mathbb{R}}$,

(ii) For each $z \in \mathbb{R}$, if $P \subseteq \mathbb{R} \times \mathbb{R}$ is in $\Delta(z)$ and $Q(x) \Leftrightarrow \{y : P(x,y)\}$ is not meager, then Q is also in $\Delta(z)$.

The pointclasses resembling Π_1^1 include:

(i) Π_{2n+1}^1 for all n, assuming PD;

(ii) The class of sets semirecursive in ${}^3E$, the class of inductive sets and the pointclass $(\Sigma_1^2)^{L(\mathbb{R})}$, assuming $L(\mathbb{R}) \models AD$. If Γ resembles Π_1^1, then so does $\Gamma(x)$ for every real x.

If Γ is a pointclass resembling Π_1^1 and $\varphi : \wp \xrightarrow{\text{onto}} \kappa$ is a regular Γ-norm on a complete Γ set then $\kappa = \underset{\sim}{\delta} = \sup\{\xi : \xi$ is the length of a $\underset{\sim}{\Delta}$ prewellordering of $\mathbb{R}\}$. Moreover if for each $X \subseteq \underset{\sim}{\delta}$ we say that X is in $\exists^{\mathbb{R}}\Gamma$ provided that $X^* = \{w \in \wp : \varphi(w) \in X\}$ is in $\exists^{\mathbb{R}}\Gamma$, then it follows from Harrington-Kechris [5] that this notion is intrinsic, i.e., independent of the choice of φ, assuming certain games of complexity somewhat higher than those in $\underset{\sim}{\Gamma}$ are determined. Let us call this class of games $\underset{\sim}{\Gamma}^*$. For a precise description of $\underset{\sim}{\Gamma}^*$ see [5]. For example if $\Gamma = \Pi^1_{2n+1}$ then we can certainly take $\underset{\sim}{\Gamma}^*$ = Projective, while if $\underset{\sim}{\Gamma} \subseteq L(\mathbb{R})$ we can certainly take $\underset{\sim}{\Gamma}^* \subseteq L(\mathbb{R})$. We have as a consequence of our main theorem:

<u>Theorem</u>. Let Γ be a pointclass resembling Π^1_1. Assume that all games in $\underset{\sim}{\Gamma}^*$ are determined. Then if ρ is a complete Γ set and $\bar{\varphi}$ a regular Γ-scale on ρ, then every $\exists^{\mathbb{R}}\Gamma$ subset of $\underset{\sim}{\delta}$ belongs to $L[T_\Gamma(\rho,\bar{\varphi})]$, where $T_\Gamma(\rho,\bar{\varphi})$ is the tree associated with the scale $\bar{\varphi}$ on ρ.

It has been calculated in Harrington-Kechris [5] that if Γ resembles Π^1_1 and $\bar{\psi}$ is any Γ-scale on a complete Γ-set $\mathcal{Q}$ then $T_\Gamma(\mathcal{Q},\bar{\psi})$ is $\exists^{\mathbb{R}}\Gamma$ (viewed as a subset of $\underset{\sim}{\delta}$ after some simple coding of tuples of ordinals $<\underset{\sim}{\delta}$ by ordinals $<\underset{\sim}{\delta}$), again assuming the determinacy of games in $\underset{\sim}{\Gamma}^*$. As a corollary we have:

<u>Theorem</u>. Let Γ be a pointclass resembling Π^1_1. Assume that all games in $\underset{\sim}{\Gamma}^*$ are determined. Let ρ be a complete Γ set of reals and $\bar{\varphi}$ a regular Γ-scale on ρ. Then the model $L[T_\Gamma(\rho,\bar{\varphi})]$ is independent of the choice of $\rho, \bar{\varphi}$, and will be denoted as $L[T_\Gamma]$.

Moschovakis [12], p. 262 defined a model H_Γ for each pointclass Γ that resembles Π^1_1, again assuming the determinacy of the games in $\underset{\sim}{\Gamma}^*$. By definition H_Γ is the smallest inner model of ZF containing all $\exists^{\mathbb{R}}\Gamma$ subsets of $\underset{\sim}{\delta}$. Assuming ZF + AD + V = $L(\mathbb{R})$, if $\Gamma = \Sigma^2_1$ then H_Γ is a good approximation of HOD (= class of heriditarily ordinal definable sets), in the sense that H_Γ and HOD have the same sets of rank $<\underset{\sim}{\delta}^2_1$. So general structure theorems for these models H_Γ imply structure theorems for HOD. From this point of view the models H_Γ, for general Γ resembling Π^1_1, are analogs of (a large initial segment of) HOD for a finer notion of definability, and they fit into a natural hierarchy starting from L and leading to HOD.

As an immediate corollary of the preceding theorem we now have

<u>Theorem</u>. Let Γ be a pointclass resembling Π^1_1. Assume that all games in $\underset{\sim}{\Gamma}^*$ are determined. Then

$$L[T_\Gamma] = H_\Gamma.$$

Fix a Γ that resembles Π^1_1, and consider the pointclasses $\Gamma(x)$ for reals x. The relativized versions of the above results imply that every $\exists^{\mathbb{R}}\Gamma(x)$ subset of $\underset{\sim}{\delta}$ is in $L[T_\Gamma,x]$, hence $L[T_\Gamma,x] = H_{\Gamma(x)}$, where $H_{\Gamma(x)}$ is the H-model associated with the pointclass $\Gamma(x)$. It follows from this, and in fact already from Martin's results on homogeneous trees mentioned in the introduction, that H_Γ relativizes by adjoining a real. That is, $H_{\Gamma(x)}$ is the smallest model M such that $H_\Gamma \subseteq M$ and $x \in M$. The relativized results imply also boldface results. By the Moschovakis Coding Lemma (see [12], p. 426), assuming AD, every subset of $\underset{\sim}{\delta}$ is $\exists^{\mathbb{R}}\Gamma(x)$ for some x. Hence every subset of $\underset{\sim}{\delta}$ is in $L[T_\Gamma, x]$, for some real x.

We finally collect together for the record the corollaries of these theorems for the pointclasses $\Gamma = \Pi^1_{2n+1}$. In this case one lets

$$H_{\Pi^1_{2n+1}} = H_{2n+1} \text{ and } T_{\Pi^1_{2n+1}}(\rho,\bar{\varphi}) = T_{2n+1}(\rho,\bar{\varphi}).$$

<u>Theorem</u>. Assume Projective Determinacy when $n \geq 1$. If ρ is a complete Π^1_{2n+1} set of reals and $\bar{\varphi}$ a regular Π^1_{2n+1}-scale on ρ then the model

$$L[T_{2n+1}] = L[T_{2n+1}(\rho,\bar{\varphi})]$$

is independent of the choice of $\rho, \bar{\varphi}$. Moreover every Σ^1_{2n+2} subset of $\underset{\sim}{\delta}^1_{2n+1}$ is in $L[T_{2n+1}]$, therefore

$$L[T_{2n+1}] = H_{2n+1}.$$

The case $2n+1 = 3$ provides the solution of the 3rd Victoria Delfino Problem.

Of course the relativized and boldface results also hold for the case $\Gamma = \Pi^1_{2n+1}$. Thus we have shown that, under AD, every subset of $\underset{\sim}{\delta}^1_{2n+1}$ is constructible from T_{2n+1} and a real (cf. [6]). This is a higher-level analog of Solovay's Theorem that every subset of $\aleph_1 (= \underset{\sim}{\delta}^1_1)$ is constructible from a real (see [16]).

§3. <u>Invariance at the even levels of the projective hierarchy</u>. We investigate now the question of the invariance of the universe constructible from the tree of a Δ^1_{2n+1}-scale on a complete Π^1_{2n} set. We will assume PD throughout this section.

Let $n \geq 1$ and let ρ be a complete Π^1_{2n} set and $\bar{\varphi} = \{\varphi_m\}$ a regular Δ^1_{2n+1}-scale on ρ. Let κ_m be such that $\varphi_m : \rho \xrightarrow{\text{onto}} \kappa_m$. In general these κ_m are not all the same. Let $\kappa = \sup_m \kappa_m$. Then $\kappa < \underset{\sim}{\delta}^1_{2n+1}$ (if one assumes AD, then if λ_{2n+1} is the cardinal of cofinality ω such that $(\lambda_{2n+1})^+ = \underset{\sim}{\delta}^1_{2n+1}$, then $\lambda_{2n+1} \leq \kappa < \underset{\sim}{\delta}^1_{2n+1}$ (see [12], Section 7D).) The scale $\{\varphi_m\}$ gives the following system of coding for ordinals $<\kappa$: The set of codes is (letting $\langle i, x\rangle = (i, x(0), x(1), \ldots)$)

$$\rho^* = \{\langle i,x\rangle : i \in \omega \wedge x \in \rho\}$$

and for $\langle i,x\rangle \in \rho^*$ we let

$$\varphi^*(\langle i,x\rangle) = \varphi_i(x).$$

Then our main theorem easily implies that if $X \subseteq \kappa$ and X is Σ^1_{2n+1} in the codes provided by (ρ^*, φ^*) then X is constructible from $L[T(\rho,\bar{\varphi})]$. To see this apply the theorem to $\Gamma = \Pi^1_{2n}$, taking the complete set to be ρ^* and the scale to be given by $\psi_0(\langle i,x\rangle) = \varphi_i(x)$, $\psi_{n+1}(\langle i,x\rangle) = \varphi_n(x)$. Then $X \in L[T(\rho^*,\bar{\psi})]$, But note that $T(\rho^*,\bar{\psi}) \in L[T(\rho,\bar{\varphi})]$, since

$$((a_0,\ldots,a_n),(\xi_0,\ldots,\xi_n)) \in T(\rho^*,\bar{\psi}) \Leftrightarrow \exists((b_0,\ldots,b_\ell),(\eta_0,\ldots,\eta_\ell)) \in$$
$$T(\rho,\bar{\varphi})[\ell \geq a_0 \wedge \ell \geq n+1 \wedge a_1 = b_0 \wedge a_2 = b_1 \wedge \cdots \wedge a_n = b_{n-1} \wedge \xi_0 = \eta_{a_0} \wedge$$
$$\forall_j(1 \leq j \leq n \Rightarrow \xi_j = \eta_{j-1})].$$

From this we can obtain a boldface invariance theorem as follows:

Theorem. Assume Projective Determinacy. Let ρ be a complete Π^1_{2n} set and $\bar{\varphi}$ a regular Δ^1_{2n+1}-scale on ρ. Let

$$\underset{\sim}{L}[T(\rho,\bar{\varphi})] = \cup_{x\in\mathbb{R}} L[T(\rho,\bar{\varphi}),x];$$

then $\underset{\sim}{L}[T(\rho,\bar{\varphi})]$ is independent of the choice of ρ, $\bar{\varphi}$ (i.e., $T(\rho,\bar{\varphi})$ is constructible from any other $T(\rho',\bar{\varphi}')$ and a real).

Proof. We will assume AD for convenience--the proof can be carried through in PD only using for example the techniques in [5], §8.

Let ρ, $\bar{\varphi}$ be as in the statement of the theorem and let ρ', $\bar{\varphi}'$ be any other similar pair. If κ, κ' are the associated ordinals as defined in the beginning of this section, then (by AD) κ, κ' have the same cardinality (namely λ_{2n+1}). So we can code $T(\rho',\bar{\varphi}')$ as a subset of κ, say X. By the Moschovakis Coding Lemma (see [12], p. 426) X is $\underset{\sim}{\Sigma}^1_{2n+1}$ in the codes provided by ρ^*, $\bar{\varphi}^*$ so by (relativizing) the remarks preceding the theorem $X \in L[T(\rho,\bar{\varphi}),x]$, for some real x. Thus $T(\rho',\bar{\varphi}') \in L[T(\rho,\bar{\varphi}),x]$ and we are done.

At this stage we do not know if $L[T(\rho,\bar{\varphi})]$ itself is independent of the choice of ρ, $\bar{\varphi}$. We only know this in a special case when the scales $\bar{\varphi}$ are nice in the following sense.

Definition. Let λ_{2n+1} be the ordinal $< \underset{\sim}{\delta}^1_{2n+1}$ which is a projective cardinal and $\underset{\sim}{\delta}^1_{2n+1}$ is the least projective cardinal bigger than λ_{2n+1}. (For the definition of projective cardinals see [5], §8.) Thus $\lambda_3 = u_\omega = \omega$th uniform indiscernible, while if AD holds then λ_{2n+1} is the cardinal of cofinality

ω whose successor cardinal is $\underset{\sim}{\delta}^1_{2n+1}$.

Let P be a complete Π^1_{2n} set and $\bar{\varphi} = \{\varphi_m\}$ a regular Δ^1_{2n+1}-scale on P. Say $\varphi_m : P \xrightarrow{\text{onto}} \kappa_m$ and let $\kappa = \sup_m \kappa_m$. We say that $\bar{\varphi}$ is *nice* if $\kappa = \lambda_{2n+1}$ (it is always true that $\kappa \geq \lambda_{2n+1}$), and the norms $\{\varphi_m\}$ satisfy the following bounded quantification property:

If $Q(v,x)$ is Σ^1_{2n+1} then the following relation is also Σ^1_{2n+1},

$$R(m,w,x) \Leftrightarrow w \in P \wedge \forall v \in P(\varphi_m(v) \leq \varphi_m(w) \Rightarrow Q(v,x)).$$

(This essentially means that the pointclass Σ^1_{2n+1} is closed under quantification of the form $(\forall \xi \leq \eta)$, where $\eta < \lambda_{2n+1}$ and ordinals are coded via one of the norms φ_i.)

It has been shown in Kechris-Martin [7] that such scales exist for $n = 1$. We do not know if this generalizes to all $n \geq 2$, although recent work of S. Jackson on computing the $\underset{\sim}{\delta}^1_m$'s makes such a generalization quite likely.

Notice here that if we call a Π^1_{2n+1}-scale $\bar{\varphi}$ on a complete Π^1_{2n+1} set P *nice* if it satisfies the corresponding bounded quantification property for Σ^1_{2n+2}, then by the results of Harrington-Kechris (see [5], §3) *every* such scale is nice (also each $\varphi_m : P \xrightarrow{\text{onto}} \kappa_m$ where $\kappa_m = \underset{\sim}{\delta}^1_{2n+1}$). So the theorem below is a reasonable generalization of the invariance theorem to even levels.

<u>Theorem</u>. Assume Projective Determinacy. Let P be a complete Π^1_{2n} set and $\bar{\varphi}$ a nice Δ^1_{2n+1}-scale on P. Then the model $L[T(P,\bar{\varphi})]$ is independent of the choice of P, $\bar{\varphi}$.

<u>Proof</u>. Note first that the relation

$$S(i,j,x,y) \Leftrightarrow x, y \in P \wedge \varphi_i(x) \leq \varphi_j(y)$$

is Σ^1_{2n+1}. This can be easily proved noticing that for $x, y \in P$

$$\varphi_i(x) \leq \varphi_j(y) \Leftrightarrow \forall x' \in P[\varphi_i(x') < \varphi_i(x) \Rightarrow \exists y' \in P(\varphi_j(y') < \varphi_j(y) \wedge \varphi_i(x') \leq \varphi_j(y'))],$$

and using the recursion theorem and the niceness of $\bar{\varphi}$. From this it follows easily that if $\varphi^*(i,x) = \varphi_i(x)$ then the tree $T(P,\bar{\varphi})$ is Σ^1_{2n+1} in the codes provided by φ^* (after identifying via some simple coding of tuples $T(P,\bar{\varphi})$ with a subset of λ_{2n+1}). Now let P', $\bar{\varphi}'$ be any other such pair. Then it is enough to show that $T(P,\bar{\varphi})$ is Σ^1_{2n+1} in the codes provided by $(\varphi')^*$. But this is immediate noticing

$$x \in P \wedge y \in P' \wedge \varphi_i(x) \leq \varphi'_j(y) \Leftrightarrow S'(i,j,x,y)$$

is also Σ^1_{2n+1}, and our proof is complete.

§4. The Kleene Theorem for Π^1_3. Assume that $\forall x \in \omega^\omega(x^\#$ exists) and let u_ξ be the ξth uniform indiscernible. A canonical coding system for ordinals $<u_\omega$ can be defined by letting

$$WO_\omega = \{w \in \mathbb{R} : w = \langle n,x^\#\rangle, \text{ for some } n \in \omega, x \in \mathbb{R}\},$$

$$|w| = \tau_n^{L[x]}(u_1,\ldots,u_{k_n}), \text{ for } w = \langle n,x^\#\rangle,$$

where $\tau_0, \tau_1, \ldots$ is a recursive enumeration of all terms in the language of $ZF + V = L[\dot{x}]$, $\dot{x}$ a constant for a real, taking always ordinal values. Call a relation

$$P(\xi,x),$$

where ξ varies over u_ω and x over $\mathbb{R}$, Π^1_3 if

$$P^*(w,x) \Leftrightarrow w \in WO_\omega \wedge P(|w|,x)$$

is Π^1_3.

Let $j_m : u_\omega \to u_\omega$, for $m \geq 1$, be defined by letting

$$j_m(u_\ell) = \begin{cases} u_\ell, & \text{if } \ell < m \\ u_{\ell+1}, & \text{if } \ell \geq m \end{cases},$$

and then

$$j_m(\tau_n^{L[\alpha]}(u_1 \cdots u_{k_n})) = \tau_n^{L[\alpha]}(j_m(u_1) \cdots j_m(u_{k_n})).$$

Let R be the relation on u_ω coding these embeddings, i.e.,

$$R = \{(m,\xi,\eta) : m \geq 1 \wedge \xi,\eta < u_\omega \wedge j_m(\xi) = \eta\}.$$

Put

$$\hat{u}_\omega = \langle u_\omega,<,R\rangle.$$

For the definition of the concepts of inductive definability that we need below, see Moschovakis [13]. We have now:

Theorem. Assume $\underset{\sim}{\Delta}^1_2$-Determinacy. Then for each relation $P(\xi,x)$, on $u_\omega \times \mathbb{R}$, the following are equivalent:

i) $P \in \Pi^1_3$;

ii) P is (absolutely) inductive on $\hat{u}_\omega$.

(In ii) P is of course viewed as a second order relation on u_ω.)

This generalizes the classical result of Kleene (see [12], 7C.2), which similarly identifies the Π^1_1 relations $P(n,x)$, $n \in \omega$, $x \in \mathbb{R}$ with those which are inductive on the structure $\hat{\omega} = \langle\omega,<\rangle$. Kechris-Martin [7] proved the result on Π^1_3 for relations $P(x)$ with no ordinal variables and the direction ii) $\Rightarrow$ i) in general. The question whether i) $\Rightarrow$ ii) was also raised in that paper.

Proof of the direction i) $\Rightarrow$ ii). We start with the following result of Martin-Solovay [11]:

Assume $\forall x \in \mathbb{R}(x^{\#}$ exists). Then there is a tree T_2' on $\omega \times u_\omega$ such that

i) $p[T_2'] = \{x^{\#} : x \in \mathbb{R}\}$.

ii) For each $y = x^{\#}$, if $T_2'(y) = \{u \in u_\omega^{<\omega} : (y\restriction \ell h(u), u) \in T_2'\}$, then $T_2'(y)$ has an honest leftmost branch f_y (i.e., $f_y \in [T_2'(y)]$ and if $f \in [T_2'(y)]$ then $f_y \leq f$ pointwise) and if we let $\psi_n(y) = f_y(n)$, then $\psi_n(y) = \tau_n^{L[x]}(u_1 \cdots u_{k_n})$.

iii) T_2' is (absolutely) hyperelementary on $\hat{u}_\omega$.

Briefly the definition of T_2' is as follows:

Let T_0 be a recursive tree on ω such that

$y \in [T_0] \Leftrightarrow y$ satisfies all the syntactical conditions for being the sharp of a real.

Put then

$((\ell_0,\ldots,\ell_n),(\xi_0,\ldots,\xi_n)) \in T_2' \Leftrightarrow (\ell_0,\ldots,\ell_n) \in T_0 \wedge$
$\exists f_0, \ldots, f_n\{\forall i \leq n(f_i : [\aleph_1]^{k_i} \to \aleph_1$ *and* f_i is constructible from a real *and* the ordinal of f_i in the ultrapower of all such functions by the k_i-fold cartesian product of the closed unbounded measure on $\aleph_1$ is equal to $\xi_i)\} \wedge \{\forall i,j \leq n \exists C$(C is a closed unbounded subset of $\aleph_1$ *and* C is constructible from a real *and* for any $\xi_1 < \cdots < \xi_{k_i}$, $\eta_1 < \cdots < \eta_{k_j}$ in C, if $\ulcorner\tau_i(\vec{v}) \lesseqgtr \tau_j(\vec{v}\,')\urcorner \leq n$, where $\vec{v}$, $\vec{v}\,'$ are sequences of variables interwoven the same way as $\vec{\xi},\vec{\eta}$ and $\ulcorner\varphi\urcorner$ denotes the Gödel number of φ, then

$$f_i(\vec{\xi}) \lesseqgtr f_j(\vec{\eta}) \Leftrightarrow \ell_{\ulcorner\tau_i(\vec{v})\lesseqgtr\tau_j(\vec{v}\,')\urcorner} = 0)\}.$$

From this it follows easily as in our introductory remarks in §3, that there is a tree T_2 on $\omega \times u_\omega$ such that

i) $p[T_2] = WO_\omega$.

ii) If $w \in WO_\omega$, $T_2(w)$ has an honest leftmost branch, say g_w and if $\varphi_m(w) = g_w(m)$, then for $w = \langle n,x^{\#}\rangle$ we have $|w| = \varphi_0(w) = \tau_n^{L[x]}(u_1,\ldots,u_{k_n})$ (and $\varphi_{m+1}(w) = \tau_m^{L[x]}(u_1,\ldots,u_{k_m})$).

iii) T_2 is (absolutely) hyperelementary on $\hat{u}_\omega$.

It follows from this fact and the observation that every Π^1_2 set A can be written as

$$x \in A \Leftrightarrow x^{\#}(n_A) = 0,$$

for some $n_A \in \omega$, that every Σ^1_3 set B can be written as $p[T_B]$, where T_B is a tree on $\omega \times u_\omega$ (absolutely) hyperelementary on $\hat{u}_\omega$, uniformly in (a code of) B.

Fix first $X \subseteq u_\omega$ which is Σ^1_3. We can repeat then the proof of the main theorem using the tree T_2 in the place of the tree of a scale T that was used there. Since the tree T_2 is (absolutely) hyperelementary on $\hat{u}_\omega$ we have by the direction ii) $\Rightarrow$ i) of the present theorem that T_2 is Δ^1_3, a fact which can be also verified easily directly. Thus if we define Q_n as in the proof of the main theorem and then Q^*_n by

$$(x,a_0 \cdots a_n,z_0 \cdots z_n) \in Q^*_n \Leftrightarrow x \in WO_\omega \wedge \forall i \leq n(z_i \in WO_\omega) \wedge$$
$$((a_0,\ldots,a_n),(|z_0| \cdots |z_n|)) \in T_2{}^{|x|},$$

then again Q^*_n is Δ^1_3. So the rest of the proof of the main theorem carries over mutatis mutandis in the present case, and proves that for $\xi < u_\omega$:

$$\xi \in X \Leftrightarrow \text{II has a winning strategy in } G_\xi,$$

where G_ξ is an open game determined by a set of finite sequences on u_ω, say $\mathcal{S}_\xi$, so that the mapping $\xi \mapsto \mathcal{S}_\xi$ is (absolutely) hyperelementary on $\hat{u}_\omega$. By a standard result of Moschovakis [13], p. 70 this implies that X is coinductive on $\hat{u}_\omega$.

This proof easily relativizes to any real $x \in \mathbb{R}$ and thus shows that given a Σ^1_3 relation P on $u_\omega \times \mathbb{R}$, we can assign to each (ξ,x) an open game $G_{\xi,x}$ so that the map $(\xi,x) \mapsto G_{\xi,x}$ is hyperelementary on $\hat{u}_\omega$ and such that $P(\xi,x) \Leftrightarrow$ II has a winning strategy in $G_{\xi,x}$. This shows that P is coinductive on $\hat{u}_\omega$ and we are done.

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