Heat transport in a stratified twophase fluid

E. E. Salpeter and D. J. Stevenson

Citation: Phys. Fluids **19**, 502 (1976); doi: 10.1063/1.861506
View online: http://dx.doi.org/10.1063/1.861506
View Table of Contents: http://pof.aip.org/resource/1/PFLDAS/v19/i4
Published by the American Institute of Physics.

Heat transport in a stratified two-phase fluid

E. E. Salpeter and D. J. Stevenson

Physics Department and Center for Radiophysics and Space Research, Cornell University, Ithaca, New York 14853
and California Institute of Technology, Pasadena, California
(Received 9 July 1975; final manuscript received 13 November 1975)

A self-gravitating fluid, stratified into two phases of appreciably different densities and heated from within is considered. The heat flux, viscosity, and thermal diffusivity are assumed to be small enough so that, away from the interface between the phases, the flux is mainly carried by turbulent convection with a very small superadiabaticity. Different modes are investigated for transporting the heat flux across the interface, and both possible signs of the latent heat L are considered. A thermal boundary layer, distortion of the interface, and the nucleation, growth and motion of droplets and bubbles are included. It is shown that, under a specified range of conditions, the transition region near the interface is thin with a small change in the temperature T across it. The entropy difference between the two phases is then L/T. These considerations probably apply to the interior of Jupiter.

I. INTRODUCTION

In a planet with a hot, fluid interior, thermal conduction is usually small and the dominant heat transport mechanism is turbulent convection. For Jupiter, in particular, the net outward heat flux F is determined by conditions near the surface. In the interior, at least, F is very small compared with the energy that could be transported at thermal speeds, and we shall define a dimensionless parameter ϵ_0 in Eq. (6) to characterize this inequality. Simple mixing length theory then identifies ϵ_0 with the average fractional superadiabaticity of the fluid. With $\epsilon_0 \ll 1$, the temperature gradient is almost adiabatic and the constancy of entropy is used[1,2] to infer temperatures in the interior from the observed surface temperature. The question arises whether this is a valid procedure if the fluid undergoes a first-order phase transition at some depth, with an appreciable density discontinuity and a latent heat L comparable to the thermal energy. If the entropy were almost constant across the interface between the phases, then a temperature drop ΔT comparable to the temperature T itself would occur in an intermediate "mixed" region (see Fig. 1). If, instead, the interface were relatively thin with a small temperature drop $\Delta T \ll T$, then the entropy per particle would change by about $L/T \approx k_B$ (Boltzmann's constant) across the interface. The presence of such an "almost isothermal" interface could lead to a central temperature for the planet that is lower (or higher, if the latent heat is negative) by as much as a factor of two from that given by an isentropic calculation.

In the present paper, we consider the general structure of the interface between two phases when a specified heat flux F (carried by convection far from the interface) must be carried across. The planetary applications are not discussed in detail, and we consider only a one-dimensional approximation (constant vertical gravity g). The most important parameters are the sign and value of the latent heat L, the kinematic viscosity ν, the thermal diffusivity κ, the surface energy between the phases D, and ϵ_0. For $L > 0$ (as in Fig. 1), Busse and Schubert[3] have considered a model where the total temperature drop ΔT is large enough so the entropy is about the same in the two phases. They show that convective eddies can then cross the interface without inhibition, but in their analysis the temperature gradient is substantially superadiabatic and the Prandtl number $\mathrm{Pr} \equiv \nu/\kappa$ can be very large. Their conclusions have been appropriately applied to solid-state convection in the Earth's mantle.[4,5] It is the main aim of the present paper to show that $|\Delta T|/T$ is, in fact, extremely small when ϵ_0 is extremely small compared with unity, as long as κ is not too small and Pr is not too large. Our conclusion is in agreement with Busse and Schubert[3] in the limit of slow laminar flow. However, in this paper we confine our attention to the turbulent case, which is most

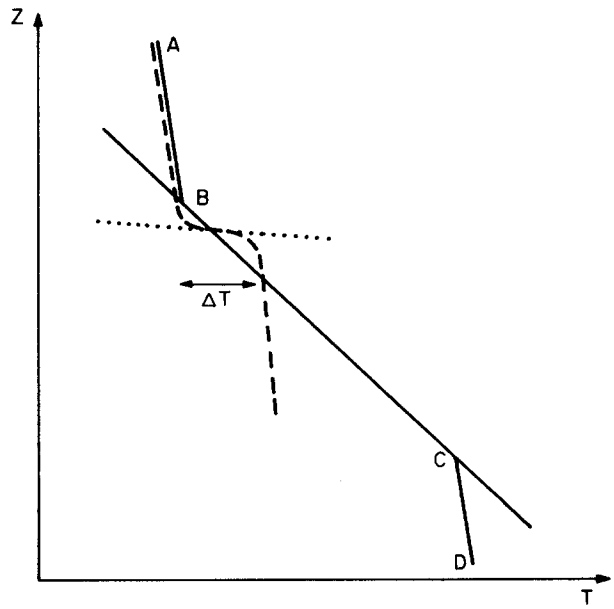

FIG. 1. Temperature versus vertical coordinate z (or pressure P) for positive latent heat and no nucleation. BC is part of the phase boundary and ABCD is an adiabat. In a fully adiabatic case, a thick "two-phase" region between B and C exists. For the conditions relevant to this paper, the actual temperature profile (- - -) is almost adiabatic except for a thin region near the interface. The temperature profile for pure conduction (. . . .) is also shown.

appropriate to astrophysical and geophysical liquid-state applications. It is important for all considerations in this paper that the phase transition be first order, with a finite surface energy D between the phases. If, instead, there were a continuous change in density and atomic structure, then no superheating or supercooling would be possible, and an element of a convective cell would change its density continuously as the pressure changes. In such a case (but still with $\epsilon_0 \ll 1$), the entropy per atom remains essentially constant as a function of height.

The assumed restrictions on the parameters are specified in Sec. II. Cases with positive latent heat are discussed in Sec. III, assuming the absence of nucleation. If dynamic effects are neglected, heat is carried through a thin boundary layer near a sharply defined, rigid interface by thermal conduction. We also consider distortion of the interface due to gravity waves excited by convection. An additional mechanism is discussed in Sec. IV: Even for a relatively small temperature drop ΔT, the heavier fluid phase in the boundary layer may be superheated enough so that bubbles of the lighter fluid phase form and cool to the equilibrium temperature. Heat flows into the bubble by thermal diffusion (efficiently, because the bubble is small compared with the boundary layer thickness) and is stored as latent heat while more molecules join the growing bubble. Bouyancy then drives the bubble with its stored energy upward and the cycle is completed by droplets, formed in the lighter fluid above, sinking and carrying negative energy content downward. Breakup and coalescence of bubbles are also considered.

Cases with negative latent heat are discussed in Sec. V. If dynamic effects are neglected, then the thermal conduction considerations of Sec. III still apply. Gravity waves at the interface supply another mechanism: Some of the heavier fluid in a wave crest changes phase, releasing hot, buoyant fluid of the lighter phase. The actual temperature drop ΔT is then less than for thermal conduction alone. It may even be opposite in sign (but still with $|\Delta T|/T \ll 1$).

In Sec. VI, our conclusions are discussed and a brief description of their probable application to Jupiter is given. The acutal modes of heat transport may be more complicated than the ones treated in this paper. Nevertheless, we hope that the main conclusion, an "almost isothermal interface", still holds.

II. DEFINITIONS AND PARAMETERS

Let v_1 and $v_2 = v_1 + \Delta v$ be the volume per particle for the two phases which coexist at temperature T and phase-boundary pressure $P_b(T)$. According to the Clausius–Clapeyron equation

$$\left(\frac{d\ln P}{d\ln T}\right)_b = \frac{L}{P\Delta v}, \tag{1}$$

where L is the latent heat per particle (defined as positive when the denser phase 1 has the smaller specific entropy at the phase boundary). We consider only a one-dimensional approximation in which the gravitational acceleration g is independent of z, the height above the average location of the phase-boundary. We assume hydrostatic equilibrium holds, to lowest order, so that the pressure scale height H_p (defined separately for each phase) is given by

$$H_p \equiv \left(\frac{d\ln P}{dz}\right)^{-1} = \frac{v_0^2}{g}$$
$$v_0 \equiv (P/\rho)^{1/2}, \tag{2}$$

where ρ is the density and v_0 is a characteristic thermal speed. We define two further thermodynamic derivatives,

$$\alpha \equiv -\left(\frac{d\ln\rho}{d\ln T}\right)_p ; \gamma \equiv \left(\frac{d\ln P}{d\ln T}\right)_s , \tag{3}$$

evaluated at constant pressure and entropy, respectively. The adiabatic temperature gradient is then

$$\left(\frac{d\ln T}{dz}\right)_s = \frac{-1}{\gamma H_p} . \tag{4}$$

For condensed matter (in contrast to a gas) α can be significantly less than unity. For example, $\alpha \approx 0.05$ in the deep interior of Jupiter. However, γ is usually slightly larger than unity. The latent heat L can have either sign and $|L|$ is assumed to be of order $k_B T$. We assume a considerable density change at the transition, $\Delta v/v_1 = \Delta\rho/\rho_2 \approx 1$. We are mainly interested in "relatively cold" bodies where thermal contributions to the internal energy are small compared with the zero temperature contribution. We thus assume

$$|L|/P\Delta v < \gamma . \tag{5}$$

(In Fig. 1, this is represented by AB having a steeper slope than BC.)

We assume a constant and given vertical heat flux F which satisfies a very strong inequality

$$\epsilon_0 \equiv (\gamma^{3/2} F/\alpha^{1/2}\rho C_p T v_0)^{2/3} \ll 1 , \tag{6}$$

where C_p is the specific heat. We define a superadiabaticity factor $\epsilon(z)$ by

$$1 + \epsilon(z) \equiv \left(\frac{dT}{dz}\right)\left(\frac{dT}{dz}\right)_s^{-1} , \tag{7}$$

where (dT/dz) is the actual temperature gradient. We assume that, far from the phase boundary ($|z| \gtrsim H_p$), most of the heat flux F is carried by efficient convection so that simple mixing length theory[6] would require $\epsilon(z)$ to be of order ϵ_0. Estimates for $\epsilon(z)$ *near* the phase boundary are the main concern of this paper. Let ν be the kinematic viscosity, κ be the thermal diffusivity, Pr be the Prandtl number, $r_{0\nu}$ be a "nominal Reynold's number," and $r_{0\kappa}$ be a corresponding number for thermal diffusion (the Peclet number),

$$\mathrm{Pr} \equiv \frac{\nu}{\kappa} ; \quad r_{0\nu} \equiv \frac{v_0 H_p}{\nu} ; \quad r_{0\kappa} \equiv \frac{v_0 H_p}{\kappa} . \tag{8}$$

If the heat flux were to be carried mainly by heat conduction, the value of $\epsilon(z)$ would be given by ϵ_κ, where

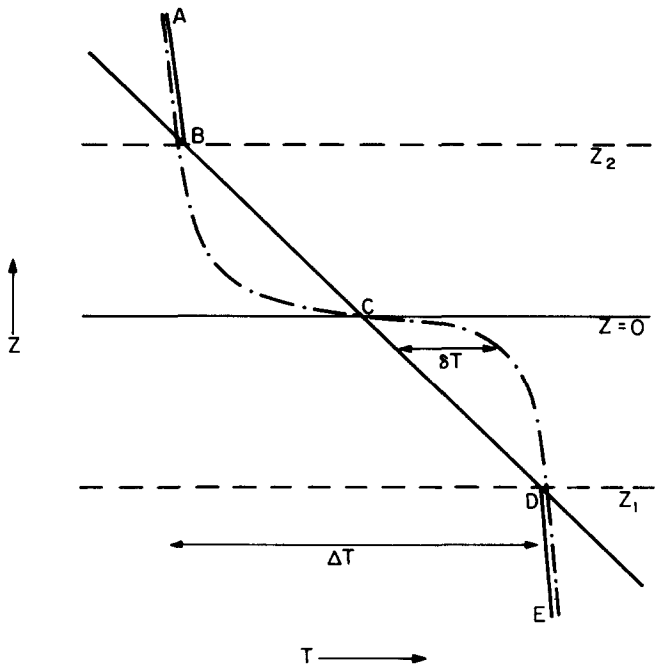

FIG. 2. Temperature profile near the interface for $L > 0$. BCD is part of the phase boundary. AB and DE are adiabats. The actual temperature profile (- · - · - ·) crosses the interface at C. The fluid between B and C is in the less dense phase and supercooled. The fluid between C and D is in the more dense phase and superheated. δT is the maximum superheating.

$$1 + \epsilon_\kappa = \alpha^{1/2}\gamma^{-1/2}\epsilon_0^{3/2}r_{0\kappa} \ . \tag{9}$$

We are interested only in cases where $\epsilon(z)$ is smaller, except possibly near the phase boundary.

There is a lower bound to $r_{0\nu}$ to insure the possibility of efficient convection with large Reynolds number far from the phase boundary. We also restrict ourselves to the more interesting case in which heat conduction can carry the full heat flux F, only for a temperature gradient in excess of that appropriate to the phase boundary (see Fig. 1). In practice, this restriction does not differ greatly from the requirement that convection (rather than conduction) dominates the heat transport far from the phase boundary. This implies a lower limit on $r_{0\kappa}$. Our calculations will also demonstrate that the temperature drop across the phase boundary region is small (i.e., $|\Delta T|/T \ll 1$) provided $r_{0\kappa}$ is not too large. (The origin of this upper bound will become more apparent in the next section.) The following inequalities are necessary and sufficient conditions for our results to apply with $|\Delta T|/T \ll 1$:

$$\frac{P\Delta v}{|L|}\gamma^{3/2}\alpha^{-1/2}\epsilon_0^{-3/2} < r_{0\kappa} \ll \min[r_*, (r_* r_{0\nu})^{1/2}]$$

$$r_{0\nu} \gtrsim 10^3\gamma^{1/2}\alpha^{-1/2}\epsilon_0^{-1/2} \ ,$$

where

$$r_* = 10^{-4}\alpha^{-1/2}(\epsilon_0/\gamma)^{-9/2} \ . \tag{10}$$

III. CASES WITH POSITIVE L WITHOUT NUCLEATION

We assume throughout a first-order phase transition between two distinct phases. In this and the following section we consider a positive latent heat of transition. We first discuss what would happen if the two phases were separated by a sharply defined, rigid horizontal boundary surface without any dynamical effects distorting the surface. We further assume, at the moment, that each phase can be superheated (or supercooled) slightly, without any phase change being initiated by nucleation or any other effect. This is illustrated qualitatively in Fig. 2: The fluid is in the heavy phase everywhere below the boundary at $z = 0$, even though it is superheated for $z_1 < z < 0$. Similarly, the fluid in the light phase above the boundary is supercooled for $0 < z < z_2$.

In this situation, the fluid on either side of the boundary behaves as a simple one-component fluid below (or above) a rigid boundary. Such cases have been discussed theoretically and experimentally[7]; we summarize the main features only to an order of magnitude, using the language of simple mixing length theory: For convection with a mixing length $l \lesssim H_p$ carrying a constant heat flux F, the superadiabaticity factor ϵ in Eq. (7) and typical convection speed V_c are given by

$$\frac{\epsilon}{\epsilon_0} \approx \left(\frac{H_p}{l}\right)^{4/3}, \ \frac{V_c}{v_0} \approx \left(\frac{l}{H_p}\right)^{1/3}\alpha^{1/2}\gamma^{-1/2}\epsilon_0^{1/2} \ , \tag{11}$$

where ϵ_0 is the very small number defined in Eq. (6). At a distance $|z| < H_p$ from the rigid boundary, one assumes a "local value of the mixing length" of $l \approx |z|$. However, this prescription breaks down in a thin boundary layer very close to the surface $z = 0$, where convection cannot be maintained efficiently. One has to onsider separately cases with $\mathrm{Pr} < 1$ and $\mathrm{Pr} > 1$.

When $\mathrm{Pr} > 1$, the flow near the boundary layer is mainly laminar and $\epsilon(z) \propto z^{-2}$ instead of $\epsilon(z) \propto z^{-4/3}$ in Eq. (11). In the boundary layer ϵ reaches the value ϵ_κ given in Eq. (9). The thickness l_c of the boundary layer is then obtained by equating the Rayleigh number R_a to its critical value[7]:

$$R_a \equiv \alpha\epsilon_\kappa l_c^4 v_0^2/\gamma\nu\kappa H_p^2 \approx 10^3 \ . \tag{12}$$

When $\mathrm{Pr} < 1$, heat leakage rather than viscosity inhibits the convection. In this case[7] the coefficient ν is replaced by an effective turbulent viscosity $V_c l_c$, with V_c given by Eq. (11), and Eq. (12) can be rewritten in the form

$$R_\kappa \equiv V_c l_c/\kappa \approx 10^3 \ . \tag{13}$$

Using Eqs. (8), (9), and (11), the boundary layer thickness l_c from either Eq. (12) or (13) can be written as

$$l_c/H_p \approx 10(\alpha\epsilon_0/\gamma)^{-3/8}r_{0\kappa}^{-3/4}(1 + \mathrm{Pr}^{1/4}) \ . \tag{14}$$

Our main interest is the temperature drop ΔT across the boundary layer. Under our present assumption of a rigid boundary layer, ΔT is close to ΔT_r, where

$$\Delta T_r/T = (l_c/\gamma H_p)(1 + \epsilon_\kappa)$$
$$\approx 10\alpha^{1/8}(\epsilon_0/\gamma)^{9/8}r_{0\kappa}^{1/4}(1 + \mathrm{Pr}^{1/4}) \ . \tag{15}$$

Outside the boundary layer, the superadiabaticity ϵ rapidly drops from $\epsilon \gtrsim 1$ to values $\epsilon_0 < \epsilon \ll 1$. The temperature derivative $(d\ln T/d\ln P)$ drops rapidly from a

value $\alpha^{1/2}\gamma^{-3/2}\epsilon_0^{3/2}\gamma_{0\kappa} \gg P\Delta v/L$ to almost the adiabatic value $\gamma^{-1} < P\Delta v/L$. The maximum value δT of the degree of superheating of the more dense phase ($z<0$) is thus close to ΔT_r and is reached near the edge of the boundary layer (see Fig. 2). The total thickness of the superheated region is

$$z_1 \approx l_c(\alpha^{1/2}\gamma^{-1/2}\epsilon_0^{3/2}\gamma_{0\kappa})(L/P\Delta v) > l_c \,. \tag{16}$$

Equation (15) should be evaluated separately for the two phases (since ϵ_0, $\gamma_{0\kappa}$, Pr, etc. change across the boundary) and the sum of the two expressions for ΔT_r gives the effective "total temperature discontinuity" across the phase-boundary. Use of Eq. (10) in Eqs. (14) and (15) shows that $(l_c/H_p) < (|L|/P\Delta v)^{1/2}(\epsilon_0/\gamma)^{1/2} \ll 1$, and that $\Delta T_r < T$ [with $\Delta T_r \approx T$ only when $\gamma_{0\kappa}$ approaches its upper limit in Eq. (10)].

In reality, the boundary surface between the two phases can be distorted by dynamical effects similar to those which have been observed in the laboratory[8] and in the sun.[9] These effects are related to "convective overshoot"[10,11] and can be thought of as gravity waves in the interface surface, excited by the kinetic energy carried by convection. A detailed description of the distortion of the interface is difficult[12] but for our present purpose, a simple dimensional argument suffices. Consider a sinusoidal deviation of the interface surface (about the $z=0$ plane) with horizontal wavelength $\lambda \quad H_p$ and vertical height amplitude $h \lesssim \lambda$. The effective gravity for the waves is $g^* = g\Delta\rho/(\rho_1+\rho_2) \sim g = v_0^2/H_p$. The frequency ω of the wave, the vertical velocity amplitude V_ω and the kinetic energy K per unit area can be written as

$$\omega \approx \frac{v_0}{(\lambda H_p)^{1/2}}, \quad \frac{V_\omega}{v_0} \approx \frac{h}{(\lambda H_p)^{1/2}}, \quad K \approx \frac{\rho v_0^2 h^2}{H_p}. \tag{17}$$

In the "universal regime" (quasi-stationary turbulence) and neglecting molecular viscosity and surface tension, the only relevant frequency is ω.[12] The energy dissipation rate F_d per unit area is then of order ωK. According to the Kolmogoroff picture of turbulence, this can be interpreted as the rate at which waves of wavelength comparable to λ are dissipated to shorter wavelengths by interaction with other waves. The kinetic energy input is no more than F_{ke}, the one-way kinetic energy flux far from the interface. Using $F_{ke} \approx \rho V_c^3 \gtrsim \omega K$ and $V_c \lesssim v_0(\alpha\epsilon_0/\gamma)^{1/2}$, we find

$$h \lesssim (\lambda H_p^3)^{1/4}(\alpha\epsilon_0/\gamma)^{3/4} \,. \tag{18}$$

If the nonlocal effects of "convective overshoot" are strong, then convective cells with size comparable to H_p could penetrate close to the boundary, and wavelengths as large as $\lambda \approx H_p$ could be present at the interface. The surface distortion could then be as large as

$$h_m \approx H_p(\alpha\epsilon_0/\gamma)^{3/4} \,. \tag{19}$$

Convective cells with scale sizes $l \ll H_p$ excite waves with $\lambda \approx l \ll H_p$, but only with amplitude $h \ll h_m$. For the longest wavelengths, on the other hand, $h \ll \lambda$, and the height amplitude mainly represents a moving up and down of the boundary layer with the actual thickness of the boundary layer itself being appreciably smaller. The total temperature drop across the phase-boundary region is then at most the sum of ΔT_r and ΔT_ω, where ΔT_ω is the temperature difference between a wave crest and a wave trough (assuming both lie on the phase boundary):

$$\frac{\Delta T_\omega}{T} \lesssim \left(\frac{P\Delta v}{L}\right)\left(\frac{h_m}{H_p}\right) = \left(\frac{P\Delta v}{L}\right)\left(\frac{\alpha\epsilon_0}{\gamma}\right)^{3/4} \ll 1. \tag{20}$$

IV. NUCLEATION FOR CASES WITH POSITIVE L

We consider next the effects of homogeneous nucleation on the superheated "heavy" fluid for $z_1 < z < 0$. (Similar considerations apply for the supercooled "light" fluid for $0 < z < z_2$.) As illustrated in Fig. 2, the superheating δT (actual temperature T at pressure P minus the phase transition temperature for this pressure) reaches a maximum $\delta T_m \simeq \Delta T$ at some height $-z \approx l_c$ below the interface surface. The value of l_c lies somewhere between the expression in Eq. (14) and h_m in Eq. (19). In the absence of nucleation, the value of ΔT is at most the larger of ΔT_r from Eq. (15) and ΔT_ω from Eq. (20). However, the rate of nucleation of "bubbles" of light fluid is a rapidly increasing function of δT (which we replace by ΔT). When ΔT reaches a critical value ΔT_c, the heat carried upward in the form of latent heat of rising bubbles reaches the total heat flux F. The actual temperature drop across the phase boundary is then the *smaller* of ΔT in Sec. III and of ΔT_c, which we estimate as follows.

A spherical bubble of the lighter phase, containing N molecules, immersed in the heavier phase, has a finite surface energy. For an approximate evaluation of the nucleation rate[13–15] it is sufficient to characterize the surface energy by a single parameter D. The latent heat for changing a bubble from N to $N+1$ molecules is assumed to be of the form $L - N^{-1/3}D$. The ratio D/L depends on the material and the temperature but for most liquids it lies between 0.1 and 1. The rate at which small seed bubbles are formed by homogeneous nucleation (per unit area per second) is roughly of order J, where

$$J = \frac{l_c v_0}{a_0^4} \exp\left(-\frac{D^3 k_B T}{2L^2(k_B \Delta T_c)^2}\right), \tag{21}$$

and where a_0 is the spacing between molecules in the liquid. Each seed bubble grows as it rises, and we shall estimate the final radius b when the bubble has risen the height l_c to reach the interface. The latent heat energy flux F_l carried upward is of order $(b/a_0)^3 LJ$ and this must equal the total heat flux F in Eq. (6). The value of ΔT_c is then given by

$$\left(\frac{T}{\Delta T_c}\right)^2 \simeq \left(\frac{2L^2 k_B T}{D^3}\right)\ln\eta, \quad \eta \equiv \frac{L l_c b^3}{k_B T a_0^4}\,\alpha^{-1/2}\left(\frac{\epsilon_0}{\gamma}\right)^{-3/2} \tag{22}$$

where we have assumed that $\rho\, C_p/k_B \simeq a_0^{-3}$.

In estimating the final radius b and upward drift velocity of a bubble, V_b, we have to distinguish various regimes, depending on the values of the parameters $R_\nu(b) \equiv V_b\, b/\nu$, $R_\kappa(b) \equiv V_b\, b/\kappa$, and $\mathrm{Pr} = R_\kappa/R_\nu$. We first consider the case where $\mathrm{Pr} \lesssim 1$ and $R_\kappa(b) \gtrsim 100$. The drag coefficient for the motion of the light bubble through the heavy fluid is then $C_d \approx 0.05$ and, using the

fact that $\Delta\rho/\rho_1 \approx 1$, one finds $V_b \approx 5(b/H_p)^{1/2} v_0$ for the terminal drift velocity. The surface of the bubble is very close to the correct phase transition temperature and is cooler than the surrounding, more dense fluid by about ΔT. Most of this temperature drop occurs across a thermal boundary layer of thickness d_κ, surrounding the bubble. Since $\nu < \kappa$, the flow speed in much of the thermal boundary layer is close to V_b and d_κ is obtained by equating the thermal diffusion time d_κ^2/κ to the flow time b/V_b, so that $d_\kappa \approx 0.4(bH_p^3)^{1/4} \gamma_{0\kappa}^{-1/2}$. Heat flows into the bubble from the surrounding superheated fluid at a rate $\rho C_p \kappa \Delta T/d_\kappa$ per unit surface area of the bubble. The bubble radius grows with a speed V_g such that the latent heat in new molecules added to the bubble equals this heat-flow rate: $V_g \approx (k_B \Delta T/L)\kappa/d_\kappa$. The final bubble radius b is then determined by the consistency requirement that the growth time b/V_g for doubling the radius be comparable to the rise time l_c/V_b. This requirement leads to the final expressions

$$b \approx (k_B \Delta T/L)^{4/7} l_c^{4/7} H_p^{3/7} \gamma_{0\kappa}^{-2/7}$$
$$\eta \approx (k_B \Delta T/L)^{5/7} (\Delta T/T) l_c^{19/7} H_p^{9/7} a_0^{-4} \gamma_{0\kappa}^{-6/7} \alpha^{-1/2} (\epsilon_0/\gamma)^{-3/2}. \quad (23)$$

Next, consider the case where $R_\nu(b) \lesssim 100$ and $R_\kappa(b) \lesssim 100$. In this case, $V_b \approx (\gamma_{0\nu}/24)(b/H_p)^2 v_0$, and the thickness d_κ in the thermal diffusion formula is replaced by the bubble radius b. Instead of Eq. (23), one finds

$$b \approx l_c^{1/4} H_p^{3/4} (k_B \Delta T/L)^{1/4} \gamma_{0\kappa}^{-1/2} \mathrm{Pr}^{1/4}$$
$$\eta \approx (k_B \Delta T/L)^{-1/4} (\Delta T/T) l_c^{7/4} H_p^{9/4} a_0^{-4} \gamma_{0\kappa}^{-3/2} \times \mathrm{Pr}^{3/4} \alpha^{-1/2} (\epsilon_0/\gamma)^{-3/2}. \quad (24)$$

Three other cases are possible but do not merit special attention unless the Prandtl number Pr differs from unity by many orders of magnitude. [For example the value of b in the case $R_\kappa(b) > R_\nu(b) \gtrsim 100$ is smaller than given by Eq. (23) by a factor of $\mathrm{Pr}^{-2/21}$.]

For all cases of practical interest, η is extremely large compared with unity and $(2L^2 k_B T/D^3)$ is at least as large as unity, so that $\Delta T_c \lesssim 0.1\ T$.

It can also be verified that for the applications of interest, the heat flux F satisfies the following double inequality

$$L(V_b/a_0^3)(b^3/l_c H_p^2) \ll F \ll L(V_b/a_0^3). \quad (25)$$

The lower bound on F insures that the total number of bubbles in the superheated fluid layer is much greater than unity. (For this purpose, the "horizontal" extent of the fluid layer is assumed to be of order H_p. This is appropriate for a spherical body.) This insures that the global heat release is steady (although the heat flux through the interface is locally intermittent).

The upper bound in Eq. (25) insures that the average separation between bubbles is much larger than the typical bubble radius b.

However, this analysis assumes that the bubbles are not appreciably distorted by their motion. A measure of this distortion is the parameter δ, where

$$\delta \equiv (V_b/v_0)^2 (b/a_0)(P\Delta v/D). \quad (26)$$

If $\delta \ll 1$, then the bubbles are essentially spherical. If $\delta \gtrsim 1$, the bubbles are highly distorted and could possibly break up. To estimate the effect of this, we consider the extreme case in which every bubble breaks in half, as soon as it grows to a size such that $\delta = 1$. Suppose that the critical bubble size is b_c. The distance that a bubble rises before reaching this size is $l_* \approx b_c(V_{b_c}/V_g)$. If $l_* \gtrsim l_c$, then no breakup occurs. If $l_* > l_c$, then one seed bubble could produce as many as $2^{(l_c/l_*)}$ progeny. To retain the correct heat flux F, it is then necessary to reduce the superheating ΔT by a factor of roughly $(l_*/l_c)^{1/2}$. This exponential avalanche of bubbles may be so great as to exhaust all the fluid that is locally available for phase change. In this limit $(l_c \gg l_*)$ one seed bubble is at most responsible for inititating a phase change in a cone of fluid, the volume of which we now estimate. Each time a bubble breaks up, its progeny may have a horizontal velocity of order V_{b_c}. However, the only horizontal force on the bubble is the drag force, so the bubble goes a horizontal distance of order b_c/C_d, where $C_d \approx 0.05$ is the drag coefficient. In contrast, the bubble goes a vertical distance $l_* \gg b_c$ before breakup. Subsequent coalescence of bubbles is not favored because of the short-range repulsive force between bubbles.[16]

An upper bound to the volume of the narrow cone formed by the rising bubbles is then $(b_c/C_d)^2(l_c/l_*)^2 l_c$. Equating the latent heat energy flux with the total heat flux then gives

$$\left(\frac{T}{\Delta T_c}\right)^2 \simeq \left(\frac{2L^2 k_B T}{D^3}\right) \ln\eta, \quad \eta = \left(\frac{L}{k_B T}\right) \frac{l_c^4 b_c^2}{a_0^4 l_*^2 C_d^2} \alpha^{-1/2} \left(\frac{\epsilon_0}{\gamma}\right)^{-3/2}. \quad (27)$$

In this case, the heat release would be intermittent and explosive [but note that ΔT_c is the same as in Eq. (22), except for logarithmic corrections].

For liquid hydrogen in Jupiter (see Sec. VI) thermal conduction can carry the heat flux at a much smaller superheating than is required for significant nucleation. However, if homogeneous nucleation were the only heat transport mechanism, then bubble breakup is possible and "explosive boiling" might occur.

V. CASES WITH NEGATIVE L

If the latent heat L is negative (heat is released in changing from the heavy to the light fluid), then $(dT/dz)_{ph} > 0$. If dynamic effects could be neglected and a rigid interface between the two fluids assumed, the situation would be as shown schematically in Fig. 3. The boundary layer thickness l_c and temperature drop ΔT_r would be exactly as given by Eqs. (14) and (15), but no suppression of nucleation need be assumed since neither fluid is superheated nor supercooled.

However, in the present case, dynamic effects interact with the phase transition in such a way as to decrease the actual temperature drop ΔT below ΔT_r (and possibly change its sign). As discussed in Sec. III, the interface between the two fluids is distorted by surface gravity waves of vertical amplitude $h \approx H_p(\alpha\epsilon_0/\gamma)^{3/4}$ and pressure variations of order $(\alpha\epsilon_0/\gamma)^{3/4} P$. No appreci-

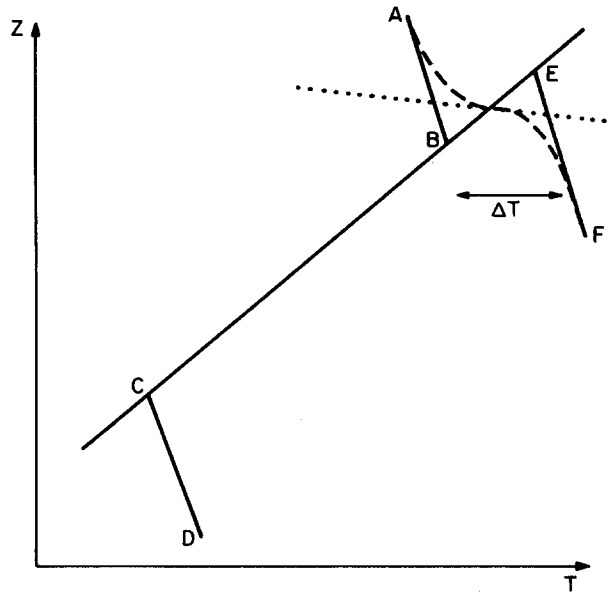

FIG. 3. The negative latent heat case with a rigid interface, showing the actual temperature profile (----) and the pure conduction profile (. . . .). ABCD is an adiabat and CBE is part of the phase boundary. EF is also an adiabat.

able amount of superheating or supercooling can be maintained at a macroscopic surface, so the temperature at each point on the distorted interface is then very close to the phase-transition temperature T_b appropriate to the local pressure. At the crest of a surface wave, the temperature is then higher than the mean temperature at the $z=0$ plane by an amount ΔT_ω, where

$$\frac{\Delta T_\omega}{T} \approx \left(\frac{P\Delta v}{L}\right)\left(\frac{\alpha\epsilon_0}{\gamma}\right)^{3/4} \tag{28}$$

and the troughs are cooler by a similar amount. As the surface rises up toward a crest, a small amount of dense fluid is changed into light fluid, with the latent heat heating the newly released light fluid. The sign of the present effect (it would be opposite for $L>0$) is such as to generate convective heat flow upward, with newly released, hotter, light fluid rising from a crest (and colder, heavy fluid sinking from a trough). The actual temperature drop is then less then ΔT_r in Eq. (15).

According to the inequality in Eq. (6), the quantity ΔT_ω in Eq. (28) always satisfies $\Delta T_\omega \ll T$. In some cases, where $\Delta T_r \ll \Delta T_\omega$, a temperature inversion may be set up near the interface, as illustrated schematically in Fig. 4. However, the temperature increase $(-\Delta T)$ across the boundary will never exceed ΔT_ω: if $-\Delta T$ were as large as ΔT_ω, the temperature increase in a rising part of the surface would be less than the amount of the temperature inversion. The newly formed light fluid would be cooler than the ambient medium at this level, so that the temperature inversion would inhibit convection completely.

The value of $-\Delta T$ for a temperature also cannot exceed ΔT_c in Eq. (22) appreciably. If it did, bubbles would form in the supercooled, heavy fluid at the bottom of the temperature inversion. These bubbles would rise (as in Sec. IV), but with $L<0$, the growth and motion of bubbles and droplets would result in a net downward heat flow. Thermal conduction also tends to depress the amount of a temperature inversion. To summarize the situation for negative latent heat: The temperature change ΔT satisfies $-\Delta T_\omega < \Delta T < \Delta T_r$. The actual value of $|\Delta T|$ is probably much smaller than these limits indicate, but an actual evaluation would be difficult.

VI. DISCUSSION

For practical applications, the main results of the present paper can be summarized as follows: When the various parameters of the problem satisfy certain conditions, the heat flow is carried across the interface between the two phases with a very small temperature drop ΔT. If these conditions are met, one can use the "isothermal approximation" in a planetary model: One considers the "heavy" and "light" phases to be separated by a sharply defined interface of constant gravitational potential with no discontinuity in temperature T. In this case, the entropy per atom has a discontinuity ΔS across the interface, given by

$$\Delta S = L/T . \tag{29}$$

The latent heat L per atom can have either sign, with $|L|/k_BT$ roughly of order unity. Cases where $|L|/k_BT$ happens to be very small present no problem, since the "isothermal" and "isentropic" approximations almost coincide. Such cases are excluded from the present paper.

We summarize the assumed conditions that led to the isothermal approximation: The heat flux F across the phase boundary is not affected by conditions near this

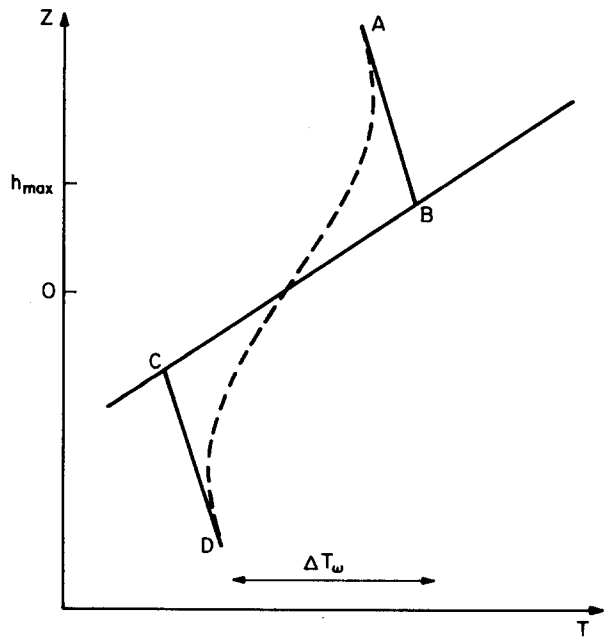

FIG. 4. Temperature profile (- - - -) near the interface for $L<0$, illustrating a possible temperature inversion. BC lies on the phase boundary. CD and AB are (different) adiabats h_{max} is the maximum gravity wave amplitude.

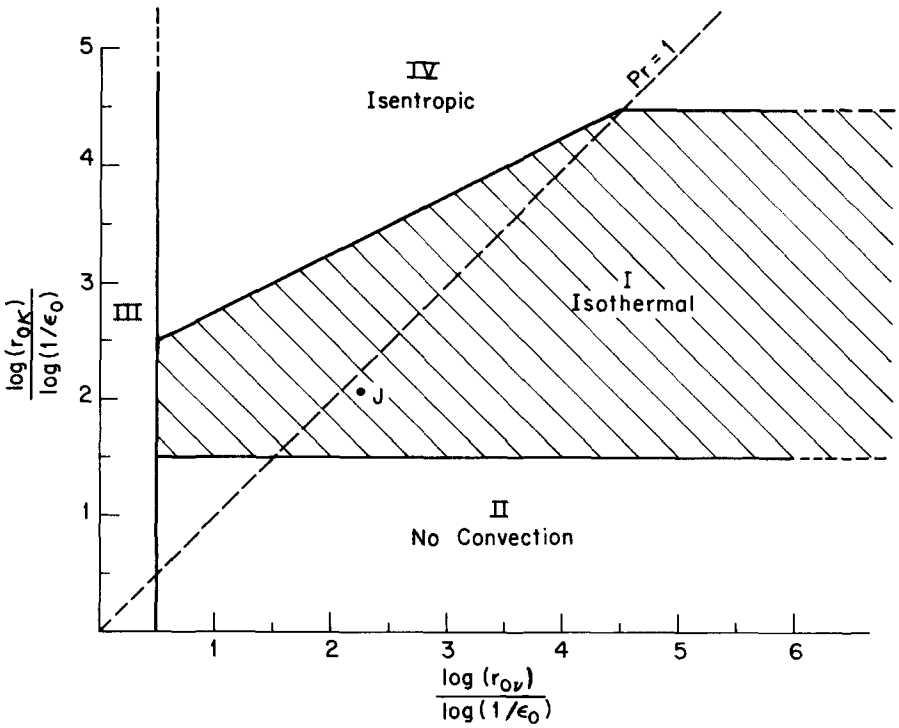

FIG. 5. The shaded region of this plot of r_{0k} versus $r_{0\nu}$ represents the parameter range for which our results apply and the "isothermal approximation" is valid. The point marked J represents the parameters appropriate to Jupiter. (This graph assumes that α, γ and $P\Delta v/|L|$ are all much greater than ϵ_0 and much less than $1/\epsilon_0$.)

boundary (at least on the average), but is specified by other considerations. For Jupiter, for instance, its age is enormous compared with dynamic or thermal timescales near the phase boundary and F is determined by opacity considerations in the atmosphere of the planet. F is assumed to be small so that the parameter ϵ_0 in Eq. (6) is extremely small compared with unity. From the kinematic viscosity ν and thermal diffusivity κ, we formed two parameters $r_{0\nu}$ and $r_{0\kappa}$ in Eq. (8).

If there is no nucleation, then the shaded region I of Fig. 5 represents the range of $r_{0\nu}$ and $r_{0\kappa}$ for which our results apply and the "isothermal approximation" is valid. In region II, the molecular thermal diffusivity is so large that the entire heat flux is transported by conduction. In region III, the molecular viscosity is large enough to prevent large scale turbulence, and our parameterization of the convective flow ceases to be appropriate. In particular, ϵ_0 is no longer a valid approximation for the fractional superadiabaticity far from the interface. This region is governed by criteria that have been developed for terrestrial mantle convection,[3-5] and which map smoothly onto our criteria in the appropriate limits. Region IV corresponds to cases where the temperature drop, across the thermal boundary layer, is comparable to the actual temperature so that the instability toward isentropic flow[3] becomes possible.

If nucleation is "easy" ($\Delta T_c/T \ll 1$ in Sec. IV), then the isothermal region I extends to larger $r_{0\kappa}$ and is limited primarily by the value of D, the surface energy. In all cases, the "isothermal approximation" is also contingent on $\epsilon_0 \ll 1$, since this insures that distortions of the interface are much smaller than a scale height.

It is important for all the considerations of this paper that the phase transition be first order with a finite surface energy D between the phases. This guarantees that a rising bubble (or falling droplet) of one phase may grow or shrink in size but cannot make a gradual change toward the other phase in its interior. This is certainly the case if at least one of the phases is solid; it is probably true for the liquid molecular to liquid metallic hydrogen transition in Jupiter (but this is not yet certain). If, instead, there were no sharp phase transition but only a continuous change in density and atomic structure, the situation would be radically different: No bubbles or droplets could form, no superheating or supercooling would be possible and an element of a convective cell would change its density continuously as its pressure changes. In such a case (but still with $\epsilon_0 \ll 1$) the entropy per atom remains essentially constant as a function of height.

The considerations of this paper are likely to apply to the interior of Jupiter: Pressures are sufficient for the phase transition from molecular to metallic (atomic) hydrogen to occur, the required pressure being a few megabars.[17] The temperature near the transition pressure is likely to be about 10^4 °K,[1,2] sufficiently large so that both phases should be fluid, but sufficiently small to be below the critical temperature so that the transition may be first order. The metallic phase is about 30% more dense than the molecular phase and $1 \lesssim (P\Delta v/k_B T) \lesssim 10$. The sign of L is not known yet, nor are $|L|/k_B T$ and $D/|L|$ (but both ratios are likely to be slightly smaller than unity). ϵ_0 is of order 10^{-8} and v_0 is about 3×10^6 cm sec^{-1} for both phases and the Prandtl number is of order unity for molecular hydrogen and about 0.05 for metallic hydrogen; thus, $r_{0\nu} \approx H_p/a_0 \approx 10^{16}$. The inequalities in Eq. (10) are then all satisfied.

For these choices of parameters, we find that $\Delta T_r/T$

≈ 10 and a corresponding thermal boundary layer thickness l_c of the order of 10 cm. This amount of superheating (for conduction at a rigid interface) is much too small to initiate nucleation. If we demand that nucleation transport the heat flux, then $\Delta T_c/T$ is at least 10^{-2} (and probably nearer 10^{-1}). The distance through which growing bubbles must rise to reach the interface would then be as large as 10^6 cm. The distance that a bubble rises before breaking up is 10^4 or 10^5 cm so that "explosive boiling" might occur with (very roughly) one explosion per second in the entire planet. It should be emphasized that the latter is a numerical illustration and not a serious description of the Jovian interior.

For $L < 0$, we find an upper bound for the temperature inversion of $\Delta T_\omega/T \approx 10^{-3}$. It is clear that for Jupiter, the "isothermal approximation" is a good one. The possible astrophysical implications of this will be discussed elsewhere. Our conclusions are not rigorous, since there is no formalism which would enable us to test the stability of all conceivable modes of energy transport in a turbulent medium. Nevertheless, the stability of the interface, for those modes which we have considered, indicates that our conclusions may be generally valid.

Our results may also be applicable to the other major planets (Saturn, Uranus, and Neptune) and to any proposed phase changes in the liquid cores of terrestrial planets, such as the earth.

ACKNOWLEDGMENTS

We are grateful to the California Institute of Technology for their hospitality and one of us (E. E. S.) is grateful for a Fairchild Scholarship. We also thank P. Gierasch for constant help, and F. Busse, G. Schubert, and D. Turcotte for discussions.

This work was supported by grants from the National Science Foundation (NSF Grants MPS 72-05056-A02 and MPS 74-17838) and National Aeronautics Space Administration.

[1]W. B. Hubbard and R. Smoluchowski, Space Sci. Rev. **14**, 599 (1973).
[2]M. Podolak and A. G. W. Cameron, Icarus **22**, 123 (1974).
[3]P. Busse and G. Schubert, J. Fluid Mech. **46**, 801 (1971).
[4]G. Schubert, D. Turcotte, and E. R. Oxburgh, Science **169**, 1075 (1970).
[5]D. L. Turcotte and G. Schubert, J. Geophys. Res. **76**, 7980 (1971).
[6]M. Schwarzchild, *Structure and Evolution of Stars* (Princeton University, Princeton, NJ, 1958), p. 47.
[7]E. A. Spiegel, Ann. Rev. Astron. Astrophys. **9**, 390 (1971).
[8]W. V. R. Malkus, Proc. R. Soc. London **225**, 196 (1954).
[9]G. Simon and N. O. Weiss, Z. Astrophys. **69**, 435 (1968).
[10]P. Gierasch, J. Atmos. Sci. **22**, 315 (1971).
[11]G. Shaviv and E. E. Salpeter, Astrophys. J. **184**, 1971 (1973).
[12]C. M. Tchen, Tellus **26**, 565 (1974).
[13]J. Feder, K. Russell, J. Lothe, and G. Pound, Adv. Phys. **15**, 111 (1966).
[14]A. C. Zettlemeyer, Ed. *Nucleation* (Dekker, New York, 1969).
[15]E. E. Salpeter, J. Chem. Phys. **58**, 4331 (1973); Astrophys. J. **193**, 579 (1974).
[16]L. M. Milne-Thompson, *Theoretical Hydrodynamics* (MacMillan, New York, 1960), p. 501.
[17]M. Ross, J. Chem. Phys. **60**, 3634 (1974).