ON THE THEORY OF Π_3^1 SETS OF REALS

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1. An ordinal basis theorem. Assuming that $\forall x \in \omega^\omega$ ($x^\#$ exists), let u_α be the αth uniform indiscernible (see [3] or [2]). A canonical coding system for ordinals $< u_\omega$ can be defined by letting $WO_\omega = \{w \in \omega^\omega : w = \langle n, x^\# \rangle$, for some $n \in \omega, x \in \omega^\omega\}$ and for $w = \langle n, x^\# \rangle \in WO_\omega$, $|w| = \tau_n^{L[x]}(u_1, \ldots, u_{k_n})$, where τ_n is the nth term in a recursive enumeration of all terms in the language of $ZF + V = L[\dot{x}]$, $\dot{x}$ a constant, taking always ordinal values. Call a relation $P(\xi, x)$, where ξ varies over u_ω and x over ω^ω, Π_k^1 if $P^*(w, x) \Leftrightarrow w \in WO_\omega \wedge P(|w|, x)$ is Π_k^1. An ordinal $\xi < u_\omega$ is called Δ_k^1 if it has a Δ_k^1 notation i.e. $\exists w \in WO_\omega$ ($w \in \Delta_k^1 \wedge |w| = \xi$).

THEOREM 1 ($ZF + DC + $ DETERMINACY (Δ_2^1)). *Every nonempty Π_3^1 subset of u_ω contains a Δ_3^1 ordinal.*

COROLLARY 2 ($ZF + DC + $ DETERMINACY (Δ_2^1)). *Π_3^1 is closed under quantification over ordinals $< u_\omega$ i.e. if $P(\xi, x)$ is Π_3^1 so are $\exists \xi P(\xi, x)$, $\forall \xi P(\xi, x)$.*

COROLLARY 3 ($ZF + DC + AD$). *The class of Π_3^1 sets of reals is closed under $< \delta_3^1$ intersections and unions.*

Martin [3] has proved the corresponding result for Δ_3^1.

2. A Kleene theory for Π_3^1. Kleene has characterized the Π_1^1 relations as those which are inductive (see [7]) on the structure $\langle \omega, < \rangle = \mathcal{Q}_1$. Let $j_m : u_\omega \to u_\omega$, $m \geqslant 1$, be defined by letting

$$j_m(u_i) = \begin{cases} u_i, & \text{if } i < m, \\ u_{i+1}, & \text{if } i \geqslant m, \end{cases}$$

and then

$$j_m(\tau_n^{L[x]}(u_1, \cdots u_{k_n})) = \tau_n^{L[x]}(j_m(u_1) \cdots j_m(u_{k_n})).$$

Let R be the relation on u_ω coding these embeddings, i.e.

$$R = \{(m, \alpha, \beta) : m \in \omega \wedge \alpha, \beta < u_\omega \wedge j_m(\alpha) = \beta\}.$$

Put $\mathcal{Q}_3 = \langle u_\omega, <, R \rangle$.

THEOREM 4. $(ZF + DC + \text{DETERMINACY } (\Delta_2^1))$. *A set of reals is Π_3^1 iff it is absolutely inductive on the structure Q_3.*

In the second part of the above characterization a relation on reals is viewed as a second order relation on u_ω and absolutely inductive means that only parameters from ω are allowed in the definitions (see [7]).

It should be mentioned here that Q_3 is up to absolute hyperelementary equivalence the same as $\langle u_\omega, <, T^2 \rangle$, where T^2 is the tree (on $\omega \times u_\omega$) coming from the Martin and Solovay [4] analysis of Π_2^1 sets (see [3] for the definition of T^2).

One also obtains the analog for Π_3^1 of the Souslin-Kleene representation of Π_1^1 sets in terms of well-founded trees.

THEOREM 5 $(ZF + DC + \text{DETERMINACY } (\Delta_2^1))$. *A set of reals P is Π_3^1 iff there is a tree T on $\omega \times u_\omega$ which is recursive in the structure Q_3 and $P(x) \Leftrightarrow T(x)$ is well founded.*

For the notation see [2]. The fact that every Π_3^1 set can be so represented is a well-known result of Martin and Solovay [4], the converse being new here.

Let $Q_3^- = \langle u_\omega, <, \{u_n\}_{n<\omega} \rangle$. Then we also have the context of full AD, in which case $u_n = \aleph_n$, $\forall n \leqslant \omega$.

THEOREM 6 $(ZF + DC + AD)$. *A set of reals is Π_3^1 iff it is Π_1^1 on the structure Q_3^-.*

3. Explaining the Q-theory. The results in §2 provide a nice explanation for the Q-theory (see [5], [1]) at level 3, which accounts for the structural differences between Π_3^1 and Π_1^1 sets. For example, a real is Δ_3^1 iff it is absolutely hyperelementary on Q_3 while it is in Q_3 iff it is hyperelementary (i.e. parameters $< u_\omega$ are allowed) on Q_3. Also if y_0 is the first nontrivial Π_3^1 singleton then y_0 is hyperelementary-in-Q_3 equivalent to the complete inductive-in-Q_3 subset of u_ω.

4. Higher level analogs of L. Assuming Projective Determinacy (PD), let T^3 be the tree (on $\omega \times \delta_3^1$) associated with an arbitrary Π_3^1-scale on a complete Π_3^1 set (see [6] and [2]). Let also C_4 be the largest countable Σ_4^1 set. The next result proves a conjecture of Moschovakis and shows that $L[T^3]$ is a correct higher level analog of L for level 4.

THEOREM 7 $(ZF + DC + \text{DETERMINACY } (L[\omega^\omega] \cap \text{power } (\omega^\omega)))$. *For any T^3 as above, $L[T^3] \cap \omega^\omega = C_4$. In particular $L[T^3] \cap \omega^\omega$ is independent of the tree T^3.*

Open problem. Is $L[T^3]$ independent of T^3?

Further applications of the methods developed here to the theory of Π_3^1 sets as well as details and proofs of the results announced here will appear elsewhere.

REFERENCES

1. A. S. Kechris, *The theory of countable analytical sets*, Trans. Amer. Math. Soc. 202 (1975), 259–297.

2. ———, *On projective ordinals*, J. Symbolic Logic 39 (1974), 269–282.

3. D. A. Martin, *Projective sets and cardinal numbers; some questions related to the continuum problem*, J. Symbolic Logic (to appear).

4. D. A. Martin and R. M. Solovay, *A basis theorem for Σ^1_3 sets of reals*, Ann. of Math. 89 (1969), 138–160.

5. ———, *Basis theorems for Π^1_{2k} sets of reals* (to appear).

6. Y. N. Moschovakis, *Uniformization in a playful universe*, Bull. Amer. Math. Soc. 77 (1970), 731–736.

7. ———, *Elementary induction on abstract structures*, North-Holland, Amsterdam, 1974.

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