Séminaire Initiation à l'Analyse
G. CHOQUET, G. GODEFROY, M. ROGALSKI, J. SAINT RAYMOND
30e Année, 1990/1991, n° 22, 2 p.

THE CLASSIFICATION OF HYPERFINITE BOREL EQUIVALENCE RELATIONS

par A.S. Kechris

Let X be a standard Borel space and E a Borel equivalence relation on X. We call E hyperfinite if there is a Borel automorphism T of X such that $xEy \Leftrightarrow \exists\, n \in \mathbb{Z}(T^n x = y)$.

For Borel equivalence relations E, F on X, Y resp. we write

$$E \sqsubseteq F \Leftrightarrow \exists\, f : X \to Y(f \text{ Borel, injective with } E = f^{-1}[F])$$

$$E \approx F \Leftrightarrow E \sqsubseteq F \text{ and } F \sqsubseteq E$$

$$E \cong F \Leftrightarrow \exists\, f : X \to Y(f \text{ a Borel isomorphism with } E = f^{-1}[F])$$

A Borel equivalence relation E on X is called smooth if there is a Borel map $f : X \to Y$ (Y some standard Borel space) with $xEy \Leftrightarrow f(x) = f(y)$.

THEOREM 1. *(Dougherty-Jackson-Kechris). Let E, F be two non-smooth, hyperfinite Borel equivalence relations. Then $E \approx F$.*

A hyperfinite E is called aperiodic if every E-equivalence class of E is infinite. Given such an E, we denote by $\mathcal{E}(E)$ the space of E-ergodic, invariant probability measures. (A measure is E-ergodic if every E invariant Borel set is null or conull and E-invariant if it is T-invariant for a Borel automorphism T that induces E - this is independent of T).

THEOREM 2. *(Dougherty-Jackson-Kechris). Let E, F be aperiodic, non-smooth hyperfinite Borel equivalence relations. Then*

$$E \cong F \Leftrightarrow \text{ card } (\mathcal{E}(E)) = \text{ (card } (\mathcal{E}(F)).$$

This has been conjectured by M.G. Nadkarni, who proved first the case when the above cardinality is countable.

It follows that up to Borel isomorphism the only aperiodic, non-smooth hyperfinite Borel equivalence relations are

E_t (on $2^{\mathbb{N}}$, where $xE_ty \Leftrightarrow \exists\, n\, \exists\, m\, \forall\, k(x_{n+k} = y_{m+k})$

$E_0 \times \Delta(n)$ (where E_0 on $2^{\mathbb{N}}$ is given by $xE_0y \Leftrightarrow \exists\, n\, \forall\, m \geq n(x_m = y_m)$ and $\Delta(n)$ is the equality relation on n elements, for $1 \leq n \leq \aleph_0$)

E_S^* (where E_S^* is the aperiodic part of the equivalence relation induced by the shift on $2^{\mathbb{Z}}$).

The above results will appear in a forthcoming paper by the author entitled : The structure of hyperfinite Borel equivalence relations.

Department of Mathematics
CALTECH
Pasadena
CA 91125 U.S.A.