ON THE LINEARIZED DYNAMICS OF TWO-DIMENSIONAL BUBBLY FLOWS OVER WAVE-SHAPED SURFACES

by

L. d'Agostino, C. E. Brennen and A. J. Acosta

California Institute of Technology
Pasadena, California 91125

1. INTRODUCTION

In the last decades the technological importance of bubbly flows has generated considerable efforts to achieve a better understanding of their properties, [1],[2]. However, the presence of two interacting phases so much increases the complexity of the problem that a satisfactory mathematical model of these flows has been possible only in special cases under fairly restrictive simplifying assumptions. The main purpose of the present note is to investigate the effects due to the inclusion of bubble dynamic response in two-dimensional flows over wave-shaped surfaces.

The earlier studies of bubbly flows based on space averaged equations for the mixture in the absence of relative motion between the two phases, [5], [6], do not consider bubble dynamic effects. This approach simply leads to an equivalent compressible homogeneous medium and has been used to analyze the behaviour of one-dimensional bubbly flows through converging-diverging nozzles.

In order to account for bubble dynamic response, in a classical paper by Foldy, [7], each individual bubble is described as a randomly distributed point scatterer. Assuming that the system is ergodic, the collective effect of bubble dynamic response on the flow is then obtained by taking the ensemble average over all possible configurations. An alternative way to account for bubble dynamic effects would be to include the Rayleigh- Plesset equation in the space averaged equations. Both methods have been successfully applied to describe the propagation of one-dimensional perturbances through liquids containing small gas bubbles, [8], [9], [10], [11].

However, because of their complexity, there are not many reported examples of the application to specific flow geometries of the space averaged equations which include the effects of bubble response, [12]. In an earlier note, [13], we considered the one-dimensional time dependent linearized dynamics of a spherical cloud of bubbles. The results clearly show that the motion of the cloud is critically controlled by bubble dynamic effects. Specifically, the dominating phenomenon consists of the combined response of the bubbles to the pressure in the surrounding liquid, which results in volume changes leading to a global accelerating velocity field. Associated with this velocity field is a pressure gradient which in turn determines the pressure encountered by each individual bubble in the mixture. Furthermore, it can be shown that such global interactions usually dominate any pressure perturbations experienced by one bubble due to the growth or collapse of a neighbor (see section 5).

In the present note the same approach is applied to the two-dimensional case of steady flows over wave-shaped surfaces (for which there exist well established solutions for compressible and incompressible flow), with the aim, as previously stated, of assessing the effects due to the introduction of bubble dynamic response. Despite its intrinsic limitations, the following linear analysis indicates some of the fundamental phenomena involved in such flows and provides a useful basis for the study of the same flows with non-linear bubble dynamics, which we intend to discuss in a later publication. The present extention to the case of bubbly flows over arbitrarily shaped surfaces also constitutes the starting point for the investigation of such flows, a problem of considerable technical interest, for example in cavitating flows past lifting surfaces.

2. BASIC EQUATIONS

Following the same approach previously indicated in our earlier note [13], several simplifying assumptions are introduced to obtain a soluble set of equations which still reflects the effects of bubble dynamic response. The relative motion of the two phases is neglected; the limitations this imposes are discussed later. The liquid is assumed inviscid and incompressible, with density, ρ, and constant population, β, of bubbles per unit liquid volume. Then, if external body forces are unimportant, the velocity, $\underline{u}(\underline{x})$, and the pressure, $p(\underline{x})$ (defined as the corresponding quantities in the liquid in the absence of local perturbations due to any neighbouring bubble), satisfy the continuity and momentum equations in the form:

$$\nabla \cdot \underline{u} = \frac{\beta}{1+\beta\tau} \frac{D\tau}{Dt} \tag{1}$$

$$\rho \frac{D\underline{u}}{Dt} = -(1+\beta\tau) \nabla p \tag{2}$$

where $\tau(\underline{x})$ is the individual bubble volume and D/Dt indicates the Lagrangian derivative. Finally, under the additional hypothesis that the bubbles remain spherical, it follows that $\tau = 4\pi R^3/3$, with the bubble radius, $R(\underline{x})$, determined by the Rayleigh-Plesset equation, [3], [4]:

$$\frac{p_B - p}{\rho} = R\frac{D^2R}{Dt^2} + \frac{3}{2}\left[\frac{DR}{Dt}\right]^2 + \frac{2S}{\rho R} \qquad (3)$$

Here S is the surface tension and p_B is the bubble internal pressure, which consists of the partial pressures of the vapor, p_v, and non-condensable gas, p_G. Neglecting thermal and mass diffusion effects in the bubbles, p_v is assumed constant and p_G is expressed by the polytropic relation of index q: $p_G = p_{G_0}(R_o/R)^{3q}$, where p_{G_0} is the gas partial pressure at a reference radius, R_o.

Equations (1), (2) and (3), together with suitable boundary conditions, represent in theory a complete system of equations for $\underline{u}(\underline{x})$, $p(\underline{x})$ and $\tau(\underline{x})$. However in practice their highly non-linear nature requires further simplifications for a closed form solution to be attained even for very simple flows.

3. DYNAMICS OF BUBBLY FLOWS OVER WAVE-SHAPED SURFACES

We now consider the problem of a two-dimensional flow of a bubbly liquid over a wave-shaped surface, as shown in Fig. 1. Let the wall profile be defined by the equation: $\eta(x) = Im\{\varepsilon exp(ikx)\}$ with $k\varepsilon << 1$ and let the subscript 0 indicate the unperturbed conditions corresponding to $\varepsilon = 0$. Then the flow velocity is U_o and, assuming, for simplicity, that all the bubbles have the same size, the undisturbed pressure in the liquid is:

$$p_o = p_{G_0} + p_v - \frac{2S}{R_o} \qquad (4)$$

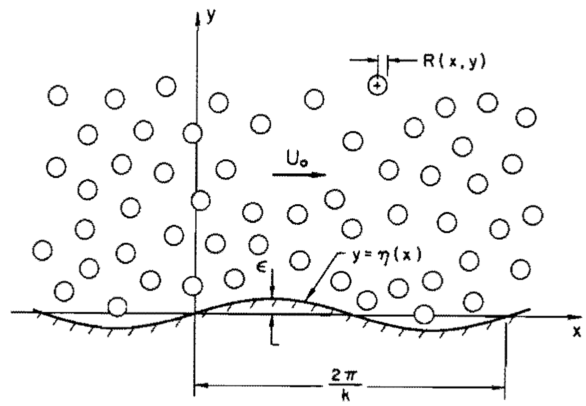

Fig. 1. Schematic of a bubbly liquid flow over a wave-shaped surface.

We limit our analysis to the case of very low void fraction ($\beta\tau<<1$), so that the expression $1+\beta\tau$ in (1) and (2) can be approximated as unity. We also make use of first order small perturbation theory, writing the velocity components as $u(x,y) = U_o + u'(x,y)$ and $v(x,y) = v'(x,y)$, with u' and v' much smaller than U_o. Then, with usual notations, equations (1), (2) and (3) reduce to:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 4\pi\beta U_o R^2 \frac{\partial R}{\partial x} \qquad (5)$$

$$\rho U_o \frac{\partial u'}{\partial x} = -\frac{\partial p}{\partial x} \qquad (6)$$

$$\rho U_o \frac{\partial v'}{\partial x} = -\frac{\partial p}{\partial y} \qquad (7)$$

$$p = p_v + p_{G_0}\left[\frac{R_o}{R}\right]^{3q} - \frac{2S}{R} - \rho U_o^2\left[R\frac{\partial^2 R}{\partial x^2} + \frac{3}{2}\left[\frac{\partial R}{\partial x}\right]^2\right] \qquad (8)$$

Finally, eliminating u' and v' from (5), (6) and (7), and using (8) one obtains the following equation for $R(x,y)$:

$$\nabla^2\left\{R\frac{\partial^2 R}{\partial x^2} + \frac{3}{2}\left[\frac{\partial R}{\partial x}\right]^2 + \frac{1}{\rho U_o^2}\left[\frac{2S}{R} - p_{G_0}\left[\frac{R_o}{R}\right]^{3q}\right]\right\} + $$

$$-\frac{4\pi\beta}{3}\frac{\partial^2 R^3}{\partial x^2} = 0 \qquad (9)$$

where ∇^2 is the two-dimensional Laplacian. Furthermore, the linearized kinematic condition at the wall $v(x,y_w)/U_o = d\eta/dx$ results in the following boundary condition for $R(x,y)$:

$$\frac{\partial}{\partial y}\left\{R\frac{\partial^2 R}{\partial x^2} + \frac{3}{2}\left[\frac{\partial R}{\partial x}\right]^2 + \frac{1}{\rho U_o^2}\left[\frac{2S}{R} - p_{G_0}\left[\frac{R_o}{R}\right]^{3q}\right]\right\}_{y=y_w} = \frac{d^2\eta}{dx^2} \qquad (10)$$

where y_w is the mean ordinate of the wall. In addition, the solution is required to be periodic in x with wave number k.

In their present form equations (9) and (10) are still non-linear and do not have any known analytical solution. In order to investigate their fundamental behaviour, we therefore examine the linearized form of these equations for small changes of the bubble radius: $R(x,y) = R_o[1+\varphi(x,y)]$, where $\varphi(x,y) \ll 1$. Then, to the first order in φ:

$$\nabla^2\left[\frac{\partial^2\varphi}{\partial x^2} + \frac{\omega_B^2}{U_o^2}\varphi\right] - 4\pi\beta R_o\frac{\partial^2\varphi}{\partial x^2} = 0 \qquad (11)$$

$$\frac{\partial}{\partial y}\left[\frac{\partial^2\varphi}{\partial x^2} + \frac{\omega_B^2}{U_o^2}\varphi\right]_{y=y_w} = \frac{1}{R_o^2}\frac{d^2\eta}{dx^2} \qquad (12)$$

where ω_B is the natural frequency of oscillation of a single bubble in a liquid, [3], [4]:

$$\omega_B^2 = 3q\frac{p_{G_0}}{\rho R_o^2} - \frac{2S}{\rho R_o^3} \qquad (13)$$

If the bubbles are in stable equlibrium in their mean or unperturbed state, then $3qp_{G_0} > 2S/R_o$ and ω_B is real. The linearized solution of (11) for the case of an unbounded bubbly flow over a wave-shaped wall takes the form:

$$\varphi(x,y) = Im\left\{k\varepsilon\frac{U_o^2/R_o^2}{\omega_B^2 - k^2U_o^2}\frac{e^{ikx-Gky}}{G}\right\}; \quad y\geq 0 \qquad (14)$$

Here G is the principal value of the square root of:

$$G^2 = 1 - M^2 = 1 - \frac{4\pi\beta U_o^2 R_o}{\omega_B^2 - k^2U_o^2} \qquad (15)$$

where M can be shown to be the flow Mach number based on the sonic speed corresponding to the frequency kU_o experienced by each bubble. Another possible solution involving $exp(ikx - Gky)$ has been eliminated, since in the subsonic regime $\varphi(x,y)$ must be finite as $y \to +\infty$ and in the supersonic regime no disturbance can propagate from the wall in the upstream direction. Therefore in the domain $y \geq 0$:

$$R(x,y) = R_o + R_o Im\left\{ k\varepsilon \, \frac{U_o^2/R_o^2}{\omega_B^2 - k^2U_o^2} \, \frac{e^{ikx-Gky}}{G}\right\} \quad (16)$$

$$u(x,y) = U_o + U_o Im\left\{ k\varepsilon \, \frac{e^{ikx-Gky}}{G}\right\} \quad (17)$$

$$v(x,y) = U_o Im\left\{ ik\varepsilon \, e^{ikx-Gky}\right\} \quad (18)$$

$$p(x,y) = p_o - \rho U_o^2 Im\left\{ k\varepsilon \, \frac{e^{ikx-Gky}}{G}\right\} \quad (19)$$

In the case of a channel of semiwidth b, symmetry about the centerline $y=b$ requires that the linearized solution of (11) take the form:

$$R(x,y) = R_o + R_o Im\left\{ k\varepsilon \, \frac{U_o^2/R_o^2}{\omega_B^2 - k^2U_o^2} \, \frac{\cosh Gk(b-y)}{\sinh Gkb} \, \frac{e^{ikx}}{G}\right\} \quad (20)$$

$$u(x,y) = U_o + U_o Im\left\{ k\varepsilon \, \frac{\cosh Gk(b-y)}{\sinh Gkb} \, \frac{e^{ikx}}{G}\right\} \quad (21)$$

$$v(x,y) = U_o Im\left\{ ik\varepsilon \, \frac{\sinh Gk(b-y)}{\sinh Gkb} \, e^{ikx}\right\} \quad (22)$$

$$p(x,y) = p_o - \rho U_o^2 Im\left\{ k\varepsilon \, \frac{\cosh Gk(b-y)}{\sinh Gkb} \, \frac{e^{ikx}}{G}\right\} \quad (23)$$

The entire flows have therefore been determined in terms of the prescribed quantities: R_o, β, k, U_o, and ε.

4. RESULTS

We now examine the nature of the above solutions. From (16) and (20) note that the bubble response is singular when:

(i) $G^2 = 0$ and therefore the flow Mach number is equal to unity (sonic condition);

(ii) $(kU_o/\omega_B)^2 = 1$, namely the exciting frequency experienced by each bubble is equal to the natural frequency of an individual bubble in an infinite liquid (bubble resonance condition). In addition, the channel flow is also singular when:

(iii) $Gkb = n\pi i$; $n = 0, \pm 1, \pm 2, \ldots$

The above conditions can be interpreted in two different ways according to whether the free stream velocity or wall wave number is assumed to be the independent variable. The former is the natural approach to the analysis of a given geometrical configuration at different flow regimes; the latter reflects the point of view used in deducing the solution for a more complex wall shape in terms of linear superposition of the different wave numbers of the wall geometry.

Since the population is related to the initial void fraction $\alpha_o = \beta\tau_o/(1+\beta\tau_o)$ (which must be much less than unity for the validity of the analysis), G^2 can be considered as a function of the reduced frequency kU_o/ω_B and one of the following two parameters: $3\alpha_o/k^2R_o^2$ when the wall geometry is fixed, or $3\alpha_o U_o^2/\omega_B^2 R_o^2$ when the flow velocity is constant. Consequently, (i) and (ii) can be used to deduce either the free stream velocities or the wall wave numbers which respectively correspond to sonic and bubble resonance conditions:

$$U_o^{*2} = \frac{\omega_B^2/k^2}{1+3\alpha_o/k^2R_o^2}; \quad k^{*2} = \frac{\omega_B^2}{U_o^2}\left[1 - \frac{3\alpha_o U_o^2}{\omega_B^2 R_o^2}\right] \quad (24)$$

$$U_{oB}^2 = \frac{\omega_B^2}{k^2}; \quad k_B^2 = \frac{\omega_B^2}{U_o^2} \quad (25)$$

Similarly, for channel flow, equation (iii) leads to infinite series of free stream velocities or wall wave numbers:

$$U_{on}^2 = \frac{\omega_B^2}{k^2}\left[1 + \frac{3\alpha_o/k^2R_o^2}{1+(n\pi/kb)^2}\right] \quad (26)$$

$$k_n^2 = \frac{1}{2}\left\{k^{*2} - \frac{n^2\pi^2}{b^2} + \left[\left[k^{*2} - \frac{n^2\pi^2}{b^2}\right]^2 + 4\,\frac{n^2\pi^2}{b^2}\,\frac{\omega_B^2}{U_o^2}\right]^{\frac{1}{2}}\right\} \quad (27)$$

where $n = 0, 1, 2, \ldots$. For large values of n these series respectively converge to the free stream velocity and the wall wave number corresponding to bubble resonance conditions. Further, the lower terms of such series will in general extend to values much smaller than the ones given by (iii) if the above two parameters are of order unity or larger. On the other hand, when the reverse is the case, all the terms of these series are contained in a small range below bubble resonance conditions.

Let us consider first the case of fixed wall geometry and variable free stream velocity. Then the behaviour of the parameter G^2 as a function of the reduced frequency is shown in Fig. 2 for some typical values of $3\alpha_o/k^2R_o^2$. Note that all curves start from unity at the origin and tend to move away from their horizontal asymptote $G^2=1$ (corresponding to the incompressible flow case) as the value of $3\alpha_o/k^2R_o^2$ increases. Also note that G^2 always changes sign twice at sonic and bubble resonance conditions, therefore dividing the flow solutions in three different regimes, namely: subsonic ($0<G^2<1$), super-resonant ($G^2>1$). The corresponding amplitudes of bubble radius oscillations at the wall ($kx=\pi/4$ and $y=0$) for the case of a semi-infinite flow with $k\varepsilon/2\pi=.01$ are shown in Fig. 3, which also shows the migration of the sonic singularity from the bubble resonance condition towards the origin for increasing values of the parameter $3\alpha_o/k^2R_o^2$. Finally, the bubble radius response in the case of channel flow is significantly different, as illustrated in Fig. 4, because of the presence of the additional resonancies (iii) introduced by the finite spacing between the boundaries.

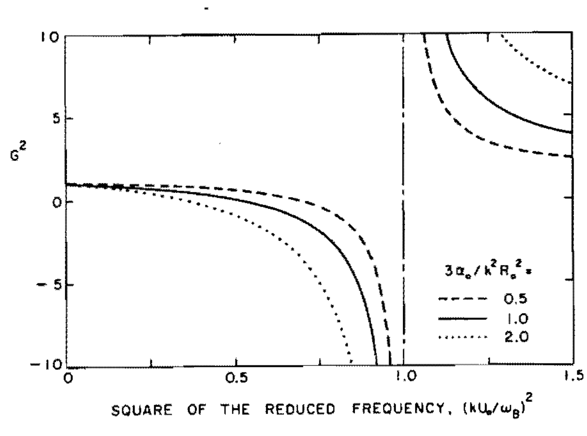

Fig. 2. Parameter G^2 v/s the square of the reduced frequency, $(kU_o/\omega_B)^2$, for different values of $3\alpha_o/(kR_o)^2$ = .5 (broken line), 1 (solid line) and 2 (dotted line).

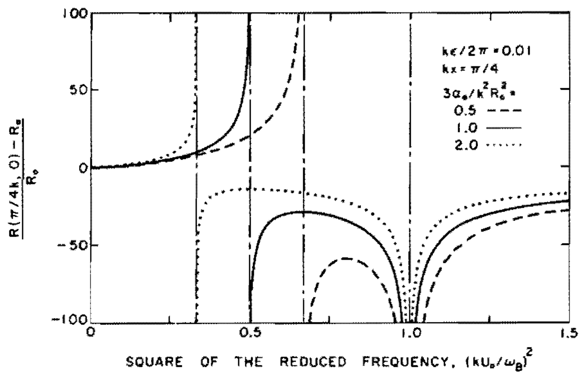

Fig. 3. Response of a semi-infinite bubbly flow over a wave-shaped wall as a function of the square of the reduced frequency, $(kU_0/\omega_B)^2$. Normalized amplitudes of the bubble radius oscillations at the wall ($kx = \pi/4$ and $y = 0$) are shown for different values of the parameter $3\alpha_0/(kR_0)^2$ = .5 (broken line), 1 (solid line) and 2 (dotted line).

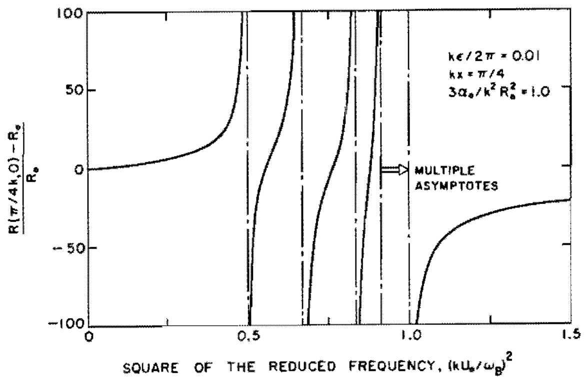

Fig. 4. Response of a bubbly flow in a symmetric wavy wall channel as a function of the square of the reduced frequency, $(kU_0/\omega_B)^2$. Normalized amplitudes of the bubble radius oscillations at the wall ($kx = \pi/4$ and $y = 0$) are shown for $3\alpha_0/(kR_0)^2 = 1$ and $kb = \pi$.

We assume next that the free stream velocity is fixed and the wall wave number is allowed to vary. In this case the curves representing the parameter G^2 as a function of the reduced frequency, now shown in Fig. 5, display again the tendency to move away from the horizontal asymptote $G^2=1$ as the value of the parameter $3\alpha_0 U_0^2/\omega_B^2 R_0^2$ increases. However, their value at the origin now depends on the free stream velocity and can be either positive, zero or negative according to whether the flow is respectively subsonic, sonic and supersonic. The normalized bubble radius response amplitudes at the wall ($kx = \pi/4$ and $y = 0$) for the case of a semi-infinite flow with $k\varepsilon/2\pi = .01$ are shown in Fig. 6. Here, as expected from the above considerations, when the free stream velocity increases the sonic resonance moves from the bubble resonance conditions to the origin and finally disappears for supersonic flows.

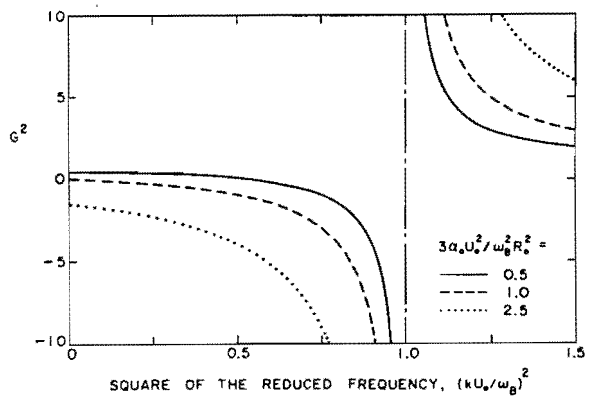

Fig. 5. Parameter G^2 v/s the square of the reduced frequency, $(kU_0/\omega_B)^2$ for different values of $3\alpha_0 U_0^2/\omega_B^2 R_0^2$ = .5 (solid line), 1 (broken line) and 2.5 (dotted line).

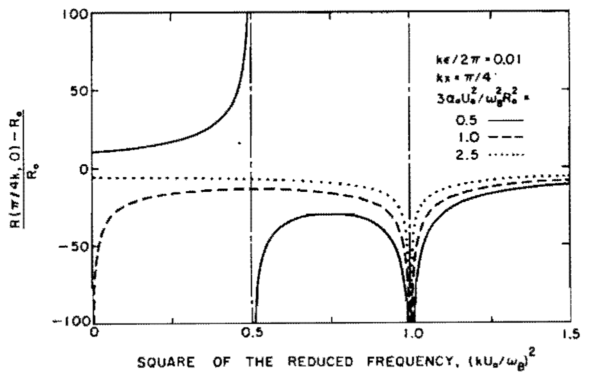

Fig. 6. Response of a semi-infinite bubbly flow over a wave-shaped wall as a function of the square of the reduced frequency, $(kU_0/\omega_B)^2$. Normalized amplitudes of the bubble radius oscillations at the wall ($kx = \pi/4$ and $y = 0$) are shown for different values of the parameter $3\alpha_0 U_0^2/\omega_B^2 R_0^2$ = .5 (solid line), 1 (broken line) and 2.5 (dotted line).

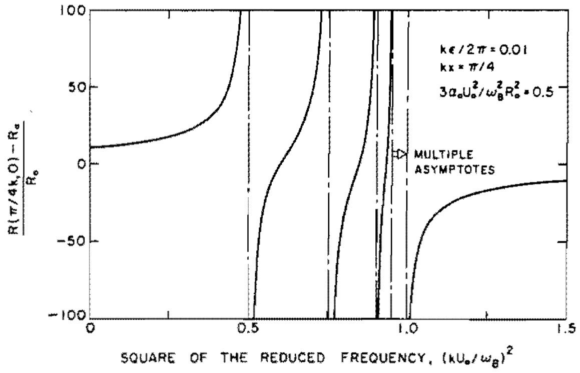

Fig. 7. Response of a bubbly flow in a symmetric wavy wall channel as a function of the square of the reduced frequency, $(kU_o/\omega_B)^2$. Normalized amplitudes of the bubble radius oscillations at the wall ($kx = \pi/4$ and $y=0$) are shown for $3\alpha_o U_o^2/\omega_B^2 R_o^2 = .5$ and $kb = \pi$.

The parameter G^2 also controls another important aspect of the flow, namely the penetration of wall induced disturbances in the y-direction, which equations (16) and (20) show to be inversely proportional to Gk. In effect, when G is considerably larger than unity, the response of the layer of bubbles near the wall essentially shields the rest of the mixture, with the result that the penetration of the disturbances induced by the wall is significantly reduced. On the other hand, as G tends to zero approaching sonic conditions, the perturbations due to the wall tend to affect the whole flow.

Significant analogies exist between the results shown here for the case of bubbly flows over wave-shaped surfaces and the ones previously obtained for the linearized dynamics of clouds of bubbles, [13]. For instance, in both flows the dispersive behaviour due to bubble dynamic effects is controlled by similar parameters, respectively G^2 and λ^2, which depend on the bubble population. These parameters also determine the elliptic or hyperbolic nature of the solution and the occurrence of the natural frequencies of the flow.

Due to the linear nature of the problem, the present theory can readily be generalized to the case of arbitrarily shaped boundaries. When the wall geometry, $\eta(x)$, can be Fourier transformed:

$$H(k) = \int_{-\infty}^{\infty} \eta(x) e^{-ikx} dx \tag{28}$$

the linearized solutions of (11) for the cases of unbounded and channel flow respectively admit the following integral representations:

$$\varphi(x,y) = Re\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{kH(k)/R_o^2}{\sqrt{k_B^2-k^2}\sqrt{k^{*2}-k^2}} e^{ikx-Gky} dk\right] \tag{29}$$

$$\varphi(x,y) = Re\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{kH(k)e^{ikx}/R_o^2}{\sqrt{k_B^2-k^2}\sqrt{k^{*2}-k^2}} \frac{\cosh Gk(b-y)}{\sinh Gkb} dk\right] \tag{30}$$

The integrands in the above equations have square root type singularities corresponding to sonic conditions $k^2=k^{*2}$ and bubble resonance conditions $k^2=k_B^2$. In (30), the additional singularities due to the finite wall spacing $k^2=k_n^2$ behave as simple poles. However, the singularities at bubble resonance regime are essential singularities since G^2 also appears in the argument of the exponential functions Therefore the integrals in (29) and (30) do not converge in the current form. In physical terms this reflects the fact that the present theory allows the bubble radius response to grow without limits at bubble resonance conditions. This difficulty would be eliminated by the introduction of dissipation effects which would bound the bubble response and remove the singularity from the real axis since G^2 is then a complex quantity. Then the integrals in (29) and (30) have integrable singularities and could in principle be evaluated to solve for the flow over a slender bump.

Finally, as a concluding remark, it is easily verified that the above theory reduces (as expected) to the first order perturbation solutions for incompressible and homogeneous flows in the limits for zero void fraction or free stream velocity and for zero wall wave number, respectively.

5. LIMITATIONS

We now briefly examine the restrictions imposed to the previous theory by the various simplifying assumptions that have been made. Specifically, we will discuss the limitation due to the introduction of the continuum model of the flow, that due to the use of the linear perturbation approach in deriving the solution and that due to the neglect of relative motion between the phases and of local pressure perturbations in the neighborhood of each individual bubble.

The perturbation approach simply requires that $\varphi \ll 1$ in equation (14), a condition which is satisfied far from the sonic and bubble resonances for proper choice of $k\varepsilon$ and $U_o/\omega_B R_o$.

For the continuum approach to be valid, the two phases must be minutely dispersed with respect to the shortest characteristic length of the flow, here either the wall wave length or the penetration of wall disturbances in the y-direction. Hence the bubble equilibrium radius is required to satisfy the most restrictive of the two conditions: $kR_o \ll 1$ and $GkR_o \ll 1$.

In order to estimate the error associated to the neglect of local pressure effects due to the dynamic response of each individual bubble, we consider the pressure perturbation experienced by one bubble as a consequence of the growth or collapse of a neighbor:

$$\Delta p = \rho\left\{\frac{R}{s}\left[R\frac{D^2R}{Dt^2} + 2\left[\frac{DR}{Dt}\right]^2\right] - \frac{R^4}{2s^4}\left[\frac{DR}{Dt}\right]^2\right\} \tag{31}$$

where $s \simeq R_o/\alpha_o^{1/3}$ is the average bubble spacing and $R=R_o(1+\varphi)$ is given by (16) or (20). To the same order of approximation used to develop the present analysis, comparison with the global pressure change expressed by (19) or (23) then shows that local pressure perturbations are unimportant if:

$$\alpha_o^{1/3}\left|\frac{k^2U_o^2/\omega_B^2}{1-k^2U_o^2/\omega_B^2}\right| \ll 1 \tag{32}$$

Far from bubble resonance regime, this condition is generally satisfied in low void fraction flows.

Finally, in order to assess the error introduced by the neglect of the relative velocity between the two phases, consider the equation of motion for a bubble of negligible mass, [14], with Stokes' viscous drag:

$$\frac{Du}{Dt} - \frac{1}{3}\frac{Du_B}{Dt} + \frac{1}{R}\frac{DR}{Dt}(u - u_B) = \frac{2\nu}{R^2}(u - u_B) \quad (33)$$

where ν is the liquid kinematic viscosity and u and u_B are respectively the velocity of the liquid and the bubble. Linearizing as before and assuming for both the relative velocity $u_r = u - u_B$ and the velocity of the liquid a solution in the form $Im[V(y)exp(ikx)]$, one obtains:

$$\left| \frac{u_r}{u - u_o} \right| = \frac{2}{\sqrt{1+(6\nu/kR_o^2 U_o)^2}} \ll 1 \quad (34)$$

Hence, relative motion effects are unimportant when $kR_o^2 U_o/\nu \ll 1$. This may, potentially, be the most limiting restriction of the present model.

6. ACKNOWLEDGEMENT

The authors would like to thank Carol Kirsch for her help in organizing the manuscript. This work was supported by Naval Sea Systems Command General Hydromechanics Research Program Administered by the David Taylor Naval Ship Research and Development Center under Contract No. N0014-75-C-0378.

7. REFERENCES

1. van Wijngaarden, L., "One-dimensional flow of liquids containing small gas bubbles", *Ann. Rev. Fluid Mech.*, 1972, Vol. 4, pp. 369-396.

2. van Wijngaarden, L., "On the equations of motion of mixtures of liquid and gas bubbles", *J. Fluid Mech.*, 1968, Vol. 33, part 3, pp. 465-474.

3. Plesset, M.S. and Prosperetti, A., "Bubble dynamics and cavitation", *Ann. Rev. Fluid Mech.*, 1977, Vol. 9, 145-85.

4. Knapp, R.T., Daly, J.W. and Hammit, F.G., "*Cavitation*", McGraw Hill, 1970.

5. Tangren, R.F., Dodge, C.H. & Seifert, H.S., "Compressibility effects in two-phase flows", *J. Appl. Phys.*, 1949, Vol. 20, pp. 637-645.

6. Muir, J.F. & Eichhorn, "Compressible flow of an air-water mixture through a vertical two-dimensional converging-diverging nozzle", *Proc. Heat Trans.* Fluid Mech. Inst., 1963, Standford, Standford Univ. Press.

7. Foldy, L.L., "The multiple scattering of waves", *Phys. Rev.*, 1945, Vol. 67, pp. 107-119.

8. Carstensen, E.L. & Foldy, L.L., "Propagation of sound through a liquid containing bubbles", *J. Acoust. Soc. Am.*, 1947, Vol. 19, pp. 481-501.

9. Fox, F.E., Curley, S.R. & Larson, G.S., "Phase velocity and absorption measurements in water containing air bubbles", *J. Acoust. Soc. Am.*, May 1955, Vol. 27, No. 3, pp. 534-539.

10. Macpherson, J.D., "The effect of gas bubbles on sound propagation in water", *Proc. Phys. Soc. London*, 1957, Vol. 70 B, pp. 85-92.

11. Silberman, E., "Sound velocity and attenuation in bubbly mixtures measured in standing wave tubes", *Journal of Acoustical Society of America*, Vol. 18, No. 8, August 1957, pp. 925-933.

12. van Wijngaarden, L., "On the collective collapse of a large number of gas bubbles in water", *Proc. 11th Int. Cong. Appl. Mech.*, 1964, Springer- Verlag, Berlin, pp. 854-861.

13. d'Agostino, L. and Brennen, C.E., "On the acoustical dynamics of bubble clouds", *ASME Cavitation and Multiphase Flow Forum*, 1983, pp. 72-75.

14. Voinov, O.V., "Force acting on a sphere in an inhomogeneous flow of an ideal incompressible fluid", *1975 Plenum Publishing Corp.*, transl. from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 182-184, July-August 1973.

FED - Vol. 9

CAVITATION and MULTIPHASE FLOW FORUM - 1984

presented at

ENERGY SOURCES TECHNOLOGY CONFERENCE
NEW ORLEANS, LOUISIANA
FEBRUARY 12-16, 1984

sponsored by

THE POLYPHASE FLOW COMMITTEE AND
THE FLUID MACHINERY COMMITTEE OF
THE FLUIDS ENGINEERING DIVISION, ASME

edited by

J. W. HOYT
SAN DIEGO STATE UNIVERSITY
SAN DIEGO, CALIFORNIA

THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
United Engineering Center 345 East 47th Street New York, N.Y. 10017