MATHEMATICAL THEORY OF SOUND RANGING.

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1. *Introduction.*—The development of the art of concealing large guns so that they can not be easily seen by hostile airmen or observers in kite balloons has brought into prominence the study of methods of locating powerful guns by means of observations of the time of arrival of the sound of their gunfire at one or more observing stations. There are really two distinct problems to be discussed.

(1) The simple case when the flash is seen and the distance of the gun is to be determined from the observed interval of time between the instants when the flash is seen and the report is heard at a single station.

(2) The more complex case when the flash is not seen. The sound of the report must now be timed at three or four observing stations and the position of the gun estimated from the observed differences in time. A small error in the timing of the sound is more disastrous in the second case than in the first, consequently an accurate method of timing the arrival of the sound is very necessary for the successful application of the second method.

Artillery chronoscopes have been used for the simple method of ranging, and some improvements in design have been made during the present war. Whereas formerly a chronoscope enabled time to be measured accurately to a fifth of a second, a modern instrument will record hundredths of a second.[1] Chronoscopes of this type are used in testing the nerves of would-be air pilots and it is found that a successful candidate will stop the pointer in fifteen-hundredths of a second after hearing a sound, while an unsuccessful candidate may allow twenty or even thirty hundredths of a second to pass. Now, it is unlikely that the personal equation of an observer will remain absolutely constant and an error of one-tenth of a second would certainly be disastrous in the second method of sound ranging and troublesome even in the first, consequently it is advisable to have instrumental means of recording automatically the time at which the sound arrives and methods of measuring time with great accuracy have been skillfully developed.

The possibility of utilizing sound to locate a large gun is to some extent due to the nature of the sound produced by gunfire. In front of the gun the sound is much more intense than it is behind, in fact, as far as the production of sound is concerned, the gun acts something like a searchlight, directing its beam along the line of fire. As a result of this the sound is much more intense in the forward direction than it would be if it were produced by the explosion of a shell. Thus to the German generals behind the lines every shot from the British guns appeared to stand out above the dull heavy roar of their own guns as a sharp staccato note like a loud drum tattoo, whence the name drum-fire (Trommelfeuer).[2] Again, the reports which have been heard in England seem to have come from German guns; for instance, the sound of gunfire, which was heard very distinctly on the evening of July 10, 1917, was attributed to the German bombardment on the Nieuport front, which commenced at 5 p. m.[3]

In both methods of sound ranging it is necessary to take into account the meteorological conditions, for the velocity of sound depends on the temperature, humidity, and composition of the air, while the wind affects the mode of propagation. The velocity of sound may be calculated from the formula [4]

$$V = V_0\left(\frac{TH}{T_0(H-0.378h)}\right)^{\frac{1}{2}}$$

where T is the absolute temperature, H is the barometric pressure, and h is the vapor pressure of water vapor, while V_0 is the velocity of sound in dry air at temperature T_0. The variation in composition can generally be neglected under ordinary conditions unless the sound ascends to a great height before it reaches the observer, but it is conceivable that on the battle field the presence of large quantities of smoke may affect the sound in a manner which is not quite negligible. A further difficulty arises on account of the fact that the very intense waves caused by the detonation of explosives have a velocity distinctly greater than that of ordinary sound,[5] and it must not be assumed without experimental verification that the velocity varies with the temperature and humidity of the air in the same way as the ordinary velocity of sound. It seems advisable, then, to invent a method of sound ranging in which the velocity of sound is eliminated altogether, and this is one of the objects of the present paper.

Whether it is of importance in military operations or not, the meteorological aspect of the problem of sound ranging is of some theoretical interest, because there are indications that it may eventually be possible to obtain some knowledge of the conditions in the upper atmosphere by a method of timing sound signals.

At present the mathematical theory is based on the idea of sound rays and is more accurate for sound of high frequency than for sound of low frequency. The work which is being done in the development of methods of producing and recording sounds of high frequency will thus be valuable for the above purpose, while a standard phone or sound generator may also be useful.

2. *Sound ranging from observations at one station when the sound travels horizontally.*—Let O (fig. 1) represent the position of the observer, G the position of the gun. The direction OG is known from the observation of the flash, while the time, t, which the sound takes to travel from G to O is also known. Let v be the velocity of the wind, AO the direction of the wind; then if the length AO represents vt, the length AG will represent Vt, and since the angle AOG is known, the position of G may be found by a simple geometrical construction or by triangulation.

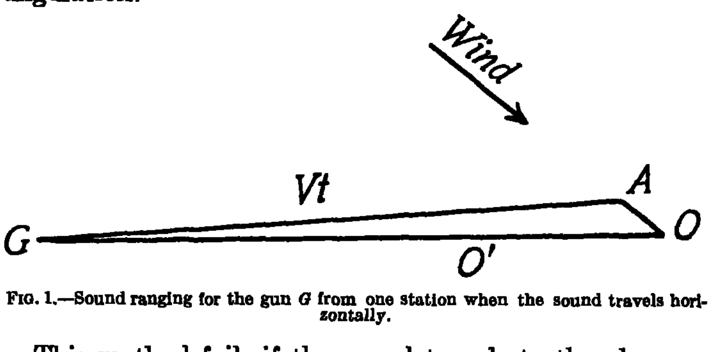


FIG. 1.—Sound ranging for the gun G from one station when the sound travels horizontally.

This method fails if the sound travels to the observer by a path through the upper air, and so it is useful to have a test which will enable an observer to find out

[1] An artillery chronoscope with a balance beating hundredths of a second has been brought out. See Engineering, May 19, 1916, p. 485.
[2] Compare *G. F. Sleggs* in Nature, London, Aug. 23, 1917.
[3] See Nature, London, July 19, 1917.

[4] *O. D. Chwolson*. Lehrbuch der Physik. v.2, p. 30.
[5] For a discussion of this matter and references see *L. V. King*, in Journal of the Franklin Institute, March, 1917, p. 274.

when this occurs. The following test is based on the theory developed in §4.

Let a second observation be made at a point O' on OG, then if the sound travels through the upper air, the interval between the times of travel at O and O' should be less than it would be if the sound traveled in a horizontal direction. This test will, of course, fail if the sound produced by the firing of the gun travels with a velocity greater than the ordinary velocity of sound.

If an object reflects sound back to the observer, its distance may be estimated by noting the interval between the production of the sound at O and the arrival of the echo. If T denotes this interval, the distance of the object is given by the formula—

$$2d=\frac{VT\left(1-\frac{v^2}{V^2}\right)}{\sqrt{1-\frac{v^2}{V^2}\sin^2\theta}}$$

where θ is the angle between the direction of the wind and the direction of the object. If v is small compared with V so that $\frac{v^2}{V^2}$ may be neglected, this method may be used to find the distance of the object when its direction is not known.

If it were possible to produce at O an intense sound having a frequency equal to one of the natural frequencies of the gun and obtain a return sound from the gun, this method might be used with advantage and a gun located by means of observations at two stations before it was even fired.

3. *Sound ranging from observations at a number of stations when the sound travels horizontally.*—When the sound travels over the surface of the earth and this is treated as flat, the problem of locating a gun from observations at three stations A', B', C', may be solved as follows: the first step is to reduce the problem to the case when there is no wind. Let t_a, t_b, t_c, be the times at which the sound is recorded at these stations and let T be any convenient time. Draw lines $A'A$, $B'B$, $C'C$, in a direction opposite to the wind and let the lengths of these lines represent the distances $v(t_a-T)$, $v(t_b-T)$, $v(t_c-T)$, respectively, where v is the velocity of the wind. We must now determine the position of a point G at which a gun can be fired so as to be heard at A, B, and C, at times t_a, t_b, and t_c, respectively, when there is no wind. Let $GA=V(t_a-T_0)$, then if we draw a line GG' in a direction opposite to the wind to represent the length $v(T-T_0)$, the point G will indicate the position of the gun. To determine the point G we can make use of the hyperbolas H and H_1 defined by the equations $GA-GB=V(t_a-t_b)$, and $GB-GC=V(t_b-t_c)$, respectively, or we can draw circles of radii $V(t_a-T)$, $V(t_b-T)$, $V(t_c-T)$, with their centers at A, B, C, respectively. The point G is then the center of a circle which has the same kind of contact with each. If the latter method is to be used the computer may find it useful to have a set of metal disks whose radii differ successively by small amounts, the circle which touches the three circles may then be found very quickly by trial.

When the distance AB is small compared with the distance of the gun the hyperbola H may be replaced by its asymptote which bisects AB and makes an angle with AB whose cosine is the ratio of t_a-t_b to the time which sound would take to travel from A to B. When this method is used it is useful to have observations at two pairs of stations, one pair of stations being at a considerable distance from the other.

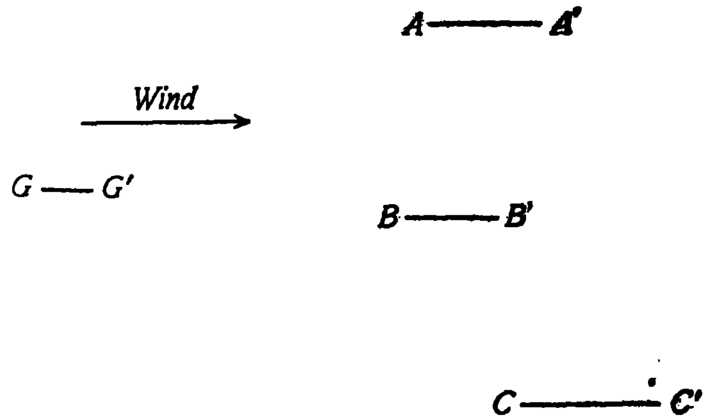
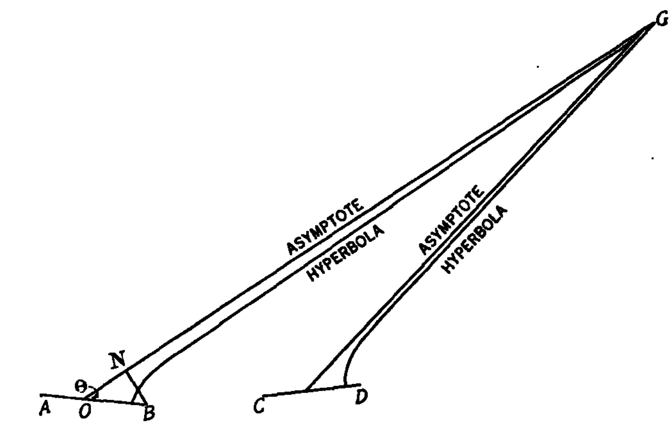


FIG. 2.—Sound ranging for the gun G' from three stations, A', B', C', when the sound travels over the flat surface of the earth.

When the velocity of the sound is unknown use may be made of observations at two trials of stations $A'B'C'$ and $P'Q'R'$. Let t_1, t_2, t_3, be the times at which the sound arrives at A, B, and C, respectively; and let x_1, y_1; x_2, y_2; x_3, y_3, be the rectangular co-ordinates of the derived points A, B, C, in the reduced problem (fig. 4); also let r_1, r_2, r_3, be the distances of an arbitrary point from A, B, and C, respectively; then the point G lies on the circular cubic whose equation is

$$r_1(t_2-t_3)+r_2(t_3-t_1)+r_3(t_1-t_2)=0.$$



$\cos\theta=V(t_a-t_b)/AB$

FIG. 3.—Method using two pairs of stations, AB being small in comparison with the distance OG.

If the distances between A, B, and C are small compared with the distance of the gun, the circular cubic may be replaced by its asymptote whose equation, when the origin is at the centre of the circle ABC, is

$$x[x_1(t_2-t_3)+x_2(t_3-t_1)+x_3(t_1-t_2)]+y[y_1(t_2-t_3)+y_2(t_3-t_1)+y_3(t_1-t_2)]$$
$$=\frac{2\Delta^2(t_2-t_3)(t_3-t_1)(t_1-t_2)}{[x_1(t_2-t_3)+x_2(t_3-t_1)+x_3(t_1-t_2)]^2+[y_1(t_2-t_3)+y_2(t_3-t_1)+y_3(t_1-t_2)]^2}$$

where Δ is the area of the triangle ABC. The denominator on the right hand side may also be written

$$a^2(t_1-t_2)(t_1-t_3)+b^2(t_2-t_3)(t_2-t_1)+c^2(t_3-t_1)(t_1-t_2),$$

where a, b, and c are the lengths of the sides of the triangle ABC. The asymptotes of the two cubics associated with ABC and PQR, respectively, meet in a point G from which an approximate position of the point G' may be derived.

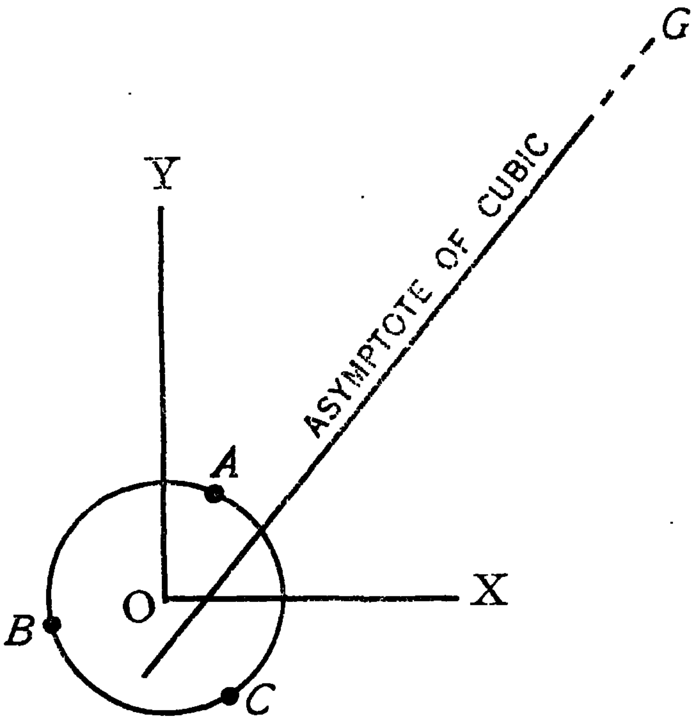


FIG. 4.—Method using two triads of stations, the gun G lying on the circular cubic or its asymptote.

4. *The general theory of rays of sound.*—Let (u, v, w) be the component velocities of the air at a point with rectangular coordinates (x, y, z) when there are no sound waves passing through the atmosphere, ρ the density, and V the local velocity of sound. Let $u+u'$, $v+v'$, $w+w'$, $\rho(1+\epsilon)$ be the values of u, v, w, ρ, when sound waves are present, then if viscosity, thermal conduction and radiation are neglected and the atmosphere is supposed to be in a state of convective equilibrium, and of the same composition throughout, the hydrodynamical equations of motion take the form

$$\frac{du'}{dt}+u'\frac{\partial u}{\partial x}+v'\frac{\partial u}{\partial y}+w'\frac{\partial u}{\partial z}+V^2\frac{\partial \epsilon}{\partial x}=0,$$

$$\frac{dv'}{dt}+u'\frac{\partial v}{\partial x}+v'\frac{\partial v}{\partial y}+w'\frac{\partial v}{\partial z}+V^2\frac{\partial \epsilon}{\partial y}=0,$$

$$\frac{dw'}{dt}+u'\frac{\partial w}{\partial x}+v'\frac{\partial w}{\partial y}+w'\frac{\partial w}{\partial z}+V^2\frac{\partial \epsilon}{\partial z}=0,$$

$$\rho\frac{d\epsilon}{dt}+\frac{\partial}{\partial x}(\rho u')+\frac{\partial}{\partial y}(\rho v')+\frac{\partial}{\partial z}(\rho w')=0,$$

$$\frac{d}{dt}\equiv\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}.$$

In obtaining these equations we have assumed that u', v', w', ϵ, are small, that all changes take place adiabatically, and that no condensation of water vapor occurs in the atmosphere.

If the disturbance which produces the sound waves is initially confined to a limited region, a wave of discontinuity of some type will travel outward. The front of this wave will satisfy the partial differential equation of the characteristics of the above system of equations.[6] The simplest way of obtaining this equation is to assume that $u', v', w', u, v, w, \rho$, are continuous but that the first derivatives of u', v', w', ϵ, suddenly change in value as we cross the moving surface $F(x, y, z, t)=0$. We thus obtain the equations

$$\frac{\partial u'}{\partial F}\frac{dF}{dt}+V^2\frac{\partial \epsilon}{\partial F}\frac{\partial F}{\partial x}=0,\quad \frac{\partial v'}{\partial F}\frac{dF}{dt}+V^2\frac{\partial \epsilon}{\partial F}\frac{\partial F}{\partial y}=0,$$

$$\frac{\partial w'}{\partial F}\frac{dF}{dt}+V^2\frac{\partial \epsilon}{\partial F}\frac{\partial F}{\partial z}=0,$$

$$\rho\frac{\partial \epsilon}{\partial F}\frac{dF}{dt}+\rho\left(\frac{\partial u'}{\partial F}\frac{\partial F}{\partial x}+\frac{\partial v'}{\partial F}\frac{\partial F}{\partial y}+\frac{\partial w'}{\partial F}\frac{\partial F}{\partial z}\right)=0.$$

from which we find that

$$\left(\frac{dF}{dt}\right)^2=V^2\left\{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2\right\}.$$

When a solution of this equation has been found, the rays are given by the equations of the bicharacteristics, namely,

$$=\frac{dx}{u\frac{dF}{dt}-V^2\frac{\partial F}{\partial x}}=\frac{dy}{v\frac{dF}{dt}-V^2\frac{\partial F}{\partial y}}=\frac{dz}{w\frac{dF}{dt}-V^2\frac{\partial F}{\partial z}}=\frac{dt}{\frac{dF}{dt}}$$

If, however, the solution is given as a complete integral

$$F=t-f(x, y, z, \alpha, \beta)=0$$

involving the two arbitrary constants α and β, the rays are obtained by combining the above equation with the equations

$$\frac{\partial f}{\partial \alpha}=0,\qquad \frac{\partial f}{\partial \beta}=0,$$

as in the Hamiltonian theory for rays of light.[7]

When there is no wind, we have the general theorem of Straubel,[8] which may be enunciated as follows: Let P and Q be two points on a ray and let rays consecutive to PQ and forming a small pencil of solid angle $d\omega$ with its vertex at P cut out an area dS' on a plane through Q perpendicular to the ray, while a similar small pencil of rays of solid angle $d\omega'$ with its vertex at Q cuts out an area dS on a plane through P perpendicular to the ray, then we have the relation

$$\frac{d\omega\, dS}{V^2}=\frac{d\omega'\, dS'}{V'^2}.$$

This theorem has not yet been generalized so as to be applicable to the case in which a wind is blowing.

A transformation theory based on the fundamental quadratic form

$$(dx-udt)^2+(dy-vdt)^2+(dz-wdt)^2=V^2dt^2$$

may be used to group together various distributions of u, v, w, V, in which the problem of determining the sound rays is soluble by means of function of a given type. Inversion is a particular transformation which may be used with advantage.

When the air is stratified in horizontal layers so that the wind is blowing horizontally and V depends only on

[6] See *Hadamard:* Propagation des ondes. Paris, 1903.

[7] See *Herman*, Geometrical Optics, Chapter XIII.

[8] *R. Straubel* in Phys. Zeitschr., 1903, 4: 114–117; *T. Levi-Civita* in Rend., Ac. dei Lincei, 1915, 24: 666.

one coordinate Z, the sound rays may be found in a well-known manner as follows:[7]

Let us write

$$\frac{\partial F}{\partial x}=\alpha, \qquad \frac{\partial F}{\partial y}=\beta,$$

then, if $F=t-f(x,y,z,\alpha,\beta)$ as before, we find that

$$t=f(x,y,z,\alpha,\beta)=\int\frac{dz}{V}\sqrt{(1+u\alpha+v\beta)^2-V^2(\alpha^2+\beta^2)}-\alpha x-\beta y$$

and the equations of the rays are obtained by writing

$$\frac{\partial f}{\partial \alpha}=0, \quad \frac{\partial f}{\partial \beta}=0.$$

The same result may also be obtained as follows:[9] Let (θ, ϕ) be the spherical polar coordinates of the wave-normal at a point with rectangular coordinates (x, y, z), then we have the equations

$$\left.\begin{aligned}&\phi=\phi_0,\\&V\mathrm{cosec}\theta+u\cos\phi+v\sin\phi=V_0\mathrm{cosec}\theta_0+u_0\cos\phi+v_0\sin\phi\\&\qquad=\lambda \text{ (say)}\end{aligned}\right\}(1)$$

which express that the velocity and direction of the line or intersection with the horizon of the tangent plane to the wave front remain constant. The suffixes are used here to indicate values of quantities *at the level of the ground.*

The ray velocity is obtained by compounding the wind velocity with the velocity of sound directed along the normal to the wave front, hence the equations determining the rays are

$$\left.\begin{aligned}dx&=udt+Vdt\sin\theta\cos\phi,\\dy&=vdt+Vdt\sin\theta\sin\phi,\\dz&=Vdt\cos\theta,\end{aligned}\right\}(2)$$

The range and time of passage of sound which travels up into the air and down again are thus given by the equations

$$\left.\begin{aligned}x&=2\int_0^Z\frac{V^2\cos\phi+us}{\sqrt{s^2-V^2}}\cdot\frac{dz}{V},\\y&=2\int_0^Z\frac{V^2\sin\phi+vs}{\sqrt{s^2-V^2}}\cdot\frac{dz}{V},\\t&=2\int_0^Z\frac{s}{\sqrt{s^2-V^2}}\cdot\frac{dz}{V},\end{aligned}\right\}(3)$$

where

$$s=\lambda-u\cos\phi-v\sin\phi,$$

and Z is defined by the equation $s=V$.

These equations give

$$\lambda t-x\cos\phi-y\sin\phi=2\int_0^Z\sqrt{s^2-V^2}\,\frac{dz}{V}. \qquad (4)$$

Now let the initial direction of the ray vary slightly so that θ and ϕ become $\theta+d\theta$ and $\phi+d\phi$, then

$$\lambda dt+td\lambda-dx\cos\phi-dy\sin\phi+(x\sin\phi-y\cos\phi)d\phi$$
$$=2\int_0^Z\frac{s(d\lambda+u\sin\phi d\phi-v\cos\phi\, d\phi}{\sqrt{s^2-V^2}}\cdot\frac{dz}{V},$$

or

$$\lambda dt=dx\cos\phi+dy\sin\phi. \qquad (5)$$

It is clear from symmetry that the line joining the two end points of the ray is parallel to the tangent at the vertex, let the displacement dx,dy, be made in this direction; then, if u, v, and V refer to the vertex of the ray, we shall have

$$\frac{dx}{u+V\cos\phi}=\frac{dy}{v+V\sin\phi}=\frac{dx\cos\phi+dy\sin\phi}{\lambda}=dt. \qquad (6)$$

If, on the other hand, the sound travels horizontally the above equations are replaced by

$$dt_0=\frac{dx}{u_0+V_0\cos\phi_0}=\frac{dy}{v_0+V_0\sin\phi_0}$$
$$=\frac{dx\cos\phi+dy\sin\phi}{u_0\cos\phi+v_0\sin\phi+V_0\cos(\phi-\phi_0)}.$$

Now

$$\lambda=u_0\cos\phi+v_0\sin\phi+V_0\mathrm{cosec}\theta_0$$
$$>u_0\cos\phi+v_0\sin\phi+V_0\cos(\phi-\phi_0),$$

hence $dt<dt_0$. This means that the time of travel of the sound increases more rapidly with the range when the sound travels horizontally than when it travels by a curved path through the upper air.

If it is found by means of the test based on this inequality, that the sound does travel to the observer through the upper air we are met with the difficulty that the wind velocity to be used in reducing the problem of sound ranging to the case of no wind, is the wind velocity at the vertex of the ray and this is unknown. It is reasonable to assume, however, that when the vertex is high this velocity is about equal to the gradient-wind velocity and is roughly the same for all the rays that come into consideration. With this assumption, our method of reduction is legitimate and the position of G may be found by the method depending on the use of two triads of observations, for if we make two independent displacements (dx,dy) and (dx_1,dy_1) the equations

$$\lambda dt=dx\cos\phi+dy\sin\phi,$$
$$\lambda dt_1=dx_1\cos\phi+dy_1\sin\phi,$$

are of the same form but with different values of λ whether the sound travels horizontally or through the upper air; also the quantity λ, like the velocity of sound, is eliminated in the above method.

The gradient-wind velocity may, of course, be derived from the weather map when this is available.

When the vertex of a ray is not very high the difference in the times of travel of this ray and of one that is supposed to travel horizontally between the same end points, is generally so small as to be almost negligible and probably the best plan is to make the calculations just as if all the sound traveled horizontally.

It should be noticed that if the velocity of the wind is less than the velocity of sound there are always two

[9] *Lord Rayleigh*, Theory of Sound, v. 2, p. 133.

directions, ϕ, for a ray which has its vertex at a given level, for which the relation $\phi = \phi_0$ is satisfied. These directions are given by the equation

$$uv_0 - u_0 v + (Vv_0 - V_0 v)\cos\phi + (uV_0 - u_0 V)\sin\phi = 0,$$

and are real because the inequality

$$(Vv_0 - V_0 v)^2 + (uV_0 - u_0 V)^2 > (uv_0 - u_0 v)^2$$

is satisfied since it may be written in the form

$$[(u_0^2 + v_0^2)V - (uu_0 + vv_0)V_0]^2 > (uv_0 - u_0 v)^2[u_0^2 + v_0^2 - V_0^2].$$

It generally happens, however, that for some values of ϕ the rays that start horizontally at a given height Z bend upward instead of downward, and so are not rays that start from the ground. Using primes to denote differentiations with regard to z, we find on applying the equation

$$V\operatorname{cosec}\theta + u\cos\phi + v\sin\phi = \lambda$$

to the level Z and a consecutive level, that

$$(V' + u'\cos\phi + v'\sin\phi)dz + V(\sec d\theta - 1) = 0;$$

hence a ray bends upward or downward according as

$$V' + u'\cos\phi + v'\sin\phi \lesseqgtr 0$$

When there are directions for both upward and downward rays, the two angular regions are separated by two radii whose directions are given by the equation

$$V' + u'\cos\phi + v'\sin\phi = 0.$$

These directions are real if $u'^2 + v'^2 > V'^2$. In this case the height Z may be called *anticlastic* (symbol $\pm$). If $u'^2 + v'^2 \leqq V'^2$ the height Z is said to be *positively synclastic* when rays which start horizontally bend downward, and *negatively synclastic* when rays which start horizontally bend upward (symbols + and −).

Let us now consider a ray which leaves the ground horizontally, then bends upward and finally returns to the ground. If this ray also possesses the property $\phi = \phi_0$ it follows from equation (4) that the time of travel is greater than if the sound traveled horizontally between the two end points of the ray. For since $\phi = \phi_0$ the quantity λ is the same in both cases and so also is $x\cos\phi + y\sin\phi$, but in the first case we have

$$\lambda t - x\cos\phi - y\sin\phi > 0$$

while in the second case the left-hand side is zero. This proves the theorem.

Now let us gradually increase the range, then since the time of travel for a ray through the upper air increases less rapidly for a ray which travels through the upper air than for a ray which travels horizontally, it follows that the times gradually become equal and eventually the time of travel through the upper air may become less than the time that sound would take to travel horizontally.

When the wind blows uniformly in one direction the condition $\phi = \phi_0$ is satisfied in the case of a ray which starts either in the direction of the wind or in the opposite direction.

When there is no wind blowing the condition $\phi = \phi_0$ is satisfied for every ray and when V increases upward every ray which goes up into the air is brought down to the ground again, while the time of travel through the upper air is always less than if the sound traveled horizontally between the end points of the ray. To prove this we notice that if R denotes the range, equations (1) and (5) give

$$\frac{dR}{dt} = V = V_0 \operatorname{cosec}\theta_0 > V_0, \qquad \text{therefore } \frac{R}{V} > t.$$

An interesting case arises when we adopt H. Mohn's assumption [10] that V is a linear function of z to a first approximation. The equations $V = V_0 + \sigma z$,

$$t = 2\int_0^Z \frac{\lambda dz}{V\sqrt{\lambda^2 - V^2}}, \qquad R = 2\int_0^Z \frac{Vdz}{\sqrt{\lambda^2 - V^2}},$$

then give

$$\sigma R = 2\sqrt{\lambda^2 - V_0^2}, \qquad \sigma t = 2\cosh^{-1}\frac{\lambda}{V_0},$$

$$\sigma R = 2V_0 \sinh\frac{\sigma t}{2},$$

and it is clear from this equation that $R > V_0 t$. The problem of sound ranging under such conditions is rather interesting, for when timings are made at two stations A_1 and A_2 the locus of the gun is given by the equations

$$\sigma R_1 = 2V_0 \sinh\frac{\sigma t_1}{2}, \qquad \sigma R_2 = 2V_0 \sinh\frac{\sigma t_2}{2},$$

which determine a bicircular quartic with A_1 and A_2 as foci when the difference $t_1 - t_2$ is given. As this difference varies we get a system of bicircular quartics (fig. 5) with two, but not four, real foci in common and this corresponds to the system of confocal hyperbolas in the ordinary problem of sound ranging.

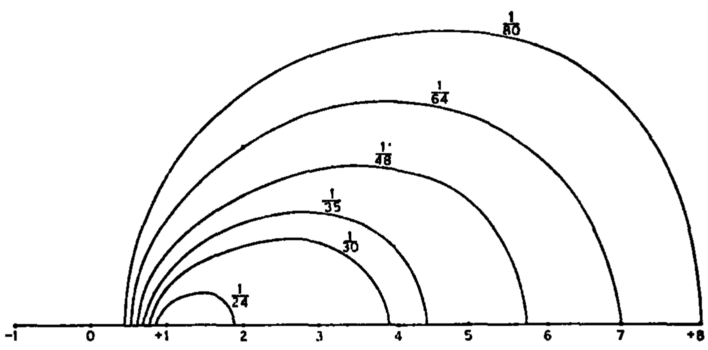


FIG. 5.—System of bicircular quartics with two real foci in common at −1 and +1, and the parametric equations $r_1 = 16\sinh(\theta + \alpha)$, $r_2 = 16\sinh\theta$; the numbers indicate the value of $\sinh^2 \frac{1}{2}\alpha$ for the different quartics.

Although the present problem of sound ranging is only of theoretical interest, it may be remarked that when observations are made at three stations A_1, A_2, A_3, the gun may be located by using charts on which the bicircular quartics of the required type are drawn, one system being associated with A_1, A_2, and another system with A_2, A_3. It is important to notice that the same charts can be used with different values of σ by simply altering the dimensions of the figure A_1A_2, B_1B_2 keeping its form unaltered.

[10] *H. Mohn* in Annalen der Hydrographie, 1892, 1893, 1895.

If in the present circumstances an observer hears first by a direct ray and afterward by a ray reflected from the ground, then if t and t' are the two intervals between the firing of the gun and the observations we have

$$\sigma R = 2V_0 \sinh\frac{\sigma t}{2}, \qquad \sigma R = 4V_0 \sinh\frac{\sigma t'}{4}.$$

To get an idea of the magnitude of the interval $t'-t$, let us consider the following ways of satisfying the equation $2\sinh\frac{\sigma t}{4} = \sinh\frac{\sigma t}{2}$:

1) $\frac{\sigma t'}{4} = 0.22, \qquad \frac{\sigma t}{2} = 0.43018, \qquad \sigma(t'-t) = 0.01964;$

2) $\frac{\sigma t'}{4} = 0.1, \qquad \sigma(t'-t) = \frac{2}{1019}.$

The sound which is reflected takes slightly longer to reach the observer and the difference in time is about 0.0223 of the total time of travel in the first case and about 0.00497 in the second. Thus if the sound takes 10 seconds to travel, the interval is about $\frac{1}{5}$ second in the first case and about $\frac{1}{20}$ second in the second. The value of σ in the first case is 0.088, representing an increase in V of 8.8 meters per second in 100 meters; in the second case σ is 0.04 and represents an increase in V of 4 meters per second in 100 meters. These rates of increase in V are not abnormally large, for instance at Drexel, Nebr., on May 8, 1916, at 6:21 a. m. the temperatures at altitudes of 396 and 500 meters were 8° and 11.8° C., respectively, while the wind velocity and humidity between these levels were practically constant. A calculation of the velocity of sound indicates that in this case V increases by about 4.5 meters per second in 104 meters.

5. *The initial radius of curvature of a ray which starts horizontally.*—Let us write $u = u_0 - az$, $v = v_0 - bz$, $V = V_0 - \sigma z$, then for a ray which starts in a horizontal direction and then bends upward

$$S = V_0 + a\cos\phi + b\sin\phi,$$

and we have to a first approximation

$$x = \int_0^z \frac{(V_0\cos\phi + u_0)\,dz}{\sqrt{2V_0 z(a\cos\phi + b\sin\phi + \sigma)}} = 2\sqrt{z}\cdot\frac{V_0\cos\phi + u_0}{\sqrt{2V_0(a\cos\phi + b\sin\phi + \sigma)}}$$

$$y = \int_0^z \frac{(V_0\sin\phi + v_0)\,dz}{\sqrt{2V_0 z(a\cos\phi + b\sin\phi + \sigma)}} = 2\sqrt{z}\cdot\frac{V_0\sin\phi + v_0}{\sqrt{2V_0(a\cos\phi + b\sin\phi + \sigma)}}$$

The initial radius of curvature is thus

$$\rho = \frac{(V_0\cos\phi + u_0)^2 + (V_0\sin\phi + v_0)^2}{2V_0(a\cos\phi + b\sin\phi + \sigma)}$$

It should be noticed that ρ becomes infinite when $a\cos\phi + b\sin\phi + \sigma = 0$, while maximum and minimum values of ρ are roots of the equation

$$(u_0 - \rho a)^2 + (v_0 - \rho b)^2 = \left(\rho\sigma - \frac{V_0^2 + u_0^2 + v_0^2}{2V_0}\right).$$

When a ray does not start in a horizontal direction but goes up into the air and returns to the earth, the radius of curvature at its vertex may be found by means of a similar formula using of course the values of $u_0, v_0, V_0, a, b, \sigma$, for the level of the vertex.

6. *The case in which the gun and the observer are at different levels.*—In this case the limits of integration in the equations (3) are altered so that if (x_1, y_1, z_1) are the coordinates of a gun, (x_2, y_2, z_2) those of the observer, we have for a ray without a vertex

$$\lambda(t_2 - t_1) - (x_2 - x_1)\cos\phi - (y_2 - y_1)\sin\phi = \int_{z_1}^{z_2} \sqrt{s^2 - V^2}\frac{dz}{V},$$

while for a ray with a vertex

$$\lambda(t_2 - t_1) - (x_2 - x_1)\cos\phi - (y_2 - y_1)\sin\phi = \int_{z_1}^{Z} \sqrt{s^2 - V^2}\frac{dz}{V} + \int_{z_2}^{Z} \sqrt{s^2 - V^2}\frac{dz}{V}.$$

Now let the positions of the gun and observer vary, then we find that

$$\lambda dt_2 - dx_2\cos\phi - dy_2\sin\phi \mp dz_2\cot\theta_2 = \lambda dt_1 - dx_1\cos\phi - dy_1\sin\phi - dz_1\cot\theta_1. \quad (7)$$

This relation is analagous to one which occurs in the general theory of geodesics.[11] It is important to notice that if the observer and gun are in motion so that

$$dx_2 = a_2 dt_2, \qquad dy_2 = b_2 dt_2, \qquad dz_2 = c_2 dt_2,$$
$$dx_1 = a_1 dt_1, \qquad dy_1 = b_1 dt_1, \qquad dz_1 = c_1 dt_1,$$

the above equations give

$$\frac{dt_2}{dt_1} = \frac{\lambda - a_1\cos\phi - b_1\sin\phi - c_1\cot\theta_1}{\lambda - a_2\cos\phi - b_2\sin\phi \mp c_2\cot\theta_2},$$

$$= \frac{V_1\operatorname{cosec}\theta_1 + (u_1 - a_1)\cos\phi + (v_1 - b_1)\sin\phi - c_1\cot\theta_1}{V_2\operatorname{cosec}\theta_2 + (u_2 - a_2)\cos\phi + (v_2 - b_2)\sin\phi \mp c_2\cot\theta_2}.$$

This is the form which Doppler's principle assumes for an atmosphere stratified in parallel planes. It should be noticed that this formula differs slightly from the empirical formula suggested in my previous note[12] and gives $dt_1 = dt_2$ when the source of sound and observer are both stationary. In this case, then, there is no change of frequency, as might be expected. The above formula does not, however, quite cover the case for which the empirical formula was suggested, because now the wind velocity and velocity of sound are the same at the locations of the gun and the observer, if these are at the same level, whereas in the case referred to the wind velocity was supposed to be different in the two places.

When a source of sound is higher than the observer, or is moving through the air, it is possible that some meteorological data may be obtained by noting the change in pitch when the sound is heard at the earth's surface and applying our general formula for Doppler's principle. C. E. Stromeyer[13] has already suggested that gustiness might be recorded by observing what in

[11] *T. Levi-Civita*, Rendiconti, Accad. dei Lincei, 1915, 24: 666.
[12] *H. Bateman*, Doppler's principle for a windy atmosphere. MONTHLY WEATHER REVIEW, Sept., 1917, 45: 441.
[13] *Stromeyer* in Nature, London, Apr. 26, 1917.

German is called "wimmern," that is, a variation in the sounds heard from church bells during gusty weather.

7. *Derivation of meteorological data by means of sound ranging.*—When the positions of the gun and the observer are known, the timing of the sound furnishes some information with regard to the structure of the atmosphere.

In the first place, if the sound travels through the upper air and the gun and the observer are at the same level, the equations

$$\lambda dt = dx\cos\phi + dy\sin\phi, \qquad \lambda\delta t = \delta x\cos\phi + \delta y\sin\phi,$$

corresponding to two different displacements of the observer, determine the angle ϕ and the quantity λ. Since the direction of the gun is known a displacement can be made directly away from the gun and then the equations (6) determine the quantities

$$u + V\cos\phi, \qquad v + V\sin\phi,$$

for the level of the vertex of the ray. Now, let the source of sound be raised a small distance dz, then it follows from equation (7) that

$$\lambda(dt_2 - dt_1) = -dz_1\cot\theta_1.$$

This equation determines the angle θ_1 which is denoted below by θ_0.

Returning to the case in which both the gun and observer are on the ground, let us vary the initial direction of a ray in such a manner that Z remains constant. We then have

$$dx = -2\int_0^Z \frac{Vdz}{(s^2 - v^2)^{\frac{3}{2}}}[(s^2 - V^2)\sin\phi.d\phi + (u + s\cos\phi)ds].$$

Now dx must be finite, hence ds must vanish when $z = Z$. Using the value

$$s = V_0\operatorname{cosec}\theta_0 + (u_0 - u)\cos\phi + (v_0 - v)\sin\phi,$$

we find that we must have

$$V_0\operatorname{cosec}\theta_0.\cot\theta_0.d\theta_0 = [(u - u_0)\sin\phi - (v - v_0)\cos\phi]d\phi.$$

Now θ_0, ϕ, and $u\sin\phi - v\cos\phi$, are known from the previous observations and V_0, u_0, v_0, can be determined from meteorological observations at the level of the ground, hence we can determine the relation between $d\theta_0$ and $d\phi$. This means that we can trace on the ground a curve which is the locus of the extremities of rays which start at a given point and have their vertices at a given height.

By using observations at two points of this curve for which the values of ϕ are ϕ_1 and ϕ_2, respectively, we can calculate the quantities

$$u + V\cos\phi_1, \qquad u + V\cos\phi_2,$$
$$v + V\sin\phi_1, \qquad v + V\sin\phi_2,$$

and so determine the values of u, v, and V.

Drawing curves for different altitudes of the vertex of a ray, we may obtain an idea of the structure of the atmosphere as far as the variations of u, v, and V, are concerned, but unfortunately we can not assign the value of Z for each curve.

An idea of the structure of the atmosphere may also be obtained in a rough way by a study of the regions of audibility after a great explosion. Data of the required type have been collected on several occasions [14] and some types of sound areas have been correlated with certain weather conditions. In Japan, according to Prof. Omori, 9 out of 11 recent Asama-yama explosions with double sound-areas occurred in the winter, while 10 out of 11 explosions with single sound-areas occurred in the summer months.

8. *Some numerical data.*—At Prof. C. F. Marvin's suggestion I have made some numerical calculations, using the data obtained at Drexel, Nebr., in 1916. The velocity of sound in the following table is calculated on the assumption that the velocity for dry air at 0°C is 333.4 meters per second, the value given by Chwolson. In column 10 the symbols indicate the characteristics of the different levels as explained in § 4; it will be seen that in most cases a level is anticlastic, but when the temperature decreases upward it is quite common for all rays which start horizontally at a given level, to be refracted upward. In order that rays may go up into the air and down again it is necessary that the total change in wind velocity in ascending to some level should be greater in magnitude than the decrease or negative increase in the velocity of sound. So far no case has been found, in the summer, in which rays starting nearly horizontally are brought down to the ground for every horizontal direction; but in January when the wind velocity is low such cases do occur.

TABLE 1.—*Velocities of sound and characteristics of different levels in six cases based on actual data secured at Drexel Aerological Station, Nebraska.*

Time.	Altitude.	Temperature.	$\frac{\Delta t}{100}$.	Pressure.	Vap. press.	Velocity of sound.	Wind. Dir.	Wind. Vel.	Symbol.*	Remarks.
	m.	*°C.*	*°C.*	*mbar.*	*mbar.*	*m. p. s.*		*m./s.*		
(1) April 11, 1916. (Supplement No. 7, p. 9.)										
A. M. 9:39	396	19.0	....	966.7	7.69	345.326	se	8.9	......	All rays refracted upward.
....	500	18.5	....	955.0	7.67	345.036	se	9.1	−	
....	750	17.2	....	927.1	7.26	344.252	se	9.7	−	
(2) May 13, 1916. (Supplement No. 7, p. 27.)										
P. M. 12:45	396	12.4	....	964	13.82	341.815	ese	6.3	......	Some rays are brought down to the ground.
....	500	11.8	....	952	13.29	341.431	ese	8.0	±	
12:50	743	10.5	0.55	924.7	12.06	340.592	se	12.0	±	
....	750	12.6	....	924	13.86	341.979	se	10.1	±	
1:15	871	14.4	−3.05	910.6	15.42	343.180	sse	8.5	±	
(3) May 5, 1916. (Supplement No. 7, p. 22.)										
A. M. 6:08	396	13.2	....	965.3	8.50	341.935	sw	5.4	......	Some rays are brought down to the ground.
6:10	487	20.5	−8.02	955.1	10.61	346.419	sw	14.1	±	
....	500	20.5	....	954	10.37	346.404	sw	13.9	±	
....	750	20.5	....	926.1	8.20	346.231	sw	11.4	±	
(4) May 19, 1916. (Supplement No. 7, p. 29.)										
P. M. 5:49	396	13.6	....	967.8	13.55	342.511	se	4.0	......	Some rays are brought down to the ground.
....	500	12.8	....	955.6	13.01	342.008	se	4.9	±	
....	750	10.7	....	927.7	11.45	340.667	se	7.2	±	
6:51	937	9.2	0.81	907.2	10.48	339.710	se	8.9	±	
6:52	983	10.4	−2.28	902.2	11.35	340.502	se	8.9	+	
(5) Jan. 3, 1916. (Supplement No. 5, p. 6.)										
A. M. 8:54	396	−4.1	....	980.4	4.07	331.147	s	4.9	......	All rays refracted downward.
....	500	−1.4	....	967.2	4.62	332.847	s	5.5	+	
9:01	669	2.9	−2.56	947.4	5.27	335.519	s	6.6	+	
(6) May 4, 1916. (Supplement No. 7, p. 21.)										
A. M. 6:49	396	10.0	....	959.5	7.98	339.986	ssw	10.3	......	Some rays are brought down to the ground.
....	500	14.4	....	948.1	9.51	342.730	ssw	17.2	±	
6:51	613	19.1	−4.19	935.1	11.28	345.655	ssw	24.6	±	

* Z is positively (+) or negatively (−) synclastic, or anticlastic (±). (See p. 8.)

[14] See for example, S. *Fujiwhara* in Bull., Central met. obs'y., Tokyo, 1916, pt. 2; and *C. Davison* in Quarterly review, July, 1917.

9. *The magnitude of the error in sound ranging.*—Let us consider the reduced problem in which the time is given for two stations A and B, then G lies on one branch of a hyperbola with A and B as foci. Let a be the semi-major axis of this hyperbola and let $AB=2c$, then the equation of the hyperbola, when the origin is taken at 0, the middle point of AB, is—

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1. \tag{8}$$

If the times vary slightly on account of errors in timing, while A and B remain fixed, a varies and the hyperbola is changed into a confocal hyperbola while the increments of x,y,a, are connected by the relation

$$\frac{2xdx}{a^2}-\frac{2ydy}{c^2-a^2}=2ada\left[\frac{x^2}{a^4}+\frac{y^2}{(c^2-a^2)^2}\right]=2\frac{da}{a}\left[1+\frac{c^2y^2}{(c^2-a^2)^2}\right].$$

The total displacement of G is least when

$$dx=\lambda\frac{x}{a^2},\ dy=-\frac{\lambda y}{c^2-a^2},\ \lambda=ada,$$

and then

$$ds=\sqrt{(dx)^2+(dy)^2}=ada\left[\frac{x^2}{a^4}+\frac{y^2}{(c^2-a^2)^2}\right]^{\frac{1}{2}}=da\left[1+\frac{c^2y^2}{(c^2-a^2)^2}\right]^{\frac{1}{2}}.$$

If, on the other hand, $dx=0$, we have

$$dy=-\frac{c^2-a^2}{ay}da\left[1+\frac{c^2y^2}{(c^2-a^2)^2}\right].$$

To get an idea of the magnitude of these quantities let us take the case when $c^2=2a^2$,

$$\text{then}\quad ds=da\left[1+\frac{4y^2}{c^2}\right]^{\frac{1}{2}},\qquad dy=-\frac{a}{y}da\left[1+\frac{4y^2}{c^2}\right].$$

If $V=1{,}100$ feet a second, an error in timing of 1/100 second may mean an error in $2a$ of 11 feet.

The following table then gives the magnitude of the error in ranging for different values of the ratio $y:c$.

TABLE 2.—*Magnitude of error in ranging.*

$y:c$	ds	dy
	Feet.	*Feet.*
$\sqrt{2}$	$16\frac{1}{2}$	$24\frac{3}{4}$
$\sqrt{12}$	$38\frac{1}{2}$	55
$\sqrt{42}$	$71\frac{1}{2}$	101
$\sqrt{420}$	$225\frac{1}{2}$	315

When the wind is blowing, the points A and B are displaced slightly from their true positions on account of the error in timing, but if the wind velocity is as large as 20 feet a second the displacement caused by an error of 1/100 second in timing is only 2 feet.

Let us now estimate the magnitude of the error introduced when the asymptote of the hyperbola is used instead of the hyperbola. Since the equation of the asymptotes is

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=0,$$

we find on subtracting from (8) that $2ydy=c^2-a^2$. Writing $c^2=2a^2$ as before, we find that if $AB=2c=1{,}000$ feet, $y=10{,}000$ feet, $dy=6\frac{1}{4}$ feet.

The error introduced by using the asymptote of the circular cubic, in the method in which the velocity of sound is eliminated, is more difficult to determine; it is probably larger than in the case just discussed but still not large enough to be important.

To get some idea of the error in time arising from the circumstance that sound may travel through the upper air instead of along a horizontal path, let us consider the simple case in which there is no wind and the velocity of sound increases upward. The range is now given by the formula

$$R=\frac{2V_0}{\sigma}\sinh\frac{\sigma t}{2}$$

Taking $\sigma=0.04$, $t=10$ seconds, we have

$$\frac{\sigma t}{2}=0.2,\ \sinh\frac{\sigma t}{2}=0.20134,\ R=55{,}000\times0.20134,$$

$$\frac{R}{V_0}-t=50\times0.00134=0.067.$$

The difference in time is thus about 7/100 second in a range of about 2 miles and may cause a serious error in sound ranging if no attention is paid to the effect of the meteorological conditions.

MEAN VALUES OF FREE-AIR BAROMETRIC AND VAPOR PRESSURES, TEMPERATURES, AND DENSITIES OVER THE UNITED STATES.

By WILLIS RAY GREGG, Meteorologist in Charge.

[Dated: Division of Aerological Investigations, Weather Bureau, Jan. 31, 1918.]

Although numerous free-air observations have been made, and are being made, in different parts of the United States, mean values of certain reduced data have thus far not been published. Temperature, humidity, and wind data, as observed at Mount Weather, Va., have been summarized in the Bulletin of the Mount Weather Observatory, 1913, v. 6, pts. 4 and 5, and similar summaries of observations by means of sounding balloons at different points in this country have been presented in the same publication, v. 4, pt. 4, and in the MONTHLY WEATHER REVIEW, July, 1914, and May, 1916. A paper on "The Planetary System of Convection" by Dr. (now Major) Wm. R. Blair, appeared in the MONTHLY WEATHER REVIEW, April, 1916. The conclusions reached in the latter paper were based on all available free-air observations made in the United States and in other parts of the world. More recently a brief summary, with special reference to the needs of aeronauts, has been prepared by Maj. W. R. Blair and published by the National Advisory Committee for Aeronautics as Report No. 13. None of these summaries has included free-air pressures and densities. As a knowledge of these data is of great importance in connection with aviation and the firing of projectiles, it has been thought best to publish in brief tabular form their average values as determined from all available observations thus far made by the U. S. Weather Bureau in this country. At a later time similar tables will be furnished for additional stations which are now being established.

Table 1, below, gives mean monthly, seasonal, and annual free-air pressures, temperatures, vapor pressures, and densities as observed at Mount Weather, Va. Pressures and vapor pressures are expressed in both metric and dynamic units, temperatures on the centigrade and approximate absolute (273+ t°C) scale, and