Boussinesq and Anelastic Approximations Revisited: Potential Energy Release during Thermobaric Instability

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ABSTRACT

Expressions are derived for the potential energy of a fluid whose density depends on three variables: temperature, pressure, and salinity. The thermal expansion coefficient is a function of depth, and the application is to thermobaric convection in the oceans. Energy conservation, with conversion between kinetic and potential energies during adiabatic, inviscid motion, exists for the Boussinesq and anelastic approximations but not for all approximate systems of equations. In the Boussinesq/anelastic system, which is a linearization of the thermodynamic variables, the expressions for potential energy involve thermodynamic potentials for salinity and potential temperature. Thermobaric instability can occur with warm salty water either above or below cold freshwater. In both cases the fluid may be unstable to large perturbations even though it is stable to small perturbations. The energy per mass of this finite-amplitude instability varies as the square of the layer thickness. With a 4-K temperature difference and a 0.6-psu salinity difference across a layer that is 4000 m thick, the stored potential energy is ~ 0.3 $m^2\,s^{-2}$, which is comparable to the kinetic energy of the major ocean currents. This potential could be released as kinetic energy in a single large event. Thermobaric effects cause parcels moving adiabatically to follow different neutral trajectories. A cold fresh parcel that is less dense than a warm salty parcel near the surface may be more dense at depth. Examples are given in which two isopycnal trajectories cross at one place and differ in depth by 1000 m or more at another.

1. Introduction

The dependence of density on three variables—temperature, pressure, and salinity—allows a column of seawater to store a finite amount of potential energy for later release, in a process called thermobaric instability. In this paper the Boussinesq approximation (section 2) is used to calculate this stored potential energy, which is the maximum amount of energy that could be released during a single event without forcing or dissipation. Energy relations for the Boussinesq approximation when the density depends on three variables seem not to have been calculated before, and so I present three derivations: from the equations of motion (section 3), from parcel theory (section 4), and from the potential energy of the column (section 5). I discuss the errors introduced by the Boussinesq approximation (section 6),

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and I give examples of thermobaric effects both in an idealized situation (section 7) and in the real ocean (section 8).

Spiegel and Veronis (1960) is the classic paper on the Boussinesq approximation, which they derive by linearizing the thermodynamic variables in the equations of motion. They do not discuss potential energy, and they do not discuss what happens when the density depends on three variables instead of two. Two classic papers on the anelastic approximation are Ogura and Phillips (1962) and Gilman and Glatzmaier (1981). They also use a linearization, and like Spiegel and Veronis they use an ideal gas equation of state. Ingersoll and Pollard (1982) give a simple derivation of the anelastic equations for arbitrary equation of state, and Lilly (1996) compares the Boussinesq and anelastic approximations. These authors present energy integrals, but they do not discuss what happens when the density depends on three variables instead of two. Both approximations are introduced to filter out sound waves from the equations of motion.

Akitomo (1999) gives a scaling argument for the ver-

tical velocity during thermobaric instability. Converted into kinetic energy, Akitomo's Eq. (14) is consistent with the stored potential energy that is derived here. Akitomo considers the efficiency of energy release, whereas I estimate the maximum possible energy release. Davis (1994), Bacon and Fofonoff (1996), and de Szoeke (2000) define a quantity analogous to the thermodynamic potential for salinity. I introduce the analogous quantity for potential temperature. These potentials are needed when the thermal expansion coefficient is a function of depth; they appear in our expression for stored potential energy. Several authors (e.g., Garwood et al. 1994; Løyring and Weber 1997; Kay 2001; McPhee 2003) consider thermobaric convection in various contexts, but they do not give the expressions for stored potential energy.

Not all approximate systems of equations conserve energy during adiabatic, inviscid motion. Here energy conservation means a pair of equations for the time rate of change of kinetic and potential energy density, respectively. Both equations have a term representing the divergence of an energy flux. Both equations have an energy conversion term; it is the same expression in both equations but has opposite sign. By adding this pair of equations the conversion term drops out, and by integrating the result over the whole domain the divergence term drops out, leaving the time derivative of the global integral of total (kinetic plus potential) energy. This time derivative is zero for adiabatic, inviscid motion, meaning that total energy is conserved.

Ordinary convective instability sets in immediately whenever the potential energy is positive—that is, whenever there is a state with lower energy that can be reached with adiabatic inviscid motions. In contrast, thermobaric instability requires a finite perturbation to get past an energy barrier, but then it releases a finite amount of stored potential energy. In this respect, it is analogous to moist convection in the atmosphere. An essential element of thermobaric instability is the increase of the thermal coefficient of expansion α with pressure. The salinity contraction coefficient β is largely independent of pressure, which means that the effect of temperature on the density tends to dominate at depth and the effect of salinity tends to dominate near the surface. Consider a column made up of cold freshwater (CFW) and warm salty water (WSW) that is stable to small perturbations (Fig. 1). If the CFW is above the WSW, a parcel of CFW from the surface may find itself denser than its surroundings when it reaches the bottom—it gains negative buoyancy. If the CFW is below the WSW, a parcel of CFW from the bottom may find itself less dense than its surroundings when it reaches the surface—it gains positive buoyancy. These
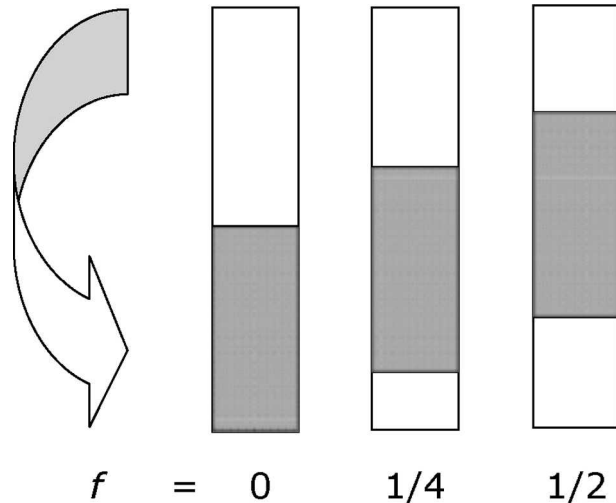

FIG. 1. Thermobaric instability when cold freshwater (CFW, white) is initially above warm salty water (WSW, gray). The three boxes show the distribution of water masses when a fraction f of the CFW has been transferred to the bottom. In the initial ($f = 0$) state, density is continuous at the interface, and so the fluid is neutrally stable to small perturbations. Finite displacements (curved arrow) lead to states ($f = 1/4$ and $f = 1/2$) with successively lower potential energy than the initial state. The same diagram turned upside down describes thermobaric instability when CFW is initially below WSW. Then f is the fraction of CFW that has been transferred to the top.

finite-amplitude displacements lower the potential energy of the system, even though infinitesimal displacements raise it. Thus the system may be stable to small perturbations and unstable to large perturbations. The CFW may be on the bottom or on the top, but in both cases the energy release is associated with large displacements of the CFW and small displacements of the WSW.

2. Boussinesq and anelastic approximations

Following Spiegel and Veronis (1960), the Boussinesq/anelastic approximations are derived by linearizing the thermodynamic terms in the equations of motion. The basic state is isentropic, chemically homogeneous, and hydrostatic, and so the basic-state potential temperature, salinity, pressure, and density satisfy $\theta_0 =$ constant, $S_0 =$ constant, and $dP_0/dz = -\rho_0 g$, respectively. In the anelastic approximation both g and ρ_0 may depend on the vertical coordinate z; in the Boussinesq approximation they are constant, but otherwise the two approximations are the same. What I am calling the Boussinesq and anelastic approximations are sometimes called the strong and weak Boussinesq approximations, respectively. In both cases the small depar-

tures from the basic-state variables are $\delta\theta$, δS, and δP. These, together with the velocity $\mathbf{v}$, are the system's dependent variables, which depend on the spatial coordinates and time.

We linearize the density as follows: $\rho = \rho_0 + \delta\rho = \rho_0(1 - \alpha_\theta\delta\theta + \beta\delta S + \gamma\delta P)$, where $\alpha_\theta = -\rho_0^{-1}(\partial\rho/\partial\theta)_{S,P}$ is the thermal coefficient of expansion based on potential temperature, $\beta = \rho_0^{-1}(\partial\rho/\partial S)_{\theta,P}$ is the salinity contraction coefficient, and $\gamma = \rho_0^{-1}(\partial\rho/\partial P)_{\theta,S}$ is the adiabatic compressibility. Here ρ_0, α_θ, β, and γ are basic-state quantities that depend only on the vertical coordinate. Alternately, γ is $1/(\rho_0 c^2)$, where $c^2 = (\partial P/\partial\rho)_{\theta,S}$ is the square of the speed of sound. Because the basic state is isentropic and hydrostatic, the scale height of basic-state density is c^2/g:

$$\frac{1}{\rho_0}\nabla\rho_0 = \frac{1}{\rho_0}\left(\frac{\partial\rho_0}{\partial P}\right)_{\theta,S}\nabla P_0 = \gamma\nabla P_0 = -\gamma\rho_0 g\hat{k} = -\frac{g\hat{k}}{c^2},$$

(1)

where $\hat{k}$ is the vertical unit vector. With $P = P_0 + \delta P$, $\rho = \rho_0 + \delta\rho$, and $\nabla P_0 = -\rho_0 g\hat{k}$, the linearized forms of the terms $-\nabla P/\rho - g\hat{k}$ in the momentum equation become

$$-\frac{1}{\rho_0}\nabla\delta P - g\hat{k}\frac{\delta\rho}{\rho_0}$$

$$= -\frac{1}{\rho_0}\nabla\delta P - g\hat{k}(-\alpha_\theta\delta\theta + \beta\delta S + \gamma\delta P). \quad (2)$$

The two terms on the right that involve δP may be combined with the aid of Eq. (1) into a single term $-\nabla(\delta P)/\rho_0$. The inviscid form of the momentum equation is then

$$\frac{D\mathbf{v}}{Dt} = -\nabla\left(\frac{\delta P}{\rho_0}\right) + g\hat{k}(\alpha_\theta\delta\theta - \beta\delta S), \quad (3)$$

where $D/Dt = (\partial/\partial t + \mathbf{v}\cdot\nabla)$ is the material derivative. Equation (3) reduces to the Boussinesq form when ρ_0 is constant.

The Boussinesq and anelastic forms of the continuity equation require that $\delta\rho/\rho_0$ be small. The buoyancy terms—the second group on the right of Eq. (3)—are of order $g\delta\rho/\rho_0$. The time-derivative and advection terms on the left are of order v^2/L, where v is the typical velocity and L is the typical length scale, which is comparable to the depth. Therefore $\delta\rho/\rho_0 \sim v^2/(gL)$. Moreover if N^2, which is the Brunt–Väisälä frequency squared, is of order $g\delta\rho/(\rho_0 L)$, then $v^2 \sim N^2L^2$. So $N^2L^2 \ll gL$ ensures that $\delta\rho/\rho_0 \ll 1$, and I can neglect the density anomaly $\delta\rho$ in the continuity equation, which becomes

$$\nabla\cdot(\rho_0\mathbf{v}) = 0. \quad (4)$$

This equation reduces to the Boussinesq form when ρ_0 is a constant. The anelastic and Boussinesq approximations both require $N^2L^2 \sim v^2 \ll gL$. In addition, the Boussinesq approximation requires $gL \ll c^2$, which ensures that the depth L is small in comparison with the density scale height. Note that the $\gamma\delta P$ term in Eq. (2) is negligible when $gL \ll c^2$.

The two remaining equations are the conservation laws for salt and heat. I define $F = -(\nabla\cdot\mathbf{F}_F)/\rho$ as the fractional rate of freshwater convergence (mass of freshwater per unit time per unit mass of seawater), where $\mathbf{F}_F$ is the freshwater mass flux (mass per unit area per unit time). Then the salt equation becomes

$$\frac{D\delta S}{Dt} = -S_0 F. \quad (5)$$

The heat equation is $TD\eta/Dt = Q$, where T is absolute temperature, η is entropy/mass, and Q is the rate of heating per unit mass, that is, $Q = -(\nabla\cdot\mathbf{F}_H)/\rho$, where $\mathbf{F}_H$ is the heat flux. There are additional terms in Q that arise from the dissipation and divergence of the salt flux (Davis 1994; Bacon and Fofonoff 1996). However, these terms are of order $P\delta V \sim PV\beta\delta S \sim PV\alpha\delta T$, whereas the thermal energy term is of order $c_P\delta T$. The latter is ~5000 times the former, and so Q is usually treated simply as a heating term (Bacon and Fofonoff 1996). I return to this point at the end of section 5.

To express the heat equation in terms of θ, the relation between θ and η is needed. Potential temperature is equal to the real temperature when pressure is equal to the reference pressure, so that $Td\eta/dT = \theta d\eta/d\theta = c_{P0}$ at the reference pressure. Here c_{P0} is the specific heat at the reference pressure, evaluated at (θ_0, S_0). Thus c_{P0} does not depend on the vertical coordinate z. It is a basic-state quantity, and so the small variations in specific heat from changes in potential temperature and salinity are neglected. For example, c_{P0} varies by 0.1% when θ and S vary by 10°C and 1.0 psu, respectively, which follows from the United Nations Educational, Scientific, and Cultural Organization (UNESCO) equation of state (obtained online at http://ioc.unesco.org/oceanteacher) based on Fofonoff (1985). In terms of potential temperature, the heat equation becomes

$$\frac{D\delta\theta}{Dt} = \frac{D\eta}{Dt}\frac{d\theta}{d\eta} = \frac{Q\theta_0}{T_0 c_{P0}}. \quad (6)$$

Bacon and Fofonoff (1996) obtain the same equation and go through the same reasoning to justify it. The specific heat c_{P0} enters in the definition of α_θ as well:

$$\alpha_\theta = \frac{1}{V}\left(\frac{\partial V}{\partial \theta}\right)_{P,S} = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{P,S}\left(\frac{\partial T}{\partial \eta}\right)_{P,S}\left(\frac{\partial \eta}{\partial \theta}\right)_{P,S}$$

$$= \alpha\frac{Tc_{P0}}{\theta c_P}, \tag{7}$$

where $c_{P0} = \theta(\partial\eta/\partial\theta)_{P,S}$ and $c_P = T(\partial\eta/\partial T)_{P,S}$.

Equations (3)–(6) determine the evolution of the four dependent variables $\mathbf{v}$, δP, $\delta\theta$, and δS. The advection term in the material derivative is the only nonlinearity. The basic-state quantities, which generally have a zero subscript, are either constants like θ_0, S_0, and c_{P0} or else they are known functions, like T_0, ρ_0, α_θ, and β, of the vertical coordinate z. As stated in the introduction, the focus is on the adiabatic, inviscid equations. One could add Coriolis terms, but they would not enter in the energy equations.

3. Energy in the Boussinesq/anelastic approximation

To obtain an equation for mechanical energy, take the dot product of $\rho_0\mathbf{v}$ with Eq. (3) and define $K = \mathbf{v}\cdot\mathbf{v}/2$ as kinetic energy per unit mass. Combine with Eq. (4) to obtain

$$\frac{\partial(\rho_0 K)}{\partial t} + \nabla\cdot[\mathbf{v}(\rho_0 K + \delta P)] = \rho_0\mathbf{v}\cdot\hat{k}g(\alpha_\theta\delta\theta - \beta\delta S). \tag{8}$$

The divergence term includes the flux of kinetic energy and the rate of work done by pressure forces. The right side is the rate of conversion of potential energy into kinetic energy. It is positive when low-density fluid is rising and high-density fluid is sinking.

The equation for potential energy should have a time derivative and a divergence term on the left and a conversion term on the right that is the same as in Eq. (8) except for sign. To produce these conversion terms, one needs to multiply Eqs. (5) and (6) by quantities whose gradients are equal to $g\beta\hat{k}$ and $-g\alpha_\theta\hat{k}$, respectively. Thus I define

$$\psi_0 = -\int^z \alpha_\theta g\,dz' \quad \text{and} \quad \mu_0 = \int^z \beta g\,dz'. \tag{9}$$

The zero subscript indicates that ψ_0 and μ_0 are basic-state quantities. The lower limit of integration depends on the definition of the zero-energy state; for now I leave it indefinite. In sections 4 and 5 I show that $\psi_0\delta\theta$ and $\mu_0\delta S$ are the work that a parcel does against buoyancy and are the first terms in an expansion of the enthalpy. The process of multiplying Eq. (5) by $\rho_0\mu_0$ and Eq. (6) by $\rho_0\psi_0$ and then adding the resulting equations yields

$$\frac{\partial\rho_0(\psi_0\delta\theta + \mu_0\delta S)}{\partial t} + \psi_0\rho_0\mathbf{v}\cdot\nabla\delta\theta + \mu_0\rho_0\mathbf{v}\cdot\nabla\delta S$$

$$= \rho_0\psi_0\frac{\theta_0 Q}{T_0 c_{p0}} - \rho_0\mu_0 S_0 F. \tag{10}$$

The basic-state quantities ρ_0, ψ_0, and μ_0 depend only on the vertical coordinate, which allows us to move them inside the time derivatives. When one integrates the $\rho_0\mathbf{v}\cdot\nabla$ terms by parts, the $\nabla\cdot(\rho_0\mathbf{v})$ term vanishes because of Eq. (4), leaving terms proportional to $-\nabla\psi_0 = g\alpha_\theta\hat{k}$ and $-\nabla\mu_0 = -g\beta\hat{k}$. With these terms on the right side, Eq. (10) becomes

$$\frac{\partial(\rho_0\psi_0\delta\theta + \rho_0\mu_0\delta S)}{\partial t} + \nabla\cdot[\rho_0\mathbf{v}(\psi_0\delta\theta + \mu_0\delta S)]$$

$$= -\rho_0\mathbf{v}\cdot\hat{k}g(\alpha_\theta\delta\theta - \beta\delta S) + \rho_0\psi_0\frac{\theta_0 Q}{T_0 c_{p0}} - \rho_0\mu_0 SF. \tag{11}$$

Equation (11) describes potential energy. The conversion term is the same as in Eq. (8) but has the opposite sign. One adds Eqs. (8) and (11) and integrates from bottom to top to get the equation for the rate of change of total energy per unit area:

$$\frac{\partial}{\partial t}\int_{z_b}^{z_a}(K + \psi_0\delta\theta + \mu_0\delta S)\rho_0\,dz$$

$$= \int_{z_b}^{z_a}\left(\psi_0\frac{\theta_0 Q}{T_0 c_{p0}} - \mu_0 S_0 F\right)\rho_0\,dz. \tag{12}$$

Here z_b and z_a are the heights of the bottom and the air–water interface, respectively. The expression on the right is simply the integrated divergence of the heat flux, as shown at the end of section 5.

Bringing the vertical integral inside the time derivative in Eq. (12) generates three terms, $-\rho_0 K w_a$, $-\rho_0\psi_0\delta\theta w_a$, and $-\rho_0\mu_0\delta S w_a$, which are evaluated at the upper boundary, where $w_a = dz_a/dt$ is the velocity of the free surface. These terms exactly cancel terms that arise from the integrated divergence—the second terms on the left of Eqs. (8) and (11). The one term from the integrated divergence that does not cancel is $w_a\delta P$, but it is zero if the atmospheric pressure is constant such that $\delta P = 0$ at the upper free surface.

Equation (12) implies that the potential energy per unit area δE is

$$\delta E = \int_{z_b}^{z_a}(\psi_0\delta\theta + \mu_0\delta S)\rho_0\,dz. \tag{13}$$

Because ψ_0 decreases with altitude and μ_0 increases with altitude according to Eq. (9), the column has more potential energy when the (high θ, low S) fluid is on the bottom and the (low θ, high S) fluid is on top. Equations (10) and (11) indicate that cooling ($Q < 0$) or removing freshwater ($F < 0$) at the surface increases the potential energy of the column.

In the special case when α_θ and β are independent of depth, ψ_0 and μ_0 are proportional to $\int g \, dz'$, which is the geopotential Φ. Then the potential energy δE is $\int (-\alpha_\theta \delta\theta + \beta \delta S)\Phi\rho_0 \, dz'$ and is entirely gravitational. As pointed out by McDougall (2003), in the general case when α_θ and β vary with depth the fluid has both gravitational and internal energy.

Not all simplified equations have energy conservation relations like Eqs. (8), (11), and (12). For example, using the full equation of state $\delta\rho_f = \rho(\theta, S, P) - \rho(\theta_0, S_0, P)$ for the buoyancy term, instead of the linearized form as in Eq. (3), does not lead to energy conservation. In that case there is no potential energy equation, constructed from Eqs. (5) and (6), that has a conversion term that matches the one in the kinetic energy equation constructed from Eq. (3). On the other hand, using the potential density anomaly $\delta\rho_p = \rho(\theta, S, P_a) - \rho(\theta_0, S_0, P_a)$ does lead to an energy conservation equation, but it ignores the dependence of density (and α_θ) on pressure. Here P_a is the reference pressure, usually 1.0 bar. Lighthill (1978) advocates this approach and claims the errors are small. However, because potential density implies a constant α_θ evaluated at the reference pressure, this approach cannot describe thermobaric instability.

Equations (3)–(6) are compatible with the conventional definition of the Brunt–Väisälä frequency if one assumes a stratified fluid at rest plus a small perturbation—that is, $\delta\theta = \overline{\delta\theta}(z) + \delta\theta'$, $\delta S = \overline{\delta S}(z) + \delta S'$, and $\mathbf{v} = \mathbf{v}'$. Then the resulting equation for the perturbation vertical velocity is

$$[\rho_0^{-1}(\rho_0 w'_{tt})_z]_z + \nabla_h^2 w'_{tt} + N^2 \nabla_h^2 w' = 0, \qquad (14)$$

where ∇_h^2 is the horizontal Laplacian, and $N^2 = g(\alpha_\theta \overline{\delta\theta}z - \beta\overline{\delta S}_z)$. As before, α_θ and β depend on the basic-state quantities θ_0, S_0, and $P_0(z)$ but do not depend on $\delta\theta$, δS, and δP. If one had used the potential density anomaly $\delta\rho_p$ in the buoyancy term, α_θ and β would have been constants, independent of z, equal to their values at the reference pressure. That approach cannot describe thermobaric instability.

4. Potential energy from parcel theory

In this section I show that δE is the energy needed to assemble the column whose final state is $\theta_0 + \delta\theta(z)$ and

$S_0 + \delta S(z)$, starting from an initial state θ_0 and S_0. Consider a parcel that is destined for level z. Inject it at the surface where the altitude is z_a, and remove a basic-state parcel of equal mass from the same altitude. The net work/mass is $P_a\delta V_a$, where δV_a is the difference in volume/mass of the two parcels when they are at the pressure P_a:

$$P_a\delta V_a = P_a\left[\delta\theta\left(\frac{\partial V}{\partial\theta}\right)_{S,P} + \delta S\left(\frac{\partial V}{\partial S}\right)_{\theta,P}\right]. \qquad (15)$$

The partial derivatives are evaluated at the surface pressure, but $\delta\theta$ and δS are evaluated at the final destination level z. Now lower the parcel adiabatically to that level. The buoyancy force/mass is $g\delta V/V_0$, and so the work/mass during this step is

$$-\left[\int_{z_a}^z g\delta V(z')/V_0 \, dz'\right] = \left[\int_{z_a}^z g(-\alpha_\theta\delta\theta + \beta\delta S) \, dz'\right]$$

$$= -\delta\theta\int_{z_a}^z \alpha_\theta g \, dz'$$

$$+ \delta S\int_{z_a}^z \beta g \, dz'. \qquad (16)$$

We evaluate g, α_θ, and β at level z', which is the height of the parcel when it is in transit. However, $\delta\theta$ and δS are evaluated at level z, and so they are taken outside the integrals. There are no terms proportional to δP because the parcel is at the same pressure P' as the surroundings. The energy/mass is the sum of Eqs. (15) and (16), which can be written ($\psi_0\delta\theta + \mu_0\delta S$) provided

$$\psi_0 = -\int_{z_a}^z \alpha_\theta g \, dz' + P_a\left(\frac{\partial V}{\partial\theta}\right)_{S,P} \quad \text{and}$$

$$\mu_0 = \int_{z_a}^z \beta g \, dz' + P_a\left(\frac{\partial V}{\partial S}\right)_{\theta,P}. \qquad (17)$$

The expressions in Eq. (17) are the same as those in Eq. (9), except now the lower limit of integration is defined. The final step is to assemble the column from bottom to top so that each parcel passes through basic-state fluid on the way to its final level z. This step involves adding the contribution of each layer, whose mass/area is $\rho_0(z)dz$. The expression for the energy/area needed to assemble the column is therefore the same as δE in Eq. (13), which was derived from the Boussinesq/anelastic approximation.

Adkins et al. (2005) give an expression for potential energy based on parcel theory that does not use buoyancy forces or linearization. At the ocean surface,

where the pressure is P_a, parcel A, which is destined for pressure P, is injected and a basic-state parcel of equal mass is removed. The energy/mass is $P_a \delta V_a$, as before. On the way down, parcel A moves through infinitesimal pressure ranges dP' and exchanges places with basic-state parcels of equal mass. During each exchange, parcel A moves down a distance equal to the thickness of the basic-state parcel, and the basic-state parcel moves up a distance equal to the thickness of parcel A. The net increase in gravitational energy per unit mass GPE/m is $g\delta z$, where $\delta z = \delta V dP'/g$ is the thickness of parcel A minus that of the basic-state parcel, δV is the difference in specific volumes, and dP'/g is mass/area of the pressure interval through which the parcel moves. Thus GPE/$m = \delta V dP'$.

As the parcels exchange places, their combined internal energy also changes and they do work on the overlying fluid. However, these terms cancel because the process is adiabatic and the decrease in internal energy of the two parcels is equal to the work that they do, which goes into increasing the GPE of the overlying fluid. The internal energy of the overlying fluid does not change, and so the only change in the *total* potential energy—internal plus gravitational—of the whole column is associated with the change in GPE/m, which is $\delta V dP'$ as was shown earlier.

This process of exchanging parcels is repeated until the parcel has moved down a finite amount, from pressure P_a to pressure P. The energy/mass needed to do this is $\int \delta V \, dP'$. The integral goes from P_a to P. Adding up all the mass/area dP/g then gives the total energy/area needed to assemble the column:

$$\delta E = \int_{P_a}^{P_b} \left(\int_{P_a}^{P} \delta V \, dP' + \delta V_a P_a \right) dP/g. \quad (18)$$

Here $\delta V = V[\theta(P), S(P), P'] - V(\theta_0, S_0, P')$. The $\delta V_a P_a$ term represents the energy/mass needed to inject the parcel at the surface. Equation (18) resembles McDougall's (2003) expression for available potential energy (APE)—that is, his Eq. (E3)—although his reference state is the minimum-energy state after adiabatic rearrangement of a baroclinic fluid and ours is simply a state with constant S and θ. To compare with Eq. (13) assume that $\delta\theta$ and δS are small, such that $\delta V = (\partial V/\partial\theta)\delta\theta + (\partial V/\partial S)\delta S$. The partial derivatives are evaluated at pressure P', whereas $\delta\theta$ and δS are evaluated at pressure P. Substituting this linearization into Eq. (18) yields

$$\delta E \approx \int_{P_a}^{P_b} (\psi_0\delta\theta + \mu_0\delta S) \, dP/g,$$

where

$$\psi_0 = \int_{P_a}^{P} \left(\frac{\partial V}{\partial\theta}\right) dP' + P_a\left(\frac{\partial V}{\partial\theta}\right)_a \quad \text{and}$$

$$\mu_0 = \int_{P_a}^{P} \left(\frac{\partial V}{\partial S}\right) dP' + P_a\left(\frac{\partial V}{\partial S}\right)_a. \quad (19)$$

These equations are the same as Eqs. (13) and (17) with the substitution $\partial V/\partial\theta = V_0\alpha_\theta$, $\partial V/\partial S = -V_0\beta$, and $V_0 dP' = -g dz'$.

5. Potential energy from first principles

In this section I derive Eq. (13) from first principles; that is, I show that $\int(\psi_0\delta\theta + \mu_0\delta S)\rho_0 \, dz$ is the total potential energy—the sum of internal energy and gravitational potential energy relative to the basic state—in the limit of small $\delta\theta$ and small δS. I first show that ψ_0 is $(\partial H/\partial\theta)_{P,S}$ and μ_0 is $(\partial H/\partial S)_{P,\theta}$, where H is specific enthalpy. Thus ψ_0 and μ_0 are the thermodynamic potentials corresponding to variations of θ and S. From thermodynamics, $(\partial H/\partial P)_{\theta,S} = V$, since constant θ is the same as constant η. Therefore

$$\psi_0 = -\int^z \alpha_\theta g \, dz = \int^P \alpha_\theta V \, dP = \int^P \frac{\partial V}{\partial\theta} \, dP$$

$$= \int^P \frac{\partial^2 H}{\partial P\partial\theta} \, dP = \frac{\partial H}{\partial\theta} \quad (20)$$

and

$$\beta_0 = \int^z \beta g \, dz = -\int^P \beta V \, dP = \int^P \frac{\partial V}{\partial S} \, dP$$

$$= \int^P \frac{\partial^2 H}{\partial P\partial S} \, dP = \frac{\partial H}{\partial S}. \quad (21)$$

Thus I have shown that

$$\psi = \left(\frac{\partial H}{\partial\theta}\right)_{P,S}, \quad V = \left(\frac{\partial H}{\partial P}\right)_{S,\theta}, \quad \text{and} \quad \mu = \left(\frac{\partial H}{\partial S}\right)_{P,\theta}. \quad (22)$$

This derivation of the thermodynamic potential for salinity follows de Szoeke (2000). Equation (22) implies

$$\nabla H = \psi\nabla\theta + V\nabla P + \mu\nabla S. \quad (23)$$

For the reference-state variables, $\nabla\theta_0 = 0$, $\nabla S_0 = 0$, and $V_0\nabla P_0 = -\nabla\Phi$. Thus from Eq. (23) it follows that $(H_0 + \Phi) = $ constant, although both H_0 and Φ vary with height.

The total potential energy per unit area E is the sum of the internal energy of the water, the gravitational potential energy of the water, and the gravitational potential energy of the air above. Let $U = $ internal energy

per unit mass. The column is in hydrostatic balance, and so $V dP = -d\Phi$. The gradient of Φ is $g\hat{k}$, but I do not assume $g = $ constant. Thus

$$E = \int_{z_b}^{z_a} U\rho \, dz + \int_{z_b}^{z_a} \Phi\rho \, dz + z_a P_a. \quad (24)$$

The last term is the potential energy associated with the atmosphere–ocean interface, where the pressure is P_a, a constant. When the interface height z_a increases, the ocean does work on the atmosphere.

To compare with the Boussinesq and anelastic approximations, consider small perturbations of Eq. (24) from an adiabatic reference state. Both S and θ are constant with height in the reference state. Using the definition of enthalpy, I replace $U\rho$ in Eq. (24) with $H\rho - P$. I perturb H, P, ρ, and z_a, assuming P_a, z_b, and Φ do not change, which yields

$$\delta E = \int_{z_b}^{z_a} [\rho_0 \delta H + (H_0 + \Phi)\delta\rho - \delta P] \, dz$$
$$+ (H_0 + \Phi)\rho_{0a}\delta z_a. \quad (25)$$

Varying z_a, the upper limit of the integral in Eq. (24), produces two terms: One is the last term in Eq. (25) and the other is $-P_a\delta z_a$, which cancels the last term in Eq. (24). I do not allow horizontal convergence, and so the mass of the column is conserved:

$$\delta \int_{z_b}^{z_a} \rho \, dz = \int_{z_b}^{z_a} \delta\rho \, dz + \rho_{0a}\delta z_a = 0. \quad (26)$$

This means that the terms involving the constant $(H_0 + \Phi)$ drop out of Eq. (25), leaving

$$\delta E = \int_{z_b}^{z_a} (\rho_0 \delta H - \delta P) \, dz. \quad (27)$$

We substitute $\delta H = \psi_0 \delta\theta + \mu_0 \delta S + V_0 \delta P$ and note that $\rho_0 V_0 \delta P$ exactly cancels $-\delta P$. The resulting expression is exactly the same as Eq. (13), which was derived from the Boussinesq/anelastic equations. Using $\delta\eta$ instead of $\delta\theta$ gives an equation like Eq. (13) except that $T_0 \delta\eta$ replaces $\psi_0 \delta\theta$. The derivation uses the fact that $T = (\partial H/\partial\eta)_{P,S}$ is the thermodynamic potential for entropy. Combining this fact with Eq. (22) and with the definition following Eq. (7) yields

$$\psi_0 = \left(\frac{\partial H}{\partial\eta}\right)_{P,S} \left(\frac{\partial\eta}{\partial\upsilon}\right)_{P,S} = T_0 \frac{c_{P_0}}{\theta_0}. \quad (28)$$

Therefore the energy source term on the right side of Eq. (12) is $\int (Q + \mu_0 S_0 F)\rho_0 \, dz$. This is simply the integrated divergence of the heat flux, since Q contains a term that cancels the $\mu_0 S_0 F$ term, as stated in the discussion following Eq. (5).

6. Accuracy of the Boussinesq approximation

The Boussinesq/anelastic equations conserve a masslike quantity—the global integral of $\rho_0 dx dy dz$, where $dx dy dz$ is the element of volume. With proper choice of ρ_0 the global integral can be made equal to the true mass of the fluid, but the spatial distributions of mass and mass flux only approximate the true distributions. In this respect the Boussinesq equations are like the primitive equations—the equations of numerical weather prediction. The latter conserve a masslike quantity—the global integral of $dx dy dP/g$, where $dx dy dP$ is the element of volume when pressure is the vertical coordinate. The primitive equations use the hydrostatic approximation, and so the spatial distributions of mass and mass flux only approximate the true distributions.

An advantage of hydrostatic models is that they do not require linearization of the thermodynamic variables. The potential energy per unit mass is precisely H, and the rate of conversion per unit mass from potential energy to kinetic energy is precisely $-\omega V$, where pressure is the vertical coordinate and $\omega \equiv DP/Dt$ is the vertical velocity (Haltiner and Williams 1980). One can use the full equation of state for $H(\theta, S, P)$ and $V(\theta, S, P)$ and still satisfy an energy conservation law. As de Szoeke and Samelson (2002) point out, once the hydrostatic approximation is made the Boussinesq approximation offers no further mathematical simplification to the equations of motion. Each by itself filters out sound waves. However, the hydrostatic models require that the ratio of vertical to horizontal scales be small. They are not an accurate description of convection, and like the Boussinesq models they conserve a masslike quantity that is only approximately equal to the true mass. Treating the velocity as a mass flux divided by a reference density (Greatbatch et al. 2001; McDougall et al. 2002) does not eliminate this basic limitation.

The question is, how large is the error in the Boussinesq approximation and is it acceptable? In section 2 I derived Eq. (4), the Boussinesq/anelastic form of the continuity equation, by neglecting terms of order $\delta\rho/\rho_0$, which are of order $N^2 L/g$. In addition, the Boussinesq approximation involves neglecting terms of order gL/c^2, which is the ratio of the depth L to the density scale height c^2/g. In the main thermocline, where $N \sim 10^{-2}$ s^{-1} and $L \sim 1000$ m, $\delta\rho/\rho_0$ is of order 10^{-2} and $gL/c^2 \sim 0.5 \times 10^{-2}$. In the deep ocean $\delta\rho/\rho_0$ is several orders of magnitude smaller and gL/c^2 is larger, for example, 2×10^{-2}. Thus the error in the Boussinesq approximation is of order $\pm 10^{-2}$ both in the deep ocean and in the main thermocline. The error in the anelastic approximation is $\pm 10^{-2}$ in the main thermocline but is several orders

of magnitude smaller in the deep ocean because N^2 is smaller there.

Whether these kinds of errors are acceptable depends on the problem. In his section 3e, Davis (1994) considers a case in which a $\pm 10^{-2}$ error is unacceptable. In his example one is using observations to infer diapycnal mixing—the small-scale mixing of a tracer across an isopycnal surface on which the large-scale value of the tracer concentration is constant. One chooses a volume that is bounded on one side by this isopycnal and measures the fluxes into and out of the other bounding sides. In steady state, the flux through the isopycnal surface is computed as a residual, and it is likely to be small. If one measures the velocity and mixing ratio (mass of tracer per unit mass of fluid) to high accuracy but uses the Boussinesq approximation for the density, one is making a $\pm 10^{-2}$ error, which is likely to be larger than the signal. The solution is to use the full equation of state for the density. Of course, there are other sources of error, such as those associated with spatial and temporal sampling. Whether these errors are significant in comparison with that introduced by the Boussinesq approximation is a separate question.

In a simulation, however, errors of $\pm 10^{-2}$ may be acceptable. McDougall et al. (2002) discuss using Boussinesq models to compute tracer transport and mixing by turbulent eddies. The $\pm 10^{-2}$ error due to the Boussinesq approximation is likely to be small in comparison with errors arising from parameterizations of small-scale processes. In modeling turbulence, the Boussinesq equations are no worse than other sets of equations. Thus it seems premature to abandon the Boussinesq models (Greatbatch et al. 2001; Lu 2001; de Szoeke and Samelson 2002) unless the *fractional* error in the quantities being measured and modeled is less than $\pm 10^{-2}$. In most simulations, fractional errors in the dependent variables $\delta\theta$, δS, δP, and $\mathbf{v}$ are larger than the $\pm 10^{-2}$ error introduced by the Boussinesq approximation.

7. Specific examples

For a quantitative discussion of energy release, let α_θ increase linearly with pressure, that is, $\alpha_\theta = \alpha_1 + \alpha_2(P - P_a)$, with $L \ll c^2/g$, so that the density has the form

$$\rho = \rho_0[1 - \alpha_1\delta\theta - \alpha_2(P - P_a)\delta\theta + \beta\delta S], \quad (29)$$

where ρ_0 is constant. The thermobaric effect depends on the coefficient α_2. I express the energy release in terms of $\Delta K = -g\Delta E/(P_b - P_a)$, which has units of meters squared per second squared or joules per kilogram. Thus ΔK represents the kinetic energy per unit mass that is generated in going from the initial state to the final state, where ΔE is the potential energy per unit area in the final state minus that in the initial state. Because g and ρ_0 are constant, pressure is a measure of mass per unit area and increases linearly with depth.

Consider an initial state that has two well-mixed fluids, CFW and WSW, of equal mass. The CFW may be on the top or on the bottom; the energy release is the same. The temperature difference is $\delta\theta_v$. The salinity difference counteracts the temperature difference, and so density is continuous at the interface, which is at $P_m = (P_a + P_b)/2$. Thus the initial state is neutrally stable to small perturbations ($N^2 = 0$), and so

$$\beta\delta S_v = [\alpha_1 + \alpha_2(P_m - P_a)]\delta\theta_v. \quad (30)$$

In the final state some fraction f of the CFW that was initially on the top (bottom) has been transferred to the bottom (top), with the WSW lifted (lowered) and sandwiched in between, as shown in Fig. 1 (CFW on the bottom is the same as Fig. 1 turned upside down). I calculate the energy release using Eqs. (9) and (13). Because the initial state is neutrally stable, the usual source of convective instability involving $\alpha_1\delta\theta_v - \beta\delta S_v$ is absent. The terms that remain involve α_2 only, and ΔK has the form

$$\Delta K = (P_b - P_a)^2 V_0\alpha_2\delta\theta_v[f(1 - f)/16]. \quad (31)$$

The dependence on P_b, V_0, α_2, and $\delta\theta_v$ is consistent with Akitomo's Eq. (14), but the factor in square brackets is new. That factor is zero at $f = 0$ and $f = 1$. It reaches a maximum value of 1/64 when $f = 1/2$. Equation (31) says that the kinetic energy is zero initially ($f = 0$) and is zero when the column has been turned upside down ($f = 1$), that is, when all the CFW has been transported to the bottom (top). The maximum kinetic energy is produced when one-half of the CFW has been transferred ($f = 1/2$). Figure 1 shows this case. The WSW that was initially in the lower (upper) half of the column is now sandwiched in the middle. Of interest is that the kinetic energy is also zero when the final state is homogeneous, with θ and S equal to the averages of their initial values (we are ignoring the small increase in entropy when the two fluids mix).

The CFW has to be forced down (up) across the upper (lower) interface to the midpoint $P = P_m$. After that point, it is negatively (positively) buoyant and will sink (rise) spontaneously to the lower (upper) interface. In other words, this is a finite-amplitude instability. There is a buoyancy barrier that has to be overcome. The fact that the kinetic energy released is exactly zero when the final state is homogeneous means that the kinetic energy released when $f = 1/2$ is just capable of overcoming this barrier. Small-amplitude convection at the interface could provide the trigger,

and the finite-amplitude instability could sustain itself provided no kinetic energy is lost to dissipation or wave radiation. Akitomo (1999) considers the efficiency of thermobaric energy release, but that is beyond the scope of this paper, which describes adiabatic, inviscid motion.

According to Eq. (31), kinetic energy per unit mass goes as the square of the pressure and the first power of the temperature and/or salinity difference. At low temperatures, $\alpha_1 \approx 8 \times 10^{-5}$ K^{-1} and $\alpha_2 \approx 3 \times 10^{-5}$ K^{-1} (100 bars)$^{-1}$. With $P = 400$ bars (4×10^7 Pa), $\delta\theta_v = 4$ K, and $\beta = 0.9 \times 10^{-3}$ (psu)$^{-1}$, Eq. (30) implies that $\delta S_v = 0.62$ psu. For $f = 1/2$, Eq. (31) yields $\Delta K = 0.3$ m^2 s^{-2}. This is a large quantity of kinetic energy for the oceans and is capable of doing a large amount of mixing. It is comparable to the kinetic energy of the major ocean currents. The fact that it is small in comparison with the heating associated with a 4-K temperature change is irrelevant. In the oceans, mechanical energy scales as $P\delta V \approx PV\alpha\delta T$, whereas thermal energy scales as $c_p\delta T$. The latter is ~5000 times the former, but mechanical energy is what drives ocean currents and mixing.

The same two cases arise in a continuously stratified fluid. Let θ increase (decrease) linearly with pressure:

$$\delta\theta = \delta\theta_v(P - P_a)/(P_b - P_a), \qquad (32)$$

where $\delta\theta_v$ is a positive (negative) quantity. The fluid is neutral to small perturbations ($N^2 = 0$), and so S increases (decreases) with pressure as

$$\beta dS/dP = \alpha_\theta d\theta/dP = [\alpha_1 + \alpha_2(P - P_a)]\delta\theta_v/(P_b - P_a). \qquad (33)$$

Here the CFW is above (below) the WSW. The α_2 term requires that S be quadratic in P even though θ is linear in P. At each stage of the process, some CFW moves from the surface (bottom) at $P = P_a$ ($P = P_b$) to the interface at $P = fP_a + (1 - f)P_b$ [the interface at $P = fP_b + (1 - f)P_a$], which is the pressure of the fluid that was originally on the bottom (top). Thus f is the fractional height to which the WSW has risen (sunk). The CFW must be forced down (up) until it reaches its level of neutral buoyancy, after which it sinks (rises) spontaneously to the interface with the WSW.

The kinetic energy per unit mass (potential energy release) has the same form as Eq. (31) but with a different dependence on f:

$$\Delta K = (P_b - P_a)^2 V_0 \alpha_2 \delta\theta_v[f(1 - f)^3/6]. \qquad (34)$$

The potential energy release is zero initially ($f = 0$). It is also zero when the column has been completely turned upside down ($f = 1$). As before, the maximum potential energy release is when the interface is 1/4 of
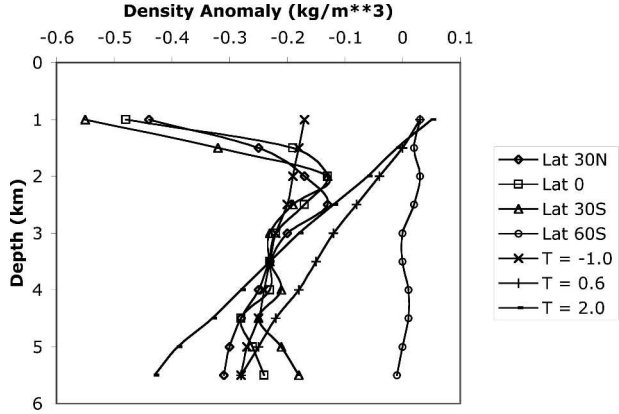

FIG. 2. Vertical profiles of density anomaly, defined as the difference between the computed density and that of a reference adiabat that has surface temperature $T = \theta = -1.9°$C and salinity $S = 34.396$ psu. The first four profiles are from the Atlantic Ocean (Levitus et al. 1994) and are labeled by latitude. The other three profiles are adiabats with $(\theta, S) = (-1.0, 34.259)$, $(0.6, 34.670)$, and $(2.0, 34.861)$, respectively.

the way off the bottom. Then $f = 1/4$, and the factor in square brackets is $9/(8 \times 64)$. For the two-fluid problem [Eq. (31)], it was 1/64. Therefore the maximum kinetic energy for the continuously stratified case is slightly more (by a factor 9/8) than that for the two-fluid case. Again, this kinetic energy is capable of completely mixing the water column, assuming 100% efficiency, because the kinetic energy released is exactly zero when the final state is homogeneous.

8. Thermobaric effects in the real oceans

Thermobaric effects are important in the modern deep ocean. When a WS parcel and a CF parcel move adiabatically, it is possible to have $\rho_{WS} > \rho_{CF}$ at the surface and $\rho_{CF} > \rho_{WS}$ at the bottom. The adiabats cross at some intermediate depth. This situation means that concepts like neutral surfaces, sloping isopycnals, and outcropping locations are nonunique. As Davis (1994) points out following his Eq. (8a), a particle that is always moving in neutral directions—that is, perpendicular to $\alpha_\theta \nabla\theta - \beta\nabla S$—can return to its starting latitude and longitude at a substantially different depth.

Figure 2 illustrates some of this nonuniqueness. All the curves are density anomalies—differences between the computed density and that of a reference adiabat. The differences between the curves are what matter. Subtracting the reference adiabat has no effect on these differences and was done only to fit the curves on the figure. The reference adiabat is the same for all curves and refers to a relatively CF parcel—surface temperature $T = -1.9°$C and $S = 34.396$ psu. The curves la-

beled lat = 30°N, 0°, 30°S, and 60°S were computed from contour plots of T–S cross sections (observed annual and zonal means) for the Atlantic Ocean (Levitus et al. 1994). The error in reading from the plots is several times 0.01 kg m^{-3}. The other curves are adiabats, expressed as anomalies relative to the reference adiabat, and are labeled by the value of surface temperature in degrees Celsius. Densities were computed from the UNESCO equation of state. For the adiabats, temperatures at depth were computed from the adiabatic lapse rate $dT/dz = -\alpha gT/C_P$.

The observed profiles are stable: Those labeled 30°N, 0°, and 30°S have density increasing with depth faster than their local adiabat, which is labeled $T = 2.0$ and has the same T and S values as the profiles at 3.5 km. In a similar way, the profile labeled 60°S has density increasing faster with depth than its local adiabat, which is labeled $T = 0.6$ and has the same T and S values as that profile at 1.0 km.

Although the profiles are stable relative to their own local adiabats, they are neutral relative to other adiabats. The 30°N, 0°, and 30°S profiles have density increasing with depth at the same rate as the $T = -1.0$ adiabat, and the 60°S profile has density increasing with depth at the same rate as the $T = -1.9$ reference adiabat (the latter is not plotted in Fig. 2; it would appear as a vertical line with density anomaly of zero). The $T = -1.0$ and $T = -1.9$ adiabats are typical of parcels in the Southern Ocean: The surface water south of 70°S has $T < -1.0°$C and $S < 34.4$ according to the Levitus et al. (1994) charts. Thus it is possible that these CF parcels are setting the density structure for the entire ocean through a combination of adiabatic vertical motion at high southern latitudes and adjustment of mass throughout the rest of the ocean to eliminate horizontal pressure gradients.

The nonuniqueness of sloping isopycnals is illustrated by the $T = 0.6$ and $T = 2.0$ adiabats. They both cross the 60°S curve at the same 1.0-km depth, but the $T = 0.6$ curve intersects the 30°N, 0°, and 30°S profiles below 5 km, and the $T = 2.0$ curve intersects the 30°N, 0°, and 30°S profiles at 3.5 km. The two isopycnals disagree by more than 1 km in height and more than 0.1 kg m^{-3} in density. The disagreement is much larger than the errors—tens of meters in height and 0.005 kg m^{-3} in density— that Reid (1994) and Jackett and McDougall (1997) achieved in their global fitting of isopycnal surfaces. This fact suggests that the properties along different paths in the real ocean, at least on large scales, differ by less than properties along the $T = 0.6$ and $T = 2.0$ adiabats. However, Fig. 2 shows that differences greater than 1.0 km are possible.

Figure 2 illustrates a possible mechanism to explain the interleaving of North Atlantic deep water and Antarctic deep waters. The $T = 2.0$ curve has $S = 34.861$, which is typical of North Atlantic source water. The $T = -1.0$ curve has $S = 34.259$, which is typical of Antarctic source water. The two curves cross at a depth of 3.5 km. Below this depth the Antarctic water is more dense, but above this depth the North Atlantic water is more dense. This means that the Atlantic water will sink to its level of neutral buoyancy and then spread out. For this example, that level is 3.5 km, but one could choose other adiabats that cross at other depths. The mechanism depends on the thermobaric effect because the two curves have to cross for it to work.

Thermobaric effects could lead to sudden changes in ocean circulation and surface temperatures, if there were a mechanism to build up the stored potential energy (Adkins et al. 2005). Cooling at the surface or warming from below, starting from a state that is initially stabilized by salt, is one such mechanism. Removal of freshwater at the surface, starting from a state that is initially stabilized by temperature, is another mechanism. In both cases, the forcing drives the stored potential energy from negative to positive. When this threshold is crossed, the column is still stable to small perturbations. As the forcing continues, the potential energy increases until a second threshold is crossed when the column becomes unstable to ordinary convection. It is possible then that the whole column will convect, releasing the entire stored potential energy. This would be a major event, with an abrupt change in sea surface temperatures and ocean turnover rates, both of which could affect weather and climate. Adkins et al. (2005) propose this mechanism to explain the grouping of Dansgaard–Oeschger events into Bond cycles during the last glacial period.

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