HEAT KERNELS OF METRIC TREES AND APPLICATIONS[*]

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Abstract. We consider the heat semigroup generated by the Laplace operator on metric trees. Among our results we show how the behavior of the associated heat kernel depends on the geometry of the tree. As applications we establish new eigenvalue estimates for Schrödinger operators on metric trees.

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1. Introduction. The relation between functional inequalities, like Sobolev inequalities for functions on (smooth or singular) manifolds, and geometric properties of this manifold, like volume growth, has been studied extensively. The crucial link between these two fields is the heat kernel of the Laplace operator on the manifold. Indeed, many functional inequalities have an equivalent form as heat kernel bounds; see, e.g., the book [Da1], the surveys [Cou, Gr, SC1], and the references therein. What is less known is that there is a close relation between heat kernel bounds and so-called Lieb–Thirring and Cwikel–Lieb–Rozenblum inequalities. The latter are inequalities about eigenvalues of Schrödinger operators and are closely related to Sobolev-like inequalities for systems of functions. These inequalities go back to [LT] and their relation to heat kernel bounds are discussed, for instance, in [LS, FLS].

Our goal in this paper is to explore the abstract connection between heat kernel bounds and Lieb–Thirring inequalities in the concrete context of metric trees. In particular, we shall derive precise bounds on the heat kernel which reflect the geometric properties of the graph. A metric graph is a combinatorial graph where the edges are considered as one-dimensional intervals. The fact that the edges are represented by nondegenerate line segments allows us to introduce various differential operators on metric graphs, such as Laplace and Schrödinger operators. Motivated by the fact that these operators on metric graphs appear in a number of models from mathematical physics (where they are often called *quantum graphs*), their analysis has recently attracted a lot of interest; see, e.g., the proceedings [EKKST, BCFK] and the references therein. Various functional inequalities for Laplace operators on metric graphs were recently studied in [EHP, NS1, F, EFK1, H]. The papers [NS2, K, EFK2, DH, So] contain inequalities about eigenvalues of Schrödinger operators on metric graphs. For a recent study of heat kernels on combinatorial graphs we refer to [KLW].

Most (but not all) of our results are valid for a special class of metric graphs, namely, *sparse*, symmetric metric trees. A metric *tree* is a metric graph on which any

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two points can be connected by a unique path. We will also suppose that the tree has a *root*. By *symmetric* we mean that the length of each edge and the branching number of each vertex depend only on the distance to the root. In the literature on metric graphs, symmetric trees are also referred to as "regular trees."

Finally, under *sparse* we understand an infinite tree with unbounded edge lengths. (Although this last assumption is not required in all of our theorems, it is a typical property of the trees that are covered by our methods.) Later on we will quantify this growth condition and introduce the notion of a global dimension.

We introduce the Neumann Laplacian $-\Delta_\Gamma$ on a rooted metric tree as the self-adjoint operator in $L_2(\Gamma)$ which acts as the usual one-dimensional Laplacian on the edges and satisfies the Kirchhoff matching conditions at the vertices and a Neumann boundary condition at the root. (Most of our results extend to the easier case of a Dirichlet condition at the root.) The heat kernel is the integral kernel of the operator $e^{t\Delta_\Gamma}$, $t > 0$, i.e.,

$$(1.1) \qquad k(x, y, t) := e^{t\Delta_\Gamma}(x, y), \qquad x, y \in \Gamma,$$

and, as we explained, we are going to study its dependence on the time t and the volume growth of the metric tree. The latter is encoded in the so-called branching function

$$(1.2) \qquad g_0(r) = \#\{x \in \Gamma : |x| = r\}, \quad r \geq 0,$$

where $|x|$ denotes the distance between x and the root of Γ.

The particular feature of this setup is that Γ represents a structure of *mixed dimensionality*. On one hand, it is locally one dimensional and therefore one should expect a $t^{-1/2}$ singularity of $k(x, x, t)$ for small times. In fact, we show that (Theorem 2.2), up to a multiplicative constant, the power function $t^{-1/2}$ serves as a uniform upper bound on $k(x, y, t)$. This result does not use the assumption that Γ is a tree.

On the other hand, since the global dimensionality of Γ depends on the behavior of g_0 at infinity, it is natural to expect that the large time decay of the heat kernel (1.1), for fixed x and y, will be determined by the growth of g_0. In order to be able to quantify the decay rate precisely, we assume that Γ is a symmetric tree. One of our main results (Theorem 2.3) says that, if g_0 does not grow too fast, then

$$(1.3) \qquad k(x, x, t) \leq \frac{C\, g_0(|x|)}{\sqrt{t}\, g_0(|x| + \sqrt{t})}$$

for all $x \in \Gamma$, $t \geq 0$, and some constant $C > 0$. Moreover, we show that this bound is order-sharp in the decay rate with respect to t for fixed $x \in \Gamma$. Inequality (1.3) is in the spirit of the works of Grigor'yan and Saloff-Coste on Riemannian manifolds and our proof relies on their approach. In the special case where the branching function grows like a constant times r^{d-1} for some $d \geq 1$ (i.e., Γ has global dimension d; see Definition 2.1) we obtain an upper bound on $k(x, x, t)$ by a constant times $g_0(|x|)t^{-d/2}$. We also obtain bounds in certain cases where g_0 grows faster than polynomially, e.g., exponentially as in the case of a homogeneous tree.

As an application of these heat kernel bounds we will derive inequalities for the negative eigenvalues of Schrödinger operators

$$-\Delta_\Gamma - V \qquad \text{in} \quad L_2(\Gamma)$$

with an electric potential $-V$ decaying at infinity. Using our new heat kernel bounds we are able to extend our previous results from [EFK2] in two ways. First, we are

able to remove the assumption that V has to be radially symmetric. Second, while in [EFK2] we treated the different dimensionality by proving a *one-parameter family* of inequalities, we now establish a stronger *single inequality* which takes separately account of the regions where V is large (in this case only the local dimension counts) and where V is small (in this case only the global dimension counts). As a sample of our result we assume again that Γ is a symmetric metric tree whose branching function grows like a constant times r^{d-1} for some $d > 1$. Then for all $\gamma \geq 1/2$ and all $\beta > 0$ there exist constants $L_d(\beta, \gamma)$ and $\tilde{L}_d(\beta, \gamma)$ such that

$$\operatorname{tr}\left(-\Delta_\Gamma - V\right)_-^\gamma \leq L_d(\beta, \gamma) \int_{g_0(|x|)^{\frac{2}{d-1}} V(x)_+ < \beta} V(x)_+^{\gamma + \frac{d}{2}} g_0(|x|)\, dx$$

$$+ \tilde{L}_d(\beta, \gamma) \int_{g_0(|x|)^{\frac{2}{d-1}} V(x)_+ \geq \beta} V(x)_+ g_0(|x|)^{\frac{1-2\gamma}{d-1}}\, dx\,.$$

We obtain similar inequalities also for a certain range of $\gamma < 1/2$ and, in particular, for $\gamma = 0$, that is, for the number of negative eigenvalues, provided $d > 2$. It is remarkable that Lieb–Thirring and Cwikel–Lieb–Rozenblum inequalities hold on metric trees in a form not too different from their Euclidean form despite the fact that the spectrum of the Laplacian $-\Delta_\Gamma$ is purely singular [BF].

2. Main results.

2.1. Preliminaries. We denote by $\mathcal{E}$ the set of edges and by $\mathcal{V}$ the set of vertices of a rooted tree graph Γ. For a vertex $v \in \mathcal{V}$ we define its generation $gen(v)$ as the number of vertices (including the starting point but excluding the end point) which lie on the unique path connecting v with the root o. We call Γ *symmetric* if all the vertices at the same distance from the root have equal branching numbers and all the edges emanating from these vertices have equal length.

In this case for a vertex of generation l we denote by b_l its (forward) branching number and by r_l its distance to the root. We also set $b_0 = 1$ and $r_0 = 0$, and assume that $b_l \geq 2$ for any $l \geq 1$. We emphasize that this assumption implies that Γ has no other leaves than o. With this notation, the branching function introduced in (1.2) becomes

$$(2.1) \qquad\qquad g_0(r) := b_0\, b_1 \cdots b_l \qquad \text{if} \quad r_l < r \leq r_{l+1}, \quad l \in \mathbb{N}_0.$$

Note that g_0 is an increasing function and that $g_0(r) \geq 1$ for every $r \geq 0$. Throughout we use the notation

$$|x| := \operatorname{dist}(x, o).$$

The Neumann Laplacian $-\Delta_\Gamma$ is defined as the nonnegative self-adjoint operator in $L_2(\Gamma)$ associated with the closed quadratic form

$$(2.2) \qquad\qquad \int_\Gamma |\varphi'(x)|^2\, dx, \quad \varphi \in H^1(\Gamma).$$

Here $H^1(\Gamma)$ consists of all continuous functions φ such that $\varphi \in H^1(e)$ on each edge $e \in \mathcal{E}$ and

$$\int_\Gamma \left(|\varphi'(x)|^2 + |\varphi(x)|^2\right)\, dx < \infty.$$

The operator domain of $-\Delta_\Gamma$ consists of all continuous functions φ such that $\varphi'(o) = 0$, $\varphi \in H^2(e)$ for each edge e of Γ and such that at each vertex $v \neq o$ of generation k the Kirchhoff matching conditions

$$(2.3) \qquad \varphi_-(v) = \varphi_1(v) = \cdots = \varphi_{b_k}(v), \quad \varphi'_-(v) = \varphi'_1(v) + \cdots + \varphi'_{b_k}(v)$$

are satisfied. Here φ_- denotes the restriction of φ on the edge terminating in v and φ_j, $j = 1, \ldots, b_k$, denote the restrictions of φ to the edges emanating from v.

From (2.2) and the Beurling–Deny theorem (see, e.g., [RS4, section XIII.12]) it follows that for any $t > 0$, the operator $e^{t\Delta_\Gamma}$ is a positivity preserving contraction on $L_p(\Gamma)$ for every $1 \leq p \leq \infty$. We denote the integral kernel of this operator by $k(\cdot, \cdot, t)$, that is,

$$\left(e^{t\Delta_\Gamma} f\right)(x) = \int_\Gamma k(x, y, t) f(y) \, dy \, .$$

Following standard arguments about regularity of the heat kernel, [Da1, section 5.2], one can show that $k(\cdot, \cdot, t)$ is continuous on $\Gamma \times \Gamma$; see, e.g., [H].

A special class of symmetric trees are those for which the branching function g_0 has a power-like growth at infinity. For such trees, following [K], we define their global dimension as follows.

DEFINITION 2.1. *A symmetric tree Γ has global dimension $d \geq 1$ if*

$$(2.4) \qquad 0 < \inf_{r \geq 0} \frac{g_0(r)}{(1+r)^{d-1}} \leq \sup_{r \geq 0} \frac{g_0(r)}{(1+r)^{d-1}} < \infty \, .$$

2.2. Main results. Our first result gives an upper bound on the heat kernel not only for trees, but for a very general class of graphs.

THEOREM 2.2. *Let Γ be a connected graph of infinite volume. Then for all $x \in \Gamma$ and all $t > 0$ it holds*

$$k(x, x, t) \leq (\pi t)^{-1/2} \, .$$

The constant in this estimate is best possible, as can be easily seen from the example $\Gamma = \mathbb{R}_+$, where $k(x, x, t) = (4\pi t)^{-1/2}(1 + e^{-x^2/t})$.

Theorem 2.2 gives an upper bound on $k(x, x, t)$ which is uniform with respect to x. However, as explained above, for a fixed $x \in \Gamma$ the large time decay of $k(x, x, t)$ should depend on the growth of g_0. This phenomenon is reflected in our next result.

THEOREM 2.3. *Let Γ be a symmetric tree. Assume that the branching function g_0 of Γ satisfies*

$$(2.5) \qquad g_0(2r) \leq C_0 \, g_0(r) \qquad \text{for all } r \in [0, \infty)$$

for some constant C_0. Then there is a constant $c > 0$ such that for all $x \in \Gamma$ and all $t > 0$ it holds that

$$(2.6) \qquad \frac{1}{c \sqrt{t} \, g_0(|x| + \sqrt{t})} \leq k(x, x, t) \leq \frac{c \, g_0(|x|)}{\sqrt{t} \, g_0(|x| + \sqrt{t})} \, .$$

Heat kernel estimates in terms of volume growth of metric graphs were recently obtained in [H] for short times, that is, for t on finite intervals. Theorem 2.3, on the other hand, gives a two-sided estimate which is uniform in $t > 0$.

One can obtain less precise bounds than (2.6) by using the monotonicity of g_0. Bounding $g_0(|x|) \leq g_0(|x| + \sqrt{t})$ we see that (2.6) yields the one-dimensional estimate $k(x, x, t) \leq c\, t^{-1/2}$ from Theorem 2.2. However, Theorem 2.2 is valid for a larger class of graphs and gives an explicit (and sharp) value of the constant c.

A different use of the monotonicity, namely, $g_0(|x| + \sqrt{t})^{-1} \leq g_0(\sqrt{t})^{-1}$, yields the bound

$$k(x, x, t) \ \leq \ \frac{c\, g_0(|x|)}{\sqrt{t}\, g_0(\sqrt{t})}\,.$$

One can control the term $g_0(\sqrt{t})$ if one assumes that Γ has a global dimension in the sense of Definition 2.1. Noting that this assumption implies condition (2.5), we obtain the following corollary.

COROLLARY 2.4. *Let Γ be a symmetric tree with global dimension d. Then for some $C > 0$, any $x \in \Gamma$, and any $t > 0$ it holds that*

$$(2.7) \qquad k(x, x, t) \ \leq \ C\ t^{-d/2}\, g_0(|x|).$$

In view of the lower bound in (2.6) and of (2.4) the decay rate $t^{-d/2}$ for fixed $x \in \Gamma$ in the above upper bound is best possible.

To summarize our analysis so far, we have seen that the behavior of the heat kernel for small and large times is determined by the local and global dimensions of Γ, respectively. Indeed, for a tree of global dimension d Theorem 2.3 gives

$$(2.8) \qquad k(x, x, t) \simeq t^{-1/2} \quad \text{as } t \to 0, \qquad k(x, x, t) \simeq t^{-d/2} \quad \text{as } t \to \infty.$$

There are of course symmetric trees which do not satisfy condition (2.5). This happens typically when the branching function g_0 grows too fast. To cover such cases we have the following theorem.

THEOREM 2.5. *Let Γ be a symmetric tree and assume that*

$$(2.9) \qquad S_\Gamma^{-1}(\delta) := \sup_{r>0} \left(\int_0^r g_0(s)\, ds \right)^{(\delta-2)/\delta} \left(\int_r^\infty \frac{ds}{g_0(s)} \right) < \infty$$

for some $\delta > 2$. Then for any $x \in \Gamma$ and $t > 0$ it holds that

$$(2.10) \qquad k(x, x, t) \ \leq \ \left(\frac{\delta}{2\tilde{S}_\Gamma(\delta)} \right)^{\delta/2} t^{-\delta/2}\, g_0(|x|),$$

where

$$(2.11) \qquad \tilde{S}_\Gamma(\delta) := \left(\frac{(\delta-2)^{\delta-2}\delta^\delta}{(2(\delta-1))^{2(\delta-1)}} \right)^{1/\delta} S_\Gamma(\delta).$$

It is easily seen that if Γ has a global dimension $d > 2$, then Theorem 2.5 is applicable with $\delta = d$ and (2.10) recovers the bound from Corollary 2.4 (with an explicit constant).

Contrary to Theorem 2.3, estimate (2.10) is applicable also when $g_0(r) \simeq \exp(Cr^\alpha)$ for some $\alpha > 0$. In this case Theorem 2.5 says that $k(x, x, t) = \mathcal{O}(t^{-n})$ as $t \to \infty$ for any $n \in \mathbb{N}$. Moreover, it is not difficult to check that for $0 < \alpha < 1$ we have $\inf \text{spec}(-\Delta_\Gamma) = 0$ and therefore the heat kernel decay should be subexponential for such values of α.

On the other hand, for $\alpha = 1$ it turns out that $\inf \operatorname{spec}(-\Delta_\Gamma) > 0$, which shows that $k(x, y, t)$ must decay exponentially fast in this case. As an example of such trees we will consider so-called *homogeneous* trees. A tree Γ_b is called homogeneous if all the edges have the same length τ and if the branching number $b_k = b \geq 2$ is independent of k. By scaling we may assume that $\tau = 1$. The branching function $g_0 =: g_b$ then reads

$$g_b(r) = b^j, \qquad j < r \leq j + 1, \quad j \in \mathbb{N}_0 \,.$$

The corresponding Laplacian $-\Delta_{\Gamma_b}^N$ is positive definite and

$$(2.12) \qquad \lambda_b := \inf \operatorname{spec}\left(-\Delta_{\Gamma_b}^N\right) = \left(\arccos \frac{1}{R_b}\right)^2, \qquad R_b = \frac{b^{\frac{1}{2}} + b^{-\frac{1}{2}}}{2} \,;$$

see [SS].

THEOREM 2.6. *Let Γ_b be a homogeneous tree with edge length 1 and branching number $b \geq 2$. Then there is a constant C_b such that for any $x \in \Gamma_b$ and any $t > 0$ it holds that*

$$(2.13) \qquad k(x, x, t) \ \leq \ C_b \ e^{-\lambda_b t} \, t^{-3/2} \, (1 + |x|)^2 \,.$$

We conclude this section with a brief outline of the rest of the paper. In section 3 we prove the general bound from Theorem 2.2. The proof of Theorems 2.3, 2.5, and 2.6 is based on the fact [Ca, NS1] that the Laplace operator on a symmetric tree can be decomposed into a direct sum of weighted Laplace operators on half-lines. In section 4 we show that the diagonal element $k(x, x, t)$ can be bounded from above in terms of the heat kernel of the weighted manifold $([0, \infty), g_0 dr)$; see Theorem 4.4. In section 5 we then establish a suitable estimate on the heat kernel of $([0, \infty), g_0 dr)$; see Proposition 5.3.

As we have explained in the introduction, one important consequence of the heat kernel bounds from Theorems 2.2, 2.3, 2.5, and 2.6 are eigenvalue estimates for Schrödinger operators on metric trees. In section 6 we will state these inequalities and we will see how the heat kernel method from this section allows us to fundamentally improve upon our previous results in [EFK2].

3. Proof of Theorem 2.2. We begin by proving the general heat kernel bound from Theorem 2.2 for arbitrary graphs. The key ingredient is the following logarithmic Sobolev inequality.

PROPOSITION 3.1 (logarithmic Sobolev inequality). *Let Γ be a connected graph of infinite volume. Then for any $u \in H^1(\Gamma)$ and any $a > 0$*

$$\frac{a^2}{\pi} \int_\Gamma |u'|^2 \, dx \geq \int_\Gamma |u|^2 \ln \left(\frac{|u|^2}{\|u\|_{L_2(\Gamma)}^2}\right) \, dx + (1 + \ln(a/2)) \int_\Gamma |u|^2 \, dx \,.$$

Proof. For any function u on Γ vanishing at infinity we consider its rearrangement u^*. By definition, this is the unique nonincreasing function on $[0, \infty)$ with the same distribution function as $|u|$. In particular, this property implies that

$$\int_\Gamma |u|^2 \, dx = \int_0^\infty |u^*|^2 \, dr \quad \text{and} \quad \int_\Gamma |u|^2 \ln |u|^2 \, dx = \int_0^\infty |u^*|^2 \ln |u^*|^2 \, dr \,.$$

Moreover, the same argument as in [F] yields that

$$\int_\Gamma |u'|^2 \, dx \geq \int_0^\infty |(u^*)'|^2 \, dr \,.$$

Therefore, the assertion follows from the Euclidean logarithmic Sobolev inequality (applied to symmetric functions); see, e.g., [LL, Thm. 8.14]. □

Proof of Theorem 2.2. By Proposition 3.1 and the Carlen–Loss argument as presented in [LL, Thm. 8.18] it follows that

$$\|e^{t\Delta_\Gamma}\|_{L_1(\Gamma)\to L_\infty(\Gamma)} = \sup_{x,y\in\Gamma} k(x,y,t) \le (\pi t)^{-1/2}$$

as required. □

4. Decomposition of the heat kernel on symmetric trees. In this section we make use of the symmetry of Γ and we shall see that it allows us to decompose the heat kernel of $-\Delta_\Gamma$ into a sum of heat kernels of one-dimensional operators. This is, of course, reminiscent of the partial wave decomposition in Euclidean space. In the context of metric trees this decomposition is due to [Ca, NS1].

We recall the definition of the branching function g_0 from (2.1) and we introduce the higher order branching functions g_l for $l \in \mathbb{N}$ by

$$(4.1) \qquad g_l(r) := \begin{cases} 0, & r < r_l, \\ 1, & r_l \le r < r_{l+1}, \\ b_{l+1}b_{l+2}\cdots b_n, & r_n \le r < r_{n+1}, \quad n > l. \end{cases}$$

4.1. Partial wave decomposition. Let v be a vertex of generation $l \in \mathbb{N}$ and denote by $\Gamma_{v,m}$, $m = 1,\dots,b_l$, the mutually disjoint (forward) subtrees rooted at v and by $\chi_{v,m}$ the corresponding characteristic functions. We shall also use the notation $\Gamma_v := \bigcup_{m=1}^{b_l} \Gamma_{v,m}$. Moreover, let

$$\omega_l := e^{2\pi i/b_l}$$

and put

$$(4.2) \qquad Y_{l,v,\sigma}(x) := \frac{1}{\sqrt{b_l\, g_l(|x|)}} \sum_{m=1}^{b_l} \omega_l^{m\sigma} \chi_{v,m}(x)$$

for $\sigma = 1,\dots,b_l - 1$ if $l \ge 1$ and

$$Y_{0,o,1}(x) := \frac{1}{\sqrt{g_0(|x|)}}.$$

If f is any function on Γ we put

$$f_{l,v,\sigma}(r) := \frac{1}{\sqrt{g_l(r)}} \sum_{|x|=r} \overline{Y_{l,v,\sigma}(x)}\, f(x).$$

Below we denote by $\sum_{l,v,\sigma}$ summation over all $l \in \mathbb{N}_0$, all v with gen $v = l$, and all σ with $1 \le \sigma \le b_l - 1$ (respectively, only $\sigma = 1$ if $l = 0$).

PROPOSITION 4.1. *For any function f on Γ one has*

$$(4.3) \qquad f(x) = \sum_{l,v,\sigma} f_{l,v,\sigma}(|x|)\, \sqrt{g_l(|x|)}\, Y_{l,v,\sigma}(x)$$

and for a.e. $r > 0$

$$(4.4) \qquad \sum_{|x|=r} |f(x)|^2 = \sum_{l,v,\sigma} |f_{l,v,\sigma}(r)|^2\, g_l(r).$$

Proof. Both equalities (4.3) and (4.4) will follow if we can show that

$$
(4.5) \qquad \sum_{l,v,\sigma} Y_{l,v,\sigma}(x)\, \overline{Y_{l,v,\sigma}(x')} = \delta_{x,x'} \quad \text{if} \quad |x| = |x'| \,.
$$

Let $|x| = |x'| = r$ and define

$$
(4.6) \qquad N = \max\left\{\operatorname{gen}(v) \, : \, x, x' \in \Gamma_v\right\}.
$$

Since (4.5) is obvious for $r < r_1$, we may assume that $N \geq 1$. Then

$$
\sum_{l,v,\sigma} Y_{l,v,\sigma}(x)\, \overline{Y_{l,v,\sigma}(x')} = \frac{1}{g_0(r)} + \sum_{l=1}^{N} \sum_{\operatorname{gen}(v)=l} \sum_{\sigma=1}^{b_l-1} Y_{l,v,\sigma}(x)\, \overline{Y_{l,v,\sigma}(x')}.
$$

(The first term on the right side is the contribution from $l = 0$.) For each $1 \leq l \leq N-1$ there is a unique vertex v of the lth generation and a unique $1 \leq m \leq b_l$ such that $x, x' \in \Gamma_{v,m}$. Hence from (4.1), (2.1), and (4.2) we get

$$
\begin{aligned}
\frac{1}{g_0(r)} + \sum_{l=1}^{N-1} \sum_{\operatorname{gen}(v)=l} \sum_{\sigma=1}^{b_l-1} Y_{l,v,\sigma}(x)\, \overline{Y_{l,v,\sigma}(x')} &= \frac{1}{g_0(r)} + \sum_{l=1}^{N-1} \sum_{\sigma=1}^{b_l-1} \frac{1}{b_l g_l(r)} \\
&= \frac{1}{g_0(r)} + \frac{1}{g_0(r)} \sum_{l=1}^{N-1} b_0 \cdots b_{l-1}(b_l - 1) \\
&= \frac{b_0 \cdots b_{N-1}}{g_0(r)} \,.
\end{aligned}
$$

In order to study the contribution from $l = N$ we suppose first that $x \neq x'$. Then there is a unique vertex w of the Nth generation and two numbers $m \neq m'$ such that $x \in \Gamma_{w,m}$ and $x' \in \Gamma_{w,m'}$. Using the fact that

$$
\sum_{\sigma=1}^{b_N} e^{\frac{2\pi i \sigma(m-m')}{b_N}} = 0
$$

we find

$$
\sum_{\operatorname{gen}(v)=N} \sum_{\sigma=1}^{b_N-1} Y_{l,v,\sigma}(x)\, \overline{Y_{l,v,\sigma}(x')} = \frac{1}{b_N\, g_N(r)} \sum_{\sigma=1}^{b_N-1} e^{\frac{2\pi i \sigma(m-m')}{b_N}} = -\frac{1}{b_N\, g_N(r)} \,.
$$

Assertion (4.5) is now a consequence of

$$
\frac{b_0 \cdots b_{N-1}}{g_0(r)} = \frac{1}{b_N\, g_N(r)} \,;
$$

see (4.1) and (2.1).

Assume now that $x = x'$. In this case we have $g_N(r) = 1$ and hence

$$
\sum_{\operatorname{gen}(v)=N} \sum_{\sigma=1}^{b_N-1} Y_{l,v,\sigma}(x)\, \overline{Y_{l,v,\sigma}(x')} = \frac{b_N - 1}{b_N} \,.
$$

Assertion (4.5) in this case is a consequence of

$$\frac{b_0 \cdots b_{N-1}}{g_0(r)} + \frac{b_N - 1}{b_N} = 1 \, ;$$

see again (4.1) and (2.1). This concludes the proof of (4.5). □
By a duality argument (4.5) is equivalent to

$$\sum_{|x|=r} Y_{l,v,\sigma}(x) \, \overline{Y_{l',v',\sigma'}(x)} = \delta_{l,l'} \, \delta_{v,v'} \, \delta_{\sigma,\sigma'}.$$

The reason for including the factor $1/\sqrt{g_l(r)}$ in the definition of $f_{l,v,\sigma}$ is that in this way $f_{l,v,\sigma}$ inherits continuity and differentiability properties of f. More precisely, one has the following proposition.

PROPOSITION 4.2. *If u is a locally absolutely continuous function on Γ, then the $u_{l,v,\sigma}$ are locally absolutely continuous. One has*

$$(4.7) \qquad u'(x) = \sum_{l,v,\sigma} u'_{l,v,\sigma}(|x|) \, \sqrt{g_l(|x|)} \, Y_{l,v,\sigma}(x)$$

and for a.e. $r > 0$

$$(4.8) \qquad \sum_{|x|=r} |u'(x)|^2 = \sum_{l,v,\sigma} |u'_{l,v,\sigma}(r)|^2 \, g_l(r) \, .$$

The first part of the assertion is easily verified; see also [NS1]. The second part is a consequence of Proposition 4.1.

4.2. The reduced operators and their heat kernels. Let A_l, $l \in \mathbb{N}_0$, be the nonnegative operators in $L_2((r_l, \infty), g_l dr)$ generated by the quadratic forms

$$(4.9) \qquad \int_{r_l}^{\infty} |u'|^2 \, g_l \, dr$$

with form domains

$$H^1((0, \infty), g_0 \, dr) \text{ if } l = 0 \qquad \text{and} \qquad H_0^1((r_l, \infty), g_l \, dr) \text{ if } l \in \mathbb{N} \, .$$

Since, by Propositions 4.1 and 4.2,

$$\|u\|_{L_2(\Gamma)}^2 = \sum_{l,v,\sigma} \int_{r_l}^{\infty} |u_{l,v,\sigma}|^2 \, g_l \, dr \qquad \text{and} \qquad \|u'\|_{L_2(\Gamma)}^2 = \sum_{l,v,\sigma} \int_{r_l}^{\infty} |u'_{l,v,\sigma}|^2 \, g_l \, dr \, ,$$

we have

$$(-\Delta_\Gamma u)_{l,v,\sigma} = A_l \, u_{l,v,\sigma} \, .$$

Hence by the spectral theorem

$$(4.10) \qquad (e^{t\Delta_\Gamma} u)_{l,v,\sigma} = e^{-tA_l} \, u_{l,v,\sigma} \, .$$

Let $k(x, y, t)$ be the heat kernel of $-\Delta_\Gamma$ and let $k_l(r, s, t)$, $l \in \mathbb{N}_0$, be the heat kernels of the operators A_l, that is,

$$(e^{t\Delta_\Gamma} u)(x) = \int_\Gamma k(x, y, t) u(y) \, dy$$

and

$$(4.11) \qquad (e^{-tA_l}f)(r) = \int_{r_l}^{\infty} k_l(r, s, t) f(s)\, g_l(s)\, ds\,.$$

From the Beurling–Deny conditions (see, e.g., [RS4, section XIII.12]) one infers that $k_l(r, s, t)$ is nonnegative. For technical reasons we extend $k_l(r, s, t)$ by zero to all negative values of $(r - r_l)$ and $(s - r_l)$. Combining (4.10) with (4.11) we find

$$
\begin{aligned}
(e^{t\Delta_\Gamma}u)(x) &= \sum_{l,v,\sigma} (\exp(t\Delta_\Gamma)\, u)_{l,v,\sigma}(|x|)\, \sqrt{g_l(|x|)}\, Y_{l,v,\sigma}(x) \\
&= \sum_{l,v,\sigma} \int_{r_l}^{\infty} k_l(|x|, s, t)\, \sqrt{g_l(s)} \sum_{|y|=s} \sqrt{g_l(|x|)}\, Y_{l,v,\sigma}(x)\, \overline{Y_{l,v,\sigma}(y)}\, u(y)\, ds \\
&= \int_0^{\infty} \sum_{|y|=s} \sum_{l,v,\sigma} k_l(|x|, |y|, t)\, \sqrt{g_l(|x|)\, g_l(|y|)}\, Y_{l,v,\sigma}(x)\, \overline{Y_{l,v,\sigma}(y)}\, u(y)\, ds \\
&= \int_\Gamma \sum_{l,v,\sigma} k_l(|x|, |y|, t)\, \sqrt{g_l(|x|)\, g_l(|y|)}\, Y_{l,v,\sigma}(x)\, \overline{Y_{l,v,\sigma}(y)}\, u(y)\, dy\,.
\end{aligned}
$$

This shows that

$$(4.12) \qquad k(x, y, t) = \sum_{l,v,\sigma} \sqrt{g_l(|x|)\, g_l(|y|)}\, Y_{l,v,\sigma}(x)\, \overline{Y_{l,v,\sigma}(y)}\, k_l(|x|, |y|, t)\,.$$

In particular, on the diagonal we have

$$(4.13) \qquad k(x, x, t) = k_0(|x|, |x|, t) + \sum_{l=1}^{\infty} \frac{b_l - 1}{b_l}\, k_l(|x|, |x|, t)\,.$$

Note that the sum involves only finitely many terms since the lth summand is nonzero only when $|x| < r_l$. A relation between the heat kernels for different values of l is given in the following lemma.

LEMMA 4.3. *For all $l \geq 1$ one has*

$$(4.14) \qquad k_l(r, s, t) \leq b_0 \cdots b_l\, k_0(r, s, t).$$

Proof. Let χ_l be the characteristic function of the interval $(0, r_l)$ and consider the family of operators $B_l^M = A_0 + M\chi_l$ in $L_2(\mathbb{R}_+, g_0 dr)$ for a constant $M > 0$. With Trotter's product formula one sees that

$$(4.15) \qquad \exp(-tB_l^M)(r, s) \leq k_0(r, s, t)\,.$$

Moreover, by monotone convergence B_l^M converges in the strong resolvent sense to the operator $0 \oplus \tilde{A}_l$ as $M \to \infty$ in $L_2((0, r_l), g_0 dr) \oplus L_2((r_l, \infty), g_0 dr)$, where $\tilde{A}_l$ is associated with the quadratic form $\int_{r_l}^{\infty} |u'|^2 g_0\, dr$ with boundary condition $u(r_l) = 0$. Therefore, the semigroup $e^{-tB_l^M}$ converges strongly to $1 \oplus e^{-t\tilde{A}_l}$. Hence by passing to the limit $M \to \infty$ in (4.15) we get

$$(4.16) \qquad \exp(-t\tilde{A}_l)(r, s) \leq k_0(r, s, t)$$

for almost every $r, s \geq r_l$. By continuity of the heat kernel we then conclude that (4.16) holds for every $r, s \geq r_l$. The operator U of multiplication by the constant $\sqrt{b_0 \cdots b_l}$

maps $L_2((r_l, \infty), g_0 dr)$ unitarily onto $L_2((r_l, \infty), g_l dr)$ and satisfies $U^* A_l U = \tilde{A}_l$. Hence $U^* \exp(-t A_l) U = \exp(-t \tilde{A}_l)$ which means

$$k_l(r, s, t) = b_0 \cdots b_l \exp(-t \tilde{A}_l)(r, s).$$

This proves inequality (4.14). □

THEOREM 4.4. *Let Γ be a symmetric tree. Then for any $x \in \Gamma$ we have*

(4.17) $$k(x, x, t) \le k_0(|x|, |x|, t) \, g_0(|x|).$$

Proof. Let $x \in \Gamma$ with $r_L < |x| \le r_{L+1}$. Combining (4.13) and Lemma 4.3 with the hypothesis that $b_0 = 1$ we obtain

$$k(x, x, t) = k_0(|x|, |x|, t) + \sum_{l=1}^{L} \frac{b_l - 1}{b_l} k_l(|x|, |x|, t)$$

$$\le k_0(|x|, |x|, t) \left(1 + \sum_{l=1}^{L} b_0 \cdots b_{l-1} (b_l - 1) \right)$$

$$= k_0(|x|, |x|, t) \, b_0 \cdots b_L = k_0(|x|, |x|, t) \, g_0(|x|),$$

as claimed. □

5. Proof of the main results. In this section we shall prove Theorems 2.3, 2.5, and 2.6. According to Theorem 4.4 the proof of upper bounds on the heat kernel of $-\Delta_\Gamma$ on the tree Γ is reduced to the proof of upper bounds on the heat kernel of the operator A_0 on the half-line. To obtain such bounds for A_0 under various assumptions on g_0 is the topic of the following three subsections.

5.1. Proof of Theorem 2.3. In order to derive our first bound on the heat kernel of A_0 we make use of a powerful equivalence theorem due to Grigor'yan and Saloff-Coste; see the surveys [Gr, SC1] and the monograph [SC2]. In these references, the metrics is assumed to be smooth but, as is well known to specialists and is easily verified by inspecting the proofs (see also [SC2, p. 113]), this regularity assumption is not necessary. According to this result, a heat kernel bound follows from a volume doubling condition and a family of Poincaré inequalities for the weighted manifold $M(g_0) = ([0, \infty), g_0 \, dr)$. These bounds will be shown below in Lemmas 5.1 and 5.2, respectively.

Let $B(z, r)$ be the ball of radius r centered in z and let $V(z, r)$ be its volume in $M(g_0)$. More explicitly,

$$B(z, r) = (\max\{z - r, 0\}, z + r), \qquad V(z, r) = \int_{B(z, r)} g_0(s) \, ds.$$

We also recall the doubling condition (2.5) on g_0.

LEMMA 5.1 (volume doubling). *Assume (2.5). Then the weighted manifold $M(g_0)$ satisfies the following volume doubling condition: for any ball $B(z, r)$ we have*

(5.1) $$V(z, r) \le 2 C_0 V(z, r/2).$$

Proof. Obviously, $\frac{d}{dr} V(z, r) = g_0(z + r) + g_0(z - r)$ (with the convention that $g_0(t) = 0$ if $t < 0$). Moreover, by (2.5) and by the fact that g_0 is nondecreasing,

$$g_0(z + r) + g_0(z - r) \le C_0 \left(g_0\left(\frac{z + r}{2} \right) + g_0\left(\frac{z - r}{2} \right) \right)$$

$$\le C_0 \left(g_0\left(z + \frac{r}{2} \right) + g_0\left(z - \frac{r}{2} \right) \right).$$

This means that for any fixed z, the function $w(r) = V(z,r)$ satisfies $w'(r) \leq C_0 w'(r/2)$. Thus

$$w(r) = \int_0^r w'(s)\, ds \leq C_0 \int_0^r w'(s/2)\, ds = 2C_0 \int_0^{r/2} w'(t)\, dt = 2C_0\, w(r/2)\,,$$

as claimed. □

LEMMA 5.2 (Poincaré inequality). *The weighted manifold $M(g_0)$ satisfies the following Poincaré inequality: for any ball $B(z,r)$ and any $f \in H^1(B(z,r))$ it holds*

$$(5.2) \qquad \inf_{\xi \in \mathbb{R}} \int_{B(z,r)} |f(s) - \xi|^2\, g_0(s)\, ds \;\leq\; 4\, r^2 \int_{B(z,r)} |f'(s)|^2\, g_0(s)\, ds\,.$$

Of course, the infimum on the left side is attained at $\xi = V(z,r)^{-1} \int_{B(z,r)} f g_0\, ds$. We emphasize that assumption (2.5) is not needed in this lemma.

Proof. The key ingredient in the proof is the inequality

$$\int_a^b |w|^2 |u|^2\, ds \leq 4 \sup_{a \leq s \leq b} \left(\int_a^s |w|^2\, dt \int_s^b \frac{dt}{|v|^2} \right) \int_a^b |v|^2 |u'|^2\, ds$$

for all u with $u(b) = 0$, which is an easy consequence of [OK, Thm. 6.2]; see also [Ma]. Given a function $f \in H^1(B(z,r))$ we apply this bound to $u(s) = f(s) - f(z+r)$ with $w = v = g_0$ and learn that

$$\inf_{\xi \in \mathbb{R}} \int_{B(z,r)} |f - \xi|^2\, g_0 ds \leq \int_{B(z,r)} |u|^2\, g_0\, ds \leq 4C \int_{B(z,r)} |u'|^2 g_0\, ds = 4C \int_{B(z,r)} |f'|^2 g_0\, ds$$

with constant

$$C = \sup_{(z-r)_+ \leq s \leq z+r} \left(\int_{(z-r)_+}^s g_0(t)\, dt \int_s^{z+r} \frac{dt}{g_0(t)} \right)\,.$$

The monotonicity of g_0 implies that

$$C \leq \sup_{(z-r)_+ \leq s \leq z+r} \left((s - (z-r)_+)\, g_0(s)\, (z+r-s)\, g_0(s)^{-1} \right) \leq r^2\,,$$

which proves the claimed bound. □

PROPOSITION 5.3 (one-dimensional heat kernel bound I). *Under condition* (2.5) *the heat kernel $k_0(r,s,t)$ of A_0 satisfies*

$$(5.3) \qquad \frac{c^{-1}}{\sqrt{t}\, g_0(r + \sqrt{t})} \;\leq\; k_0(r,r,t) \;\leq\; \frac{c}{\sqrt{t}\, g_0(r + \sqrt{t})}$$

for some $c > 0$, all $r > 0$, and all $t > 0$.

Proof. By Lemmas 5.1 and 5.2 the weighted manifold $M(g_0)$ satisfies the volume doubling condition and the Poincaré inequality. The Grigor'yan–Saloff-Coste theorem (see, e.g., [Gr, Thms. 6.1 and 6.2] or [SC2, p. 113]) then implies that

$$(5.4) \qquad \frac{c^{-1}}{V(r, \sqrt{t})} \;\leq\; k_0(r,r,t) \;\leq\; \frac{c}{V(r, \sqrt{t})}\,.$$

Clearly, we have

$$V(r, \sqrt{t}) \leq 2\sqrt{t} \, g_0(r + \sqrt{t}).$$

On the other hand, from (2.5) it easily follows that

$$V(r, \sqrt{t}) \geq \int_{r+\sqrt{t}/2}^{r+\sqrt{t}} g_0(s) \, ds \geq C_0^{-1} \int_{r+\sqrt{t}/2}^{r+\sqrt{t}} g_0(2s) \, ds = (2C_0)^{-1} \int_{2r+\sqrt{t}}^{2r+2\sqrt{t}} g_0(s) \, ds$$

$$\geq (2C_0)^{-1} \sqrt{t} \, g_0(r + \sqrt{t}).$$

In view of (5.4) this proves the statement. $\quad\square$

Proof of Theorem 2.3. The statement is an immediate consequence of Theorem 4.4, (4.13), and Proposition 5.3. $\quad\square$

5.2. Proof of Theorem 2.5. We now turn to metric trees which do not necessarily satisfy the volume doubling property. Instead, we assume that the quantity $S_\Gamma(\delta)$ in (2.9) is positive for some $\delta > 2$. Consequently, $\int_0^\infty g_0(s)^{-1} \, ds$ is finite. We also recall that $\tilde{S}_\Gamma(\delta)$ is defined in (2.11). We shall deduce the following proposition.

PROPOSITION 5.4 (Nash inequality). *Assume that g_0 satisfies* (2.9) *for some $\delta > 2$. Then for all $f \in H^1(\mathbb{R}_+, g_0) \cap L_1(\mathbb{R}_+, g_0)$ it holds*

$$(5.5) \quad \left(\int_0^\infty |f'|^2 g_0 \, dr \right)^{1/2} \left(\int_0^\infty |f| g_0 \, dr \right)^{2/\delta} \geq \tilde{S}_\Gamma^{1/2}(\delta) \left(\int_0^\infty |f|^2 g_0 \, dr \right)^{(\delta+2)/2\delta}.$$

Proof. By the one-dimensional Hardy–Sobolev inequality [OK, Thm. 6.2] (see also [Ma]),

$$\int_0^\infty |f'|^2 g_0 \, dr \geq \tilde{S}_\Gamma(\delta) \left(\int_0^\infty |f|^q g_0 \, dr \right)^{2/q},$$

where $q = 2\delta/(\delta - 2)$. By Hölder's inequality, we have for any $0 < \theta < 1$,

$$\int_0^\infty |f|^2 g_0 \, dr \leq \left(\int_0^\infty |f| g_0 \, dr \right)^\theta \left(\int_0^\infty |f|^{(2-\theta)/(1-\theta)} g_0 \, dr \right)^{1-\theta}.$$

Choosing $\theta = (q-2)/(q-1) = 4/(\delta+2)$ we obtain the assertion. $\quad\square$

PROPOSITION 5.5 (one-dimensional heat kernel bound II). *Under condition* (2.9) *for some $\delta > 2$ the heat kernel $k_0(r, s, t)$ of A_0 satisfies*

$$\sup_{r>0} k_0(r, r, t) \leq \left(\frac{\delta}{2 \, \tilde{S}_\Gamma(\delta)} \right)^{\delta/2} t^{-\delta/2}, \qquad t > 0.$$

Proof. This is a consequence of Proposition 5.4 and the classical Nash argument (see, e.g., [LL, Thm. 8.16] applied with $\gamma = \frac{\delta}{\delta+2}$). The fact that $\exp(-tA_0)$ is positivity preserving and sub-Markovian follows from the Beurling–Deny theorem [RS4, section XIII.12]. $\quad\square$

Remark 5.6. Conversely, if there exists an $S > 0$ and a $\delta > 2$ such that $\exp(-tA_0)(r, r) \leq (St)^{-\delta/2}$, then

$$\sup_{r>0} \left(\int_0^r g_0(s) \, ds \right)^{(\delta-2)/\delta} \left(\int_r^\infty \frac{ds}{g_0(s)} \right) \leq C_\delta \, S^{-1}$$

for some constant C_δ depending only on δ (and not on g_0). Indeed, the heat kernel estimate implies a Nash inequality with equivalent constants [LL, Thm. 8.16], the Nash inequality implies a Sobolev inequality with equivalent constants [BCLS], and the Sobolev inequality implies the above condition by the characterization of the one-dimensional Hardy–Sobolev inequality.

Proof of Theorem 2.5. The statement follows from Theorem 4.4 and Proposition 5.5.　□

5.3. Proof of Theorem 2.6. Finally, we discuss heat kernel bounds of homogeneous trees. In order to emphasize the dependence on the branching number $b \geq 2$ we write $A_b := A_0$ for the nonnegative operator in $L_2(\mathbb{R}_+, g_b)$ corresponding to the quadratic form

$$\int_0^\infty |f'|^2 g_b \, dr, \qquad f \in H^1(\mathbb{R}_+, g_b).$$

Moreover, let ω_b be the function on $\mathbb{R}_+$ satisfying

$$-(g_b \, \omega_b')' = \lambda_b \, g_b \, \omega_b \quad \text{in } \mathbb{R}_+ \setminus \mathbb{N},$$
$$\omega_b'(0) = 0, \quad \omega_b(j+) = \omega_b(j-), \quad \omega_b'(j-) = b \, \omega_b'(j+), \quad j \in \mathbb{N}.$$

We know from [EFK2, Lemma 4.2] that there is a positive constant c_b such that

$$(5.6) \qquad c_b^{-1} \frac{1+r}{\sqrt{g_b(r)}} \leq \omega_b(r) \leq c_b \frac{1+r}{\sqrt{g_b(r)}}.$$

Hence

$$S_b^{-1} := \sup_{r>0} \left(\int_0^r \omega_b(s)^2 g_b(s) \, ds \right)^{1/3} \left(\int_r^\infty \frac{ds}{\omega_b(s)^2 g_b(s)} \right) < \infty.$$

We write

$$\tilde{S}_b := (3/4)^{4/3} S_b.$$

PROPOSITION 5.7 (one-dimensional heat kernel bound III). *For every $b \geq 2$ the heat kernel $k_b(r, s, t)$ of A_b satisfies*

$$k_b(r, r, t) \leq \left(\frac{3}{2 \, \tilde{S}_b} \right)^{3/2} e^{-t\lambda_b} \, t^{-3/2} \, \omega_b^2(r).$$

The strategy of the following proof is similar in spirit to an argument in [Da2].

Proof. We consider the unitary operator $U : L_2(\mathbb{R}_+, g_b) \to L_2(\mathbb{R}_+, \omega_b^2 g_b)$ of multiplication by ω_b^{-1}. Integrating by parts and using the properties of ω_b we obtain the ground state representation

$$\int_0^\infty |f'|^2 \, g_b \, dr - \lambda_b \int_0^\infty |f|^2 \, g_b \, dr = \int_0^\infty |h'|^2 \, \omega_b^2 \, g_b \, dr$$

with $f = \omega_b h = U^* h$. This implies that $A_b - \lambda_b = U^* B_b U$, where B_b is the nonnegative operator in $L_2(\mathbb{R}_+, \omega_b^2 g_b)$ corresponding to the quadratic form

$$\int_0^\infty |h'|^2 \, \omega_b^2 \, g_b \, dr$$

with form domain $H^1(\mathbb{R}_+, \omega_b^2 g_b)$. This implies that $\exp(-tA_b) = e^{-t\lambda_b} U^* \exp(-tB_b)U$ which, in terms of the integral kernels, reads

$$\exp(-tA_b)(r,s) = e^{-t\lambda_b}\omega_b(r)\exp(-tB_b)(r,s)\omega_b(s)\,.$$

Hence we need to prove that

$$(5.7) \qquad \exp(-tB_b)(r,s) \leq \left(\frac{3}{2\,S_b}\right)^{3/2} t^{-3/2}\,.$$

From (5.6), the definition of $\tilde{S}_b$, and the one-dimensional Hardy–Sobolev inequality (see [OK, Thm. 6.2]), we obtain

$$\int_0^\infty |f'|^2\,\omega_b^2\,g_b\,dr \geq \tilde{S}_\Gamma \left(\int_0^\infty |f|^6\omega_b^2\,g_b\,dr\right)^{1/3}\,.$$

By Hölder,

$$\int_0^\infty |f|^2\omega_b^2\,g_b\,dr \leq \left(\int_0^\infty |f|\omega_b^2\,g_b\,dr\right)^{4/5} \left(\int_0^\infty |f|^6\omega_b^2\,g_b\,dr\right)^{1/5}\,.$$

Hence

$$(5.8) \quad \left(\int_0^\infty |f'|^2\omega_b^2\,g_b\,dr\right)^{1/2} \left(\int_0^\infty |f|\omega_b^2\,g_b\,dr\right)^{2/3} \geq \tilde{S}_b^{1/2} \left(\int_0^\infty |f|^2\omega_b^2\,g_b\,dr\right)^{5/6}\,.$$

Estimate (5.7) then follows again from Nash's argument (see, for example, [LL, Thm. 8.16]). $\quad\Box$

   Proof of Theorem 2.6. The statement follows from Theorem 4.4, Proposition 5.7, and (5.6). $\quad\Box$

   6. Applications: Spectral estimates for Schrödinger operators. Consider a Schrödinger operator $-\Delta_\Gamma - V$ in $L_2(\Gamma)$ with (minus) an electric potential $V : \Gamma \to [0,\infty)$ decaying at infinity. One of the classical problems of spectral theory is to estimate moments of negative eigenvalues of $-\Delta_\Gamma - V$ in terms of an L_p-norm of V. Estimates of the form

$$(6.1) \qquad \mathrm{tr}\,(-\Delta_\Gamma - V)_-^\gamma \leq C \int_\Gamma V(x)^{\gamma + \frac{a+1}{2}} g_0(|x|)^{\frac{a}{a-1}}\,dx, \qquad \gamma \geq 0\,,$$

were proved in [EFK2] under the assumption that $V(x) = V(|x|)$. The allowed values of γ and a in (6.1) are determined by the global dimension of Γ, and the constant C depends on a and γ but not on V. For $\gamma = 0$ the quantity

$$\mathrm{tr}\,(-\Delta_\Gamma - V)_-^0 = N(-\Delta_\Gamma - V)$$

coincides with the number of negative eigenvalues of $-\Delta_\Gamma - V$ (counted with their multiplicities). The heat kernel estimates proven in the previous section allow us to also extend some of the results obtained in [EFK2] to nonsymmetric potentials. Our approach is based on a well-known inequality due to Lieb [L] which, in combination with the identity

$$(6.2) \qquad \mathrm{tr}\,(-\Delta_\Gamma - V)_-^\gamma = \gamma \int_0^\infty \tau^{\gamma-1}\,N\,(-\Delta_\Gamma - V + \tau)\,d\tau\,,$$

yields

$$(6.3) \qquad \operatorname{tr}(-\Delta_\Gamma - V)_-^\gamma \ \leq \ M_{\beta,\gamma} \int_\Gamma \int_0^\infty k(x,x,t)\, t^{-1-\gamma}\, (t\, V(x) - \beta)_+ \, dt\, dx,$$

where $\beta > 0$ is arbitrary and

$$(6.4) \qquad M_{\beta,\gamma} = \Gamma(\gamma + 1) \left(e^{-\beta} - \beta \int_\beta^\infty s^{-1}\, e^{-s}\, ds \right)^{-1}.$$

This inequality reduces bounds on $\operatorname{tr}(-\Delta_\Gamma - V)_-^\gamma$ to bounds on the heat kernel $k(x, x, t)$.

From [EFK2] we recall the bound

$$(6.5) \qquad \operatorname{tr}(-\Delta_\Gamma - V)_-^{1/2} \ \leq \ \int_\Gamma V(x)\, dx$$

with sharp constant. By the Aizenman–Lieb argument, this also implies that

$$\operatorname{tr}(-\Delta_\Gamma - V)_-^\gamma \ \leq \ 4\, \frac{\Gamma(\gamma + 1)}{(4\pi)^{1/2}\Gamma(\gamma + 3/2)} \int_\Gamma V(x)^{\gamma + 1/2}\, dx$$

for $\gamma \geq 1/2$. When Γ has global dimension one, these bounds are essentially (possibly up to the factor 4 in front of the quotient) best possible. In the following we shall assume that Γ has global dimension larger than one.

6.1. Two-term estimates. As a first application of our heat kernel bounds we state a two-term upper bound on $\operatorname{tr}(-\Delta_\Gamma - V)_-^\gamma$ for symmetric trees with finite global dimension $d > 1$. Given a potential V and a parameter $\beta > 0$, we partition Γ as follows:

$$(6.6) \qquad \Gamma_\beta^- = \left\{ x \in \Gamma \,:\, V(x)\, g_0(|x|)^{\frac{2}{d-1}} < \beta \right\}, \qquad \Gamma_\beta^+ = \Gamma \setminus \Gamma_\beta^- \,.$$

THEOREM 6.1. *Let Γ be a symmetric tree with global dimension $d > 1$ and let $\beta > 0$.*

1. *Assume that $\gamma > 1/2$. Then*
   (6.7)
   $$\operatorname{tr}(-\Delta_\Gamma - V)_-^\gamma \ \leq \ L_d(\beta, \gamma) \int_{\Gamma_\beta^-} V(x)^{\gamma + \frac{d}{2}}\, g_0(|x|)\, dx + \tilde{L}_d(\beta, \gamma) \int_{\Gamma_\beta^+} V(x)^{\gamma + \frac{1}{2}}\, dx.$$

2. *Assume that $1 - d/2 < \gamma \leq 1/2$ if $1 < d \leq 2$ or that $0 \leq \gamma \leq 1/2$ if $d > 2$. Then*

   $$(6.8)\, \operatorname{tr}(-\Delta_\Gamma - V)_-^\gamma$$
   $$\leq L_d(\beta, \gamma) \int_{\Gamma_\beta^-} V(x)^{\gamma + \frac{d}{2}}\, g_0(|x|)\, dx + \tilde{L}_d(\beta, \gamma) \int_{\Gamma_\beta^+} V(x)\, g_0(|x|)^{\frac{1-2\gamma}{d-1}}\, dx.$$

The constants in the above estimates are given by

$$L_d(\beta, \gamma) = \frac{C\, M_{\beta,\gamma}\, \beta^{1 - \frac{d}{2} - \gamma}}{(\gamma + \frac{d}{2} - 1)(\gamma + \frac{d}{2})} \quad \text{if} \ \ \gamma \neq \frac{1}{2}, \qquad L_d(\beta, 1/2) = 2^{\frac{d+5}{2}}\, M_{\beta,\frac{1}{2}}\, \frac{C\, \beta^{\frac{1-d}{2}}}{d^2 - 1},$$

$$\tilde{L}_d(\beta, \gamma) = M_{\beta,\gamma}\, \beta^{\frac{1}{2} - \gamma} \left(\frac{\pi^{-1/2}}{|\gamma - \frac{1}{2}|} + \frac{C}{\gamma + \frac{d}{2} - 1} \right) \quad \text{if} \ \ \gamma \neq \frac{1}{2}, \qquad \tilde{L}_d(\beta, 1/2) = 2,$$

where C is the constant from Corollary 2.4.

The two terms on the right side of inequality (6.7) reflect the behavior of $\mathrm{tr}(-\Delta_\Gamma - V)_-^\gamma$ in the weak and strong coupling regime, i.e., for $V \to 0$ and $V \to \infty$, respectively. As it was shown in [EFK2] these limiting behaviors are determined by the local and global dimensions of Γ which are equal to 1 and d, respectively. Similar two-term estimates for Schrödinger operators on manifolds with different local and global dimensions were discussed recently in [EFK2, RS, MV] and the references therein.

Proof. We split the double integral on the right side of (6.3) into four integrals $I^{\sigma,\tau}$ with $\sigma, \tau \in \{\pm\}$. In the notation $I^{\sigma,\tau}$ the first index σ indicates that the x integration is over Γ_β^σ. The second index $\tau = -$ means that the t integration is restricted to $(0, g_0(|x|)^{2/(d-1)})$ and, correspondingly, for $\tau = +$.

In the integrals $I^{\sigma,-}$ with $\tau = -$ we use the bound $k(x,x,t) \le (\pi t)^{-1/2}$ from Theorem 2.2. Since $tV(x) - \beta \le 0$ for all $x \in \Gamma_\beta^-$ and $t \in (0, g_0(|x|)^{2/(d-1)})$, we have $I^{--} = 0$. Moreover,

$$I^{+-} \le \pi^{-\frac12} \int_{\Gamma_\beta^+} \int_0^{g_0(|x|)^{2/(d-1)}} t^{-\frac32-\gamma} (t V(x) - \beta)_+ \, dt \, dx$$

$$= \pi^{-\frac12} \beta^{-\gamma+\frac12} \int_{\Gamma_\beta^+} V(x)^{\gamma+\frac12} I_1(\beta^{-1}V(x)g_0(|x|)^{\frac{2}{d-1}}) \, dx \,,$$

where

$$I_1(a) = \int_1^a s^{-\frac32-\gamma} (s-1) \, ds \,, \quad a \ge 1 \,.$$

It is elementary to estimate

$$I_1(a) \le \begin{cases} (\gamma - \frac12)^{-1} & \text{if } \gamma > \frac12 \,, \\ (\frac12 - \gamma)^{-1} a^{\frac12-\gamma} & \text{if } 0 \le \gamma < \frac12 \,. \end{cases}$$

This leads to the desired bound on I^{+-}. (For $\gamma = 1/2$ we proceed differently below.)

We proceed to $I^{\sigma,+}$ with $\tau = +$ and recall the bound $k(x,x,t) \le Ct^{-d/2}g_0(|x|)$ from Corollary 2.4. We obtain

$$I^{-+} \le C \int_{\Gamma_\beta^-} g_0(|x|) \int_{g_0(|x|)^{2/(d-1)}}^\infty t^{-1-\frac{d}{2}-\gamma} (t V(x) - \beta)_+ \, dt \, dx$$

$$= C\beta^{-\gamma-\frac{d}{2}+1} \left(\int_{\Gamma_\beta^-} V(x)^{\gamma+\frac{d}{2}} g_0(|x|) \, dx\right) \left(\int_1^\infty s^{-1-\frac{d}{2}-\gamma} (s-1) \, ds\right) \,.$$

The latter integral is finite since $\gamma > 1 - d/2$. Finally, we have

$$I^{++} \le C \int_{\Gamma_\beta^+} g_0(|x|) \int_{g_0(|x|)^{2/(d-1)}}^\infty t^{-1-\frac{d}{2}-\gamma} (t V(x) - \beta)_+ \, dt \, dx$$

$$= C\beta^{-\gamma+\frac12} \int_{\Gamma_\beta^+} V(x)^{\gamma+\frac12} I_2(\beta^{-1}V(x)g_0(|x|)^{\frac{2}{d-1}}) \, dx \,,$$

where

$$I_2(a) = a^{\frac{d-1}{2}} \int_a^\infty s^{-1-\frac{d}{2}-\gamma} (s-1) \, ds \,, \quad a \ge 1 \,.$$

It is elementary to estimate

$$
I_2(a) \le \begin{cases} (\gamma + \frac{d}{2} - 1)^{-1} & \text{if } \gamma > \frac{1}{2}, \\ (\gamma + \frac{d}{2} - 1)^{-1} \, a^{\frac{1}{2} - \gamma} & \text{if } 0 \le \gamma < \frac{1}{2}. \end{cases}
$$

In view of (6.3) we have

$$
\operatorname{tr}(-\Delta_\Gamma - V)_-^\gamma \le M_{\beta,\gamma}\big(I^{++} + I^{+-} + I^{-+}\big).
$$

We have thus proven the statement of the Theorem for $\gamma > 1/2$ and for $1 - d/2 < \gamma < 1/2$.

For $\gamma = 1/2$ we proceed in a different way. First, we note that by substituting the bound for $k(x, x, t)$ from Corollary 2.4 into (6.3) we obtain

$$
(6.9) \qquad \operatorname{tr}(-\Delta_\Gamma - V)_-^{1/2} \le 4 M_{\beta,\frac{1}{2}} \, \frac{C \, \beta^{\frac{1-d}{2}}}{d^2 - 1} \int_\Gamma V(x)^{\frac{1+d}{2}} \, g_0(|x|) \, dx \, .
$$

Next, we recall that for two lower semibounded, self-adjoint operators H_1 and H_2 one has $N(H_1 + H_2) \le N(H_1) + N(H_2)$ by the variational principle. (Here again, $N(\cdot)$ denotes the number of negative eigenvalues, counting multiplicities.) Applying this with H_j replaced by $H_j + \tau/2$, i.e., $N(H_1 + H_2 + \tau) \le N(H_1 + \tau/2) + N(H_2 + \tau/2)$, and integrating with respect to τ, we find in view of (6.2)

$$
\operatorname{tr}(H_1 + H_2)_-^\gamma = \gamma \int_0^\infty N(H_1 + H_2 + \tau)\tau^{\gamma-1} \, d\tau \le 2^\gamma \left(\operatorname{tr}(H_1)_-^\gamma + \operatorname{tr}(H_2)_-^\gamma \right) .
$$

Now for given potential V and given parameter $\beta > 0$, we decompose $V = V_< + V_>$ with $V = V_<$ on Γ_β^- and $V = V_>$ on Γ_β^+. Applying the previous construction with $H_1 = -\Delta_\Gamma - 2V_<$ and $H_2 = -\Delta_\Gamma - 2V_>$ we conclude that

$$
(6.10) \qquad \operatorname{tr}(-\Delta_\Gamma - V)_-^\gamma \le \operatorname{tr}(-\Delta_\Gamma - 2V_<)_-^\gamma + \operatorname{tr}(-\Delta_\Gamma - 2V_>)_-^\gamma \, .
$$

The claimed inequality now follows by applying (6.9) and (6.5) to the first and second term, respectively. $\quad\square$

Remark 6.2. The operator $-\Delta_\Gamma - V$ has no weakly coupled eigenvalues if $d > 2$. Indeed, recall the Hardy-type inequality

$$
(6.11) \qquad -\Delta_\Gamma \ge \frac{C}{1 + |x|^2}, \qquad d > 2 \, ;
$$

see [EFK1]. In view of (6.11) it thus follows that for β small enough the first term on the right-hand side of (6.7) and (6.8) can be left out provided we modify the constant $\tilde{L}_d(\beta, \gamma)$. Indeed, using the notation of the proof of Theorem 6.1 we deduce from (6.11) that for sufficiently small β

$$
\operatorname{tr}(-\Delta_\Gamma - V)_-^\gamma = \operatorname{tr}(-\Delta_\Gamma - V_< - V_>)_-^\gamma \le \operatorname{tr}\left(-\frac{1}{2}\Delta_\Gamma + \frac{C}{2(1 + |x|^2)} - V_< - V_>\right)_-^\gamma
$$

$$
\le \operatorname{tr}\left(-\frac{1}{2}\Delta_\Gamma - V_>\right)_-^\gamma = 2^{-\gamma} \operatorname{tr}(-\Delta_\Gamma - 2V_>)_-^\gamma \, ,
$$

where we used that $V_< < \beta g_0(|x|)^{-2/(d-1)} \le C(2(1 + |x|^2))^{-1}$ if β is sufficiently small.

6.2. One-term estimates. The following statements are easy consequences of Theorem 6.1. We state them separately in order to show that they extend some of the results of [EFK2] to general, not necessarily symmetric, potentials V.

COROLLARY 6.3. *Let Γ be a symmetric tree with global dimension d. Assume that either $1 < d \leq 2$ and $0 \leq a < d - 1$, or else that $d > 2$ and $0 \leq a \leq 1$. Then for any $\gamma \geq \frac{1-a}{2}$ there is a constant $K(a, d, \gamma)$ such that*

$$(6.12) \qquad \mathrm{tr}\,(-\Delta_\Gamma - V)^\gamma_- \ \leq \ K(a, d, \gamma) \int_\Gamma V(x)^{\gamma + \frac{a+1}{2}}\, g_0(|x|)^{\frac{a}{d-1}}\, dx.$$

The only case of [EFK2, Thm. 2.7(1)] which is not covered by this corollary is $a = d - 1$ for $1 < d < 2$.

Proof. Assume first that $\gamma = (1-a)/2$. Then $1 - d/2 < \gamma \leq 1/2$ if $1 < d \leq 2$ and $0 \leq \gamma \leq 1/2$ if $d > 2$ and we are in the situation of the second item of Theorem 6.1. For any $\beta > 0$ we have $V(x)^{\gamma + \frac{d}{2}}\, g_0(|x|) \leq \beta^{\frac{d-a-1}{2}} V(x)^{\gamma + \frac{a+1}{2}}\, g_0(|x|)^{\frac{a}{d-1}}$ if $x \in \Gamma^-_\beta$ (since $d \geq a + 1$) and $V(x)\, g_0(|x|)^{\frac{1-2\gamma}{d-1}} = V(x)^{\gamma + \frac{a+1}{2}}\, g_0(|x|)^{\frac{a}{d-1}}$ if $x \in \Gamma^+_\beta$. From Theorem 6.1 we thus get inequality (6.12) with the constant

$$K(a, d, (1-a)/2) = \inf_{\beta > 0}\, \max\left\{ \beta^{\frac{d-a-1}{2}}\, L_d(\beta, (1-a)/2)\, ,\, \tilde{L}_d(\beta, (1-a)/2) \right\}.$$

For $\gamma > (1-a)/2$ the claim follows by the Aizenman–Lieb argument. $\qquad\square$

6.3. Estimates for homogeneous trees. As we have already mentioned, the infimum of the spectrum of the Laplace operator on a homogeneous tree with a branching number $b \geq 2$ is strictly positive and given by λ_b from (2.12). It is therefore natural to look for estimates on $\mathrm{tr}(-\Delta_\Gamma - \lambda_b - V)^\gamma_-$. This is the content of the following theorem.

THEOREM 6.4. *Let Γ_b be a homogeneous tree with branching number $b \geq 2$. Then for any $\gamma \geq 0$ there is a constant $L_b(\beta, \gamma)$ such that*

$$\mathrm{tr}\left(-\Delta^N_{\Gamma_b} - \lambda_b - V\right)^\gamma_- \ \leq \ L_b(\beta, \gamma) \int_{\Gamma_b} V(x)^{\gamma + \frac{3}{2}}\, (1 + |x|)^2\, dx\,.$$

Proof. The analogue of (6.3) in this case is

$$\mathrm{tr}\,(-\Delta_\Gamma - \lambda_b - V)^\gamma_- \ \leq \ \inf_{\beta > 0}\, M_{\beta, \gamma} \int_\Gamma \int_0^\infty k_b(x, x, t)\, e^{\lambda_b t}\, t^{-1-\gamma}\, (t\, V(x) - \beta)_+\, dt\, dx.$$

The result then follows by using the estimate from Theorem 2.6. $\qquad\square$

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