Office of Naval Research
Department of the Navy
Contract Nonr-220(35)

GENERAL FORMULATION OF A PERTURBATION
THEORY FOR UNSTEADY CAVITY FLOWS

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Report No. 97.7

March 1965

ABSTRACT

The problem of a two-dimensional cavity flow of an ideal fluid with small unsteady disturbances in a gravity free field is considered. By regarding the unsteady motion as a small perturbation of an established steady cavity flow, a fundamental formulation of the problem is presented. It is shown that the unsteady disturbance generates a surface wave propagating downstream along the free cavity boundary, much in the same way as the classical gravity waves in water, only with the centrifugal acceleration owing to the curvature of the streamlines in the basic flow playing the role of an equivalent gravity effect. As a particularly simple example, the surface waves in a hollow potential vortex flow is calculated by using the present theory.

INTRODUCTION

Since the pioneering contributions of Helmholtz⁽¹⁾¹ and Kirchhoff⁽²⁾, nearly a century ago, on the subject of steady, irrotational, plane flows involving free streamlines, extensive applications have been made to jets and to flows with a cavity or wake formation. In spite of such a long history and the mature state of steady free streamline theory and its wide applications to engineering problems, the subject of unsteady cavity flows has received attention only in the last seventeen years or so. Some of the difficulties involved in unsteady cavity flows can be envisaged as follows. The theoretical treatment of irrotational, two-dimensional cavitating flows of an ideal fluid is usually based on a certain proposed physical model, for example, the Kirchhoff-Helmholtz model. If the flow is steady, the exact solution of such a problem, within the assumption of the proposed model, is usually obtained by using the hodograph method, since in this case a surface of constant pressure is also one of constant speed. This property, however, no longer holds valid in the case when the flow is unsteady. Consequently, in order to investigate some of the characteristics of unsteady cavitating flows, different approaches and approximations have been introduced by various authors. Some of the early contributions have been discussed by Gilbarg⁽³⁾, Birkhoff and Zarantonello⁽⁴⁾. In order to help appraise the present state of the knowledge, a brief survey may be made here of the recent developments.

¹ Numbers in parenthesis refer to similarly numbered references in bibliography at end of paper.

In 1949 von Kármán⁽⁵⁾ treated an accelerated flow normal to a flat plate held fixed in an inertial frame such that with a certain acceleration, the flow separates from the plate to form a closed cavity of constant shape attached behind the plate, and he obtained a solution for a particular Froude number characterizing the acceleration. The entire set of flows for cavities with constant shape was later derived by Gilbarg⁽⁶⁾ for arbitrary polygonal obstacles. For cavities with varying shape, Gilbarg⁽⁶⁾ proposed the assumption that the free boundary, which is a material line, may be approximated by a streamline. As pointed out by Gilbarg, it seems physically reasonable that errors from this approximation may be quite small, at least for not too rapidly varying flows. Adopting this approximation, Woods⁽⁷⁾ treated the unsteady cavitating flow past curved obstacles with a finite cavity closed in the rear by a second fictitious body, as in the Riabouchinsky model for steady cavity flows⁽³⁾. Noticing the essential difference between the two distinct cases when the fluid at infinity is accelerating or when the body is accelerating (in an inertial frame), Yih⁽⁸⁾ treated both cases, deriving general formulae for unsteady cavity flows when the velocity potential Φ assumes the form $\Phi = U(t)f(x, y)$.

For the general case of unsteady cavity flows, another approach is to regard the unsteady part of the motion as a small perturbation of a steady cavity flow already established. With this approach Ablow and Hayes⁽⁹⁾ developed a perturbation theory which was later employed by Fox and Morgan⁽¹⁰⁾ to investigate the stability problem of some free sur-

face flows. Also, Curle⁽¹¹⁾ considered the large and small time solutions of a jet issuing from a slit. In these perturbation theories, the exact, linearized boundary conditions on the free surface are used. In this category a somewhat different perturbation theory has been applied to several specific problems by Woods⁽¹²⁾, Parkin⁽¹³⁾, Wu⁽¹⁴⁾, Timman⁽¹⁵⁾ and Geurst^(16, 17). In these latter works the free surface of an unsteady cavity flow is approximated by a streamline, thus releasing completely the kinematic condition imposed on the free boundary. By doing so, it is hoped that such approximation can give satisfactory results, perhaps for slowly varying flows. Based on such an approximation the resulting flow has been interpreted⁽¹²⁾ to contain the effect that an unsteady disturbance applied on the solid body will produce two vortex sheets leaving the separation points, propagating downstream on the free surface of the cavity with a velocity equal to that of the free stream of the basic flow. On physical grounds it can perhaps be argued that the linearized theory based on the approximation that the material lines be replaced by streamlines would become less consistent and less accurate for moderately and rapidly varying flows. On the other hand, the approach of Ablow and Hayes seems to have not yet been fully extended to treat the general case of unsteady cavity flows. It is the purpose of the present work to present a consistent formulation of a perturbation theory for the general case, following a method rather independent of that of Ablow and Hayes.

By assuming the time-dependent part of the flow to be small, a perturbation theory is developed here by a systematic linearization in the physical plane, without assuming that the displaced free surface of the

cavity be approximated by a streamline. From this general formulation it is seen that the unsteady motion of the solid body produces in general free surface dynamic waves propagating along the cavity boundary, much the same as the gravity waves generated by a floating body in motion. The centrifugal force owing to the curvature of the streamlines in the basic flow now plays the role of an equivalent gravity in the classical water wave problem. In this sense, the unsteady cavity flows are similar in nature to the radiation of gravity waves over a flat water surface, only now in a much more complex form since the centrifugal acceleration varies along the cavity surface. Such a dynamic wave phenomenon cannot be found in the theory using the streamline-approximation mentioned previously. A simple illustration of the present formulation is carried out for the surface waves over a hollow vortex first treated by Lord Kelvin⁽¹⁸⁾. Numerical results of typical unsteady cavity flows by using the present theory generally involve extensive analytical details; such results will be presented in a later work. It is the hope of this paper to stimulate further interest in developing this important and interesting subject, and in making applications for engineering purposes.

GENERAL THEORY

We suppose that for the time $t < 0$ a steady, irrotational, two-dimensional flow past a solid body has been established (in a gravity-free field), its solution being assumed to be known. For $t > 0$ the solid body is given an unsteady small disturbance, whose magnitude is char-

acterized by a small parameter ϵ . The resulting flow will be assumed to remain irrotational in a region containing the body-cavity system. We shall establish a perturbation theory, to the first order in ϵ , by regarding the time-dependent part of the flow as a small perturbation of the basic steady flow.

Under this assumption the flow possesses a velocity potential $\varphi(x, y, t; \epsilon)$ which may be expanded for $t > 0$ as

$$\varphi(x, y, t; \epsilon) = \varphi_0(x, y) + \epsilon \varphi_1(x, y, t) + O(\epsilon^2). \quad (1)$$

where x, y are the Cartesian coordinates of the physical plane, $\varphi_0(x, y)$ is the velocity potential of the basic steady flow, $\varphi_1(x, y, t)$ is the perturbation potential, being independent of ϵ . It may be noted that in the present formulation the space variables (x, y) are not perturbed. Strictly speaking, the function $\varphi_0(x, y)$ is defined only at points within the region of the basic steady flow, whereas $\varphi(x, y, t; \epsilon)$ may exist at points outside that region as dictated by the perturbed flow configuration. Under such circumstances it is assumed that the basic flow potential $\varphi_0(x, y)$ may be continued analytically into the region wherever needed. It is clear that $\varphi(x, y, t; \epsilon)$, $\varphi_0(x, y)$ and $\varphi_1(x, y, t)$ are all harmonic functions of x, y . We may further introduce the complex variable $z = x + iy$, the complex potential $f = \varphi + i\psi$, and the complex velocity $w = u - iv$, defined by:

$$w = \frac{df}{dz} = qe^{-i\theta}, \quad (2)$$

with q denoting the velocity magnitude and θ the flow inclination with the positive x -axis. The corresponding expansions of f and w are

$$f(z, t; \epsilon) = f_0(z) + \epsilon f_1(z, t) + O(\epsilon^2), \quad (3)$$

$$w(z, t; \epsilon) = w_0(z) + \epsilon w_1(z, t) + O(\epsilon^2). \quad (4)$$

Here $f_0(z) = \varphi_0(x, y) + i\psi_0(x, y)$ is the complex potential of the basic flow, $f_1(z, t) = \varphi_1(x, y, t) + i\psi_1(x, y, t)$ is the complex perturbation potential, both being analytic functions of z .

The pressure p is given by the Bernoulli equation

$$\frac{p}{\rho} + \frac{\partial \varphi}{\partial t} + \frac{1}{2} (\nabla \varphi)^2 = C \quad (5)$$

where ρ is the constant fluid density, C may be a function of t only, which, after being absorbed by the term $\partial \varphi / \partial t$, may be taken a constant.

Consistent with the above perturbation scheme, p is written in the form

$$p(x, y, t; \epsilon) = p_0(x, y) + \epsilon p_1(x, y, t) + O(\epsilon^2). \quad (6)$$

The pressure p_0 of the basic flow satisfies the steady form of (5).

$$\frac{1}{\rho} p_0 + \frac{1}{2} (\nabla \varphi_0)^2 = \frac{1}{\rho} p_\infty + \frac{1}{2} U^2 = \frac{1}{\rho} p_c + \frac{1}{2} q_c^2, \quad (7)$$

where p_∞ , U are respectively the free stream pressure and velocity, p_c the constant cavity pressure, q_c the constant flow speed on the cavity boundary of the basic flow, which is characterized by the cavitation number σ defined by

$$\sigma \equiv (p_\infty - p_c) / \left(\frac{1}{2} \rho U^2 \right) = (q_c / U)^2 - 1. \quad (8)$$

To facilitate the subsequent analysis, it is convenient to introduce a set of intrinsic coordinate (s, n) as an alternative space variable,

where s is the arc length measured along a streamline in the direction of the basic flow, and n the distance measured normal to a streamline in the direction of increasing ψ_0 , as shown in Figure 1. Thus, the functions $s(\varphi_0, \psi_0)$, $n(\varphi_0, \psi_0)$ can be defined by

$$d\varphi_0 = q_0(\varphi_0, \psi_0) ds, \quad d\psi_0 = q_0(\varphi_0, \psi_0) dn, \quad (9a)$$

with ds measured along $\psi_0 = \text{constant}$ and dn along $\varphi_0 = \text{constant}$.

Consequently, the differentiations with respect to s and n are defined as

$$\frac{\partial}{\partial s} = q_0 \left(\frac{\partial}{\partial \varphi_0} \right)_{\psi_0}, \quad \frac{\partial}{\partial n} = q_0 \left(\frac{\partial}{\partial \psi_0} \right)_{\varphi_0}. \quad (9b)$$

In terms of (s, n) , the continuity equation and irrotationality condition become respectively

$$\frac{\partial q_0}{\partial s} + q_0 \frac{\partial \theta}{\partial n} = 0, \quad (10a)$$

$$\frac{\partial q_0}{\partial n} - q_0 \frac{\partial \theta}{\partial s} = 0. \quad (10b)$$

The boundary conditions of this problem are as follows:

(i) There are two boundary conditions on the free surface of the cavity, one being kinematic and the other dynamic in nature. Let the displacement of the perturbed free surface of the cavity, S_f' , from that of the steady basic flow, S_f , be denoted by

$$F(s, n, t; \epsilon) = n - \epsilon h(s, t) = 0, \quad (11)$$

so that S_f is given by $n = 0$ (see Figure 1). Then the kinematic condition that the fluid particles on the free surface will remain on it requires

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \frac{\partial \varphi}{\partial s} \frac{\partial F}{\partial s} + \frac{\partial \varphi}{\partial n} \frac{\partial F}{\partial n} = 0 \quad \text{on} \quad n = \epsilon h(s, t). \quad (12)$$

By noting the definition of $s(\varphi_0, \psi_0)$, $n(\varphi_0, \psi_0)$ given by (9), one finds from (11) that

$$\frac{\partial F}{\partial n} = 1 - \epsilon \frac{\partial h}{\partial s} \left(\frac{\partial s}{\partial n} \right)_{\varphi_0}, \quad \frac{\partial F}{\partial s} = \left(\frac{\partial n}{\partial s} \right)_{\psi_0} - \epsilon \frac{\partial h}{\partial s}. \quad (13)$$

Also from (9) one readily derives that

$$\frac{\partial n}{\partial s} = q_0 \left(\frac{\partial n}{\partial \varphi_0} \right)_{\psi_0} = q_0 \int_0^{\psi_0} \frac{\partial}{\partial \varphi_0} \left(\frac{1}{q_0} \right) d\psi_0 = q_0 \int_0^n \frac{\partial}{\partial s} \left(\frac{1}{q_0} \right) dn; \quad (14a)$$

and similarly,

$$\frac{\partial s}{\partial n} = q_0 \int_0^s \frac{\partial}{\partial n} \left(\frac{1}{q_0} \right) ds. \quad (14b)$$

Substituting (13), (14) into (12), and using the expansion (1), one obtains, up to the order ϵ ,

$$\frac{\partial \varphi_1}{\partial n} = \frac{\partial h}{\partial t} + q_0 \frac{\partial h}{\partial s} + h \frac{\partial q_0}{\partial s} \quad \text{on} \quad n = \epsilon h(s, t), \quad (15)$$

in which use has been made of the general relationship $\partial \varphi_0 / \partial s = q_0$, and $\partial \varphi_0 / \partial n = 0$. After expanding the quantities in (15) about the undisturbed free surface, or $n = 0$, it is obvious that the same expression as above holds valid on $n = 0$. Now, by further applying the boundary condition that on $n = 0$, $q_0 = q_c$, which is a constant, the kinematic condition

finally becomes

$$\frac{\partial \varphi_1}{\partial n} = \frac{\partial h}{\partial t} + q_c \frac{\partial h}{\partial s} \quad \text{on} \quad n = 0. \quad (16)$$

For the dynamic condition, we assume here that the perturbed cavity boundary is subject to a prescribed unsteady, but uniform pressure perturbation,

$$p(s, n, t; \epsilon) = p_c + \epsilon p_1^*(t) \quad \text{on} \quad n = \epsilon h(s, t). \quad (17)$$

Substituting (17) and (1) into the Bernoulli equation (5), and using the property $\partial \varphi_0 / \partial n = 0$, one finds

$$\epsilon \frac{1}{\rho} p_1^*(t) + \epsilon \frac{\partial \varphi_1}{\partial t} + \frac{1}{2} \left(\frac{\partial \varphi_0}{\partial s} \right)^2 + \epsilon \frac{\partial \varphi_0}{\partial s} \frac{\partial \varphi_1}{\partial s} + 0 (\epsilon^2) = \frac{1}{2} q_c^2 \quad \text{on} \quad S_f'.$$

Expanding various quantities in the above equation about $n = 0$, and using $(\partial \varphi_0 / \partial s) = q_c$ on $n = 0$, one obtains, up to the order ϵ ,

$$\frac{\partial \varphi_1}{\partial t} + q_c \left(\frac{\partial \varphi_1}{\partial s} + h \frac{\partial q_0}{\partial n} \right) = - \frac{1}{\rho} p_1^*(t) \quad \text{on} \quad n = 0.$$

Now, from the irrotationality condition (10b) for the basic flow,

$$\left(\frac{\partial q_0}{\partial n} \right)_{n=0} = \left(q_0 \frac{\partial \theta}{\partial s} \right)_{n=0} = \mp \frac{q_c}{R}$$

where R is the radius of curvature of the steady cavity boundary, the - (or +) sign holds for the upper (or lower) branch of the cavity wall.

These signs are necessary to make R always a positive quantity. For a steady cavity flow it is assumed that the cavity pressure is a minimum pressure in the flow field, which implies that the cavity surface of the basic flow is concave when viewed from the cavity, hence $\partial \theta / \partial s$ is negative on the

upper cavity boundary AI and positive on the lower boundary BI (cf. Figure 1). The first order dynamic condition is therefore

$$\frac{\partial \varphi_1}{\partial t} + q_c \frac{\partial \varphi_1}{\partial s} + \frac{q_c^2}{R} h = -\frac{1}{\rho} p_1^*(t) \quad \text{on} \quad n = 0. \quad (18)$$

Equations (16) and (18) are two conditions on the cavity free surface; they can be combined into one for φ_1 by eliminating h , giving

$$\frac{1}{R} D(RD\varphi_1) + \frac{q_c^2}{R} \frac{\partial \varphi_1}{\partial n} = -\frac{1}{\rho} \left(\frac{R'}{R} p_1^* + \frac{dp_1^*}{dt} \right) \quad \text{on} \quad n = 0, \quad (19)$$

where

$$D \equiv \frac{\partial}{\partial t} + q_c \frac{\partial}{\partial s}, \quad R' \equiv dR/ds.$$

At this point it is of interest to note that if q_c^2/R is regarded as an equivalent gravitational acceleration g and the s -coordinate is rectilinear, then (16) and (18), or equivalently (19), are in the same form as those boundary conditions in the classical water wave problems in a gravity field, with g pointing towards the interior of the flow. Thus, the centrifugal acceleration q_c^2/R due to the curvature of the basic flow streamline now plays the role of the restoring force, much the same as gravity in water waves, in producing and propagating the surface waves along the curved cavity boundary. It may be remarked here that, if the perturbed free surface is approximated by its unperturbed steady free streamline boundary, thereby releasing the kinematic condition (16) and also the term with h in (18), then the essential restoring force is altogether dropped out. On physical grounds, it may therefore be expected that the present formulation will yield result fundamentally different

from a theory using the approximation that the perturbed free surface be replaced by a streamline.

The free surface condition (19) may also be expressed in a complex variable form. First we note from (4) that on S_f of the basic flow,

$$\theta = i \log(w_o/q_c).$$

Hence,

$$\frac{1}{R} = \bar{\tau} \frac{\partial \theta}{\partial s} = \bar{\tau} i \frac{1}{w_o} \frac{\partial w_o}{\partial s} = \bar{\tau} i \frac{q_c}{w_o} \frac{\partial w_o}{\partial \varphi_o} = \bar{\tau} i \frac{q_c}{w_o} \frac{dw_o}{df_o}.$$

For compactness, we impose the normalization that $q_c = 1$. Then, by using (9b) and the Cauchy-Riemann equation $\partial \varphi_1 / \partial \psi_o = -\partial \psi_1 / \partial \varphi_o$, (19) can be finally written as

$$\begin{aligned} \operatorname{Re} \left[\left\{ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial f_o} \right)^2 - \left[\frac{d}{df_o} \log \left(\frac{1}{w_o} \frac{dw_o}{df_o} \right) \right] \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial f_o} \right) - \left(\frac{1}{w_o} \frac{dw_o}{df_o} \right) \frac{\partial}{\partial f_o} \right\} f_1(f_o, t) \right] \\ = \frac{-1}{\rho} \left[p_1^* \frac{d}{df_o} \log \left(\frac{1}{w_o} \frac{dw_o}{df_o} \right) + \frac{dp_1^*}{dt} \right] \quad \text{on } S_f, \end{aligned} \quad (20)$$

which is the free surface condition in a complex variable form.

(ii) At the solid surface the normal component of the flow velocity relative to the moving boundary must vanish. Again using the intrinsic coordinates (s, n) , let the displacement of the wetted side of the solid body, S_o' , from its basic position S_o be prescribed by

$$F(s, n, t; \epsilon) = n - \epsilon h(s, t) = 0 \quad \text{on } S_o' \quad (21)$$

over a range of s covered by the solid surface in the steady flow. Then it is clear that the kinematic condition (15) also holds valid on the solid

surface S_0 (or $n = 0$), that is,

$$\frac{\partial \varphi_1}{\partial n} = \frac{\partial h}{\partial t} + \frac{\partial}{\partial s} (q_0 h) \quad \text{on } S_0, \quad (22)$$

h being here a known function of s and t . The equivalent complex form of (22) is readily seen to be

$$\text{Im} \frac{\partial f_1(f_0, t)}{\partial f_0} = \frac{-1}{q_0} \left[\frac{\partial h}{\partial t} + \frac{\partial}{\partial s} (q_0 h) \right] \quad \text{on } S_0. \quad (23)$$

Note that the speed q_0 of the basic flow on the solid surface S_0 is not a constant.

(iii) The condition at the point at infinity depends on the free stream velocity and on whether or not the cavity volume is permitted to change with time. If the free stream velocity has a prescribed small perturbation $U_1(t)$ (U_1 may be complex), so that the free stream velocity is $U_\infty = U + \epsilon U_1(t)$, then we require that

$$w_1 = \frac{\partial f_1}{\partial z} \rightarrow U_1(t) \quad \text{as } |z| \rightarrow \infty. \quad (24)$$

Furthermore, when changes in the cavity volume are involved, then an appropriate representation of the flow can be made by introducing a fluid source at the point at infinity, as discussed by Wu⁽¹⁹⁾. For such cases, a finite cavity model for the basic flow is required to incorporate the source at infinity into the flow problem, as shown previously by the present authors⁽²⁰⁾. These points have been further justified from a physical and mathematical ground by Benjamin⁽²¹⁾. Furthermore, from Kelvin's theorem on the conservation of circulation, the circulation

around the point at infinity cannot be changed in unsteady flow for $t < \infty$. Therefore, in addition to condition (24), it is required that

$$\oint_{\Gamma} \frac{df_1}{dz} dz = i Q_1(t) \quad (25)$$

where Γ is a contour around the point at infinity and $Q_1(t)$ is equal to the time rate of increase in cavity volume, which is supposed to be prescribed.

Finally, we state that if the problem is of the initial value type, then no radiation condition is needed for the surface waves; these waves will turn out to propagate automatically towards downstream. However, in the case of simple harmonic motions, when being treated as a quasi-steady flow, then the so-called "radiation condition" will be needed to ensure that the waves generated by body motion will not propagate upstream on the cavity surface. This completes our formulation of the problem.

SURFACE WAVES ON A HOLLOW VORTEX

This relatively simple problem was chosen to demonstrate an application of this general theory to a special case; this application is partly meant for a verification of the complicated expression of the free surface boundary condition, since the problem has already been solved by Lord Kelvin⁽¹⁸⁾ in a completely different way.

The basic flow is an irrotational, circulating motion about a point, say $z = 0$, as center. The free surface will be denoted by

$|z| = a$ on which the speed of flow is normalized to unity. The velocity potential is simply

$$f_0(z) = -ia \log z \quad (26)$$

and therefore,

$$w_0(z) = -\frac{ia}{z}. \quad (27)$$

The transformation

$$\zeta = i \frac{1 - w_0}{1 + w_0} \quad (28)$$

maps the entire basic flow in the hodograph w_0 -plane, $|w_0| \leq 1$, into the upper half of the ζ -plane (see figure 2).

We assume that the cavity pressure be kept at constant, that is, $p_1(t) = 0$, then the boundary conditions of this problem are

$$\operatorname{Re}[L(f_1)] = 0 \quad \text{on} \quad \eta = 0^+, \quad -\infty < \xi < \infty, \quad (29)$$

where $L = \left(\frac{\partial}{\partial f_0} + \frac{\partial}{\partial t}\right)^2 - \left\{\frac{d}{df_0} \log \left(\frac{1}{w_0} \frac{dw_0}{df_0}\right)\right\} \left(\frac{\partial}{\partial f_0} + \frac{\partial}{\partial t}\right) - \frac{1}{w_0} \frac{dw_0}{df_0} \frac{\partial}{df_0}$,

$$\left|\frac{\partial f_1}{\partial z}\right| = 0 \quad \text{as} \quad |z| \rightarrow \infty \quad (30)$$

and

$$\oint_{\Gamma} \left(\frac{\partial f_1}{\partial z}\right) dz = 0, \quad (31)$$

where Γ is a contour around the point at infinity, or, $\zeta = i$. A particular solution of the above boundary value problem is

$$L(f_1) = 0. \quad (32)$$

The complementary solution can be written as

$$L(f_1) = \sum_{n=-\infty}^{\infty} [c_n(t)(\zeta - i)^n - \overline{c_n(t)}(\zeta + i)^n], \quad (33)$$

where $c_n(t)$ are unknown functions of t . In order to satisfy equations (30) and (31), that is, $L(f_1) = O[(\zeta - i)^a]$, $a > 0$, as $\zeta \rightarrow i$, and $|L(f_1)| < \infty$ at every point on the free surface it is necessary that $c_n = 0$ for all n . Therefore, the solution of this boundary value problem is given by (32) alone.

By use of equations (26) and (27), equation (32) can be changed into

$$\left(\frac{z^2}{a^2} \frac{\partial^2}{\partial z^2} + \frac{2z}{a^2} \frac{\partial}{\partial z} - \frac{2i}{a} z \frac{\partial^2}{\partial z \partial t} - \frac{\partial^2}{\partial t^2} \right) f_1(z, t) = 0. \quad (34)$$

Since $f_1(z, t)$ should be regular everywhere outside $|z| = a$, we may write

$$f_1(z, t) = \sum_{n=1,2,\dots}^{\infty} A_n(t) \frac{1}{z^n}. \quad (35)$$

Substitution of equation (35) into equation (34) gives

$$\ddot{A}_n(t) - \frac{2in}{a} \dot{A}_n(t) - \frac{n}{a^2} (n-1) A_n(t) = 0,$$

where dot represents the differentiation with respect to t . The solution of the last equation is

$$A_n = a_n e^{\frac{i}{a}(n \pm \sqrt{n})t},$$

where a_n are constants. Finally, we have

$$f_1(z, t) = \sum_{n=1,2,\dots}^{\infty} \left[a_n e^{\frac{i}{a}(n + \sqrt{n})t} + \beta_n e^{\frac{i}{a}(n - \sqrt{n})t} \right] \frac{1}{z^n}, \quad (36)$$

where the constants α_n and β_n can be determined by appropriate initial conditions. The above result agrees with that obtained by Lord Kelvin⁽¹⁸⁾.

ACKNOWLEDGMENT

This work is sponsored in part by the Office of Naval Research of the U. S. Navy, under contract Nonr-220(35). Reproduction in whole or in part is permitted for any purpose of the United States Government.

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Figure 1. Illustration of an unsteady perturbation flow.

Figure 2. Hollow vortex and its conformal mapping planes.

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