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Derivation and Application of the Fokker–Planck Equation to Discrete Nonlinear Dynamic Systems Subjected to White Random Excitation

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The Fokker–Planck equation is derived and applied to discrete nonlinear dynamic systems subjected to white random excitation. For the class of problems in which the nonlinearities involve only the displacements of the system, it is shown that exact solutions can be constructed for the stationary Fokker–Planck equation. It is further shown that if stationary solutions exist they are unique.

INTRODUCTION

RECENT developments in jet and rocket propulsion have given rise to new problems in mechanical and structural vibrations. The pressure fields generated by these devices fluctuate in a random manner and contain a wide spectrum of frequencies that may result in severe vibration in the aircraft or missile structure. As more data are gathered on strong-motion earthquakes, it is becoming apparent that earthquakes are examples of random processes that may excite severe vibration and even failure in buildings and other structures. Measurements of the motion of ships in a confused sea or aircraft flying through turbulent air reveal that such motions can be described only statistically.

The examples given above have two things in common: (a) they involve the response of mechanical systems to random excitation; (b) in general, they involve nonlinear behavior, since almost all real physical systems exhibit nonlinearity for sufficiently large motions.

The theory of linear systems subjected to random excitation is well-developed,[1–4] and, though there still remain many unanswered questions, we can answer most of the questions of practical interest. In the case of nonlinear systems, however, the standard techniques of linear analysis cannot be applied, though approximate methods have been developed to extend linear analysis to certain systems containing small nonlinearities.[5–7] The purpose of this paper is to deal with a different approach to the analysis of linear and nonlinear systems, based on the theory of Markoff random processes. We show that the behavior of discrete dynamic systems subjected to white random excitation are examples of continuous multidimensional Markoff processes. Such processes are completely characterized by their transitional or conditional probability law, which is obtained as the fundamental solution to the Fokker–Planck equation appropriate to the dynamic system. It is shown that exact stationary solutions may be constructed for a class of nonlinear problems in which the nonlinearity is a function only of the displacements. Furthermore, it is shown that if stationary solutions exist they are unique.

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[7] T. K. Caughey, "Random Excitation of a Loaded Nonlinear String," J. Appl. Mech. **27**, 575–578 (1960).

I. FOKKER–PLANCK EQUATION

A. Basic Concepts of Probability Theory

Roughly speaking, what is meant by a random excitation is one in which the forcing function does not depend in a completely definite way on the independent variable, time, as in a casual process. On the contrary, one gets in different observations different functions of time, so that only the probability is directly observable. The following set of probability distributions completely defines a random function[8]:

$p_1(yt)dy$= probability of finding y in the range from y to $y+dy$ at time t.

$p_2(y_1t_1,y_2t_2)dy_1dy_2$= joint probability of finding y in the range from y_1 to y_1+dy_1 at time t_1 and in the range from y_2 to y_2+dy_2 at time t_2.

$p_3(y_1t_1,y_2t_2,y_3t_3)dy_1dy_2dy_3$= joint probability of finding y in the range from y_1 to y_1+dy_1 at time t_1, in the range from y_2 to y_2+dy_2 at time t_2, and in the range from y_3 to y_3+dy_3 at time t_3.

The higher probability densities p_n, where $n=4$, 5, 6, $\cdots$, are defined in a similar manner. Each p_n must satisfy the following conditions:

(a) $p_n \geqq 0$.

(b) p_n is symmetric in y_1t_1, y_2t_2, $\cdots$, y_nt_n.

(c) $$p_k=\underbrace{\int\cdots\int}_{n-k} p_n dy_{k+1}\cdots dy_n.$$

Condition (c) is the important equation for determining a *marginal* probability.

The probability density p_n can be used as a means of classifying a random function. The simplest case is that of a purely random function. This means that the value of y at some time t_1 does not depend upon, or is not correlated with, the value of y at any other time t_2. The probability distribution $p_1(yt)dy$ completely describes the function in this case, since the higher distributions are found from the following equation:

$$p_n(y_1t_1,y_2t_2,\cdots,y_Nt_N)=\prod_{i=1}^{N} p_1(y_it_i). \qquad (1.1)$$

The next more complicated case is where the probability density p_2 completely describes the functions. This is the so-called *Markoff process*. To define a Markoff process more precisely, we introduce the idea of the *conditional* probability. We define $p_{c2}(y_1|y_2,t)dy_2$ as probability that, for a given $y=y_1$ at $t=0$, we find y in the range from y_2 to y_2+dy_2 at a time t later. We find p_{c2} by the relation

$$p_2(y_1t_1,y_2t_2)=p_1(y_1t_1)p_{c2}(y_1|y_2,\ t_2-t_1). \qquad (1.2)$$

Equation (1.2) is the analogous to the joint probability of two dependent events:

$$P(AB)=P(A)P(A|B),$$

where $P(AB)$ is the probability of events A and B both occurring, $P(A)$ the probability of events A occurring, and $P(A|B)$ the probability of event B occurring on the given condition that event A has already occurred.

Then, $P(AB)$ is the analog of p_2, $P(A)$ is the analog of p_1, and $P(A|B)$ is the analog of p_{c2}.

The function p_{c2} must satisfy the conditions

(d) $p_{c2}(y_1|y_2t)\geqq 0$;

(e) $$\int p_{c2}(y_1|y_2t)dy_2=1;$$

(f) $$p_1(y_2t_2)=\int p_1(y_1t_1)p_{c2}(y_1|y_2,\ t_2-t_1)dy_1.$$

We can now define the Markoff process to mean that the conditional probability that y lies in the interval, from y_1 to y_1+dy_1 at t_1, from y_2 to y_2+dy_2 at t_2, $\cdots$ from y_{n-1} to $y_{n-1}+dy_{n-1}$ at t_{n-1}, depends only on the values of y at t_n and t_{n-1}. That is, for a Markoff process

$$p_{cn}(y_1t_1,y_2t_2,\cdots,y_{n-1}t_{n-1}|y_nt_n) = p_{c2}(y_{n-1}t_{n-1}|y_nt_n). \qquad (1.3)$$

It is now possible to derive p_3, p_4, $\cdots$ from p_2 and Eq. (1.2). For example:

$$p_3(y_1t_1,y_2t_2,y_3t_3)=p_2(y_1t_1,y_2t_2)p_{c2}(y_2t_2|y_3t_3) = \frac{p_2(y_1t_1,y_2t_2)p_2(y_2t_2,y_3t_3)}{p_1(y_2t_2)}; \qquad (1.4)$$

$$p_4(y_1t_1,y_2t_2,y_3t_3,y_4t_4)=p_3(y_1t_1,y_2t_2,y_3t_3)p_{c2}(y_3t_3|y_4t_4) = \frac{p_2(y_1t_1,y_2t_2)p_2(y_2t_2,y_3t_3)}{p_1(y_2t_2)} \times\frac{p_2(y_3t_3,y_4t_4)}{p_1(y_3t_3)}, \qquad (1.5)$$

the latter from Eqs. (1.4) and (1.2).

In addition to conditions (d)–(f) on p_{c2}, it must also satisfy the condition

$$p_{c2}(y_1|y_2t)=\int p_{c2}(y_1|y\tau)p_{c2}(y|y_2,\ t-\tau)dy \qquad (0\leqq\tau<t). \qquad (1.6)$$

This equation has been alternately called the *Smoluchowski equation* and the Chapman–Kolmogorov

[8] M. C. Wang and G. E. Uhlenbeck, "On the Theory of Brownian Motion II," Rev. Mod. Phys. **17**, 323–342 (1945). [Also N. Wax *et al.*, *Selected Papers on Noise and Stochastic Processes* (Dover Publications, Inc., New York, 1954), pp. 113–132.]

equation. It implies that, when y follows any path from y_1 at time zero to y_2 at time t later, the particular path of y at time τ is unimportant. Equation (1.6) is seen to integrate the probability over any path selected.

The next step would be to consider processes that are completely described by $p_3, p_4, \cdots$. Physically, there are few examples studied that involve these higher-order processes. Sometimes, when a process is not Markovian, we can find another variable z, which combined with y, makes a Markoff process. The variable z may be $\dot{y}=dy/dt$ or another coordinate. In this case, the Smoluchowski equation becomes

$$p_{c2}(y_1z_1|y_2z_2,t)=\int\int p_{c2}(y_1z_1|yz,t_1)\times p_{c2}(yz|y_2z_2,t-t_1)dydz. \quad (1.7)$$

Equation (1.7) may be generalized to problems involving N coordinates in place of the two used above. The general form of the equation is then

$$p_{c2}(\mathbf{y}_1|\mathbf{y}_2,t)=\int_{N\text{-fold}}\cdots\int\prod_{i=1}^{N}p_{c2}(\mathbf{y}_1|\mathbf{z},t_1)\times p_{c2}(\mathbf{z}|\mathbf{y}_2,t-t_1)dz_i, \quad (1.8)$$

where $\mathbf{y}$ is the position vector of a point in N-dimensional phase space and the integral is extended over all the phase space.

B. Derivation of the Fokker–Planck Equation

In order to derive the Fokker–Planck equation, the following assumptions must be made. The first and second incremental statistical moments of the displacement of the phase point in an infinitesimal period of time are

$$A_i(\mathbf{y},t)=\int_{N\text{-fold}}\cdots\int(z_1-y_1)p_{c2}(\mathbf{y}|\mathbf{z},\Delta t)\prod_{i=1}^{N}dz_i; \quad (1.9)$$

$$B_{ij}(\mathbf{y},t)=\int_{N\text{-fold}}\cdots\int(z_i-y_i)(z_j-y_j)\times p_{c2}(\mathbf{y}|\mathbf{z},\Delta t)\prod_{i=1}^{N}dz_i; \quad (1.10)$$

where $(i, j=1, 2,\cdots, N)$. The assumption is made that as $\Delta t\rightarrow 0$ only these moments of the displacement of the phase point become proportional to Δt, so that the following limits exist:

$$a_i(\mathbf{y},t)=\lim_{\Delta t\to 0}\frac{A_i}{\Delta t};\qquad b_{ij}(\mathbf{y},t)=\lim_{\Delta t\to 0}\frac{B_{ij}}{\Delta t}; \quad (1.11)$$

and that the higher moments are of the order of (Δt). Physically, this implies that in a small time interval the coordinates of the phase point can change only by small amounts, and that is tantamount to the assumption of a Gaussian or normal probability distribution for the disturbances acting on the system.

Let us consider the Smoluchowski equation written in the form

$$p_c(\mathbf{x}|\mathbf{y}, t+\Delta t)=\int_{N\text{-fold}}\cdots\int\prod_{i=1}^{N}dz_i\times p_c(\mathbf{x}|\mathbf{z},t)p_c(\mathbf{z}|\mathbf{y},\Delta t), \quad (1.12)$$

where $\mathbf{z}$ at time t is a point in phase space on any path of $\mathbf{y}$ from $\mathbf{x}$ at time zero to $\mathbf{y}$ at time $t+\Delta t$ later. Now, let $R(\mathbf{y})$ be an arbitrary scalar function of the variables $y_1, y_2, \cdots, y_N$, such that $R(\mathbf{y})\rightarrow 0$ as all $y_i\rightarrow\pm\infty$. Multiplying Eq. (1.12) by $R(\mathbf{y})$ and integrating over the phase space,

$$\int_{N\text{-fold}}\cdots\int R(\mathbf{y})p_c(\mathbf{x}|\mathbf{y}, t+\Delta t)\prod_{i=1}^{N}dy_i =\int_{N\text{-fold}}\cdots\int\prod_{j=1}^{N}dz_j\int_{N\text{-fold}}\cdots\int R(\mathbf{y})p_c(x|z,t)\times p_c(\mathbf{z}|\mathbf{y},\Delta t)\prod_{i=1}^{N}dy_i. \quad (1.13)$$

Developing $R(\mathbf{y})$ in a Taylor series in (y_i-z_i),

$$R(\mathbf{y})=R(\mathbf{z})+\sum_{i=1}^{N}(y_i-z_i)\frac{\partial R(\mathbf{y})}{\partial z_i}+\tfrac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(y_i-z_i)(y_j-z_j)\times\frac{\partial^2R(\mathbf{y})}{\partial z_i\partial z_j}+0(|\mathbf{y}-\mathbf{z}|^2). \quad (1.14)$$

Substituting Eq. (1.14) into Eq. (1.13) and using Eq. (1.11), Eq. (1.13) becomes

$$\frac{1}{\Delta t}\int_{N\text{-fold}}\cdots\int R(\mathbf{y})\{p_c(\mathbf{x}|\mathbf{y}, t+\Delta t)-p_c(\mathbf{x}|\mathbf{y},t)\}\prod_{i=1}^{N}dy_i =\int_{N\text{-fold}}\cdots\int R(\mathbf{y})\left\{-\sum_{i=1}^{N}\frac{\partial}{\partial y_i}[a_ip_c(\mathbf{x}|\mathbf{y},t)]+\tfrac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial^2}{\partial y_i\partial y_j}[b_{ij}p_c(\mathbf{x}|\mathbf{y},t)]\right\}\prod_{i=1}^{N}dy_i. \quad (1.15)$$

Taking the limit of the L.H. side as $\Delta t \rightarrow 0$, transposing the R.H. side, and substituting p_c for $p_c(\mathbf{x}|\mathbf{y},t)$,

$$\int_{N\text{-fold}} \cdots \int R(\mathbf{y}) \left\{ \frac{\partial p_c}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial y_i}[a_i p_c] - \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2}{\partial y_i \partial y_j}[b_{ij} p_c] \right\} \prod_{i=1}^{N} dy_i = 0. \quad (1.16)$$

Since the function $R(\mathbf{y})$ is arbitrary, the quantity in the braces must vanish identically, giving

$$\frac{\partial p_c}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial y_i}[a_i p_c] + \tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2}{\partial y_i \partial y_j}[b_{ij} p_c]. \quad (1.17)$$

This parabolic, partial differential equation is known as the *Fokker–Planck equation* and is characteristic of diffusion processes of great importance to the fields of chemistry and thermodynamics.

The required solution of Eq. (1.17) is the positive solution satisfying the initial conditions

$$p_c(\mathbf{x}|\mathbf{y},t) \rightarrow \prod_{i=1}^{N} \delta(y_i - x_i) \quad (t \rightarrow 0^+), \quad (1.18)$$

where x_i is the initial value of y_i. Thus, the probability density approaches a delta function because the probability distribution (of $\mathbf{y}$ becoming $\mathbf{x}$) approaches unity when the time t approaches zero (for $t \geqq 0$).

C. Stationary Solution

In certain problems, it may happen that with the passage of time the conditional probability $p_c(\mathbf{x}|\mathbf{y},t)$ tends to a limiting stationary probability density $p(\mathbf{y})$. Simply stated, the probability density is no longer dependent on the time and the initial conditions. A solution to $p(\mathbf{y})$, if it exists, may be obtained from the Fokker–Planck equation by letting $t \rightarrow \infty$ and writing $\partial p/\partial t = 0$. Thus, the equation for $p(\mathbf{y})$ is

$$\tfrac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2}{\partial y_i \partial y_j}[b_{ij}(\mathbf{y}) p(\mathbf{y})] - \sum_{i=1}^{N} \frac{\partial}{\partial y_i}[a_i(\mathbf{y}) p(\mathbf{y})] = 0. \quad (1.19)$$

II. APPLICATIONS

A. Single-Degree-of-Freedom System with Nonlinear Stiffness

Consider the nonlinear oscillator whose response $x(t)$ is related to the excitation $f(t)$ by the following differential equation:

$$\ddot{x} + \beta \dot{x} + F(x) = f(t), \quad (2.1)$$

where the excitation $f(t)$ is a stationary Gaussian, white random process with a mean of zero; i.e.,

$$\langle f(t) \rangle = 0; \qquad \langle f(t_1) f(t_2) \rangle = (W_0/2)\delta(t_1 - t_2). \quad (2.2)$$

$\langle \cdots \rangle$ denotes an ensemble average, W_0 is the constant or white spectral density of the excitation, β is the ratio of the linear, viscous damping coefficient to the mass, and $F(x)$ is the ratio of the nonlinear restoring force to the mass. If the nonlinear oscillator is attached to an *immovable* base, then $x(t)$ is the *absolute* displacement response of mass of the oscillator and $f(t)$ is the ratio of the exciting force (applied to the mass) to the mass. On the other hand, if the nonlinear oscillator is attached to an *oscillating* base, then $x(t)$ is the displacement response of the mass *relative* to the base and $f(t)$ is the negative of the vibratory *acceleration* of the base.

When $f(t)$ is the ratio of the exciting force to the mass, W_0 is the constant value of the mean square force (per unit mass) per unit bandwidth at frequency $\mathfrak{f}$:

$$W_{ff}(\mathfrak{f}) = \lim_{\substack{(\Delta \mathfrak{f}) \rightarrow 0 \\ t \rightarrow \infty}} \frac{\overline{\Delta \mathfrak{f}(t)^2}}{(\Delta \mathfrak{f})},$$

where $(\Delta \mathfrak{f})$ is the bandwidth in cycles per unit time. When $f(t)$ is the negative of the vibratory acceleration of the base, then W_0 is the constant value of $W_{ff}(\mathfrak{f})$, the mean square acceleration per unit bandwidth.

Writing $y_1 = x$ and $y_2 = \dot{x}$, Eq. (2.1) is equivalent to the following pair of first-order equations:

$$\dot{y}_1 = y_2; \quad \dot{y}_2 = -\beta y_2 - F(y_1) + f(t). \quad (2.3)$$

The coefficients a_i, b_{ij} in the Fokker–Planck equation may now be determined from Eq. (1.11) once A_i, B_{ij} of Eqs. (1.9) and (1.10) are determined. By knowledge of the physical system, $A_1 = \langle \Delta y_1 \rangle$; $A_2 = \langle \Delta y_2 \rangle$; $B_{11} = \langle \Delta y_1^2 \rangle$; $B_{12} = B_{21} = \langle \Delta y_1 \Delta y_2 \rangle$; $B_{22} = \langle \Delta y_2^2 \rangle$. Making these substitutions into Eq. (1.11),

$$\begin{aligned}
a_1 &= \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta y_1 \rangle}{\Delta t} = y_2 = \dot{x}; \\
a_2 &= \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta y_2 \rangle}{\Delta t}; \\
b_{11} &= \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta y_1^2 \rangle}{\Delta t} = 0; \\
b_{12} = b_{21} &= \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta y_1 \Delta y_2 \rangle}{\Delta t} = 0; \\
b_{22} &= \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta y_2^2 \rangle}{\Delta t}.
\end{aligned} \quad (2.4)$$

Substituting Eq. (2.3) for $\langle \Delta y_2 \rangle$ and $\langle \Delta y_2^2 \rangle$ and utilizing

τ as a dummy variable in t, a_2 and b_{22} become

$$a_2=\lim_{\Delta t\to 0}\frac{\left\langle[-\beta y_2-F(y_1)]\Delta t+\int_t^{t+\Delta t}f(\tau)d\tau\right\rangle}{\Delta t};$$

$$b_{22}=\lim_{\Delta t\to 0}\frac{\left\langle\left\{[-\beta y_2-F(y_1)]\Delta t+\int_t^{t+\Delta t}f(\tau)d\tau\right\}^2\right\rangle}{\Delta t}$$

$$=\lim_{\Delta t\to 0}\left\langle\left\{[-\beta y_2-F(y_1)]^2\Delta t+2[-\beta y_2-F(y_1)]\times\int_t^{t+\Delta t}f(\tau)d\tau+\frac{1}{\Delta t}\int\int f(\tau_1)f(\tau_2)d\tau_1 d\tau_2\right\}\right\rangle. \quad (2.5)$$

Applying the conditions of Eq. (2.2),

$$a_2=-\beta y_2-F(y_1)=-\beta\dot{x}-F(x);\qquad b_{22}=W_0/2. \quad (2.6)$$

In similar manner, it may be shown that higher moments are of $O(\Delta t)$ as $\Delta t\to 0$. Hence, the system satisfies the necessary conditions and is therefore governed by the Fokker-Planck equation.

Substituting Eq. (2.4) into Eq. (1.19) gives

$$\frac{W_0}{4}\frac{\partial^2 p}{\partial y_2^2}-\frac{\partial}{\partial y_1}(y_2 p)+\frac{\partial}{\partial y_2}\{[\beta y_2+F(y_1)]\}p=0. \quad (2.7)$$

This is the stationary form of *Kramers equation*[9] and has been solved independently by Uhlenbeck,[8] Caughey,[10] Chuang and Kazada,[11] Ariaratnam,[12] and Wu.[13] We present the Caughey Wu solution here.

Equation (2.7) may be rewritten in the form

$$\left[\beta\frac{\partial}{\partial y_2}-\frac{\partial}{\partial y_1}\right]\left[y_2 p+\frac{W_0}{4\beta}\frac{\partial p}{\partial y_2}\right]+\frac{\partial}{\partial y_2}\left[F(y_1)p+\frac{W_0}{4\beta}\frac{\partial p}{\partial y_1}\right]=0. \quad (2.8)$$

It is obvious that one solution of Eq. (2.8) will be obtained by requiring that $p(y_1,y_2)=p(x,\dot{x})$ satisfy the two equations

$$y_2 p+\frac{W_0}{4\beta}\frac{\partial p}{\partial y_2}=0$$

and

$$F(y_1)p+\frac{W_0}{4\beta}\frac{\partial p}{\partial y_1}=0, \quad (2.9)$$

from which one readily obtains

$$p(y_1,y_2)=p(x,\dot{x})=C\exp\left\{-\frac{4\beta}{W_0}\left[\frac{y_2^2}{2}+\int_0^{y_1}F(\zeta)d\zeta\right]\right\}, \quad (2.10)$$

where C is a normalizing constant and ζ is a dummy variable in x or y_1. We see from this equation that the displacement and the velocity are statistical independent; i.e.,

$$p(y_1,y_2)=p(y_1)\cdot p(y_2). \quad (2.11)$$

It is observed from Eq. (2.10) that $p(y_1,y_2)$ is Gaussian in the velocity y_2. Indeed, it is identical with that for the linear problem $F(y_1)=\omega_0^2 y_1$. Further, it is noted that the probability may be written $p(y_1,y_2)=C\exp\{-4\beta E/W_0\}$, where E is the total energy per unit mass of the system. This is simply the Maxwell–Boltzmann distribution for an undamped autonomous oscillator whose mean kinetic energy per unit mass is $\langle T\rangle=W_0/8\beta$.

An interesting consequence of Eq. (2.10) is that the system satisfies the virial theorem for a rigid body moving under a conservative force[14]:

$$\bar{T}=\tfrac{1}{2}\overline{y_1\frac{\partial V}{\partial y_1}},$$

where the bar denotes a time average and V is the potential energy per unit mass. If the process is ergodic, we may replace ensemble averages with time averages. Thus, $\bar{T}=\langle T\rangle=\langle\frac{1}{2}y_2^2\rangle=W_0/8\beta$. Using Eq. (2.10) in a like manner, $\overline{y_1(\partial V/\partial y_1)}=\langle y_1F(y_1)\rangle=W_0/4\beta$. Thus, $\bar{T}=\frac{1}{2}\overline{y_1(\partial V/\partial y_1)}$, satisfying the virial theorem.

Example

To illustrate the application of Eq. (2.10), let us now show that the mean square displacement in a so-called "hardening spring" oscillator, whose characteristics are shown in Fig. 1, is always less than that of the corresponding linear oscillator. Let

$$F(y_1)=F(x)=\omega_0^2[x+\epsilon g(x)], \quad (2.12)$$

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[12] S. T. Ariaratnam, "Random Vibration of Non-Linear Suspensions," J. Mech. Eng. Sci. 2, 3, 195–201 (1960).

[13] R. E. Oliver and T. Y. Wu, "Sled–Track Interaction and a Rapid Method for Track-Alignment Measurement," Aeronautical Engineering Research Inc. Tech. Rept. 114, Part 2 (30 June 1958).

[14] H. Goldstein *Classical Mechanics* (Addison-Wesley Publishing Co., Reading, Mass., 1959), p. 70.

where (a) $\epsilon>0$ and has dimensions of $x/g(x)$, (b) $g(x)=-g(-x)$, (c) $xg(x)>0$ for $|x|>0$, and ω_0 is the undamped natural frequency of the corresponding linear system.

Let us evaluate the mean square displacement:

$$\langle x^2\rangle=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}x^2p(x,\dot{x})dxd\dot{x}. \tag{2.13}$$

Substituting Eqs. (2.10) and (2.12) into Eq. (2.13) and integrating over $\dot{x}$,

$$\langle x^2\rangle=\int_{-\infty}^{+\infty}\{x[x+\epsilon g(x)]-x\epsilon g(x)\}p(x)dx, \tag{2.14}$$

where

$$p(x)=C\exp\left\{-\frac{1}{\sigma_x^2}\int F(\zeta)d\zeta\right\}$$

$$=C\exp\left\{-\frac{1}{\sigma_x^2}\left[\frac{x^2}{2}+\epsilon G(x)\right]\right\}; \tag{2.15}$$

$$G(x)=\int_0^x g(\zeta)d\zeta. \tag{2.16}$$

The mean square displacement and velocity of the corresponding *linear* system is $\sigma_x^2=W_0/4\beta\omega_0^2$ and $\sigma_{\dot{x}}^2=W_0/4\beta$, respectively. [In Eq. (2.13), x and $\dot{x}$ need not be statistically independent in order to integrate over $\dot{x}$.]

Integrating the first term of Eq. (2.14) by parts,

$$\langle x^2\rangle=\sigma_x^2-\int_{-\infty}^{+\infty}x\epsilon g(x)p(x)dx$$

$$=\sigma_x^2-\epsilon\langle xg(x)\rangle. \tag{2.17}$$

Now, from (c) and (d) above,

$$\langle x^2\rangle<\sigma_x^2, \tag{2.18}$$

where σ_x^2 is the mean square displacement for $\epsilon=0$. Hence, under conditions (a)–(c) above, the mean square displacement of a "hardening spring" nonlinear system is always less than that for the corresponding linear system.

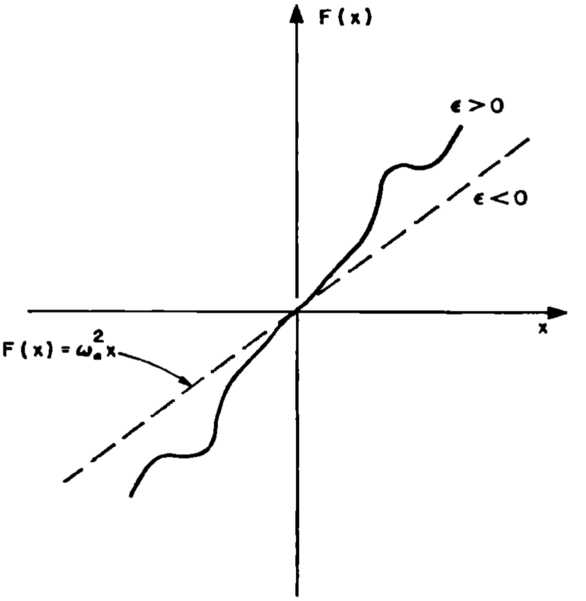

FIG. 1. Force vs displacement characteristic for a hardening spring oscillator ($\epsilon>0$).

B. Extension to n-Degree-of-Freedom Systems

Under certain restrictions, the foregoing theory may be extended to n-degree-of-freedom systems. Consider the system of equations

$$\ddot{x}_i+\beta_i\dot{x}_i+\frac{1}{M_i}\frac{\partial\mathcal{V}}{\partial x_i}=f_i(t)\quad(i=1,2,\cdots,n). \tag{2.19}$$

Letting $y_i=x_i$ and $y_{i+n}=\dot{x}_i$, Eqs. (2.19) may be replaced by the $2n$ system of equations:

$$\dot{y}_i=y_{i+n}$$

and

$$\dot{y}_{i+n}=-\beta_iy_{i+n}-\frac{1}{M_i}\frac{\partial\mathcal{V}}{\partial y_i}+f_i(t), \tag{2.20}$$

where $\mathcal{V}=\mathcal{V}(y_1,y_2,\cdots,y_n)$ is the potential energy of the system, $f_i(t)$ are uncorrelated Gaussian, white random processes with means of zero; i.e.,

$$\langle f_i(t)\rangle=0;$$
$$\langle f_i(t)f_i(t+\tau)\rangle=W_{ii}\delta(\tau)/2; \tag{2.21}$$
$$\langle f_i(t)f_j(t+\tau)\rangle=0\quad(i\neq j).$$

The coefficients a_i, b_{ij} can be calculated using Eqs. (1.9)–(1.11), where $(i,j=1,2,\cdots,2n)$:

$$a_i=y_{i+n};$$
$$a_{i+n}=-\beta_iy_{i+n}-\frac{1}{M_i}\frac{\partial\mathcal{V}}{\partial y_i};$$
$$b_{ij}=0\quad(i\neq j); \tag{2.22}$$
$$b_{i+n,j+n}=0\quad(i\neq j);$$
$$b_{ii}=0;$$
$$b_{i+n,i+n}=W_{ii}/2.$$

The Fokker–Planck equation for the stationary probability density function $p(y_1,y_2,\cdots y_{2n})$ is given by

$$\sum_{i=1}^n\frac{W_{ii}}{4}\frac{\partial^2p}{\partial y_{i+n}^2}-\sum_{i=1}^n\frac{\partial}{\partial y_i}(y_{i+n}p)$$
$$+\sum_{i=1}^n\frac{\partial}{\partial y_{i+n}}\left[\left(\beta_iy_{i+n}+\frac{1}{M_i}\frac{\partial\mathcal{V}}{\partial y_i}\right)p\right]=0. \tag{2.23}$$

This equation may conveniently be rewritten in the Caughey–Wu form:

$$\sum_{i=1}^{n}\left[\left(\beta_i\frac{\partial}{\partial y_{i+n}}-\frac{\partial}{\partial y_i}\right)\left(y_{i+n}p+\frac{W_{ii}}{4\beta_i}\frac{\partial p}{\partial y_{i+n}}\right)+\frac{\partial}{\partial y_{i+n}}\left(\frac{1}{M_i}\frac{\partial \mathcal{V}}{\partial y_i}p+\frac{W_{ii}}{4\beta_i}\frac{\partial p}{\partial y_i}\right)\right]=0. \quad (2.24)$$

If we assume that $W_{ii}M_i/4\beta_i=K$, then

$$p(y_1,y_2\cdots,y_{2n})=C\exp\left\{-\frac{1}{K}\left[\tfrac{1}{2}\sum_{i=1}^{n}M_iy_{i+n}^2+\mathcal{V}\right]\right\} \quad (2.25)$$

As previously shown, Eq. (2.25) may also be written

$$p(y_1,y_2,\cdots y_{2n})=C\exp\{-\mathcal{E}/K\}, \quad (2.26)$$

where $\mathcal{E}$ is the total energy stored in the system. It is noted from Eq. (2.24) that the probability is Gaussian in the velocities. The marginal probability is obtained by integrating over the velocities and is given by

$$p(y_1,y_2,\cdots y_n)=C'\exp\{-\mathcal{V}/K\}. \quad (2.27)$$

It is interesting to note that as a consequence of Eq. (2.24) the system satisfies the general virial theorem[14]:

$$\bar{\mathcal{T}}=\tfrac{1}{2}\sum_{i=1}^{n}\overline{y_i\frac{\partial \mathcal{V}_i}{\partial y_i}}.$$

To prove this, we note that

$$\bar{\mathcal{T}}=\langle\mathcal{T}\rangle=\tfrac{1}{2}\sum_{i=1}^{n}M_i\langle y_{i+n}^2\rangle=\tfrac{1}{2}nK;$$

$$\sum_{i=1}^{n}\overline{y_i\frac{\partial \mathcal{V}}{\partial y_i}}=\sum_{i=1}^{n}\left\langle y_i\frac{\partial \mathcal{V}}{\partial y_i}\right\rangle=nK.$$

Thus, the system satisfies the virial theorem.

Example

To illustrate the use of Eq. (2.25), let us prove that the mean square displacements in a nonlinear n-degree-of-freedom system are smaller than those for the corresponding linear system when the nonlinearities are of the "hardening spring" type. Consider

$$\ddot{x}_i+\beta\dot{x}_i+\omega_i^2x_i+\mu\frac{\partial V^\dagger}{\partial x_i}=f_i(t), \quad (2.28)$$

where (a) $V^\dagger=V^\dagger(x_1,x_2,\cdots,x_n)$ is the potential energy per unit mass of the *nonlinear* terms; (b) $\mu>0$ and is dimensionless; (c) $x_i(\partial V^\dagger/\partial x_i)>0$; and (d) $f_i(t)$ are uncorrelated Gaussian, white random processes with the same spectral density W_0.

From this equation, the total potential energy per unit mass is

$$V=\sum_{i=1}^{n}\tfrac{1}{2}\omega_i^2x_i^2+\mu V^\dagger.$$

From Eq. (2.27),

$$p(x_1,x_2,\cdots,x_n)=C'\exp\{-4\beta V/W_0\}. \quad (2.29)$$

Now, the mean square displacement is

$$\langle x_j^2\rangle=\int_{-\infty}^{+\infty}\underset{n\text{-fold}}{\cdots}\int_{-\infty}^{+\infty}x_j^2p(x_1,x_2,\cdots,x_n)\prod_{i=1}^{n}dx_i. \quad (2.30)$$

Expressing

$$x_j^2=\frac{x_j}{\omega_j^2}\left(\omega_j^2x_j+\mu\frac{\partial V^\dagger}{\partial x_j}-\mu\frac{\partial V^\dagger}{\partial x_j}\right) \quad (2.31)$$

and substituting this and Eq. (2.29) into Eq. (2.30),

$$\langle x_j^2\rangle=\frac{1}{\omega_j^2}\left[\left\langle x_j\left(\omega_j^2x_j+\mu\frac{\partial V^\dagger}{\partial x_j}\right)\right\rangle-\left\langle \mu x_j\frac{\partial V^\dagger}{\partial x_j}\right\rangle\right]. \quad (2.32)$$

Integrating the first term of Eq. (2.32) by parts and using Eq. (2.29),

$$\langle x_j^2\rangle=W_0/4\beta\omega_j^2-\frac{\mu}{\omega_j^2}\left\langle x_j\frac{\partial V^\dagger}{\partial x_j}\right\rangle. \quad (2.33)$$

Using conditions (b) and (c) above, the second term is positive. Hence,

$$\langle x_j^2\rangle<W_0/4\beta\omega_j^2=\sigma_{x_j}^2, \quad (2.34)$$

just as is the case of the single-degree-of-freedom system of Eq. (2.18). Thus, under the above conditions, the mean square displacements of the system are smaller than those for the corresponding linear system.

C. N-Degree-of-Freedom Quasilinear System

Now consider the following system of equations in matrix notation:

$$I\{\ddot{x}\}+\beta I\{\dot{x}\}+[1+2\lambda V^*][\Omega^2]\{x\}=\{f(t)\}, \quad (2.35)$$

where $[\Omega^2]$ is an $N\times N$, symmetric, positive, definite matrix; $V^*=\frac{1}{2}\{x\}^T[\Omega^2]\{x\}$ is the potential energy per unit mass of the *linear* system; λ has dimensions of $1/V^*$; $\{f(t)\}$ is an N-column vector of uncorrelated Gaussian, white random excitations with the same spectral density W_0.

Since $[\Omega^2]$ is symmetric, there exists an orthogonal transformation that reduces $[\Omega^2]$ to diagonal form. Let

$$\{x\}=[\theta]\{\xi\} \quad (2.36)$$

be this transformation, where $[\theta]^T[\theta]=I$. Thus, $[\theta]^T[\Omega^2][\theta]=[\![\omega^2]\!]$, a diagonal matrix. From this transformation,

$$V^*=\tfrac{1}{2}\{\xi\}^T[\theta]^T[\Omega^2][\theta]\{\xi\}=\tfrac{1}{2}\{\xi\}^T[\![\omega^2]\!]\{\xi\}. \quad (2.37)$$

Hence,

$$V^*=\tfrac{1}{2}\sum_{k=1}^{N}\omega_k{}^2\xi_k{}^2 \quad (2.38)$$

Equation (2.35) may, therefore, be reduced to

$$I\{\ddot{\xi}\}+\beta I\{\dot{\xi}\}+[1+\lambda\sum_{j=1}^{N}\omega_j{}^2\xi_j{}^2][\![\omega^2]\!]\{\xi\}=\{Q(t)\}, \quad (2.39)$$

where

$$\{Q(t)\}=[\theta]^T\{f(t)\}. \quad (2.40)$$

The correlation matrix is given by

$$\begin{aligned}\langle\{Q(t_1)\}\{Q(t_2)\}^T\rangle&=[\theta]^T\langle\{f(t_1)\}\{f(t_2)\}^T\rangle[\theta]\\&=[\theta]^T(\tfrac{1}{2}W_0)\delta(t_1-t_2)I[\theta]\\&=\tfrac{1}{2}W_0\delta(t_1-t_2)I. \quad (2.41)\end{aligned}$$

Thus, the Q_i's are uncorrelated.

The ith row of Eq. (2.39) is

$$\ddot{\xi}_i+\beta\dot{\xi}_i+\omega_i{}^2[1+\lambda\sum_{j=1}^{n}\omega_j{}^2\xi_j{}^2]\xi_i=Q_i(t). \quad (2.42)$$

This may be written as

$$\ddot{\xi}_i+\beta\dot{\xi}_i+\partial V/\partial\xi_i=Q_i(t), \quad (2.43)$$

where the total potential energy per unit mass is

$$V=\tfrac{1}{2}\sum_{l=1}^{N}\omega_l{}^2\xi_l{}^2+\frac{\lambda}{4}\sum_{l=1}^{N}\sum_{m=1}^{N}\omega_l{}^2\omega_m{}^2\xi_l{}^2\xi_m{}^2. \quad (2.44)$$

Equation (2.39), therefore, satisfies all the conditions laid down in Sec. B on the n-degree-of-freedom system. Hence,

$$p(\xi_1,\xi_2,\cdots\xi_N)=C'\exp\{-4\beta V/W_0\}. \quad (2.45)$$

Example

Let us again prove that the mean square displacements are smaller than those for the corresponding linear system when the nonlinearities are of the "hardening spring" type. From Eq. (2.36),

$$x_i=\sum_{j=1}^{N}\theta_i{}^j\xi_j. \quad (2.46)$$

Therefore,

$$\langle x_i{}^2\rangle=\sum_{j=1}^{N}\sum_{k=1}^{N}\theta_i{}^j\theta_i{}^k\langle\xi_j\xi_k\rangle. \quad (2.47)$$

Similar to Eqs. (2.13) and (2.30),

$$\langle\xi_j\xi_k\rangle=\underbrace{\int\cdots\int}_{N\text{-fold}}\xi_j\xi_k p(\xi_1,\xi_2,\cdots,\xi_N)\prod_{i=1}^{N}d\xi_i. \quad (2.48)$$

But p is symmetric in ξ_j and ξ_k. Hence,

$$\langle\xi_j\xi_k\rangle=0 \quad (j\neq k). \quad (2.49)$$

Therefore,

$$\langle x_i{}^2\rangle=\sum_{j=1}^{N}(\theta_i{}^j)^2\langle\xi_j{}^2\rangle, \quad (2.50)$$

where

$$\begin{aligned}\langle\xi_j{}^2\rangle&=\underbrace{\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}}_{N\text{-fold}}\xi_j{}^2 p(\xi_1,\xi_2,\cdots,\xi_N)\prod_{i=1}^{N}d\xi_i\\&=\underbrace{\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}}_{N\text{-fold}}\frac{1}{\omega_j{}^2}\xi_j\left[\frac{\partial V}{\partial\xi_j}-2\lambda\omega_j{}^2\xi_jV^*\right]\\&\qquad\times p(\xi_1,\xi_2,\cdots,\xi_N)\prod_{i=1}^{N}d\xi_i. \quad (2.51)\end{aligned}$$

Integrating the first term of Eq. (2.51) by parts and utilizing Eq. (2.45),

$$\langle\xi_j{}^2\rangle=\sigma_{\xi_j}{}^2-\lambda\langle\xi_j{}^2(\sum_{k=1}^{N}\omega_k{}^2\xi_k{}^2)\rangle. \quad (2.52)$$

Since the second term of Eq. (2.52) is positive, then

$$\langle x_i{}^2\rangle<\sum_{j=1}^{N}(\theta_i{}^j)^2\sigma_{\xi_j}{}^2=\sigma_{x_j}{}^2, \quad (2.53)$$

which proves the intended premise. An illustration of such a system is that of a massless string carrying N identical particles of mass M. The problem was first solved by Caughey[7] by using approximate techniques. Ariaratnam[12] later solved the problem exactly by using essentially the technique outlined above.

III. UNSOLVED PROBLEMS

In this paper, a number of nonlinear problems have been discussed and solved. The impression should not be conveyed, however, that all problems relating to the response of nonlinear systems with random excitation have been solved, for this is not the case. I should like to discuss a few problems that remain to be solved.

(*a*) *Problems in which the nonlinearities involve velocity as well as displacement.* For a number of years now, my students and I have worked diligently on this problem, with singular lack of success. It is apparent that the product-type solution as used in this paper is inadmissable for the case where the nonlinearity is velocity-dependent.

(*b*) *Problems in multidegree-of-freedom systems in which the exciting forces are correlated and in which the spectral density is not related to the damping.*

(*c*) *Problems in which the exciting forces do not exhibit white spectra.* In this case, it is not generally possible to solve the Fokker–Planck equation for the system by techniques implied in this paper. For linear systems, it has been possible to construct an equation similar to the Fokker–Planck equation governing the joint probability, but, at the present time, we have been unable to do this for any nonlinear system.

(d) *Although we have succeeded in obtaining the stationary probability law for a class of nonlinear problems, we have been generally unable to obtain the transitional probability law.* Without this transitional probability, it is generally impossible to obtain the correlation function and spectral density. Caughey and Dienes,[15] however, have managed to solve a rather trivial first-order problem in complete detail and obtain the spectral density. The techniques used in the solution of that problem do not appear to lend themselves to the solution of other nonlinear problems.

APPENDIX A: UNIQUENESS OF A STATIONARY SOLUTION TO THE FOKKER PLANCK EQUATION

Given the Fokker–Planck equation in the form

$$\frac{\partial p_c}{\partial t}+\frac{\partial}{\partial y_k}[a_k(y,z)p_c]+\frac{\partial}{\partial z_k}[c_k(y,z)p_c]-\frac{1}{2}\frac{\partial^2}{\partial z_k\partial z_i}[b_{ki}(y,z)p_c]=0,\quad \text{(A1)}$$

where $z_i=y_{i+n}$, y_k are m in number, z_k are n in number, and summation convention is implied. Under suitable assumptions, most of which were implied in the derivation of Eq. (1.17), it is possible to show that there can be no more than one stationary solution. Integrals written as double integrals over y and z are used to designate the $(n+m)$-fold Lebesgue integrals over all the z_k and y_k.

Though it is not shown here, it can be proved from the following assumptions that the order of integration is immaterial in all such integrals used herein.

A. Restrictions

Make the following requirements to Eq. (A1):

(a) $b_{ki}(y,z)$ represent the terms of a positive definite matrix.

(b) $\partial a_k(y,z)/\partial y_k$ and $\partial c_k(y,z)/\partial z_k$ exist for every y and z.

(c) $\partial^2 b_{ki}(y,z)/\partial z_k\partial z_i$ exists for every y and z.

(d) The only solution to the problem given by the following equations is $x=$constant:

(i) $a_k(y,z)[\partial x/\partial y_k]=0$;

(ii) $\partial x/\partial z_k=0 \quad (k=1, 2, \cdots, n)$;

(iii) $x\geqq 0$.

[15] T. K. Caughey and J. K. Dienes, "Analysis of a Non-Linear First-Order System with a White Noise Input," J. Appl. Phys. 32, 2476–2479 (1961).

B. Further Restrictions

We define a class of functions $\mathfrak{U}$, such that p_c is in $\mathfrak{U}$ if each of the following conditions are satisfied for all $t\geqq 0$.

(e) Each of the following terms is integrable (in the Lebesgue sense) over all y and z, and the multiple integrals may be evaluated by repeated integrations in any order. Here, no summation is implied:

(i) p_c; (iv) $c_k\dfrac{\partial p_c}{\partial z_k}$; (vii) $\dfrac{\partial b_{ki}}{\partial z_k}\dfrac{\partial p_c}{\partial z_i}$;

(ii) $a_k\dfrac{\partial p_c}{\partial y_k}$; (v) $\dfrac{\partial c_k}{\partial z_k}p_c$; (viii) $p_c\dfrac{\partial^2 b_{ki}}{\partial z_k\partial z_i}$.

(iii) $\dfrac{\partial a_k}{\partial y_k}p_c$; (vi) $b_{ki}\dfrac{\partial^2 p_c}{\partial z_k\partial z_i}$;

(f) Each of the following limits exists:

(i) $\lim_{y_k\to\infty} a_kp_c=0$; (iii) $\lim_{z_k\to\infty} b_{ki}\dfrac{\partial p_c}{\partial z_i}=0$;

(ii) $\lim_{z_k\to\infty} c_kp_c=0$; (iv) $\lim_{z_k\to\infty} p_c\dfrac{\partial b_{ki}}{\partial z_i}=0$;

where no summation is implied and limits are taken with t and all other y and z held fixed.

(g) $p_c\geqq 0$.

(h) At $t=0$, we have the relationship $\int\int p_c dydz=1$.

(i) $\partial p_c/\partial t$ is continuous in all variables.

C. Proof of Uniqueness

By utilizing all of the above restrictions, we may prove a number of theorems. The one that is important is given as follows:

Theorem 1: Given that p_{c1} and p_{c2} each belong to $\mathfrak{U}$, define p_{c3}, p_{c4}, and x by

$$p_{c3}=ap_{c1}+(1-a)p_{c2};\quad \text{(A2)}$$

$$p_{c4}=cp_{c1}+(1-c)p_{c2};\quad \text{(A3)}$$

$$x=p_{c3}/p_{c4};\quad \text{(A4)}$$

where we have $0<a<c<1$. Further, let us define

$$X(t)=\int\int x^2p_{c4}dydz.\quad \text{(A5)}$$

Then, we have the relationship

$$dX/dt=-2\int\int p_{c4}b_{ki}\frac{\partial x}{\partial z_k}\frac{\partial x}{\partial z_i}dydz.\quad \text{(A6)}$$

Proof: The proof of this theorem involves using restriction (e) to show that the integrals exist and restriction (f) to integrate by parts, after utilizing restrictions (e) and (i) to interchange differentiation and integration. By utilizing *Theorem 1*, it can be seen that, if both p_{c1} and p_{c2} are *stationary* solutions to the Fokker–Planck equation, then $X(t)$ must be constant. Hence, the integral of Eq. (A6) must be zero. Further, from the positive definite assumption on the b_{ki}'s in restriction (a) and the fact that p_{c4} will be nonnegative, we see that the integral of Eq. (A6) cannot be zero unless the integrand is identically zero (using existence of the various derivatives to imply continuity). Hence, we arrive at the following lemma.

Lemma: If p_{c1} and p_{c2} each belong to $\mathfrak{U}$, with the added conditions that

$$\partial p_{c1}/\partial t=0 \tag{A7}$$

and

$$\partial p_{c2}/\partial t=0, \tag{A8}$$

then we have the relationship that

$$p_{c4}b_{ki}\frac{\partial x}{\partial z_k}\frac{\partial x}{\partial z_i}=0. \tag{A9}$$

From here, it is a simple matter to arrive at the uniqueness. Utilizing restrictions (a) and (d), we can see that $\partial x/\partial z_k=0$ for each z_k, and hence $x=\text{constant}$. From the definition of x, we arrive at the fact that p_{c1} and p_{c2} must be related by a constant, which from restriction (h) must be unity.

Theorem 2: Given the added conditions that

$$\partial p_{c1}/\partial t=\partial p_{c2}/\partial t=0 \tag{A10}$$

and

$$p_{c1}>0, \tag{A11}$$

we have

$$p_{c1}=p_{c2}. \tag{A12}$$

Thus, if one stationary solution can be found that is nonzero everywhere, and if the above restrictions are satisfied (which they are in most problems of interest), then that solution is unique among the class of well-behaved solutions.

ACKNOWLEDGMENTS

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