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Exact results for supersymmetric abelian vortex loops in $2+1$ dimensions

Anton Kapustin,[a] **Brian Willett**[b] **and Itamar Yaakov**[c]

[a]*Department of Physics, California Institute of Technology,*
1200 E. California Blvd, Pasadena, CA
[b]*Institute for Advanced Study, Einstein Drive, Princeton, NJ*
[c]*Department of Physics, Princeton Univerity, Princeton, NJ*

E-mail: kapustin@theory.caltech.edu, bwillett@ias.edu, iyaakov@princeton.edu

Abstract: We define a class of supersymmetric defect loop operators in $\mathcal{N}=2$ gauge theories in $2+1$ dimensions. We give a prescription for computing the expectation value of such operators in a generic $\mathcal{N}=2$ theory on the three-sphere using localization. We elucidate the role of defect loop operators in IR dualities of supersymmetric gauge theories, and write down their transformation properties under the $SL(2,\mathbb{Z})$ action on conformal theories with abelian global symmetries.

Keywords: defect operator, supersymmetry, localization

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1 Introduction

Quantum field theories admit a variety of operators defined not by insertions of the fundamental fields, but by constraints which change the domain of the path integral in field space. An operator defined by such a prescription is called a defect operator. A famous example is the twist operators of 2d conformal field theory. More generally, one can define a defect operator inserted along a submanifold L by deleting L and requiring the fields to have prescribed singularities as one approaches L. The effect of the insertion can be "measured" by evaluating the expectation values of ordinary operators. When one can detect the presence of the defect from afar (for example, because some field strengths are now required to belong to a nontrivial cohomology class), the insertion is said to have created topological disorder.

The first example of a defect operator in gauge theory is probably the 't Hooft loop operator in 4d gauge theories [1] which can be used as an order parameter for Higgs phases. It also plays an important role in the context of electric-magnetic duality of $\mathcal{N}=4$ gauge theories in $3+1$ dimensions [2–4]. The duality exchanges states with electric and magnetic charge and therefore exchanges Wilson loop operators and 't Hooft loop operators.

There is a somewhat similar story for 3d gauge theories. In such theories there is a often a duality which exchanges elementary excitations in the Coulomb phase with Abrikosov-Nielsen-Olesen vortices in the Higgs phase. A 3d operator creating a very heavy vortex with world line L is a defect loop operator and may be regarded as analogous to the 't Hooft loop operator in 4d creating a monopole. Such a defect operator is defined by the fact that the gauge field has a fixed holonomy around any small loop linking L. These operators were studied in [5]. In 4d an analogous construction gives a surface defect [6].

It should be noted that the definition of the vortex loop operator is independent of the existence of the Higgs phase, or vortex solutions, or even of a dynamical gauge field. The definition makes perfect sense in the topological pure Chern-Simons theory as a defect in the gauge connection, although the defect, in that case, can be identified with a Wilson loop [7, 8]. In a theory possessing an abelian global symmetry, a vortex loop can be defined by gauging a global symmetry using a non-dynamical flat connection satisfying the holonomy condition. It is also not necessary that the defect be defined on a closed loop, however, to preserve gauge invariance, an open contour must extent to the boundary of spacetime.

Much more can be said about defect operators when the theory is supersymmetric. All of the operators mentioned above have BPS analogues in supersymmetric theories in $2+1$ and in $3+1$ dimensions. Extending the definition of a defect so as to preserve a fraction of the supersymmetry can require imposing conditions on additional fields. This is analogous to the inclusion of fields other than the connection in the definition of a supersymmetric Wilson loop. In this work, we will define the supersymmetric analogue of the vortex loop. Exact computation of the expectation value for a supersymmetric defect may be feasible by employing localization techniques ([9–11]). This was carried out for the supersymmetric version of the 't Hooft loop on $\mathbb{S}^4$ in [12]. Here, we extend previous results for localization of supersymmetric gauge theories in 2+1 dimensions (see [13] for the original derivation and [14] for a review) to include the supersymmetric vortex loop. We will also discuss the role played by supersymmetric vortex loops in the context of mirror symmetry.

In section 2, we define several versions of the abelian vortex loop. We discuss the operator's transformation under Witten's $SL(2,\mathbb{Z})$ action on conformal field theories. We then extend the definition to accommodate supersymmetry. In section 3, we employ localization to evaluate the expectation value of the supersymmetric vortex loop for a generic superconformal $\mathcal{N}=2$ gauge theory on $\mathbb{S}^3$. The result can be inferred from the $SL(2,\mathbb{Z})$ action. We provide an independent derivation using the original definition. In section 4, we demonstrate, with a few examples, the role of the vortex loop in IR duality of $\mathcal{N}=2$ gauge theories. We conclude with a discussion of possible extensions.

2 Definition of the abelian vortex loop

2.1 Vortex loop in a gauge theory

Defect loop operators in 3d gauge theories have been previously introduced in the context of Chern-Simons theories [7, 8]. Such a loop operator is specified by giving a loop γ in the 3-manifold M and an element β of the Lie algebra $\mathfrak{g}$ of the gauge group G. The holonomy around any small loop linking γ of the gauge connection A is required to approach β as we shrink the loop size to zero. With this condition, A is singular on γ. Separating A into a smooth part A' and a singular part A'', we can write

$$F_{A''} = \beta \star [\gamma] \tag{2.1}$$

where $\star[\gamma]$ is a 2-form current supported on γ whose cohomology class is the Poincarï¿œ dual of $[\gamma] \in H_1(M,\mathbb{Z})$ (we assume that M is orientable). In [6], the authors defined surface operators in $\mathcal{N}=4$ SYM theory in $3+1$ dimensions using a similar prescription. The prescription for a subset of these codimension 2 operators, the one in which only the connection is singular, coincides with the definition above (substitute α for β). As noted there, the data specifying the singularity is actually only $e^{i\beta}$, and equation 2.1 should be handled with care. If the gauge group G is $U(1)$, β is simply a real number. To avoid ambiguity it is sufficient to restrict the range of β to the interval $(-\pi, \pi)$. In what follows, we will sometimes assume this restriction. We will also set

$$q = \frac{\beta}{2\pi}, \qquad q \in \left(-\frac{1}{2}, \frac{1}{2}\right) \tag{2.2}$$

From now on we will assume that G is abelian and will call such a loop operator a gauge vortex loop.

2.2 Global vortex loops and the $SL(2,\mathbb{Z})$ action

We can define a similar loop operator if G is a global symmetry group rather than a gauge symmetry group. To this end we merely set the smooth part A' of the gauge field to zero and set the singular part (A'' above) to be a fixed flat connection whose holonomy along a loop linking γ is $e^{i\beta}$. We will call such a loop operator a global vortex loop. It appears naturally when we consider the action of Witten's $SL(2,\mathbb{Z})$ on Wilson loops.

Following [15], we consider a conformal field theory in $2+1$ dimension with a choice of an abelian global symmetry current J. We couple J to a background gauge field A, and consider the partition function as a functional of A

$$Z[A] = \int \mathcal{D}\Phi e^{iS[\Phi] + i\int \sqrt{g} d^3x J^\mu A_\mu + \ldots} \tag{2.3}$$

where "..." refers to seagull terms necessary to ensure invariance under gauge transformations of A. Here Φ and $S[\Phi]$ are short hand for the fields and action of the theory. In addition to the seagull terms, one can add extra terms which are gauge-invariant functionals of A alone. If we wish to preserve conformal symmetry, a natural choice is the abelian Chern-Simons term

$$\frac{i\alpha}{4\pi} \int A \wedge dA \tag{2.4}$$

If we want to associate A to a non-trivial principal bundle, then this is only defined for integer α. However, for now, we will assume that A is a connection on a trivial bundle, and allow arbitrary real values of α. To the triplet $(S[\Phi], J, \alpha)$ we associate the following functional of A:

$$Z_{J,\alpha}[A] = \int \mathcal{D}\Phi e^{iS[\Phi] + i\int \sqrt{g} d^3x J^\mu A_\mu + \ldots + \frac{i\alpha}{4\pi}\int A \wedge dA} \tag{2.5}$$

Witten defined an $SL(2,\mathbb{Z})$ action on such triplets [15]. The action of the T generator is merely a shift of α by 1. The action of S is defined by first promoting A to a dynamical gauge field and then replacing J with the topological $U(1)$ current for this new gauge field

$$J_{top} = \frac{1}{2\pi} \star dA \tag{2.6}$$

Together, these operations generate an action of $SL(2,\mathbb{Z})$ on the set of triples. At the level of partition functions, these operations act as follows:

$$(T \cdot Z_{J,\alpha})[A] = Z_{J,\alpha+1}[A] \tag{2.7}$$

$$(S \cdot Z_{J,\alpha})[A] = \int \mathcal{D}A'\mathcal{D}\Phi e^{iS[\Phi]+i\int \sqrt{g}d^3xJ^\mu A'_\mu+\ldots+\frac{i\alpha}{4\pi}\int A'\wedge dA'+\frac{i}{2\pi}\int A\wedge dA'} \tag{2.8}$$

Our first goal is to extend these operations to observables more general than the partition function $Z_{J,\alpha}$. Specifically, we would like to consider insertions of Wilson loops for the background gauge field A. First, it is useful to define a slight generalization of an abelian Wilson loop. Recall that the abelian Wilson loop operator is specified by a loop $\gamma : S^1 \to M$ and a charge $q \in \mathbb{R}$ and is defined as an insertion of

$$e^{iq\int_\gamma A} \tag{2.9}$$

into the path integral. This can be rewritten as

$$e^{\frac{i}{2\pi}\int \omega\wedge A} \tag{2.10}$$

where ω is a closed 2-form current with support along γ, defined so that the integrals in the previous two expressions agree for all 1-forms A. In local coordinates (r, θ, z) where the loop lies along the z axis, we can write:

$$\omega = 2\pi q\delta_\gamma \equiv q\delta(r)dr \wedge d\theta \tag{2.11}$$

Such a term can simply be added to the action, since in the abelian case we do not have to worry about path ordering the exponential.

We can consider such an insertion for more general 2-form currents ω. Invariance with respect to infinitesimal gauge transformations forces ω to be closed. Invariance with respect to "large" gauge transformations (i.e. gauge transformations which are topologically nontrivial maps from the abelian Lie group G to M) requires the de Rham cohomology class of $\omega/2\pi$ to be integral. This means that ω arises as the field strength of some connection A_ω on a $U(1)$ bundle, and we can write the Wilson loop as a BF coupling to this new background gauge field:

$$e^{\frac{i}{2\pi}\int A\wedge dA_\omega} \tag{2.12}$$

In the case

$$\omega = 2\pi q\delta_\gamma \tag{2.13}$$

where δ_γ is a 2-form current supported on γ which is Poincarï¿œ dual to γ, the integrality condition on ω reduces to the requirement that the class $q[\gamma] \in H_1(M, \mathbb{R})$ is integral. In particular, if γ is homologically trivial, there are no integrality constraints on q. Near γ the connection A_ω in suitable coordinates looks as follows:

$$A_\omega = qd\theta \tag{2.14}$$

More generally, if we take ω to be supported in a small tubular neighborhood of γ, we get a regularization of the Wilson loop along γ.

With this in mind, we define a insertion of W_ω by

$$(W_\omega \cdot Z_{J,\alpha})[A] := \int \mathcal{D}\Phi e^{iS[\Phi]+i\int \sqrt{g}d^3xJ^\mu A_\mu+\ldots+\frac{i\alpha}{4\pi}\int A\wedge dA+\frac{i}{2\pi}\int A_\omega\wedge dA} \tag{2.15}$$

Let us see how the $SL(2, \mathbb{Z})$ generators act on it. It is clear that T commutes with W_ω, so we only need to consider

$$((S^{-1}W_\omega S)\cdot Z_{J,\alpha})[A] = \int \mathcal{D}A_1\mathcal{D}A_2\mathcal{D}\Phi \exp\left(iS[\Phi]+i\int\sqrt{g}d^3xJ^\mu A_{1\mu}+...+\frac{i\alpha}{4\pi}\int A_1\wedge dA_1+\right. \quad (2.16)$$

$$\left. +\frac{i}{2\pi}\int A_2\wedge dA_1+\frac{i}{2\pi}\int A_\omega\wedge dA_2-\frac{i}{2\pi}\int A\wedge dA_2\right) \quad (2.17)$$

We can see A_2 enters only via a term $\int(A_1-A_\omega-A)\wedge dA_2$, and so the integral over A_2 produces exactly a delta function setting $A_1=A_\omega+A$ [15]. This leaves

$$\int \mathcal{D}\Phi \exp\left(iS[\Phi]+i\int\sqrt{g}d^3xJ^\mu(A_{\omega\mu}+A_\mu)+...+\frac{i\alpha}{4\pi}\int(A+A_\omega)\wedge d(A+A_\omega)\right) \quad (2.18)$$

$$= Z_{J,\alpha}[A+A_\omega] \quad (2.19)$$

Note that, even at $A=0$, this gives an insertion of an operator:

$$\exp\left(i\int\sqrt{g}d^3xJ^\mu A_{\omega\mu}+...+\frac{i\alpha}{4\pi}\int A_\omega\wedge\omega\right) \quad (2.20)$$

In the case where $\omega=2\pi q\delta_\gamma$, A_ω is a flat gauge field with a holonomy $e^{2\pi iq}$, and this operation has the same effect on the path integral as prescribing that all fields charged under the current J pick up a fixed monodromy around the loop γ. There is also an additional α-dependent phase factor related to the self-linking number of the loop (which must be regularized by specifying a framing). We see that in this special case the operation S maps the charge-q Wilson loop for the background gauge field A to the global vortex loop with holonomy $e^{2\pi iq}$. More generally, for arbitrary ω satisfying the above integrality conditions we can define an operation

$$(D_\omega\cdot Z_{J,\alpha}[A]) = Z_{J,\alpha}[A+A_\omega] \quad (2.21)$$

What we have demonstrated is that the operation S maps W_ω to D_ω:

$$S^{-1}W_\omega S = D_\omega \quad (2.22)$$

More precisely, the equality holds up to a phase factor which depends not only on ω, but also on the Chern-Simons coupling of the theory on which these operations act.

One can rephrase this result in terms of Wilson loops for dynamical gauge fields. Gauging a symmetry (without adding a Chern-Simons term) is the same as applying the operation S. The resulting theory has a new global symmetry $U(1)_J$ whose current is $\star dB/2\pi$, where we denoted by B the dynamical gauge field, to distinguish it from the background gauge field A which couples to the $U(1)_J$ current. Noting that $S^2=C$ (the charge conjugation), we get

$$SW_\omega = D_{-\omega}S. \quad (2.23)$$

Applying this to the partition function $Z^{ungaged}_{J,\alpha}[B]$, we learn that

$$<W_\omega[B]>=D_{-\omega}\cdot Z_{gauge}[A]=Z_{gauge}[A-A_\omega]. \quad (2.24)$$

In particular, setting $\omega/2\pi$ to be the delta-function supported on a loop γ, we see that a global vortex loop for the $U(1)_J$ symmetry is nothing more than an ordinary Wilson loop in the underlying gauge field. Similarly, we find

$$< D_\omega[B] >= W_\omega \cdot Z_{gauge}[A] \tag{2.25}$$

which shows that, in the absence of a Chern-Simons term, the gauge vortex loop by itself is somewhat trivial: it merely modifies the functional dependence of $Z_{gauge}[A]$ on the background gauge field A which couples to the $U(1)_J$ current.

2.3 Pure Chern-Simons theory

Before moving on to the supersymmetric version of the vortex loop, let us briefly comment on how the gauge vortex loop behaves in pure bosonic Chern-Simons theory. It was argued in [7] that such a defect operator should be equivalent to a Wilson loop. However, we have seen above that for an abelian gauge group the gauge vortex loop is somewhat trivial, its only effect being a modification of the $U(1)_J$ current by a c-number term supported on the loop. To see that this agrees with the behavior of the Wilson loops, we recall the formula for the expectation value of a product of Wilson loops in $U(1)$ Chern-Simons theory at level k [8]

$$\langle \prod_a \exp(iq_a \int_{\gamma_a} A)\rangle = \exp\left(\frac{2\pi i}{k}\sum_{a,b} q_a q_b \Phi(\gamma_a, \gamma_b)\right) \tag{2.26}$$

where $\Phi(\gamma_a, \gamma_b)$ is the linking number of the loops a and b. The latter can be written in terms of the corresponding gauge fields A_a, with $dA_a = \omega_a$ the 2-form delta function supported on γ_a, as

$$\Phi(\gamma_a, \gamma_b) = \int_{\gamma_b} A_a = \frac{1}{2\pi}\int A_a \wedge \omega_b \tag{2.27}$$

Letting

$$\omega = \frac{1}{k}\sum_a q_a \omega_a, \tag{2.28}$$

we can rewrite the expectation value as follows:

$$\langle \prod_a \exp(iq_a \int_{\gamma_a} A)\rangle = \exp\left(\frac{ik}{2\pi}\int A_\omega \wedge dA_\omega\right). \tag{2.29}$$

Thus the insertion of a collection of Wilson loops is equivalent to a phase factor which depends on k as well as on a flat connection A_ω. This is compatible with the claim that a collection of Wilson loops in pure abelian Chern-Simons theory is equivalent to a gauge vortex loop D_ω for some 2-form current ω, which in turn is trivial up to a phase.

2.4 Supersymmetric vortex loops

We would like to extend the considerations above to the supersymmetric case. Specifically, we will work with theories with $\mathcal{N} = 2$ supersymmetry (4 real supercharges). It is convenient to work in $\mathcal{N} = 2$ superspace, with fermionic coordinated θ_α and a superspace derivative D_α. For the theories of interest, the dynamical fields can be organized into chiral and vector superfields. The gauge field is part of a vector multiplet, and all fields in this multiplet take values in the adjoint representation of the gauge group. A vector superfield V satisfies $V = V^\dagger$ and contains a vector field A_μ, a real scalar σ, a complex spinor λ_α, and a real auxiliary scalar D. Matter fields live in chiral multiplets and take values in some representation of the gauge and flavor groups. A chiral superfield Φ satisfies $\bar{D}_\alpha \Phi = 0$, and contains a complex scalar ϕ, a complex two component spinor ψ_α, and an auxiliary complex scalar F.

We are interested mainly in a class of renormalizable gauge theories with abelian global symmetries. These are defined by a UV action which includes a kinetic term for the matter fields of the form

$$S_{\text{charged matter kinetic}} = -\int d^3x d^2\theta d^2\bar{\theta} \sum_i \left(\Phi_i{}^\dagger e^{2V} \Phi_i\right) \tag{2.30}$$

$$= \sum_i \int d^3x \left((D_\mu \phi)_i (D^\mu \phi)^i + i\bar{\psi}_i \gamma^\mu D_\mu \psi^i + F_i F^i - \phi_i \sigma^2 \phi^i + \phi_i D\phi^i - \bar{\psi}_i \sigma \psi^i + i\phi_i \bar{\lambda}\psi^i - i\bar{\psi}_i \lambda \phi^i \right) \tag{2.31}$$

and a supersymmetric Yang-Mills action for the fields in the gauge multiplet

$$S_{\text{Yang Mills}} = \frac{1}{g^2} \int d^3x d^2\theta d^2\bar{\theta} Tr_f \left(\frac{1}{4}\Sigma^2\right) \tag{2.32}$$

$$= \frac{1}{g^2} \int d^3x Tr_f \left(\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \sigma D^\mu \sigma + D^2 + i\bar{\lambda}\gamma^\mu D_\mu \lambda\right) \tag{2.33}$$

where Σ is a linear multiplet defined by

$$\Sigma = \bar{D}^\alpha D_\alpha V \tag{2.34}$$

$$\Sigma^\dagger = \Sigma \tag{2.35}$$

$$D^\alpha D_\alpha \Sigma = \bar{D}^\alpha \bar{D}_\alpha \Sigma = 0 \tag{2.36}$$

In addition, one can allow supersymmetric completions of Chern-Simons terms. In the abelian case, these have a very simple superspace expression

$$S_{\text{abelian Chern Simons}} = \frac{k}{4\pi} \int d^3x d^2\theta d^2\bar{\theta} Tr_f \left(V\Sigma\right) \tag{2.37}$$

$$= \frac{k}{4\pi} \int d^3x \mathrm{Tr}_f \left(\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - \bar{\lambda}\lambda + 2D\sigma\right) \tag{2.38}$$

As a slight generalization of this, we can also consider a "off-diagonal" Chern-Simons term, also known as a BF term, coupling two or more different abelian gauge fields. The supersymmetric completion of this has the form

$$S_{\text{BF}} = \frac{k_{ij}}{4\pi} \int d^3x d^2\theta d^2\bar{\theta} Tr_f \left(\Sigma^i V^j\right) \tag{2.39}$$

$$= \frac{k_{ij}}{4\pi} \int d^3x \left(\varepsilon^{\mu\nu\rho} A^j{}_\mu \partial_\nu A^i{}_\rho - \frac{1}{2}\bar{\lambda}^j \lambda^i + D^i \sigma^j\right) \tag{2.40}$$

We will consider the S^3 partition function for these theories with insertions of Wilson loop and defect loop operators. The supersymmetry transformations on S^3 are generated by Killing spinors [13]. We will need the transformations of the vector multiplet fields under the S^3 supersymmetry generated by a particular killing spinor ε

$$\delta A_\mu = -\frac{i}{2}\lambda^\dagger \gamma_\mu \varepsilon \tag{2.41}$$

$$\delta\sigma = -\frac{1}{2}\lambda^\dagger \varepsilon \tag{2.42}$$

$$\delta D = -\frac{i}{2}D_\mu \lambda^\dagger \gamma^\mu \varepsilon + \frac{i}{2}[\lambda^\dagger, \sigma]\varepsilon + \frac{1}{4}\lambda^\dagger \varepsilon \tag{2.43}$$

$$\delta\lambda = (-\frac{i}{2}\varepsilon^{\mu\nu\rho}F_{\mu\nu}\gamma_\rho - D + i\gamma^\mu D_\mu \sigma - \sigma)\varepsilon \tag{2.44}$$

$$\delta\lambda^\dagger = 0 \tag{2.45}$$

The fermionic symmetry generated by ε will be used in 3 to compute the expectation value of the defect operator on S^3 using localization.

We now attempt to generalize the $SL(2,\mathbb{Z})$ action to the supersymmetric case. To start, it is natural to define T by simply adding a supersymmetric Chern-Simons term instead of an ordinary one. For S, we should use the supersymmetric version of the BF term and integrate over the entire background vector multiplet. It is now a simple exercise to check that the $SL(2,\mathbb{Z})$ relations remain satisfied for this generalization. We will not use the $(ST)^3 = C$ property, so we omit a check of that relation, but it will be important that $S^2 = C$, so let us sketch the argument. Consider the supersymmetric version of 2.16 defined via the action 2.39 where all vector fields have been extended to $\mathcal{N} = 2$ vector multipelts V, V_1, V_2 and V_ω. When we integrate over the second gauge field, the BF coupling gives us a delta function constraint imposing that the first gauge field is the negative of the background gauge field, as before. In addition, one can see that the integration over the auxiliary fields in the second vector multiplet imposes a similar constraint on the auxiliary fields of the first vector multiplet. Thus we see that the net effect is to set the first vector multiplet equal to the negative of the background vector multiplet.

Now consider the supersymmetric generalization of the abelian Wilson loop operator. For a certain class of loops γ preserving supersymmetry, this has the form

$$\exp\left(iq \oint_\gamma (A - i\sigma d\ell) \right) \tag{2.46}$$

For example, on S^3, the loops must be great circles which are fibers of the Hopf fibration. This operator is then invariant under the supersymmetry generated by ε. We define an operator W_γ inserting a supersymmetric Wilson loop in a background vector multiplet as before. Finally, we define the supersymmetric vortex loop by the prescription $D_\gamma = S^{-1} W_\gamma S$.

As before, we can integrate out the auxiliary vector multiplets V_1 and V_2 to obtain a description of the defect terms of the original fields alone. To start, let us write the part of the action containing V_2. In terms of component fields the action (in Euclidean signature) looks as follows:

$$S[\Phi, V_1, V_2] = ... + \frac{i}{2\pi}\int (-A_2 \wedge dA_1) + \frac{i}{2\pi}\int d^3x(-\sigma_2 D_1 - \sigma_1 D_2 + \frac{1}{2}(\lambda_1^\dagger \lambda_2 + \lambda_2^\dagger \lambda_1)) + iq\int_\gamma (A_2 - i\sigma_2 d\ell) \tag{2.47}$$

For simplicity we set to zero the background vector multiplet. We see that the integrals over D_2 and λ_2 set σ_1 and λ_1 to zero, while the integrals over A_2 and σ_2 impose the constraints

$$dA_1 = 2\pi q \delta_\gamma, \quad \star D_1 = -2\pi i q \delta_\gamma \wedge d\ell. \tag{2.48}$$

Here $d\ell$ is the volume 1-form on γ and δ_γ is the 2-form Poincarï¿œ dual to $[\gamma]$, as before. Note that D_1 is purely imaginary, which violates the usual reality condition on D.

As before, we can generalize this operator to account for more general background gauge multiplet configurations. We to specialize to S^3, and pick a supercharge δ corresponding to the Killing spinor ϵ. Then the BPS condition for an abelian vector multiplet is [13]

$$0 = (i\gamma^\mu(-\star F_\mu + \partial_\mu \sigma) - (D + \sigma))\epsilon \tag{2.49}$$

We would like to find configurations for which only F and D are non-zero. Using $v^\mu \gamma_\mu \epsilon = \epsilon$, where v^μ is the Killing vector along the Hopf fibration as in [13], we see we can take:

$$F = 2\pi f(x) \star v, \quad D = -2\pi i f(x) \tag{2.50}$$

for a function $f : S^3 \to \mathbb{R}$. The normalization is for later convenience. Note that the Bianchi identity implies

$$0 = \frac{1}{2\pi} dF = d(f \star v) = df \wedge \star v = \star(v^\mu \partial_\mu f) \tag{2.51}$$

so that f must be constant along the fibers of the Hopf fibration. Equivalently, we impose that f arises from a function $\tilde{f} : S^2 \to \mathbb{R}$ by:

$$f = \tilde{f} \circ \pi \tag{2.52}$$

where $\pi : S^3 \to S^2$ is the projection map of the Hopf fibration. Thus, the operator is really labeled by the function $\tilde{f}$ on S^2. When $\tilde{f}$ approaches a delta-function on S^2, the corresponding operator approaches the supersymmetric vortex loop wrapping a fiber of the Hopf fibration.

If we apply this construction to the $U(1)_J$ symmetry, we get an operator which is a natural generalization of the supersymmetric Wilson loop:

$$\mathcal{O}_{\tilde{f}} = \exp\left(i \int_{S^3} \sqrt{g} d^3x f(x)(v^\mu A_\mu - i\sigma) \right) \tag{2.53}$$

That this operator is supersymmetric follows from

$$\delta(v^\mu A_\mu - i\sigma) = 0 \tag{2.54}$$

as can be easily checked. The condition $v^\mu \partial_\mu f = 0$ is necessary for gauge-invariance. Taking $\tilde{f}$ to be a delta function on S^2, so that f is a delta function supported along a great circle in S^3, one recovers the ordinary supersymmetric Wilson loop. Note that we have picked the normalizations so that when $\tilde{f}$ is a delta function on S^2 integrating to 1, so that f integrates to 2π on S^3, we recover the charge 1 Wilson loop.

3 Localization in the presence of a vortex loop

In this section we compute the expectation value of the global vortex loop on S^3 by localization. The global symmetry is assumed to be a $U(1)$ subgroup of the flavor symmetry group. We will present three approaches to the calculation which yield the same result. Applying the localization procedure in the presence of the defect requires some regularization and the agreement of the approaches presented below gives us confidence in the validity of the computation. We begin with a quick review of localization for 3d gauge theories. Additional details can be found in [13].

3.1 Localization of 3d gauge theories

The expectation value of BPS operators in 3d $\mathcal{N}=2$ superconformal gauge theories can be computed by localization on S^3. The relevant result, derived in [13], is that deformation invariance allows us to reduce the computation of the infinite dimensional path integral with BPS operator insertions to a matrix model with the integration domain given by the Lie algebra of the gauge group. The data entering the computation is a UV action with gauge group G, Lie algebra $\mathfrak{g}$ and Chern-Simons levels k_i, a set of representations R_i for the chiral matter multiplets and the IR conformal dimensions (equivalently R-charges) Δ_i for each chiral multiplet. The integration measure for the matrix model is then

$$\frac{1}{\mathrm{Vol}(G)} da|_{a\in\mathrm{Ad}(\mathfrak{g})} \tag{3.1}$$

The contribution of a level k Chern-Simons term (for a simple gauge group factor associated to a) is

$$e^{-i\pi k \mathrm{Tr}\left(a^2\right)} \tag{3.2}$$

A Fayet-Iliopoulos term with coefficient η contributes

$$e^{2\pi i\eta \mathrm{Tr}(a)} \tag{3.3}$$

Every dynamical gauge multiplet contributes

$$Z^{\text{gauge multiplet}}_{\text{1 - loop}}(a) = \det_{\mathrm{Ad}(\mathfrak{g})}\left(2\sinh(\pi a)\right) = \prod_{\rho\in\mathrm{roots}(\mathfrak{g})} 2\sinh(\pi\rho(a)) \tag{3.4}$$

and every dynamical chiral multiplet contributes

$$Z^{\text{chiral multiplet}}_{\text{1 - loop}}(a,\Delta) = \frac{\det\mathcal{O}_F}{\sqrt{\det\mathcal{O}_B}} = \prod_{\rho\in R} \exp\left(\ell\left(z(\rho(a),\Delta)\right)\right) \tag{3.5}$$

where ρ are the weights of R and

$$\ell(z) = -z\log\left(1-e^{2\pi i z}\right) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2\left(e^{2\pi i z}\right)\right) - \frac{i\pi}{12} \tag{3.6}$$

$$z(\rho(a),\Delta) = i\rho(a) - \Delta + 1 \tag{3.7}$$

abelian flavor parameters can be incorporated by shifting $\rho \to \rho + m$. The insertion of a supersymmetric Wilson loop in a representation R gives a factor of

$$W(a) = \frac{1}{\dim(R)}\mathrm{Tr}_R\left(e^{2\pi a}\right) \tag{3.8}$$

which for an abelian Wilson loop of charge q reduces to

$$e^{2\pi q a} \tag{3.9}$$

Integration with the measure 3.1 of the product of all relevant contributions yields the exact expectation value.

In computing 3.5 we have implicitly assumed a standard δ exact term, and hence standard kinetic operators $(\mathcal{O}_F, \mathcal{O}_B)$, for the fields in the chiral multiplet [13, 16]. In the presence of the flavor vortex loop, the kinetic term of a charged chiral multiplet is altered by a background gauge field created by the loop. The new term and the revised 1-loop contribution are derived below. The contribution of the vector multiplet is unaffected because it is not charged under flavor symmetries.

3.2 Method 1: using the $SL(2,\mathbb{Z})$ definition of D

The simplest way to extract the effect of inserting a supersymmetric defect line operator is by using the definition of the operation D as

$$D_q = S^{-1} W_q S \tag{3.10}$$

where D_q is the vortex loop with holonomy $\exp(2\pi i q)$. There are no integrality constraints on q because the large circle on S^3 is homologically trivial. Since we can perform the operations on the RHS at the level of the matrix model, it should be possible to compute the LHS indirectly this way. Explicitly, suppose we compute the partition function as a holomorphic function of a flavor deformation m (and possibly other parameters which we suppress):

$$Z(m) \tag{3.11}$$

This is the analogue of $Z[A]$ in section 2. The operation S then tells us to treat this flavor parameter as a gauge parameter, and integrate over it, with a coupling to an FI parameter η:

$$(S\cdot Z)(\eta) = \int dm Z(m) e^{2\pi i \eta m} \tag{3.12}$$

The operation W_1 tells us to insert a charge-q Wilson loop in the background field corresponding to η:

$$(W_q S\cdot Z)(\eta) = e^{2\pi q\eta}\int dm Z(m) e^{2\pi i \eta m} \tag{3.13}$$

Finally, S^{-1} tells us to integrate over η and insert a new FI term, which we will denote m', with the opposite sign:

$$(S^{-1} W_1 S\cdot Z)(m') = \int e^{-2\pi i\eta m'} e^{2\pi q\eta}\int dm Z(m) e^{2\pi i \eta m} \tag{3.14}$$

Now to integrate out the variables η and m, we simply note that the integral over η imposes a delta function which sets $m = m' + iq$. Thus we are left with:

$$(D_q\cdot Z)(m) = Z(m+iq) \tag{3.15}$$

Indeed, this result can be inferred from the 4d perspective by considering Wilson and 't Hooft loops ending on a 3d boundary [17, 18]. This argument was rather indirect; it also raises the question about the interpretation of poles in the partition function for special values of $m + iq$. We now proceed to present two more explicit derivations of this result.

3.3 Method 2: smearing the defect

We return to the smeared Wilson loop

$$\mathcal{O}_f = \exp\left(i\int_{S^3}\sqrt{g}d^3x f(x)(v^\mu A_\mu - i\sigma)\right) \tag{3.16}$$

where f is some real function on S^3 constant along the fibers of the Hopf fibration, as above. It is convenient to decompose $f(x)$ as

$$f(x) = \frac{q}{\pi} + f_o(x) \tag{3.17}$$

where

$$\int_{S^3}\sqrt{g}d^3x f_o = 0 \tag{3.18}$$

and q is constant, specifically, $q = \frac{1}{2\pi}\int_{S^3}\sqrt{g}d^3x f$. Note that this normalization agrees with the case where f is a delta function supported on a great circle, since the integral of f should give 2π times the charge of the Wilson loop.

For general f, we can decompose $\mathcal{O}_f = \mathcal{O}_q\mathcal{O}_{f_o}$, so it suffices to study them separately. Actually, we will find that it is only $\mathcal{O}_q$ which contributes to the localized path-integral. Specifically, we claim that the operator $\mathcal{O}_{f_o}$ can be absorbed into a shift of the action by a total δ-variation. To see this, let us pick a function $g : S^3 \to \mathbb{R}$ and consider:

$$\delta(\int\sqrt{g}d^3x\epsilon^\dagger\gamma^\mu(\partial_\mu g)\lambda) = \int\sqrt{g}d^3x\epsilon^\dagger\gamma^\mu(\partial_\mu g)(i\gamma^\nu(-\frac{1}{2}\epsilon_{\nu\rho\sigma}F^{\rho\sigma} + \partial_\nu\sigma) - (D+\sigma))\epsilon) \tag{3.19}$$

$$= \int\sqrt{g}d^3x(\partial_\mu g)\Big(i(g^{\mu\nu} + i\epsilon^{\mu\nu\tau}v_\tau)(-\frac{1}{2}\epsilon_{\nu\rho\sigma}F^{\rho\sigma} + \partial_\nu\sigma) - v_\mu(D+\sigma)\Big) \tag{3.20}$$

The term involving $D+\sigma$ is proportional to $v^\mu\partial_\mu g$ and vanishes if we impose that g, like f, is constant along the fibers. The remaining terms can be expanded to give:

$$\int\sqrt{g}d^3x(\partial_\mu g)\Big(-\frac{i}{2}\epsilon^{\mu\rho\sigma}F_{\rho\sigma} + F^{\mu\nu}v_\nu + i\partial^\mu\sigma - i\epsilon^{\mu\nu\rho}\partial_\nu\sigma v_\rho\Big) \tag{3.21}$$

Integrating by parts, the first and last terms can be seen to vanish, and the others give

$$\int\sqrt{g}d^3x\Big(A^\mu(\frac{1}{2}\nabla^\nu(v_\mu\partial_\nu g - v_\nu\partial_\mu g)) - i\sigma(\nabla^2 g)\Big) \tag{3.22}$$

The quantity multiplying A^μ can be expanded as:

$$\nabla^\nu v_\mu(\partial_\nu g) + v_\mu\nabla^2 g - (\nabla^\nu v_\nu)\partial_\mu g - v_\nu\nabla^\nu\partial_\mu g \tag{3.23}$$

Using $\nabla_\mu v_\nu = \epsilon_{\mu\nu\rho}v^\rho$, this can be simplified to:

$$v_\mu\nabla^2 g - \nabla_\mu(v_\nu\nabla^\nu g) \tag{3.24}$$

The second term vanishes when we impose that g is constant along the fibers, and we are left with:

$$\int\sqrt{g}d^3x(\nabla^2 g)(v^\mu A_\mu - i\sigma) \tag{3.25}$$

which agrees with the exponent of the operator $\mathcal{O}_{f_o}$ above, provided we can find a g such that:

$$f_o = \nabla^2 g. \tag{3.26}$$

This clearly requires $\int_{S^3} \sqrt{g} d^3 x f_o = 0$, so that one cannot use this trick to remove the constant part of f. However, if this condition is met, then the equation can be solved, and g will indeed be constant along the fibers.[1] This proves that the non-constant part of the operator can be discarded, as δ-exact terms do not affect the path integral, i.e. we can replace a Wilson loop localized on a loop by one that is smeared uniformly over the entire S^3; these differ only by a δ-exact insertion.

Thus without a loss of generality we can restrict to the case $f = \frac{q}{\pi}$ (a constant). Then the background vector multiplet we must couple the flavor symmetry current to is given by:

$$F = 2q \star v, \quad D = -2iq \tag{3.27}$$

Since $dv = 2 \star v$, the corresponding gauge field can be taken to be

$$A = qv \tag{3.28}$$

The δ-exact gauged action of a chiral multiplet of conformal dimension 1/2 is given by [13]

$$S_\delta = \int \sqrt{g} d^3 x \bigg(\phi^\dagger (-D_\mu D^\mu + \sigma^2 + iD + \frac{3}{4}) \phi + \psi^\dagger (i \gamma^\mu D_\mu - i\sigma)\psi + F^\dagger F \bigg) \tag{3.29}$$

Let us couple this to an ordinary BPS background, with $\sigma = -D = \sigma_o$, as well as the background vector multiplet described above, with $A = qv$ and $D = -2iq$. We find:

$$S_\delta = \int \sqrt{g} d^3 x \bigg(\phi^\dagger (-\nabla^2 - 2iqv^\mu \partial_\mu + q^2 + {\sigma_o}^2 - i\sigma_o + 2q + \frac{3}{4}) \phi + \psi^\dagger (i\gamma^\mu \nabla_\mu - q\gamma^\mu v_\mu - i\sigma_o)\psi + F^\dagger F \bigg) \tag{3.30}$$

The bosonic operator has the form:

$$\mathcal{O}_B = -\nabla^2 + aiv^\mu \partial_\mu + b \tag{3.31}$$

with $a = -2q$ and $b = q^2 + {\sigma_o}^2 - i\sigma_o + 2q + \frac{3}{4}$, which has determinant [13]

$$\sqrt{\det \mathcal{O}_B} = \prod_{\ell=0}^{\infty} \bigg(\prod_{m=-\ell/2}^{\ell/2} \bigg(\ell(\ell+2) - 2am + b \bigg) \bigg)^{\ell+1} \tag{3.32}$$

$$= \prod_{\ell=0}^{\infty} \bigg(\prod_{m=-\ell/2}^{\ell/2} \bigg(\ell(\ell+2) + 4qm + q^2 + {\sigma_o}^2 - i\sigma_o + 2q + \frac{3}{4} \bigg) \bigg)^{\ell+1} \tag{3.33}$$

Meanwhile, for the fermions, the operator has the form:

$$\mathcal{O}_F = i\gamma^\mu \nabla_\mu + ic\gamma^\mu v_\mu + d \tag{3.34}$$

with $c = -q$ and $d = -i\sigma_o$, which has determinant:

[1] This can be seen most easily by working with a mode expansion $f_o = \sum_{\ell,m,n} c_{\ell,m,n} Y_{\ell,m,n}$, where $Y_{\ell,m,n}$ are spherical harmonics on S^3, satisfying $\nabla^2 Y_{\ell,m,n} = -\ell(\ell+2) Y_{\ell,m,n}$ and $v^\mu \partial_\mu Y_{\ell,m,n} = imY_{\ell,m,n}$. Then $g = \sum_{\ell,m,n} \frac{1}{\ell(\ell+2)} c_{\ell,m,n} Y_{\ell,m,n}$, which is well defined since f has no $\ell = 0$ component by assumption, and, like f, has $c_{\ell,m,n} = 0$ for all $m \neq 0$.

$$\det \mathcal{O}_F = \prod_{\ell=0}^{\infty} \left((\ell - c - d + 3/2)(\ell + c - d + 3/2) \prod_{m=-\ell/2}^{\ell/2-1} \left(\ell(\ell+2) - 4cm - 2c + d + c^2 - d^2 + 3/4 \right) \right)^{\ell+1} \quad (3.35)$$

$$= \prod_{\ell=0}^{\infty} \left((\ell + q + i\sigma_o + 3/2)(\ell - q + i\sigma_o + 3/2) \prod_{m=-\ell/2}^{\ell/2-1} \left(\ell(\ell+2) + 4qm + 2q - i\sigma_o + q^2 + {\sigma_o}^2 + 3/4 \right) \right)^{\ell+1} \quad (3.36)$$

We see that most modes cancel, and we are left with:

$$Z_{1-loop} = \frac{\det \mathcal{O}_F}{\sqrt{\det \mathcal{O}_B}} = \prod_{\ell=0}^{\infty} \left(\frac{(\ell + q + i\sigma_o + 3/2)(\ell - q + i\sigma_o + 3/2)}{(\ell(\ell+2) - 2q\ell + 2q - i\sigma_o + q^2 + {\sigma_o}^2 + 3/4)} \right)^{\ell+1} \quad (3.37)$$

$$= \prod_{\ell=0}^{\infty} \left(\frac{(\ell + q + i\sigma_o + 3/2)(\ell - q + i\sigma_o + 3/2)}{(\ell + 1 + q + i\sigma_o + \frac{1}{2})(\ell + 1 + q - i\sigma_o - \frac{1}{2})} \right)^{\ell+1} \quad (3.38)$$

$$= \prod_{\ell=0}^{\infty} \left(\frac{\ell + i(\sigma_o + iq) + 3/2}{\ell - i(\sigma_o + iq) + 1/2} \right)^{\ell+1} \quad (3.39)$$

Note that σ_o and α appear in the combination $\sigma + iq$, so that we can just make this replacement in the ordinary one-loop determinant to find:

$$Z_{1-loop} = e^{\ell(\frac{1}{2} + i\sigma - q)} \quad (3.40)$$

In fact, this computation goes through with minimal changes for chiral multiplets of arbitrary dimension (one merely shifts σ_o by an imaginary amount), and we find:

$$Z_{1-loop} = e^{\ell(1 - \Delta + i\sigma - q)} \quad (3.41)$$

Note we have obtained the same result as in the indirect argument above. We will now turn to an even more explicit argument, where we do not smear out the defect over the sphere but instead work directly with a (regularized) delta function background.

3.4 Method 3: explicit computation in a singular background

Let us now focus on the specific case where the function $\tilde{f}$ is a delta function, corresponding to the dual of an ordinary (unsmeared) Wilson loop. Although we have argued that one can replace the delta function by a constant which has the same integral over S^2, we would like to gain better physical insight into the vortex loop by explicitly finding the modes in a singular background. For simplicity we will focus on the case where the matter has canonical dimension, although it is straightforward to generalize this.

Recall that the smeared vortex loop on S^3 can be obtained by coupling to a background $F = 2\pi f \star v$ and $D = -2\pi i f$, where f is some function on S^3 constant along the fibers of the Hopf fibration. Here we compute the modes and 1-loop determinant explicitly in the case where f is a (infinitessimally smeared) delta function supported on a single fiber.

We will work in toroidal coordinates on S^3, with $\eta \in [0, \pi/2]$ and θ and ϕ in $[0, 2\pi)$. Explicitly, we can relate these coordinates to the unit sphere $S^3 \subset \mathbb{R}^4$ via:

$$x = \cos\eta\cos\theta, \quad y = \cos\eta\sin\theta, \quad z = \sin\eta\cos\phi, \quad w = \sin\eta\sin\phi \tag{3.42}$$

The surfaces of constant η are torii, which degenerate to great circles at $\eta = 0, \frac{\pi}{2}$.

The usual round metric takes the following form in these coordinates:

$$ds^2 = d\eta^2 + \sin^2\eta d\theta^2 + \cos^2\eta d\phi^2 \tag{3.43}$$

$$\Rightarrow \nabla^2 = \frac{1}{\sin\eta\cos\eta}\frac{\partial}{\partial\eta}\sin\eta\cos\eta\frac{\partial}{\partial\eta} + \frac{1}{\sin^2\eta}\frac{\partial^2}{\partial\theta^2} + \frac{1}{\cos^2\eta}\frac{\partial^2}{\partial\phi^2} \tag{3.44}$$

The Killing vector v is given in these coordinates by:

$$v = \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi} \tag{3.45}$$

or, as a 1-form, by:

$$\tilde{v} = \sin^2\eta d\theta + \cos^2\eta d\phi \tag{3.46}$$

It satisfies:

$$dv = 2\cos\eta\sin\eta d\eta \wedge d\theta - 2\cos\eta\sin\eta d\eta \wedge d\phi = 2 \star v \tag{3.47}$$

We take the defect to be supported on the great circle at $\eta = 0$. We define a regularized delta function supported on the loop by:

$$f = \frac{g(\eta/\epsilon)}{2\pi\epsilon\sin\eta\cos\eta} \tag{3.48}$$

where $g(x)$ has support in $0 \leq x \lesssim 1$ and integrates to 1 on $[0, \frac{\pi}{2\epsilon}]$, so that f integrates to 2π. This approaches a delta function supported on the great circle at $\eta = 0$ for $\epsilon \to 0$. Then the background field-strength we need to consider is

$$F = 2\pi q\ f \star v = \frac{q}{\epsilon}g(\eta/\epsilon)d\eta \wedge (d\theta - d\phi) \tag{3.49}$$

which is solved by a background vector potential

$$A = qG(\eta/\epsilon)d\theta - q(G(\eta/\epsilon) - 1)d\phi \tag{3.50}$$

where $G' = g$ with $G(0) = 0$, so that $G(x) \to 1$ for large x, and we pick the constants so that this is everywhere well-defined. We also have a background auxiliary scalar

$$D = -2\pi i q f = -\frac{iq\ g(\eta/\epsilon)}{\epsilon\sin\eta\cos\eta} \tag{3.51}$$

3.4.1 Bosons

First consider the bosonic operator on S^3:

$$\mathcal{O}_B = -D_\mu D^\mu + \sigma^2 + iD + \frac{3}{4} \tag{3.52}$$

We will couple to a defect background, as above, as well as an ordinary $\sigma = -D = \sigma_o$ background. Using the form of D above, along with:

$$A^\mu\partial_\mu = \left(\frac{q\,G(\eta/\epsilon)}{\sin^2\eta}\frac{\partial}{\partial\theta} - \frac{q(G(\eta/\epsilon)-1)}{\cos^2\eta}\frac{\partial}{\partial\phi}\right) \tag{3.53}$$

$$A^2 = \left(\frac{q^2G(\eta/\epsilon)^2}{\sin^2\eta} + \frac{q^2(G(\eta/\epsilon)-1)^2}{\cos^2\eta}\right) \tag{3.54}$$

and looking for an eigenfunction $\mathcal{O}_B\phi = \lambda\phi$ of the form $\phi = f(\eta)e^{im\theta+in\phi}$, we get the following equation:

$$-\frac{1}{\sin\eta\cos\eta}\frac{d}{d\eta}\left(\sin\eta\cos\eta\frac{df}{d\eta}\right)+\left(\frac{(m+qG)^2}{\sin^2\eta}+\frac{(n-q(G-1))^2}{\cos^2\eta}+\frac{qg(\eta/\epsilon)}{\epsilon\sin\eta\cos\eta}+\frac{3}{4}+{\sigma_o}^2-i\sigma_o-\lambda\right)f=0 \tag{3.55}$$

Before solving this equation, let us consider the fermions, as we will see their components satisfy a very similar equation.

3.4.2 Fermions

The operator in this case is

$$\mathcal{O}_F = i\not{D} - i\sigma \tag{3.56}$$

It is convenient to use a left-invariant vielbein. One computes this in toroidal coordinates as:

$$e_i^L = \begin{cases} \sin(\theta+\phi)\dfrac{\partial}{\partial\eta} + \cos(\theta+\phi)(\cot\eta\dfrac{\partial}{\partial\theta} - \tan\eta\dfrac{\partial}{\partial\phi}) & i=1 \\ -\cos(\theta+\phi)\dfrac{\partial}{\partial\eta} + \sin(\theta+\phi)(\cot\eta\dfrac{\partial}{\partial\theta} - \tan\eta\dfrac{\partial}{\partial\phi}) & i=2 \\ \dfrac{\partial}{\partial\theta} + \dfrac{\partial}{\partial\phi} & i=3 \end{cases} \tag{3.57}$$

Then the Dirac operator can be written as:

$$i\not{\nabla} = \begin{pmatrix} i\partial_3 - \frac{3}{2} & i\partial_1 + \partial_2 \\ i\partial_1 - \partial_2 & -i\partial_3 - \frac{3}{2} \end{pmatrix} \tag{3.58}$$

which, in toroidal coordinates, becomes:

$$i\not{\nabla} = \begin{pmatrix} i\dfrac{\partial}{\partial\theta} + i\dfrac{\partial}{\partial\phi} - \dfrac{3}{2} & e^{-i(\theta+\phi)}\left(-\dfrac{\partial}{\partial\eta} + i\left(\cot\eta\dfrac{\partial}{\partial\theta} - \tan\eta\dfrac{\partial}{\partial\phi}\right)\right) \\ e^{i(\theta+\phi)}\left(\dfrac{\partial}{\partial\eta} + i\left(\cot\eta\dfrac{\partial}{\partial\theta} - \tan\eta\dfrac{\partial}{\partial\phi}\right)\right) & -i\dfrac{\partial}{\partial\theta} - i\dfrac{\partial}{\partial\phi} - \dfrac{3}{2} \end{pmatrix} \tag{3.59}$$

We should also couple to the gauge field, which amounts to the replacement $\frac{\partial}{\partial\theta} \to \frac{\partial}{\partial\theta} + iqG, \frac{\partial}{\partial\phi} \to \frac{\partial}{\partial\phi} - iq(G-1)$, as well as the constant σ_o. Then if we look for eigenspinors of the form

$$\psi = e^{im\theta+in\phi}\begin{pmatrix} \psi_1(\eta) \\ e^{i(\theta+\phi)}\psi_2(\eta) \end{pmatrix} \tag{3.60}$$

we get the following coupled first-order equations for ψ_1 and ψ_2:

$$(\frac{d}{d\eta} - (m+qG)\cot\eta + (n-q(G-1))\tan\eta)\psi_1 = -(m+n+q-\lambda-i\sigma_o+\frac{1}{2})\psi_2 \quad (3.61)$$
$$(-\frac{d}{d\eta} - (m+qG+1)\cot\eta + (n-q(G-1)+1)\tan\eta)\psi_2 = (m+n+q+\lambda+i\sigma_o+\frac{3}{2})\psi_1$$

Solving for ψ_2 using the first equation and plugging into the second one, we get a second order equation in terms of ψ_1 alone:

$$\left(-\frac{d}{d\eta} - (m+qG+1)\cot\eta + (n-q(G-1)+1)\tan\eta\right)\left(\frac{d}{d\eta} - (m+qG)\cot\eta = (n-q(G-1))\tan\eta\right)\psi_1 = \quad (3.62)$$

$$= -(m+n+q-\lambda-i\sigma_o+\frac{1}{2})(m+n+q+\lambda+i\sigma_o+\frac{3}{2})\psi_1. \quad (3.63)$$

This can be rearranged to

$$-\frac{1}{\sin\eta\cos\eta}\frac{d}{d\eta}\left(\sin\eta\cos\eta\frac{d\psi_1}{d\eta}\right) + \left(\frac{(m+qG)^2}{\sin^2\eta} + \frac{(n-q(G-1))^2}{\cos^2\eta} + \frac{qg(\eta/\epsilon)}{\epsilon\sin\eta\cos\eta} - (\lambda+i\sigma_o+\frac{1}{2})^2+1\right)\psi_1 = 0 \quad (3.64)$$

which is precisely the same equation as satisfied by the bosonic modes, with the following relation between the eigenvalues:

$$\lambda_B + (i\sigma_o+\frac{1}{2})^2 = (\lambda_F + i\sigma_o + \frac{1}{2})^2 \quad (3.65)$$

This is a quadratic equation for λ_F with two solutions $\lambda_{F\pm}$. Their product can be read off as the constant term, and one can see that this is simply λ_B.

In principle, we can take ψ_1 to be the scalar eigenfunction for a given choice of λ_B, take one of the solutions $\lambda_{F\pm}$ to the equation above, and plug these into the first equation in (3.61) to solve for ψ_2. For each λ_B and pair of solutions $\lambda_{F\pm}$ for which this procedure goes through, we can see that the contribution to the partition function, $\lambda_{F+}\lambda_{F-}/\lambda_B = 1$, is trivial.

However, there are two exceptions we must be more careful with. First, we must also allow solutions with $\psi_1 = 0$, but ψ_2 non-vanishing. Then we see that (3.61) reduces to:

$$(m+n+q-\lambda-i\sigma_o+\frac{1}{2})\psi_2 = 0$$
$$(-\frac{d}{d\eta} - (m+qG+1)\cot\eta + (n-q(G-1)+1)\tan\eta)\psi_2 = 0$$

Thus the eigenvalue in these cases is $\lambda = m+n+q-i\sigma_o+\frac{1}{2}$, and ψ_2 satisfies:

$$\frac{d}{d\eta}\log\psi_2 = -(m+qG+1)\cot\eta + (n-q(G-1)+1)\tan\eta = \begin{cases} -(m+1)\cot\eta + ... & \text{near } \eta = 0 \\ -(m+q+1)\cot\eta + (n+1)\tan\eta & \text{in the bulk} \end{cases} \quad (3.66)$$

$$\Rightarrow \psi_2 = \begin{cases} \sin^{-(m+1)}\eta + ... & \text{near } \eta = 0 \\ \sin^{-(m+q+1)}\eta\cos^{-(n+1)}\eta & \text{in the bulk} \end{cases} \quad (3.67)$$

Regularity at the endpoints implies that m and n should be negative integers. But then for $-(q+1) < m < 0$, the bulk solution is singular as it approaches the loop. We will return to this point in a moment. These solutions correspond to extra fermionic modes that we have not accounted for before, so their eigenvalues should be included in the numerator of the partition function.

The other exception occurs when the differential operator acting on ψ_1 in the first equation in (3.61) annihilates our choice of ψ_1. Then we must pick $\psi_2 = 0$, and there will not be two choices of λ_F, but only one, and so the cancellation with the corresponding bosonic mode will not be complete. We see that in this case (3.61) gives:

$$(\frac{d}{d\eta} - (m+qG)\cot\eta + (n - q(G-1))\tan\eta)\psi_1 = 0$$

$$(m+n+q+\lambda+i\sigma_o+\frac{3}{2})\psi_1 = 0$$

Now we find $\lambda = -(m+n+q+i\sigma_o+\frac{3}{2})$, and ψ_1 satisfies:

$$\frac{d}{d\eta}\log\psi_1 = (m+qG)\cot\eta - (n-q(G-1))\tan\eta = \begin{cases} m\cot\eta + ... & \text{near } \eta = 0 \\ (m+q)\cot\eta - n\tan\eta & \text{in the bulk} \end{cases} \tag{3.68}$$

$$\Rightarrow \psi_1 = \begin{cases} \sin^m\eta + ... & \text{near } \eta = 0 \\ \sin^{(m+q)}\eta\cos^n\eta & \text{in the bulk} \end{cases} \tag{3.69}$$

Now regularity at the endpoints forces m and n to be nonnegative integers, and the bulk solutions are singular for $0 \le m < -q$. These are modes for which there is only one solution, say λ_{F+}, to the equation (3.65), and so the cancellation with the bosons is not complete. The net contribution is $\lambda_{F+}/\lambda_B = 1/\lambda_{F-}$, and one can read this off from (3.63) as $\lambda_{F-} = m+n+q-i\sigma_o+\frac{1}{2}$.

Putting this together, we see all modes cancel out of the partition function except for the special cases noted above, and these give:

$$Z = \frac{\prod_{m,n<0}(m+n+q-i\sigma_o-\frac{1}{2})}{\prod_{m,n\ge 0}(m+n+q-i\sigma_o+\frac{1}{2})} \tag{3.70}$$

$$= \prod_{\ell=0}^{\infty}\left(\frac{-\ell+q-i\sigma_o-\frac{3}{2}}{\ell+q-i\sigma_o+\frac{1}{2}}\right)^{\ell+1} \tag{3.71}$$

$$= e^{\ell(\frac{1}{2}+q-i\sigma_o)} \tag{3.72}$$

Thus we have obtained the result for the 1-loop determinant without ever having to solve the second order differential equation. However, note that this function may have poles, e.g. when $q = \frac{1}{2}$. To get a better understanding of the origin of these singularities, we will now solve the differential equation explicitly.

3.4.3 Solving the eigenvalue equation

Consider the following second order equation, which has come up for both the bosons and fermions:

$$-\frac{1}{\sin\eta\cos\eta}\frac{d}{d\eta}\left(\sin\eta\cos\eta\frac{df}{d\eta}\right) + \left(\frac{(m+qG)^2}{\sin^2\eta} + \frac{(n-q(G-1))^2}{\cos^2\eta} + \frac{qg(\eta/\epsilon)}{\epsilon\sin\eta\cos\eta} + 1 - (\ell+1)^2\right)f = 0 \tag{3.73}$$

where, for later convenience, we have written the eigenvalue in terms of a parameter ℓ. This is related to the eigenvalues by:

$$\lambda_B = (\ell+1)^2 - (i\sigma_o + \frac{1}{2})^2, \quad \lambda_{F\pm} = \pm(\ell+1) - i\sigma_o - \frac{1}{2} \tag{3.74}$$

Near Loop Region Let us start by focusing on the region $0 < \eta \lesssim \epsilon$, as this will determine what boundary conditions to impose on the bulk solution. We start by defining $\xi = \eta/\epsilon$, and expanding the equation above to leading order in ϵ:

$$-\frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{df}{d\xi}\right) + \left(\frac{(m+qG(\xi))^2}{\xi^2} + \frac{qg(\xi)}{\xi}\right)f = 0 \tag{3.75}$$

One solution can be immediately obtained, independent of the functional form of g and G, by noting that this equation follows from the first order equation:

$$\frac{df}{d\xi} = \frac{m+qG}{\xi}f \tag{3.76}$$

as can be easily checked using $G' = g$. Then, using $G \to 0$ as $\chi \to 0$ and $G \to 1$ as $\chi \to \infty$, we see that:

$$f(\xi) \sim \begin{cases} \xi^m & \xi \to 0 \\ \xi^{(m+q)} & \xi \to \infty \end{cases} \tag{3.77}$$

For $m \geq 0$, this is the regular solution we want. We can match with the solution in the bulk by looking at the behavior in the the large ξ region, and we see that we should take the bulk solution which goes as $f(\eta) \sim \eta^{(m+q)}$. Note that if $q < 0$, for $0 \leq m < -q$, the bulk solution would appear to be singular right up until we reach the near loop region, at which point the presence of the defect modifies the solution to go as η^{-m} and be regular. For $m \geq -q$, we take the regular solution, as in the absence of a defect. Note that this behavior does not depend on the precise functional form of g, only that it correctly reproduces a delta function in the $\epsilon \to 0$ limit.

It remains to understand what happens when $m < 0$. Here, the first order equation does not have any non-trivial solutions regular at $\xi = 0$, so we must return to the second order equation. Then we can find a regular solution, but it appears it depends non-trivially on g and G. Specifically, we find:

$$f(\xi) \sim \begin{cases} \xi^m & \xi \to 0 \\ A(g,m,q)\xi^{|m+q|} + B(g,m,q)\xi^{-|m+q|} & \xi \to \infty \end{cases} \tag{3.78}$$

for some constants which depend non-trivially on g, m, and q, and in particular are generically both non-zero, unlike in the previous case.

Nevertheless, if we reinstate the ϵ dependence, we see that to match with the bulk, we should take the solution there to go as:

$$f(\eta) \sim A(g,m,q)\eta^{|m+q|}\epsilon^{-|m+q|} + B(g,m,q)\eta^{-|m+q|}\epsilon^{|m+q|} \tag{3.79}$$

and for $\epsilon \to 0$, the first term will dominate, and so we should take the regular solution.

Thus we should always take the regular solution, $f \sim \eta^{|m+q|}$, except in the case where the coefficient of the regular part is precisely zero, which happens only when $0 \leq m < -q$, and in these cases we take the singular solution $f \sim \eta^{-|m+q|}$.

Bulk Since $G = 1$ and $g = 0$ everywhere except an infinitessimal region near $\eta = 0$, in the bulk the equation reduces to:

$$-\frac{1}{\sin\eta\cos\eta}\frac{d}{d\eta}\left(\sin\eta\cos\eta\frac{df}{d\eta}\right)+\left(\frac{(m+q)^2}{\sin^2\eta}+\frac{n^2}{\cos^2\eta}+1-(\ell+1)^2\right)f=0 \tag{3.80}$$

It is convenient to look for an f of the form:

$$f(\eta)=\sin^{\tilde{m}}\eta\cos^{\tilde{n}}\eta h(\sin^2\eta) \tag{3.81}$$

for $\tilde{m}, \tilde{n}$ we will choose in a moment. Plugging this in, we find the following equation for $h(x)$:

$$x(1-x)h''+(\tilde{m}+1-(\tilde{m}+\tilde{n}+2)x)h'-\frac{1}{4}\left(\frac{(m+q)^2-\tilde{m}^2}{x}+\frac{(n-q)^2-\tilde{n}^2}{1-x}+(\tilde{m}+\tilde{n}+1)^2-(\ell+1)^2\right)h=0 \tag{3.82}$$

The greatest simplification is achieved by setting $\tilde{m}^2 = (m+q)^2$ and $\tilde{n}^2$, with the sign of $\tilde{m}$ and $\tilde{n}$ to be fixed later. Then we are left with the hypergeometric equation:

$$x(1-x)h''+(c-(a+b+1)x)h'-abh=0 \tag{3.83}$$

where:

$$c=\tilde{m}+1, \quad a+b=\tilde{m}+\tilde{n}+1, \quad ab=\frac{1}{4}((\tilde{m}+\tilde{n}+1)^2-(\ell+1)^2) \tag{3.84}$$

$$\Rightarrow a,b=\frac{1}{2}(\tilde{m}+\tilde{n}+1\pm(\ell+1)) \tag{3.85}$$

The solutions can be written in terms of hypergeometric functions ${}_2F_1(a,b;c;x)$:

$${}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n \tag{3.86}$$

where $(a)_n = a(a+1)...(a+n-1)$. Provided this is well-defined[2], it converges for all $|x| < 1$, and for $|x| = 1$ if $\mathrm{Re}(c-a-b) > 0$.

In general, there are two linearly independent solutions, but if we impose regularity at the boundary at $x = 1$ this restricts us to the solution:

$$h(x)={}_2F_1(a,b;a+b+1-c;1-x) \tag{3.87}$$

$$=\frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}\,{}_2F_1(a,b;c;x)+\frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}x^{1-c}\,{}_2F_1(a+1-c,b+1-c;2-c;x) \tag{3.88}$$

$$=C_1(a,b,c)\,{}_2F_1(a,b;c;x)+C_2(a,b,c)\,{}_2F_1(a+1-c,b+1-c;2-c;x) \tag{3.89}$$

we should also set $\tilde{n} = |n|$ to ensure $f(\eta)$ is regular.

Now we need to fix the behavior at the other endpoint. We have:

[2] Specifically, if c is a negative integer, we must have a or b to be a negative integer greater than or equal to c for this series to make sense.

$$f(\eta) \sim C_1(a,b,c)\eta^{\tilde{m}} + C_2(a,b,c)\eta^{-\tilde{m}} \tag{3.90}$$

Thus, depending on the behavior we want here, we should fix the relative values of the C_i which will impose a condition on ℓ and restrict us to a discrete spectrum. Actually, since we are free to choose the sign of $\tilde{m}$, we can simply pick $\tilde{m}$ so that the behavior we want is $\eta^{\tilde{m}}$, and then the condition is always $C_2 = 0$. This is ensured if the argument of one of the Gamma functions in the denominator in C_2 is set to a negative integer, which determines:

$$a, b = \frac{1}{2}(\tilde{m} + |n| + 1 \pm (\ell + 1)) = -k, \quad k \in \mathbb{Z}_{>0} \tag{3.91}$$

$$\Rightarrow \lambda_B = (\ell+1)^2 + (\sigma_o - \frac{i}{2})^2 = (\tilde{m} + |n| + 2k + 1)^2 + (\sigma_o - \frac{i}{2})^2 \tag{3.92}$$

$$\lambda_F = \pm(\ell+1) - i\sigma_o - \frac{1}{2} = \pm(\tilde{m} + |n| + 2k + 1) - i\sigma_o - \frac{1}{2} \tag{3.93}$$

Finally, we recall that the correct choice of $\tilde{m}$ was found above to be:

$$\tilde{m} = \begin{cases} -|m-q| & 0 \leq m < -q \\ |m-q| & \text{else} \end{cases} \tag{3.94}$$

Then the set of eigenvalues is given by taking the expressions above for all integers m, n and non-negative integers k.

Actually, as above, we need to be more careful with the fermions. Specifically, there will be eigenvalues in addition to these, as well as some that we have to throw out, corresponding to the cases where, respectively, the top and bottom components of the fermion vanish. These are precisely the exceptions noted in the previous section, from which the entire contribution to the partition function comes.

One interesting property of these eigenvalues is that, since $\tilde{m}$ may be negative, the eigenvalues may become zero, and the bosonic eigenvalues may even be negative. The first place these zero modes can occur is for $q = -\frac{1}{2}$, in which case for $m = k = \sigma_o = 0$, λ_B and λ_F^+ are both zero. Actually this fermionic eigenvalue is one of the spurious ones that we should throw out, and so in fact there is a single bosonic zero mode which results in a pole in the 1-loop partition function. For larger q, one finds negative bosonic eigenvalues, which become difficult to make sense of in the path integral. We will take the viewpoint that the vortex loop operator is only properly defined for $q > -\frac{1}{2}$ (and, when the matter content is in a self-conjugate representation of the relevant flavor symmetry, this also forces $q < \frac{1}{2}$), although the naive result for the determinant gives a natural analytic continuation of this quantity which may have some physical relevance.

4 Duality with Vortex Loop Operators

Let us now turn to some applications of the vortex loop operator. Since this operator is defined by an explicit procedure applied to a global $U(1)$ symmetry, then if we know what this symmetry maps to across a duality, we obtain an identification of loop operators on each side of the duality. This provides a new operator mapping across duality, although it does not provide any essentially new information beyond the mapping of global symmetries.

One inter?sting case is when a $U(1)_J$ symmetry on one side of the duality is identified with a flavor symmetry on the dual. This occurs, for example, in $3D$ mirror symmetry. Let us take the simplest case, the duality between $\mathcal{N} = 4$ SQED with one flavor and a free hypermultiplet. Note that there are Wilson loops on the SQED side, but it is less clear what the corresponding loop operators on the dual side are, since there is no gauge group.

However, the results above provide the answer: the Wilson loop on the SQED side are the same as defects in the global $U(1)_J$ symmetry. Mirror symmetry dictates that this symmetry is identified with a $U(1)_V$ flavor symmetry under which the two chirals in the hypermultiplet have charge ± 1. Then the Wilson loop in SQED simply maps to a defect operator in this flavor symmetry, supported on the same loop.

At the level of the matrix model, this follows from the identity:

$$\int d\lambda \frac{e^{2\pi i \eta \lambda} e^{2\pi q \lambda}}{2\cosh(\pi\lambda)} = \frac{1}{2\cosh \pi(\eta - iq)} \tag{4.1}$$

which simply follows from extending the usual self-Fourier-transform property of $1/\cosh$ to the entire complex plane.

It is worth noting that the integral on the LHS only converges for $|q| < \frac{1}{2}$. As remarked above, this is precisely the range in which the defect operator is also well-defined. The divergence on the LHS as $|q| \to \frac{1}{2}$ is reflected on the RHS as a bosonic zero mode developing in the defect background. The RHS gives a natural analytic continuation of this quantity to $|q| > \frac{1}{2}$, but it is not clear what, if any, physical relevance this has.

We can also consider mirror symmetry applied to SQED with N_f flavors. Here the dual is a quiver theory with gauge group $U(1)^{N_f}/U(1)_{diag} \cong U(1)^{N_f-1}$, with N_f bifundamental flavors $(q_a \tilde{q}_a)$ charged as $(1, -1, 0, ...), (0, 1, -1, ...), ..., (-1, 0, ..., 1)$, N_f neutral chirals S_a, and a superpotential $\sum_a q_a S_a \tilde{q}_a$. Here the $U(1)_J$ symmetry of $SQED$ maps to a flavor symmetry under which the all the q_a have charge $\frac{1}{N_f}$. In particular, the Wilson loop in SQED maps to a defect in this symmetry. Note that, because of the extra factor of $\frac{1}{N_f}$ in this mapping, the restriction on the defect charge now allows us to consider Wilson loops of charge up to $\pm\frac{N_f}{2}$. This coincides with the fact that, in the matrix model, the Wilson loop expectation value now converges for this wider range of charges because of the increased damping in the integrand from the factor $(2\cosh(\pi\lambda))^{-N_f}$.

Finally, we note that we can also apply this type of argument to theories with $U(N)$ gauge symmetries for $N > 1$, provided we restrict only to Wilson loops in the overall $U(1)$ of the gauge group. For example, in the $U(N)$ versions of Aharony ([19]) and Giveon-Kutasov dualities ([20]), it is known that the $U(1)_J$ current maps to itself, up to a flip of sign, and so a Wilson loop in the overall $U(1)$ of the gauge group must map to the same Wilson loop, with the opposite charge. One can also consider such abelian Wilson loops in non-abelian mirror symmetry, where they will map to flavor defects as above.

5 Discussion

We have defined a set of abelian vortex loop operators which exist in any abelian gauge theory in $2+1$ dimensions. The definition can be extended to conformal field theories with abelian global symmetries by weakly gauging the symmetry currents. Witten's $SL(2, \mathbb{Z})$ action for this class of theories naturally extends to loop operators. In fact, abelian vortex loops are the S-duals of the ordinary Wilson loops. Alternatively, a Wilson loop in a $U(1)$ factor of the gauge group can be viewed as a global vortex loop for the associated topological $U(1)_J$ symmetry.

One of our results was the definition of the supersymmetric version of the vortex loop. This was accomplished by identifying the fields in the abelian vector multiplet which needed to be turned on to create the right type of singularity. It turns out that, besides the singular gauge connection, we needed to also give an *imaginary* background value to the auxiliary scalar D. This is an interesting example of the fact that background fields need not satisfy the reality conditions usually imposed on the dynamical fields of the theory. We proceeded to evaluate the expectation value of a supersymmetric defect loop, defined on a great circle on S^3, using localization. The result could be anticipated by considering the $SL(2,\mathbb{Z})$ action and, indeed, had already been derived from the 4d perspective. We have given, by using and comparing two different regularization methods, an additional microscopic derivation.

The supersymmetric vortex loop plays a central role in mirror symmetry of 3d gauge theories. This class of dualities has the property that it exchanges flavor symmetries with the topological symmetries associated to the abelian factors of the dual gauge group. As a consequence, the duality exchanges (the supersymmetric versions of) $U(1)$ Wilson loops with abelian vortex loops. Identifying such entries in the duality dictionary is an important step towards, possibly, proving the duality for the full quantum theory. We have demonstrated that the expectation values for the dual loop operators match, in simple examples, by using localization and the matrix model.

The analysis presented here has a natural extension to non-abelian defects. The definition of such an operator can require additional steps to ensure gauge invariance. For a defect in a global non-abelian symmetry group the definition is similar to the abelian case and the results can be read off from Section 3 by conjugating the defect data into the Cartan of the flavor group. When the defect appears in a dynamical gauge field, the localization procedure for the vector multiplet is modified. The result in the case of pure Chern-Simons theory is known to require a quantization of the data entering the definition of the defect, the overall effect being, again, the insertion of a Wilson loop operator in some representation [7]. A naive analysis would imply that this result is not affected by the presence of additional charged matter. However, the mapping of such operators under mirror symmetry for non-abelian theories and under Seiberg-like dualities requires further investigation.

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