Majorana ghosts: From topological superconductor to the origin of neutrino masses, three generations and their mass mixing

Zheng-Cheng Gu[1,2]

[1]*Institute for Quantum Information and Matter, California Institute of Technology, Pasadena, CA 91125, USA*

[2]*Department of Physics, California Institute of Technology, Pasadena, CA 91125, USA*

Abstract

The existence of three generations of neutrinos and their mass mixing are the deep mysteries of our universe. The history of neutrino physics can be traced back to Majorana's elegant work on a real solution of the Dirac equation – known as the Majorana fermion. A cutting-edge step towards understanding the nature of neutrino has been taken by the experimental discovery of neutrino mass mixing during the past decade, which indicates neutrino has a small but non-vanishing mass. A natural way to explain the origin of this small mass is the so-called seesaw mechanism, which requires the neutrino to be a Majorana fermion. Recently, Majorana's spirit returns in modern condensed matter physics – in the context of Majorana zero modes in certain classes of topological superconductors(TSCs). In this paper, we attempt to investigate the topological nature of the neutrino by establishing a connection between the Majorana fermion and Majorana zero modes – assuming a relativistic Majorana fermion is made up of four Majorana zero modes. We begin with an exactly solvable 1D condensed matter model which realizes a $T^2 = -1$ time reversal symmetry protected TSC. We show that the pair of Majorana zero modes on each end will realize a $T^4 = -1$ representation of the time reversal symmetry and carry 1/4 spin. We find that a pair of Majorana zero modes can realize a $P^4 = -1$ parity symmetry as well and even a nontrivial $\overline{C}^4 = -1$ charge conjugation symmetry. The $\overline{C}PT$ symmetries for a Majorana fermion made up of four Majorana modes form a super algebra. We then generalize the $\overline{C}PT$ super algebra into quantum field theory and point out that the nontrivial charge conjugation symmetry can be promoted to a $\mathbb{Z}_2$ gauge symmetry, whose spontaneously breaking leads to the origin of the (right-handed) neutrino mass. The $\mathbb{Z}_2$ gauge symmetry indicates the existence of the fifth force in our universe, which is possible to be detected in future LHC experiment. Finally, we show that the origin of three generations of neutrinos can be naturally explained as three distinguishable ways to form a pair of complex fermions(a particle and an anti-particle) out of four Majorana zero modes, characterized by the $T^4 = -1$, $(TP)^4 = -1$ and $(T\overline{C})^4 = -1$ fractionalized symmetries that particles/anti-particles carry. Together with the $\mathbb{Z}_2$ gauge (minimal coupling) principle, we are able to determine the mass mixing matrix with no fitting parameter at leading order(in the absence of the CP violation and charged lepton contribution). We obtain $\theta_{12} = 31.7^\circ, \theta_{23} = 45^\circ$ and $\theta_{13} = 0^\circ$(known as the golden ratio pattern), which are intrinsically close to the current experimental results. We further predict an exact mass ratio for the three mass eigenstates with $m_1/m_3 = m_2/m_3 = 3/\sqrt{5}$.

I. INTRODUCTION

In a summer night, when looking at the starry sky and thinking about the origin of our beautiful universe, we may not even notice that we are surrounded by billions of neutrinos. The neutrino, first discovered in 1956[1] and named as the "ghost particle", has extremely weak interactions with other matters, and it is one of the big mysteries to us and has a deep relationship with the physics of early universe.

The theoretical perspective of neutrino physics can be traced back to Ettore Majorana's elegant work[2] on a real solution to the Dirac equation – known as the Majorana fermion. Unfortunately, for over a century, we have found that all the fundamental particles have their own anti-particles and therefore are described by Dirac fermions. However, the neutrino is still possible to be a Majorana fermion because it does not carry electric charge. In the Standard Model(SM), the neutrino is described by a left-handed chiral Weyl fermion with zero rest mass[3], but it is not clear whether the neutrino is a Dirac fermion or a Majorana fermion. The smoking gun experiment that might be able to distinguish these two cases is the so-called neutrinoless double-β decay, unfortunately, such experimental evidence is still missing so far[4–7].

A cutting-edge step towards understanding this big puzzle has been taken by the neutrino oscillation experiments during the past decade[8–18]. These experiments have confirmed that the neutrino has a nonzero mass, at energy scale of $0.1eV$. This big discovery starts to shake the foundation of modern particle physics, which is built on the well tested SM. So far, it is the first and the only new physics beyond the SM that has been observed experimentally.

The biggest challenges of the puzzles are: (1)Where does the neutrino mass come from? (2)Why there are three generations of neutrinos[19]? (3)Where do those mystery mixing angles come from? An elegant way to explain the origin of neutrino mass is to introduce a sterile right-handed neutrino that does not carry any electric-weak charge, and through the so called seesaw mechanism[20–22] – by introducing a heavy Majorana mass for the right-handed sterile neutrino, a small mass for the left-handed light neutrino can be induced. Apparently, the seesaw mechanism requires the neutrino to be a Majorana fermion, however the rest two puzzles have not been solved in a natural way so far.

On the other hand, after almost 80 years since Majorana's disappearance, his spirit returns in modern condensed matter physics [23] – in the context of Majorana zero modes in certain classes of topological superconductors(TSCs)[24, 25]. Searching for Majorana zero modes has become a fascinating subject both theoretically[26–30]and experimentally. Very recently, experimental evidences for the existence of Majorana zero modes in 1D have been observed in superconductor/semiconductor nanowire devices[31–33] based on an elegant theoretical proposal[34, 35]. Nevertheless, despite the similarity in mathematical structure, Majorana modes have nothing to do with the Majorana fermion in the SM from a traditional perspective.

In this paper, we attempt to investigate the topological nature of neutrinos by establishing a connection between a Majorana fermion and Majorana zero modes – assuming a relativistic Majorana fermion is made up of four Majorana zero modes. We begin with an exactly solvable 1D condensed matter model which realizes a $T^2 = -1$ time reversal symmetry protected TSC and show that the pair of Majorana zero modes on its ends realize a $T^4 = -1$ representation of time reversal symmetry and carry 1/4 spin. We then show that such kind of fractionalized representation for a pair of Majorana zero modes can be generalized into a $P^4 = -1$ parity symmetry and a $\overline{C}^4 = -1$ nontrivial charge conjugation symmetry as well. These fractionalized $\overline{C}PT$ symmetries allow us to define a $\overline{C}PT$ super algebra for a Majorana fermion made up of four Majorana modes. Furthermore, we find that the nontrivial charge conjugation symmetry $\overline{C}$ changes the sign of the mass term.(It is well known that the usual charge conjugation symmetry has a trivial action on a Majorana fermion.) Therefore, under the assumption that the nontrivial charge conjugation symmetry is indeed a $\mathbb{Z}_2$ guage symmetry, the origin of the (right-handed) neutrino mass can be explained by spontaneous gauge symmetry breaking through the Anderson-Higgs mechanism[36].

These new concepts can even explain the origin of three generations of neutrinos, as there are three inequivalent ways to form a pair of complex fermions(a particle and an anti-particle) out of four Majorana zero modes, characterized by the $T^4 = -1$, $(TP)^4 = -1$ and $(T\overline{C})^4 = -1$ fractionalized symmetries that the particles/anti-particles carry. Together with the $\mathbb{Z}_2$ gauge (minimal coupling) principle, we are able to derive the neutrino mass mixing matrix with *no* fitting parameters within leading order(LO) approximation(without CP violation and charged lepton contributions). The obtained mixing angles are consistent with the golden ratio(GR) pattern that has been proposed phemomelogically[37–40], which is intrinsically close to the current experimental observations. However, our mass mixing matrix has an enhanced symmetry compared to the standard GR pattern with a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$

Klein symmetry. We find that there are three $\mathbb{Z}_2$ symmetry generators U, S, R in our theory, satisfying an interesting algebra with $U^2 = S^2 = R^2 = 1$ and $US = SU, UR = RU, SR = -URS$. Within the LO approximation, the left-handed light neutrinos have an inverted hierarchy structure and satisfy a special mass ratio relation $m_1 = m_2 = \frac{3}{\sqrt{5}} m_3$. Based on the current experimental data for Δm_{23}^2, we obtain $m_1 = m_2 \simeq 0.075eV$ and $m_3 \simeq 0.054eV$. Since within LO approximation $\Delta m_{12}^2 = 0$ and $\theta_{13} = 0$, the experimentally observed small mass splitting Δm_{12}^2 and nonzero θ_{13} are purely contributed by the CP violation physics and we expect an interesting relation $|\Delta m_{12}/\Delta m_{23}| \sim \theta_{13}/\theta_{23}$. Our prediction of (approximated) neutrino masses is also consistent with the cosmological bound on neutrino masses, where $m_1 + m_2 + m_3 < 0.3eV$[41].

The paper is organized as follows: In section II, we begin with a 1D TSC protected by the $T^2 = -1$ symmetry and show why a pair of Majorana zero modes on each end must carry a $T^4 = -1$ symmetry. Then we discuss the concepts of 1/4 spin for a Majorana spinon and the vacuum polarization physics. In section III, we first discuss how to realize Majorana zero modes in higher dimensions, then we show that a relativistic dispersion and an $SU(2)$ spin can emerge at quantum criticality with proliferated Majorana zero modes. In section IV, we propose a $P^4 = -1$ parity symmetry, a $\overline{C}^4 = -1$ nontrivial charge conjugation symmetry for a pair of Majorana zero modes and show that the $\overline{C}PT$ symmetries for a Majorana fermion made up of four Majorana zero modes form a super algebra. In section V, we generalize the $\overline{C}PT$ super algebra into the relativistic quantum field theory and discuss the origin of (right-handed) neutrino mass. In section VI, we give a simple explanation of the origin of three generations of neutrinos and show how to use quantum field theory to describe the three generations of neutrinos. In section VII, we derive the neutrino mass mixing matrix with no fitting parameters. Then we analyze the symmetry of mass mixing matrix and discuss the CP violation physics. Finally, we summarize the new concepts proposed in this paper and discuss other possible new physics along this line of thinking.

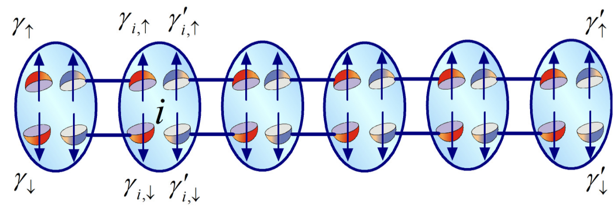

FIG. 1: (color online)A 1D topological superconductor protected by the $T^2 = -1$ time reversal symmetry can be constructed by two copies of Kitaev's Majorana chains with opposite spin species. We note that each physical site consists of four Majorana modes, or two Majorana spinons. The dangling Majorana spinons on both ends become zero modes protected by the time reversal symmetry.

II. THE $T^4 = -1$ TIME REVERSAL SYMMETRY FOR MAJORANA ZERO MODES

A. 1D Majorana chain with $T^2 = -1$ time reversal symmetry

To begin, we consider a 1D topological superconductor protected by the time reversal symmetry $T^2 = -1$, which realizes a special symmetry protected topological(SPT) phases[42] in 1D. Literally, such a 1D TSC has been originally proposed in a 1D free fermion system with a $T^2 = -1$ symmetry(the DIII class)[43, 44]. The simplest model that realizes such a 1D topological superconductor is just two copies of Kitaev's Majorana chains[24] with opposite spin species, as seen in Fig.1, described by the following Hamiltonian:

$$H = \sum_{i=1}^{N} \sum_{\sigma} i\sigma \gamma'_{i,\sigma} \gamma_{i+1,\sigma}, \tag{1}$$

The Majorana operators $\gamma_{i,\sigma}$ and $\gamma'_{i,\sigma}$ satisfy:

$$\{\gamma_{i,\sigma}, \gamma'_{i',\sigma'}\} = 0; \quad \{\gamma_{i,\sigma}, \gamma_{i',\sigma'}\} = 2\delta_{ii'}\delta_{\sigma\sigma'} \tag{2}$$

In terms of the complex fermion operators:

$$c_{i,\uparrow} = \frac{1}{2}(\gamma_{i,\uparrow} + i\gamma'_{i,\uparrow}); \quad c_{i,\downarrow} = \frac{1}{2}(\gamma_{i,\downarrow} - i\gamma'_{i,\downarrow}) \tag{3}$$

We can rewrite the above Hamiltonian as:

$$H = \sum_{i=1}^{N}\sum_{\sigma}\left(c_{i,\sigma} - c^\dagger_{i,\sigma}\right)\left(c_{i+1,\sigma} + c^\dagger_{i+1,\sigma}\right) \tag{4}$$

Under the time reversal symmetry, the bulk complex fermion operators transform as usual:

$$\begin{aligned} TiT^{-1} = -i; \quad Tc_{i,\uparrow}T^{-1} &= -c_{i,\downarrow}; \quad Tc_{i,\downarrow}T^{-1} = c_{i,\uparrow} \\ Tc^\dagger_{i,\uparrow}T^{-1} &= -c^\dagger_{i,\downarrow}; \quad Tc^\dagger_{i,\downarrow}T^{-1} = c^\dagger_{i,\uparrow}, \end{aligned} \tag{5}$$

According to Eq.(3), it is clear that Majorana spinons $(\gamma_{i,\uparrow}, \gamma_{i,\downarrow})$ and $(\gamma'_{i,\uparrow}, \gamma'_{i,\downarrow})$ on a single site should transform in the same way:

$$\begin{aligned} TiT^{-1} &= -i; \quad T\gamma_{i,\uparrow}T^{-1} = -\gamma_{i,\downarrow}; \quad T\gamma_{i,\downarrow}T^{-1} = \gamma_{i,\uparrow} \\ TiT^{-1} &= -i; \quad T\gamma'_{i,\uparrow}T^{-1} = -\gamma'_{i,\downarrow}; \quad T\gamma'_{i,\downarrow}T^{-1} = \gamma'_{i,\uparrow} \end{aligned} \tag{6}$$

Although the model Hamiltonian Eq.(1) is very simple, it describes a nontrivial time reversal symmetry protected TSC, characterized by topological zero modes and the symmetry fractionalization on its ends. On the other hand, Eq.(1) also describes a fixed point Hamiltonian with zero correlation length, therefore all its nontrivial topological properties could be applied to generic models describing the same SPT phase.

As seen in Fig. 1, a pair of dangling Majorana modes with opposite spins($\gamma_\uparrow \equiv \gamma_{1,\uparrow}, \gamma_\downarrow \equiv \gamma_{1,\downarrow}$ for left end and $\gamma'_\uparrow \equiv \gamma'_{N,\uparrow}, \gamma'_\downarrow \equiv \gamma'_{N,\downarrow}$ for right end) form a Majorana spinon on each end, and Eq.(6) implies that the fermion mass term $i\gamma_\uparrow\gamma_\downarrow(i\gamma'_\uparrow\gamma'_\downarrow)$ changes sign under the time reversal. Thus, the pair of Majorana modes are stable against T-preserving interactions and the Hamiltonian Eq.(1) describes a time reversal symmetry protected TSC. Recent progress on the classification of $1D$ SPT phases[45, 46] further pointed out that the edge Majorana modes indeed carry the $T^4 = -1$ projective representation of time reversal symmetry, rather than the usual $T^2 = -1$ representation. A simple reason why we need such a $T^4 = -1$ representation can be explained as following: If we assume a Majorana spinon carries the same $T^2 = -1$ representation as Kramers doublets, the total time reversal symmetry action on a single physical site will carry a $T^2 = 1$ representation as it contains two Majorana spinons. Therefore, a $T^2 = -1$ representation for the complex spinon on a single physical site prohibits the same $T^2 = -1$ representation for a Majorana spinon.

To understand the origin of the $T^4 = -1$ representation, we need to investigate the precise meaning of $T^2 = -1$ time reversal symmetry for interacting fermion systems. Indeed, the local Hilbert space on a single site for the above $T^2 = -1$ TSC is a Fock-space which involves both fermion parity odd states $c^\dagger_{i,\uparrow}|0\rangle, c^\dagger_{i,\downarrow}|0\rangle$ and parity even states $|0\rangle, c^\dagger_{i,\uparrow}c^\dagger_{i,\downarrow}|0\rangle$. It is clear that the fermion parity odd basis carries a projective representation of time reversal symmetry $T^2 = -1$ while the fermion parity even basis carries a linear representation $T^2 = 1$. As a result, the time reversal symmetry group for interacting fermion systems has been extended over the $\mathbb{Z}_2$ fermion parity symmetry group $\{I, P_f\}$, and the total symmetry group should consist of four group elements $\{I, T, T^2, T^3\}$ with $T^4 = 1$, which is a $\mathbb{Z}_4$ group. We note that the $\mathbb{Z}_2$ fermion parity symmetry can not be broken in *local* interacting fermion systems, hence such a group extension can not be avoided. Since 1D SPT phases are classified by the projective representation of the corresponding symmetry group[45, 46], the Majorana spinon $(\gamma_\uparrow, \gamma_\downarrow)$ and $(\gamma'_\uparrow, \gamma'_\downarrow)$ on both ends must carry the projective representation of the bulk $\mathbb{Z}_4$ antiunitary symmetry with $T^4 = 1$, which leads to the $T^4 = -1$ representation.

Possible experimental realization of such an interesting TSC has been proposed by several groups recently[47–50]. In the following, we show how to write down an explicit time reversal operator to realize the fractionalized $T^4 = -1$ symmetry for a Majorana spinon.

B. $T^4 = -1$ time reversal symmetry

For the pair of Majorana zero modes $\gamma_\uparrow$ and $\gamma_\downarrow$ on the left end, let us define the anti-unitary operator T by $T = UK$, where U is a unitary operator:

$$U = \frac{1}{\sqrt{2}}(1 + \gamma_\uparrow\gamma_\downarrow) = e^{\frac{\pi}{4}\gamma_\uparrow\gamma_\downarrow} \tag{7}$$

Since $(\gamma_\uparrow\gamma_\downarrow)^\dagger = \gamma_\downarrow\gamma_\uparrow = -\gamma_\uparrow\gamma_\downarrow$, we have:

$$U^\dagger = \frac{1}{\sqrt{2}}(1 - \gamma_\uparrow\gamma_\downarrow) = e^{-\frac{\pi}{4}\gamma_\uparrow\gamma_\downarrow} \tag{8}$$

It is straightforward to verify that U is a unitary operator:

$$UU^\dagger = \frac{1}{2}(1 + \gamma_\uparrow\gamma_\downarrow)(1 - \gamma_\uparrow\gamma_\downarrow) = 1 \tag{9}$$

Furthermore, this new definition of time reversal operator gives rise to the correct transformation law for $\gamma_\uparrow$ and $\gamma_\downarrow$:

$$\begin{aligned} T\gamma_\uparrow T^{-1} &= \frac{1}{2}(1+\gamma_\uparrow\gamma_\downarrow)\gamma_\uparrow(1-\gamma_\uparrow\gamma_\downarrow) = -\gamma_\downarrow \\ T\gamma_\downarrow T^{-1} &= \frac{1}{2}(1+\gamma_\uparrow\gamma_\downarrow)\gamma_\downarrow(1-\gamma_\uparrow\gamma_\downarrow) = \gamma_\uparrow, \end{aligned} \tag{10}$$

However, we notice that $T^2 = \gamma_\uparrow\gamma_\downarrow \neq -1$ and satisfies:

$$T^4 = (\gamma_\uparrow\gamma_\downarrow)^2 = -1 \tag{11}$$

We call the two Majorana modes that carry the above $T^4 = -1$ representation as Majorana doublets, which can be viewed as a square rooted representation of the usual Kramers doublets. With such a definition of time reversal symmetry operator for a pair of Majorana modes, the symmetry protected nature becomes manifested, since a $T^4 = -1$ projective representation can not be destroyed by time reversal preserving *local* interactions.

Similarly, for the pair of Majorana zero modes $\gamma'_\uparrow, \gamma'_\downarrow$ on the right end, T can be defined by $T = U'K$ with:

$$U' = \frac{1}{\sqrt{2}}(1+\gamma'_\uparrow\gamma'_\downarrow) = e^{\frac{\pi}{4}\gamma'_\uparrow\gamma'_\downarrow} \tag{12}$$

The above definition of $T^4 = -1$ time reversal operators on both ends can be applied to any physical site i which contains two Majorana spinons $(\gamma_{i,\uparrow}, \gamma_{i,\downarrow})$ and $(\gamma'_{i,\uparrow}, \gamma'_{i,\downarrow})$. The total time reversal action is defined by $T = U_i \otimes U'_i K$ with $U_i = e^{\frac{\pi}{4}\gamma_{i,\uparrow}\gamma_{i,\downarrow}}$ and $U'_i = e^{\frac{\pi}{4}\gamma'_{i,\uparrow}\gamma'_{i\downarrow}}$. We have:

$$T^2 = \gamma_{i,\uparrow}\gamma_{i,\downarrow}\gamma'_{i,\uparrow}\gamma'_{i,\downarrow} = P^f_i = P^f_{i,L}P^f_{i,R} \tag{13}$$

with

$$P^f_{i,L} = -i\gamma_{i,\uparrow}\gamma_{i,\downarrow}; \quad P^f_{i,R} = i\gamma'_{i,\uparrow}\gamma'_{i,\downarrow}, \tag{14}$$

Here P^f is the total fermion parity for a single physical site and $P^f_L(P^f_R)$ is fermion parity operators for the left(right) pair of Majorana spinon. The above definition of time reversal symmetry operator satisfies the requirement of $T^2 = -1$ for fermion parity odd states while it satisfies $T^2 = 1$ for fermion parity even states.

C. Representation theory of the $T^4 = -1$ time reversal symmetry

In the above, we use an algebraic way to construct the $T^4 = -1$ symmetry, which will be very helpful for us to understand the underlying physics and provides us a simple way to do calculations. Now let us work out the explicit representation theory for the $T^4 = -1$ time reversal symmetry. We note that the two pairs of Majoran spinons on both ends allow us to define two complex fermions c_L and c_R:

$$c_L = \frac{1}{2}(\gamma_\uparrow + i\gamma_\downarrow); \quad c_R = \frac{1}{2}(\gamma'_\uparrow - i\gamma'_\downarrow) \tag{15}$$

where $c_{L(R)}$ transforms nontrivially under the $T^4 = -1$ symmetry. We have:

$$\begin{aligned} Tc_L T^{-1} &= -ic_L^\dagger; \quad Tc_R T^{-1} = ic_R^\dagger \\ Tc_L^\dagger T^{-1} &= ic_L; \quad Tc_R^\dagger T^{-1} = -ic_R \end{aligned} \tag{16}$$

Since the T operator only involves two Majorana operators, we are able to construct a precise two dimensional representation theory for the $T^4 = -1$ symmetry. On the other hand, a projective representation can not be one dimensional, hence we must have:

$$T|\widetilde{0}\rangle = UK|\widetilde{0}\rangle = U|\widetilde{0}\rangle = |\widetilde{1}\rangle \equiv c_{L(R)}^\dagger|\widetilde{0}\rangle \tag{17}$$

where $|\widetilde{0}\rangle$ is the vacuum of $c_{L(R)}$ fermion satisfying $c_{L(R)}|\widetilde{0}\rangle = 0$ and $|\widetilde{1}\rangle \equiv c_{L(R)}^\dagger|\widetilde{0}\rangle$. We also assume that the global phase of $|\widetilde{0}\rangle$ is fixed in such a way that the complex conjugate K has a trivial action on it. From the relation Eq.(16), it is straightforward to derive:

$$\begin{aligned} T|\widetilde{1}\rangle &= UKc_{L(R)}^\dagger|\widetilde{0}\rangle = Uc_{L(R)}^\dagger|\widetilde{0}\rangle \\ &= Tc_{L(R)}^\dagger T^{-1}T|\widetilde{0}\rangle = \pm ic_{L(R)}c_{L(R)}^\dagger|\widetilde{0}\rangle = \pm i|\widetilde{0}\rangle \end{aligned} \tag{18}$$

Here the $+$ sign corresponds to c_L and the $-$ sign corresponds to c_R. Thus, in the basis $|\widetilde{0}\rangle$ and $|\widetilde{1}\rangle$, we can derive the representation theory $T = UK$ with:

$$U = \begin{pmatrix} 0 & 1 \\ \pm i & 0 \end{pmatrix}, \tag{19}$$

Clearly, the above representation satisfies $T^4 = -1$.

D. 1/4 spin and vacuum polarization

Although the $SU(2)$ spin rotational symmetry is broken in Hamiltonian Eq.(1), it still has a residual $U(1)$ spin rotational symmetry along the y-axis:

$$[H, S^y_{\text{total}}], \quad S^y_{\text{total}} = \sum_i S^y_i \equiv \frac{i}{2}\sum_i (c^\dagger_{i,\uparrow} c_{i,\downarrow} - c^\dagger_{i,\downarrow} c_{i,\uparrow}) \tag{20}$$

Hence, all the eigenstates of the Hamiltonian Eq.(1) are labeled by $S^y_{\text{total}} = \frac{m^y}{2}, m^y = 0, \pm 1, \pm 2, \ldots$ which are consistent with a spin 1/2 system. Interestingly, the dangling Majorana spinons on both ends actually carry a 1/4 spin instead of the usual 1/2 spin.

For a pair of Majorana modes $\gamma_{i,\uparrow}$ and $\gamma_{i,\downarrow}$ on site i, the corresponding spin operator can be defined by(Since Majorana spinon is a real spinon, we can only define an $SO(2)$ spin instead of the $SU(2)$ spin for a complex fermion.):

$$\begin{aligned} S_{i,L} &= \frac{iS}{2}\sum_{\sigma\sigma'} \gamma_{i,\sigma}\epsilon_{\sigma\sigma'}\gamma_{i,\sigma'} \\ &= \frac{S}{2}\sum_{\sigma\sigma'} \gamma_{i,\sigma}\sigma^y_{\sigma\sigma'}\gamma_{i,\sigma'} = \frac{iS}{2}(\gamma_{i,\uparrow}\gamma_{i,\downarrow} - \gamma_{i,\downarrow}\gamma_{i,\uparrow}) = iS\gamma_{i,\uparrow}\gamma_{i,\downarrow} = -SP^f_{i,L} \end{aligned} \tag{21}$$

It is easy to check that:

$$TS_{i,L}T^{-1} = -S_{i,L} \tag{22}$$

We see that the definition of Majorana spin operator Eq.(21) has the correct transformation law under the time reversal symmetry. Similarly, we can define $S_{i,R}$ as $S_{i,R} = -SP^f_{i,R}$. Since the fermion parity operator $P^f_{i,L(R)}$ takes eigenvalues ± 1, S represents the spin carried by a Majorana spinon. In the following, we will show $S = 1/4$ rather than 1/2.

On each physical site i, we have:

$$\begin{aligned} S^y_i &= \frac{i}{2}(c^\dagger_{i,\uparrow} c_{i,\downarrow} - c^\dagger_{i,\downarrow} c_{i,\uparrow}) \\ &= \frac{i}{8}\left[(\gamma_{i,\uparrow} - i\gamma'_{i,\uparrow})(\gamma_{i,\downarrow} - i\gamma'_{i,\downarrow}) - (\gamma_{i,\downarrow} + i\gamma'_{i,\downarrow})(\gamma_{i,\uparrow} + i\gamma'_{i,\uparrow})\right] = -\frac{1}{4}(P^f_{i,L} + P^f_{i,R}) \end{aligned} \tag{23}$$

The above expression implies that the two fermion occupied state $c^\dagger_{i,L}c^\dagger_{i,R}|0\rangle$ has spin polarization 1/2 in the y direction while the two fermion vacuum $|0\rangle$(defined by $c_{i,L}c_{i,R}|0\rangle = 0$) has spin polarization $-1/2$ in the y direction. Therefore, the complex fermion $c_{i,L}$ formed by the Majorana spinon $(\gamma_{i,\uparrow}, \gamma_{i,\downarrow})$ and the complex fermion $c_{i,R}$ formed by $(\gamma'_{i,\uparrow}, \gamma'_{i,\downarrow})$ effectively carry a 1/4 spin. More precisely, we can express S^y_i as a summation of two $SO(2)$ spins:

$$S^y_i = S_{i,L} + S_{i,R} = -S(P^f_{i,L} + P^f_{i,R}) \tag{24}$$

Comparing Eq.(23) and Eq.(24), we obtain $S = 1/4$. For the $T^2 = -1$ TSC described by Eq. (1), the 1/4 spin can be directly measured for the dangling Majorana spinon on both ends. Let us compute the expectation value of S^y for the left end:

$$\langle G|S_1^y|G\rangle = \langle G|S_{1,L}|G\rangle + \langle G|S_{1,R}|G\rangle = \langle G|S_{1,L}|G\rangle \equiv \langle G|S_L|G\rangle = \pm\frac{1}{4}, \quad (25)$$

where S_L is the spin operator for Majorana zero modes on left end, defined by $S_L = -SP_L^f = iS\gamma_\uparrow\gamma_\downarrow$. We note that $|G\rangle$ is one of the four possible ground states of Hamiltonian Eq.(1), and for any ground state, the contribution from the second term $\langle G|S_{1,R}|G\rangle$ vanishes. Similar calculation leads to the same results for the right end. Actually, the 1/4 spin physics for Majorana zero modes is similar to the presence of half-charge zero energy solutions in Jackiw and Rebbi's soliton-monopole systems[51], which is a consequence of vacuum polarization.

Although each end of a $T^2 = -1$ TSC carries 1/4 spin, the whole system still carries spin 1/2. We denote the four fold degenerate ground states as: $|\widetilde{00}\rangle, |\widetilde{11}\rangle \equiv c_L^\dagger c_R^\dagger|\widetilde{00}\rangle, |\widetilde{10}\rangle \equiv c_L^\dagger|\widetilde{00}\rangle$ and $|\widetilde{01}\rangle \equiv c_R^\dagger|\widetilde{00}\rangle$, where $|\widetilde{00}\rangle$ is the vacuum of c_L and c_R, defined by $c_Lc_R|\widetilde{00}\rangle = 0$. It is straight forward to derive $\langle\widetilde{00}|S^y_{\text{tot}}|\widetilde{00}\rangle = -1/2, \langle\widetilde{11}|S^y_{\text{tot}}|\widetilde{11}\rangle = 1/2, \langle\widetilde{10}|S^y_{\text{tot}}|\widetilde{10}\rangle = 0$ and $\langle\widetilde{01}|S^y_{\text{tot}}|\widetilde{01}\rangle = 0$.

The vacuum polarization also leads to an interesting property of the time reversal symmetry in the ground state subspace: $|\widetilde{00}\rangle$ and $|\widetilde{11}\rangle$ form a $T^2 = -1$ Kramers doublet while $|\widetilde{01}\rangle$ and $|\widetilde{10}\rangle$ form a $T^2 = 1$ representation. We note that Eq. (26) and Eq. (18) imply:

$$T|\widetilde{00}\rangle = |\widetilde{11}\rangle; \quad T|\widetilde{11}\rangle = Tc_L^\dagger T^{-1}Tc_R^\dagger T^{-1}T|\widetilde{00}\rangle = c_Lc_R|\widetilde{11}\rangle = c_Lc_Rc_L^\dagger c_R^\dagger|\widetilde{00}\rangle = -|\widetilde{00}\rangle \quad (26)$$

Thus, in the basis $|\widetilde{00}\rangle$ and $|\widetilde{11}\rangle$, we can derive the representation $T = UK$ with:

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (27)$$

which implies $|\widetilde{00}\rangle$ and $|\widetilde{11}\rangle$ form a $T^2 = -1$ Kramers doublet and is consistent with a 1/2 spinon. Similarly, from Eq.(26) and Eq.(18), we obtain:

$$T|\widetilde{10}\rangle = i|\widetilde{01}\rangle; \quad T|\widetilde{01}\rangle = i|\widetilde{10}\rangle \quad (28)$$

Thus, in the basis $|\widetilde{10}\rangle$ and $|\widetilde{01}\rangle$, we can derive the representation $T = UK$ with:

$$U = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad (29)$$

We see that $T^2 = 1$ in the basis $|\widetilde{10}\rangle$ and $|\widetilde{01}\rangle$, which is also consistent with the fact that $c_{L(R)}$ carries the $SO(2)$ spin instead of $SU(2)$ spin.

III. MAJORANA ZERO MODES IN HIGHER DIMENSIONS AND EMERGENT RELATIVISTIC DISPERSION, $SU(2)$ SPIN AT QUANTUM CRITICALITY

A. Majorana zero modes in higher dimensions

In the previous section, we discuss a simple example of a 1D $T^2 = -1$ TSC with Majorana zero modes on its ends. Indeed, Majoranoa zero modes exist in DIII class TSC in higher dimensions as well. In 2D, it is well known that a single Majorana zero mode can emerge in the vortex core of a $p + ip$ or $p - ip$ TSC[25], however, the time reversal symmetry is broken in this class of *chiral* TSC. Nevertheless, the DIII class TSC in 2D that is realized as a composition of a $p+ip$ and a $p-ip$ TSC with opposite spins[52] can preserve the $T^2 = -1$ time reversal symmetry. Apparently, the vortex core of such a TSC has a pair of Majorana zero modes $\gamma_\uparrow$ and $\gamma_\downarrow$ with opposite spins. In the following, we argue that they also carry a $T^4 = -1$ representation of time reversal symmetry. As having been discussed in Ref. [52], a time reversal action on a single vortex core will change the *local* fermion parity of the complex fermion zero mode $c_L = \gamma_\uparrow + i\gamma_\downarrow$ for the ground state wavefunction, therefore we expect the same representation theory Eq.(26),Eq.(18) and Eq.(19) for the zero modes inside the vortex core, which satisfies $T^4 = -1$. For the anti-vortex core with Majorana modes $\gamma'_\uparrow$ and $\gamma'_\downarrow$, we can define a complex fermion mode $c_R = \gamma_\uparrow - i\gamma_\downarrow$ and derive the $T^4 = -1$ representation theory as well. Now we see that the c_L/c_R complex fermion is similar to the two complex fermion modes defined on the left/right end of the 1D $T^2 = -1$ TSC. The $T^4 = -1$ time reversal operators for the Majorana spinons $(\gamma_\uparrow, \gamma_\downarrow)$ and $(\gamma'_\uparrow, \gamma'_\downarrow)$ can be defined by Eq.(7) and Eq.(12).

The 3D analogy of the vortex would be a hedgehog and the possibility of the emergence of a Majorana zero mode on the hedgehog has been proposed recently[30]. However, there is an important difference in 3D. Since the classical configuration of a hedgehog will have a divergent energy, the only way to introduce a UV cutoff is to couple the system to a gauge field, e.g., an $SU(2)$ gauge field[53]. By turning on the $SU(2)$ gauge field, a single Majorana mode will suffer from the Witten anormally[54] and the only way to cancel this anormally is to introduce a pair of Majorana zero modes. Therefore, the Majorana zero modes are unstable in the absence of time reversal symmetry(a mass term can be dynamically generated) and the analogy of $p + ip$ TSC does not exist in 3D. However, in the presence

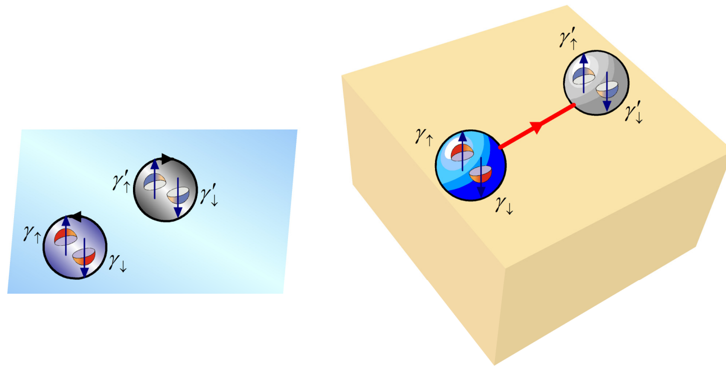

FIG. 2: (color online)Majorana zero modes in 2D and 3D can be realized as the bound states on the vortex/anti-vortex core and hedgehog/anti-hedgehog core of DIII class TSC. The red line in (b) represents a quantized flux line that connects a pair of hedgehog and anti-hedgehog.

of $T^2 = -1$ time reversal symmetry, the pair of Majorana zero modes $\gamma_\uparrow$ and $\gamma_\downarrow$ on the hedgehog can be stabilized(similar to the 1D and 2D case, the mass term is forbidden by the time reversal symmetry) and we argue that they also carry a $T^4 = -1$ time reversal symmetry according to the same reason as in 2D – the time reversal action changes the local fermion parity of the complex fermion mode $c_L = \gamma_\uparrow + i\gamma_\downarrow$ for the ground state wavefunction. The DIII class TSC in 3D labeled by odd integers(there is a $\mathbb{Z}$ classification[43, 44] for free fermion system in this case) could be good candidates to realize a pair of Majorana zero modes on its hedgehog/anti-hedgehog. Detailed discussions of these interesting 3D models are beyond the scope of this paper and will be presented elsewhere. Finally, we point out an important difference for the Majorana zero modes between 1D and higher dimensions. In 1D, for a generic Hamiltonian, the zero modes are only well defined in the infinite long chain limit. However, in 2(3)D, the distance between vortex(hedgehog) and anti-vortex(anti-hedgehog) can be finite(but much larger than penetration depth) since the zero modes are well defined bound states and they can be regarded as *local* particles.

B. Deconfined Majorana zero modes in 1D

In the following we will show that relativistic dispersion and $SU(2)$ spin rotational symmetry will emerge at a quantum critical point where Majorana zero modes are proliferated. Let us begin with a 1D model by adding a chemical potential term into the Hamiltonian Eq.(1):

$$H' = \sum_{i=1}^{N}\sum_{\sigma}\left(c_{i,\sigma} - c_{i,\sigma}^\dagger\right)\left(c_{i+1,\sigma} + c_{i+1,\sigma}^\dagger\right) - 2\mu\sum_{i=1}^{N}\sum_{\sigma}(c_{i,\sigma}^\dagger c_{i,\sigma} - \frac{1}{2}), \tag{30}$$

As having been discussed in Ref. [24], a phase transition occurs at $\mu = 1$ and the system becomes a trivial superconductor when $\mu > 1$. In terms of Majorana operators, we will have a simple picture to visualize the above phase transition.

$$H' = \sum_{i=1}^{N}\sum_{\sigma} i\sigma\gamma'_{i,\sigma}\gamma_{i+1,\sigma} + \mu\sum_{i=1}^{N}\sum_{\sigma} i\sigma\gamma'_{i,\sigma}\gamma_{i,\sigma}, \tag{31}$$

As seen in Fig. 3, in the limit where the on site hopping $t_2 \equiv \mu$ is dominant, the above Hamiltonian describes a trivial superconductor, while in the limit where the inter site hopping $t_1(=1)$ is dominant, it describes a topological superconductor and Majorana modes are confined on both ends. At the phase transition point $t_2 = t_1 = 1$, the Majorana spinon with 1/4 spin and $T^4 = -1$ time reversal symmetry becomes deconfined. The analogy of such a deconfined quantum critical phenomenon has been known for long time in 1D spin chain models with $T^2 = 1$ time reversal symmetry, e.g., in certain spin-1 chain systems[55], 1/2 spinon with $T^2 = -1$ time reversal symmetry on its ends become deconfined at the phase transition point.

At low energy, the critical theory has emergent relativistic dispersion and $SU(2)$ (pseudo) spin. In terms of $c_{L(R)}$ fermions, we can rewrite the critical Hamiltonian as:

$$H_{1D} = i\sum_{i}(c_{i,L}^\dagger c_{i+1,R} - c_{i+1,R}^\dagger c_{i,L}) + i\sum_{i}(c_{i,L}^\dagger c_{i,R} - c_{i,R}^\dagger c_{i,L}) \tag{32}$$

In momentum space, the above Hamiltonian can be diagonalized by:

$$H_{1D} = \sum_{k}(c_L^\dagger(k), c_R^\dagger(k))\begin{pmatrix} 0 & i(1+e^{ik}) \\ -i(1+e^{-ik}) & 0 \end{pmatrix}\begin{pmatrix} c_L(k) \\ c_R(k) \end{pmatrix} \tag{33}$$

It has one positive energy mode and one negative energy mode with $E_k = \pm 2t|\cos\frac{k}{2}|$. The dispersion relation is relativistic around the momentum points $k = \pm\pi$. The particle and

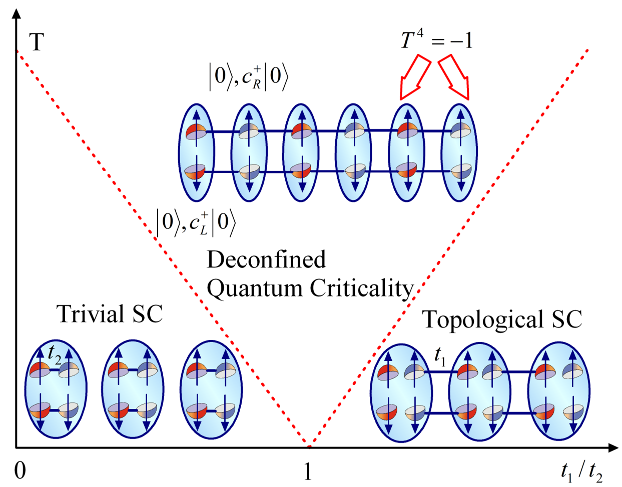

FIG. 3: (color online) At the deconfined quantum critical point, the proliferating of Majorana zero modes lead to the emergence of relativistic dispersion and $SU(2)$ (pseudo) spin rotational symmetry.

hole excitations in such a systems form an $SU(2)$ doublet. However, for a 1D chain, the $SU(2)$ (pseudo) spin rotational symmetry does not carry angular momentum and is a purely internal symmetry.

C. Deconfined Majorana zero modes in 3D and emergent relativistic dispersion, $SU(2)$ spin

Now, let us construct a quantum critical model in 3D. First, we construct a 3D cubic lattice model consisting of hedgehog/anti-hedgehog, with hedgehog occupied sublattice A and anti-hedgehog occupied sublattice B, as seen in Fig 4. We use red dots to represent the pair of Majorana modes $(\gamma_{\uparrow}, \gamma_{\downarrow})$ on the hedgehog and blue dots to represent the pair of Majorana modes $(\gamma'_{\uparrow}, \gamma'_{\downarrow})$ on the anti-hedgehog. Similar to the 1D cases, we then turn on the hoping among those Majorana modes and consider the following Hamiltonian:

$$\begin{aligned} H_{3D} &= -\sum_{\mathbf{i}\in A;\mathbf{j}=\mathbf{i}\pm\hat{\mathbf{x}}} (i\gamma_{\mathbf{i},\uparrow}\gamma'_{\mathbf{j},\downarrow} + i\gamma_{\mathbf{i},\downarrow}\gamma'_{\mathbf{j},\uparrow}) + \sum_{\mathbf{i}\in A;\mathbf{j}=\mathbf{i}\pm\hat{\mathbf{y}}} (i\gamma_{\mathbf{i},\uparrow}\gamma'_{\mathbf{j},\uparrow} - i\gamma_{\mathbf{i},\downarrow}\gamma'_{\mathbf{j},\downarrow}) \\ &\quad + \sum_{\mathbf{i}\in A;\mathbf{j}=\mathbf{i}+\hat{\mathbf{z}}} (i\gamma_{\mathbf{i},\uparrow}\gamma_{\mathbf{j},\downarrow} - i\gamma_{\mathbf{i},\downarrow}\gamma_{\mathbf{j},\uparrow}) + \sum_{\mathbf{i}\in B;\mathbf{j}=\mathbf{i}+\hat{\mathbf{z}}} (i\gamma'_{\mathbf{i},\uparrow}\gamma'_{\mathbf{j},\downarrow} - i\gamma'_{\mathbf{i},\downarrow}\gamma'_{\mathbf{j},\uparrow}) \end{aligned} \tag{34}$$

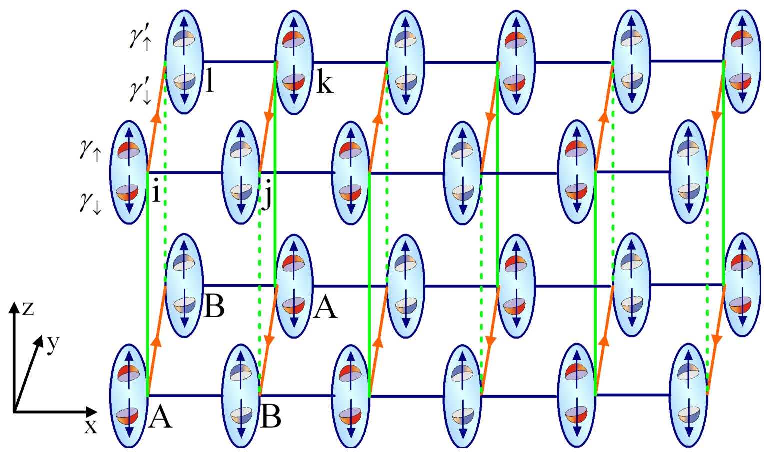

FIG. 4: (color online)A 3D hedgehog/anti-hedgehog cubic lattice. Red dots represent the pair of Majorana modes $(\gamma_\uparrow, \gamma_\downarrow)$ on the hedgehog and blue dots represent the pair of Majorana modes $(\gamma'_\uparrow, \gamma'_\downarrow)$ on the anti-hedgehog. Solid/dashed lines represent the hopping amplitude $1/-1$. Lines with arrows represent the hopping amplitudes $\pm i$. Multiplications of the hopping amplitudes surround a square surface give rise to -1, e.g., $t_{ij}t_{jk}t_{kl}t_{li} = -1$. Such a hopping amplitudes pattern is the so called π-flux pattern.

In terms of complex fermions $c_{\mathbf{i},L} = \gamma_{\mathbf{i},\uparrow} + i\gamma_{\mathbf{i},\downarrow}$ and $c_{\mathbf{i},R} = \gamma'_{\mathbf{i},\uparrow} - i\gamma'_{\mathbf{i}\downarrow}$, we have:

$$\begin{aligned} H_{3D} &= \sum_{\mathbf{i}\in A;\mathbf{j}=\mathbf{i}\pm\hat{\mathbf{x}}} \left(c^\dagger_{L,\mathbf{i}}c_{R,\mathbf{j}} + c^\dagger_{R,\mathbf{j}}c_{L,\mathbf{i}}\right) + i\sum_{\mathbf{i}\in A;\mathbf{j}=\mathbf{i}\pm\hat{\mathbf{y}}} \left(c^\dagger_{L,\mathbf{i}}c_{R,\mathbf{j}} - c^\dagger_{R,\mathbf{j}}c_{L,\mathbf{i}}\right) \\ &+ \sum_{\mathbf{i}\in A;\mathbf{j}=\mathbf{i}+\hat{\mathbf{z}}} \left(c^\dagger_{L,\mathbf{i}}c_{L,\mathbf{j}} + c^\dagger_{L,\mathbf{j}}c_{L,\mathbf{i}}\right) - \sum_{\mathbf{i}\in B;\mathbf{j}=\mathbf{i}+\hat{\mathbf{z}}} \left(c^\dagger_{R,\mathbf{i}}c_{R,\mathbf{j}} + c^\dagger_{R,\mathbf{j}}c_{R,\mathbf{i}}\right) \end{aligned} \tag{35}$$

The special hopping pattern in the above Hamiltonian is one way to realize the so called π-flux pattern, namely, a pattern with the enclosed flux π on each face of the cubic lattice. The Hamiltonian is invariant under the time reversal symmetry $\widetilde{T} = T^{(-)^{i_z}}$. Without such a twisted definition of the time reversal symmetry, the fermion hopping in the z-direction will change sign under the time reversal. It is clear that such a twisted definition is allowed because we can choose either T or T^{-1} as the definition of the time reversal symmetry.

In momentum space, we have:

$$H_{3D} = \sum_{\mathbf{k}} (c^\dagger_L(\mathbf{k}), c^\dagger_R(\mathbf{k})) \begin{pmatrix} 2\cos k_z & 2\cos k_x + 2i\cos k_y \\ 2\cos k_x - 2i\cos k_y & -2\cos k_z \end{pmatrix} \begin{pmatrix} c_L(\mathbf{k}) \\ c_R(\mathbf{k}) \end{pmatrix} \tag{36}$$

The above Hamiltonian has one positive energy mode and one negative energy mode with:

$$E_{\mathbf{k}} = \pm 2\sqrt{\cos^2 k_x + \cos^2 k_y + \cos^2 k_z}, \tag{37}$$

Around the momentum point $\mathbf{k}_0 = (\pi/2, \pi/2, \pi/2)$, the above Hamiltonian describes a chiral Weyl fermion:

$$H^{eff}_{(\pi/2,\pi/2,\pi/2)} = 2\sum_{\mathbf{k}} (c_L^\dagger(\mathbf{k}), c_R^\dagger(\mathbf{k})) \begin{pmatrix} \bar{k}_z & \bar{k}_x + i\bar{k}_y \\ \bar{k}_x - i\bar{k}_y & -\bar{k}_z \end{pmatrix} \begin{pmatrix} c_L(\mathbf{k}) \\ c_R(\mathbf{k}) \end{pmatrix} \tag{38}$$

where $\mathbf{k} = \mathbf{k}_0 + \bar{\mathbf{k}}$. It is clear that the above Hamiltonian has a relativistic dispersion $E_{\mathbf{k}} = \pm 2|\bar{\mathbf{k}}|$ and an emergent $SU(2)$ spin carrying angular momentum.

In the above, we construct a particular 3D hedgehog/anti-hedgehog lattice model with proliferated Majorana zero modes. Other models with deconfined Majorana modes have also been considered recently, e.g., the fermion dimer model[56] and the Majorana flat bands model in certain gapless TSC[57]. However, one of the most important features in our model is that it has a sublattice structure, and the sublattice degeneracy naturally leads to an $SU(2)$ spin degree of freedom at low energy. Actually, our model can be viewed as the 3D analogy of the 2D graphene system, where the valley degeneracy becomes the emergent $SU(2)$ spin at low energy. But why hedgehog/anti-hedgehog lattice models with a sublattice structure is more natural than those models without sublattice structure? One possible reason is that the hedgehog and anti-hedgehog pair are always confined in a superconductor[58], therefore any stable 3D hedgehog/anti-hedgehog lattice model must contain a hedgehog and anti-hedgehog pair per unit cell.

Our analysis for condensed matter systems implies that the presence of $SU(2)$ spin at low energy has a deep relationship with the sublattice structure at cutoff scale. A very interesting question is that whether the $SU(2)$ spin for all the fundamental particles arises from a similar discrete structure at cutoff scale. Unfortunately, it is very difficult to examine the above idea theoretically since a quantum field theory with an explicit cutoff is absent so far. Although lattice models could be thought as a natural venue to regulate the theory, any pre-assuming lattice structure for space-time will break the Lorentz invariance seriously. To overcome this difficulty, a discrete topological non-linear sigma model with a dynamic background is a promising candidate. Important progress along this direction has been made recently[59, 60], even with super-coordinates[61]. It would be very interesting to examine these ideas in future.

IV. $P^4 = -1$ PARITY SYMMETRY, $\overline{C}^4 = -1$ CHARGE CONJUGATION SYMMETRY AND SUPER $\overline{C}PT$ ALGEBRA FOR MAJORANA FERMION

So far, we have constructed concrete condensed matter models with emergent Majorana zero modes carrying $T^4 = -1$ time reversal symmetry on point like defects in certain classes of TSC. Furthermore, we have also shown that the proliferating of Majorana zero modes will lead to a chiral Weyl fermion with emergent relativistic dispersion and $SU(2)$ spin at low energy. Since neutrinos are described by the chiral Weyl fermion, it is very natural to ask if they can be interpreted as the (proliferated) Majorana zero modes. However, such a conjecture could be very challenging as it requires a strongly correlated vacuum instead of the trivial vacuum which we have assumed for the traditional quantum field theory. Nevertheless, in the semiclassical limit, it is still possible to investigate other fractionalized (discrete) symmetries carried by Majorana zero modes and to discuss the interesting physical consequence. In this section, we limit our discussion at the single particle level, and the generalization into the quantum field theory will be presented in the next section.

As having been discussed in last the section, the confinement of hedgehog and anti-hedgehog pair in 3D superconductor suggests that the four Majorana zero modes $\gamma_\uparrow, \gamma'_\uparrow, \gamma_\downarrow$ and $\gamma'_\downarrow$ identify the local degrees of freedom with respect to translational symmetry.(For a lattice model, that are the degrees of freedom in a unit cell.) On the other hand, a relativistic Majorana fermion is a four component Lorentz spinon, hence, it is natural to investigate the full symmetry properties of the four dimensional zero energy subspace expanded by the four Majorana zero modes $\gamma_\uparrow, \gamma'_\uparrow, \gamma_\downarrow$ and $\gamma'_\downarrow$. Particularly, we will discuss the other two fundamental discrete symmetries – parity and charge conjugation.

A. $P^4 = -1$ parity symmetry

For a single particle, we only consider the parity symmetry as a $\mathbb{Z}_2$ action on the internal degrees of freedom, and in quantum field theory, we will include its action on coordinates as well. Interestingly, in the zero energy subspace expanded by four Majorana zero modes, we can define a $P^4 = -1$ symmetry for each parity pair of Majorana zero modes $\gamma_\uparrow, \gamma'_\uparrow$ or $\gamma_\downarrow, \gamma'_\downarrow$. The reason why we can have such a fractionalized parity symmetry for Majorana zero modes is the same as the reason for time reversal symmetry. The parity symmetry for an interacting spin-1/2 fermion system is actually $P^2 = P^f$. Therefore, for the Fock basis $c_\uparrow^\dagger|0\rangle, c_\downarrow^\dagger|0\rangle$ and $|0\rangle, c_\uparrow^\dagger c_\downarrow^\dagger|0\rangle$, the parity odd sector satisfies $P^2 = -1$ while the parity even sector satisfies $P^2 = 1$. Here the complex fermion operator $c_\uparrow$ and $c_\downarrow$ are defined by:

$$c_\uparrow = \gamma_\uparrow + i\gamma_\uparrow'; \quad c_\downarrow = \gamma_\downarrow - i\gamma_\downarrow', \tag{39}$$

which give rise to a natural notion of spin basis out of four Majorana zero modes.

The explicit construction of $P^4 = -1$ operator for a pair of Majorana zero modes is very similar to that for the $T^4 = -1$ time reversal symmetry. For the pair of Majorana modes $\gamma_\uparrow, \gamma_\uparrow'$ and $\gamma_\downarrow, \gamma_\downarrow'$, their parity operators are defined by:

$$P_{\uparrow\uparrow'} = \frac{1}{\sqrt{2}}(1 + \gamma_\uparrow\gamma_\uparrow') = e^{\frac{\pi}{4}\gamma_\uparrow\gamma_\uparrow'}; \quad P_{\downarrow\downarrow'} = \frac{1}{\sqrt{2}}(1 - \gamma_\downarrow\gamma_\downarrow') = e^{-\frac{\pi}{4}\gamma_\downarrow\gamma_\downarrow'}, \tag{40}$$

We see such a definition satisfies $P^4_{\uparrow\uparrow'(\downarrow\downarrow')} = -1$ for each pair of Majorana modes. The total parity action on the four Majorana zero modes is defined by $P = P_{\uparrow\uparrow'} \otimes P_{\downarrow\downarrow'}$. Its action on the four Majorana modes reads:

$$\begin{aligned} P\gamma_\uparrow P^{-1} &= -\gamma_\uparrow'; \quad P\gamma_\downarrow P^{-1} = \gamma_\downarrow' \\ P\gamma_\uparrow' P^{-1} &= \gamma_\uparrow; \quad P\gamma_\downarrow' P^{-1} = -\gamma_\downarrow, \end{aligned} \tag{41}$$

It is easy to verify that the complex fermion $c_\uparrow$ and $c_\downarrow$ representing the spin basis transform in an expected way:

$$\begin{aligned} Pc_\uparrow P^{-1} &= ic_\uparrow; \quad Pc_\downarrow P^{-1} = ic_\downarrow \\ Pc_\uparrow^\dagger P^{-1} &= -ic_\uparrow^\dagger; \quad Pc_\downarrow^\dagger P^{-1} = -ic_\downarrow^\dagger, \end{aligned} \tag{42}$$

We note that although the spin of a particle does not change under parity, there could be a nontrivial phase factor for the spin-1/2 particle. On the other hand, $c_L = \gamma_\uparrow + i\gamma_\downarrow$ and $c_R = \gamma_\uparrow' - i\gamma_\downarrow'$ transform like a particle and an anti-particle pair:

$$\begin{aligned} Pc_L P^{-1} &= -c_R; \quad Pc_R P^{-1} = c_L \\ Pc_L^\dagger P^{-1} &= -c_R^\dagger; \quad Pc_R^\dagger P^{-1} = c_L^\dagger \end{aligned} \tag{43}$$

Our definition of parity operator is comparable with the time reversal operator $PTP^{-1} = P_f T$ with $T = e^{\frac{\pi}{4}\gamma_\uparrow\gamma_\downarrow} e^{\frac{\pi}{4}\gamma_\uparrow'\gamma_\downarrow'} K$, and $P_f = \gamma_\uparrow\gamma_\downarrow\gamma_\uparrow'\gamma_\downarrow'$ is the total fermion parity operator.

B. $\overline{C}^4 = -1$ charge conjugation symmetry

Since the Majorana fermion describes a charge neutron particle, the charge conjugation action is trivial from a traditional perspective. Strikingly, we find a way to define a nontrivial $\overline{C}^4 = -1$ charge conjugation symmetry for a pair of Majorana zero modes. Similar to the $T^4 = -1/P^4 = -1$ time reversal/parity symmetry, for each pair of Majorana zero modes with opposite spins, we can define a $\overline{C}^4 = -1$ charge conjugation operator:

$$\overline{C}_{\uparrow\downarrow'} = \frac{1}{\sqrt{2}}(1 + \gamma_\uparrow\gamma'_\downarrow) = e^{\frac{\pi}{4}\gamma_\uparrow\gamma'_\downarrow}; \quad \overline{C}_{\downarrow\uparrow'} = \frac{1}{\sqrt{2}}(1 + \gamma_\downarrow\gamma'_\uparrow) = e^{\frac{\pi}{4}\gamma_\downarrow\gamma'_\uparrow}, \tag{44}$$

and the total action of charge conjugation symmetry on four Majorana zero modes is $\overline{C} = \overline{C}_{\uparrow\downarrow'} \otimes \overline{C}_{\downarrow\uparrow'}$. It is straight forward to verify:

$$\begin{aligned} \overline{C}\gamma_\uparrow\overline{C}^{-1} &= -\gamma'_\downarrow; \quad \overline{C}\gamma_\downarrow\overline{C}^{-1} = -\gamma'_\uparrow \\ \overline{C}\gamma'_\uparrow\overline{C}^{-1} &= \gamma_\downarrow; \quad \overline{C}\gamma'_\downarrow\overline{C}^{-1} = \gamma_\uparrow, \end{aligned} \tag{45}$$

which implies:

$$\begin{aligned} \overline{C}c_\uparrow\overline{C}^{-1} &= ic^\dagger_\downarrow; \quad \overline{C}c_\downarrow\overline{C}^{-1} = -ic^\dagger_\uparrow \\ \overline{C}c^\dagger_\uparrow\overline{C}^{-1} &= -ic_\downarrow; \quad \overline{C}c^\dagger_\downarrow\overline{C}^{-1} = ic_\uparrow, \end{aligned} \tag{46}$$

and

$$\begin{aligned} \overline{C}c_L\overline{C}^{-1} &= -ic_R; \quad \overline{C}c_R\overline{C}^{-1} = -ic_L \\ \overline{C}c^\dagger_L\overline{C}^{-1} &= ic^\dagger_R; \quad \overline{C}c^\dagger_R\overline{C}^{-1} = ic^\dagger_L, \end{aligned} \tag{47}$$

We note that for the spin basis $c_{\uparrow(\downarrow)}$, the charge conjugation acts like a particle-hole symmetry, however, for the $c_{L(R)}$ basis it acts like a charge conjugation symmetry(if we interpret c_L as a particle while c_R as an anti-particle). Similar to the commutation relation between time reversal and parity symmetry, the $\overline{C}^4 = -1$ charge conjugation symmetry also commutes with the other two symmetries up to a total fermion parity.

$$\overline{C}T\overline{C}^{-1} = P^f T; \quad \overline{C}P\overline{C}^{-1} = P^f P \tag{48}$$

C. Super $\overline{C}, P, T$ algebra

Let us summarize the closed algebraic relation of $\overline{C}, P, T$, and P^f symmetry for a Majorana fermion formed by four Majorana zero modes.

$$\begin{aligned} \overline{C}^2 &= P^f; \quad P^2 = P^f; \quad T^2 = P^f; \quad \left(P^f\right)^2 = 1 \\ TP^f &= P^f T; \quad PP^f = P^f P; \quad \overline{C}P^f = P^f\overline{C} \\ TP &= P^f PT; \quad T\overline{C} = P^f\overline{C}T; \quad P\overline{C} = P^f\overline{C}P, \end{aligned} \tag{49}$$

The above algebra satisfied by the $\overline{C}, P, T$ symmetries is indeed a super algebra, which can be regarded as a super extension of the usual charge conjugation, parity and time reversal symmetries over the fermion parity symmetry P^f. This super algebra is one of the central results of this paper. It arises from the topological nature of the Majorana zero modes and reflects the strongly correlated nature of vacuum.

In next section, we will show that the above super $\overline{C}, P, T$ algebra is also applicable for Majorana field. In quantum field theory, such a super extension is allowed because P^f is not a physical observable, or in other words, there is no way to measure the total fermion parity for a quantum state since any physical process must preserve fermion parity symmetry. From a traditional point of view, our results suggest that the $\overline{C}, P, T$ transformations for Majorana field can be different from a Dirac field, just like a scalar field and a Dirac field have very different CPT transformations. Therefore, a Majorana field with a topological origination has a completely new physical meaning and indicates a strongly correlated vacuum, despite the equivalence between Majorana representation and Weyl representation[62].

In addition to the fundamental discrete symmetries $\overline{C}, P, T$, we can also define a spin rotational symmetry in the spin basis, where $c_\uparrow^\dagger|0\rangle, c_\downarrow^\dagger|0\rangle$ carry spin-1/2 while $|0\rangle, c_\uparrow^\dagger c_\downarrow^\dagger|0\rangle$ carry spin-0. Therefore, the $SU(2)$ spin operator $\mathbf{S}$ can be naturally defined by:

$$S^\alpha = \frac{1}{2}\sum_{\sigma,\sigma'} c_\sigma^\dagger \tau^\alpha_{\sigma\sigma'} c_{\sigma'}; \alpha = x, y, z \tag{50}$$

where τ^α is the usual Pauli matrix. It is easy to verify that:

$$T\mathbf{S}T^{-1} = -\mathbf{S}; \quad P\mathbf{S}P^{-1} = \mathbf{S}; \quad \overline{C}\mathbf{S}\overline{C}^{-1} = \mathbf{S}, \tag{51}$$

The above nice property makes the $\overline{C}PT$ symmetries commute with the $SU(2)$ spin rotational symmetry and allows us to generalize the $\overline{C}PT$ super algebra into the relativistic quantum field theory.

V. $\overline{C}, P, T$ SYMMETRIES FOR MAJORANA FIELD AND THE ORIGIN OF NEUTRINO MASS

A. $\overline{C}, P, T$ symmetries for relativistic quantum field theory

Let us implement the $\overline{C}, P, T$ symmetries to a Majorana field. We choose four real gamma matrices:

$$\gamma_0 = -i\rho_z \otimes \sigma_y; \quad \gamma_1 = I \otimes \sigma_z; \quad \gamma_2 = -\rho_y \otimes \sigma_y; \quad \gamma_3 = -I \otimes \sigma_x, \tag{52}$$

where ρ and σ are Pauli matrices and I is the identity matrix. We can define a real γ_5 by:

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = i\rho_x \otimes \sigma_y \tag{53}$$

The four component Majorana field describing the pair of complex fermions c_L and c_R reads:

$$\psi_c(x) = \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix}, \tag{54}$$

where

$$\xi(x) = \begin{pmatrix} \gamma_\uparrow(x) \\ \gamma_\downarrow(x) \end{pmatrix}; \quad \eta(x) = \begin{pmatrix} -\gamma'_\uparrow(x) \\ \gamma'_\downarrow(x) \end{pmatrix}, \tag{55}$$

Here the Majorana spinon basis $\xi(x)$ and $\eta(x)$ are equivalent to complex fermions c_L and c_R, which give rise to a natural notion of particle and anti-particle.

The (equal time) canonical commutation relation reads:

$$\{\psi_c^\dagger(\mathbf{x}), \psi_c(\mathbf{y})\} = 2\delta^{(3)}(\mathbf{x} - \mathbf{y}). \tag{56}$$

In terms of $\gamma_\sigma(\mathbf{x})$ and $\gamma'_\sigma(\mathbf{x})$, we have:

$$\{\gamma_\sigma(\mathbf{x}), \gamma'_{\sigma'}(\mathbf{y})\} = 0; \quad \{\gamma_\sigma(\mathbf{x}), \gamma_{\sigma'}(\mathbf{y})\} = 2\delta^{(3)}(\mathbf{x} - \mathbf{y})\delta_{\sigma\sigma'}, \tag{57}$$

which is the continuum version of the commutation relation Eq.(2). The $\overline{C}PT$ symmetry operators can be defined by:

$$\begin{aligned}
\overline{C} &= \prod_{\mathbf{x}} e^{\frac{\pi}{4}\gamma_\uparrow(\mathbf{x})\gamma'_\downarrow(\mathbf{x})} e^{\frac{\pi}{4}\gamma_\downarrow(\mathbf{x})\gamma'_\uparrow(\mathbf{x})} = e^{\frac{\pi}{4}\int d^3x \gamma_\uparrow(\mathbf{x})\gamma'_\downarrow(\mathbf{x})} e^{\frac{\pi}{4}\int d^3x \gamma_\downarrow(\mathbf{x})\gamma'_\uparrow(\mathbf{x})} \\
P &= U_P P_0 = \prod_{\mathbf{x}} e^{\frac{\pi}{4}\gamma_\uparrow(\mathbf{x})\gamma'_\uparrow(\mathbf{x})} e^{-\frac{\pi}{4}\gamma_\downarrow(\mathbf{x})\gamma'_\downarrow(\mathbf{x})} P_0 = e^{\frac{\pi}{4}\int d^3x \gamma_\uparrow(\mathbf{x})\gamma'_\uparrow(\mathbf{x})} e^{-\frac{\pi}{4}\int d^3x \gamma_\downarrow(\mathbf{x})\gamma'_\downarrow(\mathbf{x})} P_0 \\
T &= U_T K = \prod_{\mathbf{x}} e^{\frac{\pi}{4}\gamma_\uparrow(\mathbf{x})\gamma_\downarrow(\mathbf{x})} e^{\frac{\pi}{4}\gamma'_\uparrow(\mathbf{x})\gamma'_\downarrow(\mathbf{x})} K = e^{\frac{\pi}{4}\int d^3x \gamma_\uparrow(\mathbf{x})\gamma_\downarrow(\mathbf{x})} e^{\frac{\pi}{4}\int d^3x \gamma'_\uparrow(\mathbf{x})\gamma'_\downarrow(\mathbf{x})} K \\
P^f &= \prod_{\mathbf{x}} \gamma_\uparrow(\mathbf{x})\gamma_\downarrow(\mathbf{x})\gamma'_\uparrow(\mathbf{x})\gamma'_\downarrow(\mathbf{x}) = \overline{C}^2 = T^2 = P^2
\end{aligned} \tag{58}$$

Here P_0 is the action on the spacial coordinates with $P_0\mathbf{x}P_0^{-1} = -\mathbf{x}$. It is easy to check that the above $\overline{C}PT$ symmetry operators satisfy the super algebra Eq.(49).

The transformations of the Majorana field under the above $\overline{C}PT$ symmetries can also be derived:

$$\begin{aligned}
\overline{C}\psi_c(x)\overline{C}^{-1} &= \overline{C}\begin{pmatrix}\xi(x)\\ \eta(x)\end{pmatrix}\overline{C}^{-1} = \begin{pmatrix}-\epsilon\eta(x)\\ -\epsilon\xi(x)\end{pmatrix} = -\gamma_5\psi_c(x);\\
P\psi_c(x)P^{-1} &= P\begin{pmatrix}\xi(x)\\ \eta(x)\end{pmatrix}P^{-1} = \begin{pmatrix}\eta(\widetilde{x})\\ -\xi(\widetilde{x})\end{pmatrix} = \gamma_0\gamma_5\psi_c(\widetilde{x});\\
T\psi_c(x)T^{-1} &= T\begin{pmatrix}\xi(x)\\ \eta(x)\end{pmatrix}T^{-1} = \begin{pmatrix}-\epsilon\xi(-\widetilde{x})\\ \epsilon\eta(-\widetilde{x})\end{pmatrix} = \gamma_0\psi_c(-\widetilde{x}),
\end{aligned} \tag{59}$$

where $\widetilde{x} = (t, -\mathbf{x})$. Let us consider the Majorana field Lagrangian in the massless limit:

$$\mathcal{L}_0 = \frac{1}{4}\overline{\psi}_c(x)i\gamma_\mu\partial_\mu\psi_c(x); \quad \overline{\psi}_c(x) = \psi_c^\dagger(x)\gamma_0, \tag{60}$$

Apparently, $\mathcal{L}_0$ is invariant under the $\overline{C}, P, T$ symmetries:

$$\overline{C}\mathcal{L}_0(x)\overline{C}^{-1} = \mathcal{L}_0(x); \quad P\mathcal{L}_0(x)P^{-1} = \mathcal{L}_0(\widetilde{x}); \quad T\mathcal{L}_0(x)T^{-1} = \mathcal{L}_0(-\widetilde{x}), \tag{61}$$

B. Charge conjugation as a $\mathbb{Z}_2$ gauge symmetry and its spontaneously breaking–the origin of right-handed neutrino mass

Given the new (potentially correct) definition of $\overline{C}, P, T$ symmetries for a Majorana fermion, we are ready to discuss the origin of the neutrino mass, assuming that the neutrino is a Majorana fermion. We can construct a mass term preserving time reversal symmetry, parity symmetry and spin rotational symmetry:

$$H_m = \frac{m}{2}\left[i\gamma_\uparrow(\mathbf{x})\gamma'_\uparrow(\mathbf{x}) - i\gamma_\downarrow(\mathbf{x})\gamma'_\downarrow(\mathbf{x})\right] \tag{62}$$

However, such a mass term breaks the charge conjugation symmetry since $\overline{C}H_m\overline{C}^{-1} = -H_m$.

If we elevate the charge conjugation symmetry to a $\mathbb{Z}_2$ gauge symmetry, the origin of the Majorana mass term can be explained as the spontaneous gauge symmetry breaking through the Anderson-Higgs mechanism[36]. Now we come up with the most important concept in this paper: The spontaneous breaking of the (nontrivial) charge conjugation gauge symmetry leads to a mass term of a Majorana fermion. The fundamental $\mathbb{Z}_2$ gauge field will lead to a fifth force among fundamental particles, and it is possible to detect such a new force in future LHC experiments. Finally, to be comparable with the SM, the neutrino mass discussed here should be the mass of the right-handed sterile neutrino, since a Majorana mass term for the left-handed light neutrino is not allowed and can only be induced through the seesaw mechanism[20–22].

To implement the above idea in quantum field theory, we can introduce a new real scalar field $\phi(x) = \phi(t, \mathbf{x})$ which carries $\mathbb{Z}_2$ guage charge one(thus it transforms as $\overline{C}\phi(x)\overline{C}^{-1} = -\phi(x)$) and couple it to the Majorana field. The Anderson-Higgs mechanism[36] can be realized by condensing the real scalar field $\phi(x)$. We assume that such a fundamental scalar field does not carry other gauge charge and is invariant under the P and T symmetry. The following Lagrangian preserves all the $\overline{C}, P, T$ symmetries:

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_m + \mathcal{L}_\phi + \mathcal{L}_{\mathbb{Z}_2} \\ &= \frac{1}{4}\overline{\psi}_c(x) i\gamma_\mu D_\mu \psi_c(x) + \frac{ig}{4}\phi(x)\overline{\psi}_c(x)\gamma_5\psi_c(x) + |D_\mu \phi|^2 - V(\phi) + \mathcal{L}_{\mathbb{Z}_2}\end{aligned} \tag{63}$$

If we assume that the real scalar field condenses at $\langle\phi(x)\rangle = \phi_0$, a mass term $im\overline{\psi}_c(x)\gamma_5\psi_c(x)$ arises with $m = g\phi_0/4$. Here D_μ represents the covariant derivative and $\mathcal{L}_{\mathbb{Z}_2}$ represents the action of $\mathbb{Z}_2$ gauge field. We need to regulate the field theory in a discrete space-time to write down its explicit form and we will leave these details in our future publications. Interestingly, the neutrino mass term in our approach is described by a "chiral" charge.(We note that $\gamma_5^2 = -1$ for Majorana field and it does not really have a physical meaning of chirality.) However, we can redefine the Majorana field $\psi_c'(x)$ by $\psi_c'(x) = \frac{1+\gamma_5}{\sqrt{2}}\psi_c(x)$(Again due to $\gamma_5^2 = -1$, $\frac{1+\gamma_5}{\sqrt{2}}$ is unitary transformation rather than "chiral projection".) and transform the mass term into the usual form.

$$\mathcal{L}_m = \frac{ig}{4}\phi(x)\overline{\psi}_c'(x)\psi_c'(x), \quad \overline{\psi}_c'(x) = (\psi_c')^\dagger(x)\gamma_0 \tag{64}$$

However, $\psi'(x)$ transforms differently under the $\overline{C}PT$ symmetries.

$$\overline{C}\psi_c'(x)\overline{C}^{-1} = -\gamma_5\psi_c'(x); \quad P\psi_c'(x)P^{-1} = \gamma_0\psi_c'(\widetilde{x}); \quad T\psi_c'(x)T^{-1} = -\gamma_0\gamma_5\psi_c'(-\widetilde{x}) \tag{65}$$

VI. ORIGIN OF THREE GENERATIONS OF NEUTRINOS

A. General discussion and physical pictures

The existence of three generations of neutrinos is one of the biggest mysteries in our universe. In this section, we will show that such a puzzle can be naturally resolved by assuming that a Majorana fermion is made up of four Majorana zero modes. The key observation is that there are *three* inequivalent ways to define a pair of Majorana spinons that describe a particle and an anti-particle out of four Majorana zero modes. More precisely, the pair of Majorana spinons can be made up not only by $(\gamma_\uparrow, \gamma_\downarrow), (\gamma'_\uparrow, \gamma'_\downarrow)$, but can also by $(\gamma'_\uparrow, \gamma_\downarrow), (\gamma_\uparrow, \gamma'_\downarrow)$ or $(\gamma_\uparrow, \gamma'_\uparrow), (\gamma_\downarrow, \gamma'_\downarrow)$.

Since a Majorana spinon that describes a particle or an anti-particle is equivalent to a spinless complex fermion, let us define:

$$d_L = \frac{1}{2}(\gamma_\uparrow - i\gamma'_\downarrow); \quad d_R = \frac{1}{2}(\gamma'_\uparrow - i\gamma_\downarrow), \tag{66}$$

Under the $\overline{C}, P, T$ symmetries, they transform as:

$$\begin{aligned} \overline{C} d_L \overline{C}^{-1} &= -id_L; \quad \overline{C} d_R \overline{C}^{-1} = id_R \\ P d_L P^{-1} &= -d_R; \quad P d_R P^{-1} = d_L \\ T d_L T^{-1} &= id_R^\dagger; \quad T d_R T^{-1} = id_L^\dagger, \end{aligned} \tag{67}$$

Similarly, we can define:

$$f_L = \frac{1}{2}(\gamma_\uparrow + i\gamma'_\uparrow) = c_\uparrow; \quad f_R = \frac{1}{2}(\gamma_\downarrow + i\gamma'_\downarrow) = c_\downarrow^\dagger \tag{68}$$

Under the $\overline{C}, P, T$ symmetries, they transform as:

$$\begin{aligned} \overline{C} f_L \overline{C}^{-1} &= if_R; \quad \overline{C} f_R \overline{C}^{-1} = if_L \\ P f_L P^{-1} &= if_L; \quad P f_R P^{-1} = -if_R \\ T f_L T^{-1} &= -f_R^\dagger; \quad T f_R T^{-1} = f_L^\dagger, \end{aligned} \tag{69}$$

We see that $d_{L(R)}$ and $f_{L(R)}$ fermions transform differently under the $\overline{C}, P, T$ symmetries. Especially, the local Fock space of $d_{L(R)}$ caries the $(TP)^4 = -1$ projective representation of TP symmetry while the local Fock space of $f_{L(R)}$ caries the $(T\overline{C})^4 = -1$ projective

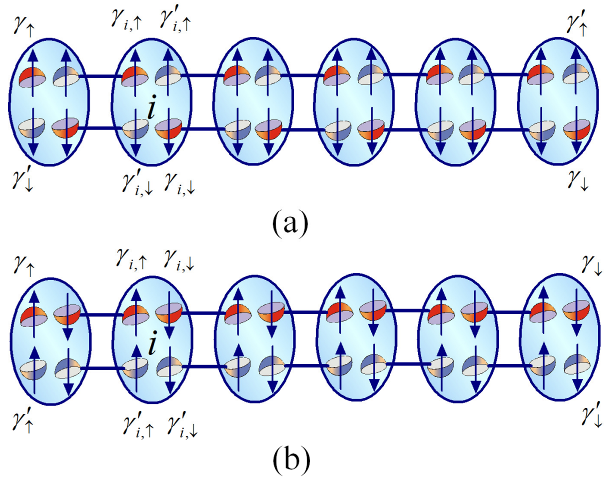

FIG. 5: (color online)The other two 1D TSC models protected by TP and $T\overline{C}$ symmetries, the Majorana modes on their ends carry the $(TP)^4 = -1$ and $(T\overline{C})^4 = -1$ projective representations.

representation of $T\overline{C}$ symmetry. We have:

$$\begin{aligned}(TP)d_L(TP)^{-1} &= -id_L^\dagger; \quad (TP)d_R(TP)^{-1} = id_R^\dagger \\ (T\overline{C})f_L(T\overline{C})^{-1} &= -if_L^\dagger; \quad (T\overline{C})f_R(T\overline{C})^{-1} = if_R^\dagger,\end{aligned} \tag{70}$$

Apparently the above TP and $T\overline{C}$ transformations for $d_{L(R)}$ and $f_{L(R)}$ fermions have the same form as Eq.(16), therefore they carry the same representation theory as Eq.(19).

From a condensed matter theory point of view, the above argument can be understood as there are three different types of point like topological defects in a TSC protected by $\overline{C}PT$ symmetries, characterized by the $T^4 = -1$, $(TP)^4 = -1$, and $(T\overline{C})^4 = -1$ projective symmetries that the corresponding Majorana zero modes carry. Since point like defects can only be created/anihilated in pairs, there is a natural notion of particle and anti-particle pair. In the following, we again construct some explicit 1D TSC models to further explain this idea.

Similar to the time reversal protected Majorana chain that has been discussed at the very beginning of this paper, we can also construct TP(Here again we only consider the internal action of P symmetry, since the symmetry protection nature of Majorana zero modes only relies on the internal action and has nothing to do with the coordinate action.) and $T\overline{C}$

protected Majorana chains explicitly. Let us consider the following Hamiltonian:

$$H_d = \sum_{i=1}^{N} \left(i\gamma'_{i,\uparrow}\gamma_{i+1,\uparrow} + i\gamma_{i,\downarrow}\gamma'_{i+1,\downarrow} \right) , \tag{71}$$

and

$$H_f = \sum_{i=1}^{N} \left(i\gamma_{i,\downarrow}\gamma_{i+1,\uparrow} + i\gamma'_{i,\downarrow}\gamma'_{i+1,\uparrow} \right) , \tag{72}$$

It is clear that H_d is invariant under the TP symmetry and H_f is invariant under the $T\overline{C}$ symmetry. In Fig. 5, we see that for H_d, the pair of Majorana modes on both ends form a $(TP)^4 = -1$ representation, while for H_f, the pair of Majorana modes on both ends form a $(T\overline{C})^4 = -1$ representation. All our discussions for the 1D model can be generalized into 3D as well, where the Majorana modes will be localized on the hedgehog/anti-hedgehog, and similar hedgehog/anti-hedgehog lattice model Eq.(35) with proliferated Majorana modes can be constructed in the same way, replacing $c_{L(R)}$ fermion by $d_{L(R)}$ and $f_{L(R)}$ fermions.

B. Possible internal structure of Majorana fermion

Although the picture of topological defect provided by condensed matter models is very promising and insightful for us to understand the origin of three generations of neutrinos, a fundamental theory does not necessarily to be emerged from any pre-assuming topological defect model. Here we provide an alternative understanding for the origin of three generations of neutrinos by proposing a possible internal structure of a Majorana fermion. As seen in Fig. 6, we conjecture that a Majorana fermion is actually made up of four Majorana zero modes located on the four vertices of a tetrahedra at cutoff scale. In this picture, the $SO(3)$ spacial rotation can be realized by the classical rotation of the tetrahedra. The origin of three generations of neutrinos can be explained by three different ways to form a pair of particle and anti-particle out of four Majorana modes, namely, $c^\dagger_{L(R)}$, $d^\dagger_{L(R)}$ and $f^\dagger_{L(R)}$, identified by the $T^4 = -1$, $(TP)^4 = -1$ and $(TC)^4 = -1$ symmetries that the particles/anti-particles carry. Indeed, both the internal structure and topological defect picture share the same spirit: the Hilbert space for each pair of Majorana modes must be spatially separated at cutoff scale to make the projective representations $T^4 = -1$, $(TP)^4 = -1$ and $(TC)^4 = -1$ meaningful. Thus, the neutrino mass mixing physics can be naturally understood as the resonance among the three different quantum states out of four Majorana zero modes, and

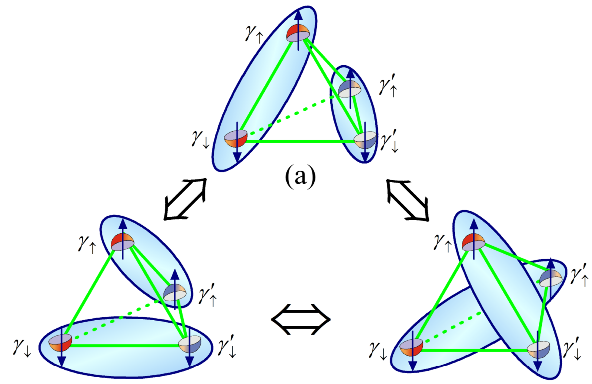

FIG. 6: (color online)A conjectured internal structure of a Majorana fermion, which consists of four Majorana zero modes located on the vertices of a tetrahedra. Such an internal structure is comparable with $SO(3)$ rotational symmetry. The internal structure of neutrino indicates that the three generations of neutrinos and anti-neutrinos can be explained by three different ways to form a pair of particle and anti-particle out of four Majorana modes.

such a picture would be very useful for us to compute the mass mixing matrix, which will be presented in next section. Unfortunately, the above single particle picture can not be generalized into quantum field theory since a rigorous way to incorporate the internal structure of a fundamental particle is absent so far, however, if we have already introduced three independent Majorana fields, there is no difficulty for us to use quantum field theory to describe them. In the following, we present the quantum field theory description for the three generations of neutrinos.

C. Quantum field theory description for three generations of neutrinos

The quantum field theory description for neutrino(anti-neutrino) made by $c_{L(R)}$ fermion has already been presented in section V. To describe neutrino(anti-neutrino) made by $f_{L(R)}$ fermion in the quantum field theory, we just need to define the Majorana fermion field $\psi_f(x) = \begin{pmatrix} \widetilde{\xi}(x) \\ \widetilde{\eta}(x) \end{pmatrix}$ with a different Majorana spinon basis:

$$\widetilde{\xi}(x) = \begin{pmatrix} \gamma_\uparrow(x) \\ \gamma'_\uparrow(x) \end{pmatrix}; \quad \widetilde{\eta}(x) = \begin{pmatrix} \gamma_\downarrow(x) \\ \gamma'_\downarrow(x) \end{pmatrix}, \tag{73}$$

The above Majorana fermion field satisfies the $\overline{C}PT$ symmetries:

$$\overline{C}\psi_f(x)\overline{C}^{-1} = -\gamma_5\psi_f(x); \quad P\psi_f(x)P^{-1} = \gamma_0\psi_f(\widetilde{x}); \quad T\psi_f(x)T^{-1} = -\gamma_0\gamma_5\psi_f(-\widetilde{x}), \tag{74}$$

It is clear that the $f_{L(R)}$ fermion transforms differently under $\overline{C}PT$ symmetries, and for the $f_{L(R)}$ fermion, its mass term takes the usual form:

$$\mathcal{L}_m = \frac{ig}{4}\phi(x)\overline{\psi}_f(x)\psi_f(x), \quad \overline{\psi}_f(x) = \psi_f^\dagger(x)\gamma_0 \tag{75}$$

Finally, for the neutrino(anti-neutrino) made by the $d_{L(R)}$ fermion, we need to choose $\bar{\gamma}_0 = R\gamma_0 R^{-1} = i\rho_x \otimes \sigma_y \equiv \gamma_5$ with:

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(1 + \gamma_0\gamma_5) \tag{76}$$

The corresponding $\gamma_{1,2,3}$ and γ_5 transform as: $\bar{\gamma}_{1,2,3} = R\gamma_{1,2,3}R^{-1} = \gamma_{1,2,3}$ and $\bar{\gamma}_5 = R\gamma_5 R^{-1} = i\rho_z \otimes \sigma_y \equiv -\gamma_0$). Indeed, this representation was first proposed by Ettore Majorana.

The quantum field theory can be obtained by defining $\psi_d(x) = \begin{pmatrix} \hat{\xi}(x) \\ \hat{\eta}(x) \end{pmatrix}$ with:

$$\hat{\xi}(x) = \begin{pmatrix} \gamma_\uparrow(x) \\ \gamma'_\downarrow(x) \end{pmatrix}; \quad \hat{\eta}(x) = \begin{pmatrix} \gamma_\downarrow(x) \\ -\gamma'_\uparrow(x) \end{pmatrix}, \tag{77}$$

Under the $\overline{C}PT$ symmetries with above definition, $\psi_d(x)$ transforms as:

$$\begin{aligned} \overline{C}\psi_d(x)\overline{C}^{-1} &= -\bar{\gamma}_5\psi_d(x) \equiv \gamma_0\psi_d(x); \\ P\psi_d(x)P^{-1} &= \bar{\gamma}_0\psi_d(\widetilde{x}) \equiv \gamma_5\psi_d(\widetilde{x}); \\ T\psi_d(x)T^{-1} &= -\bar{\gamma}_0\bar{\gamma}_5\psi_d(-\widetilde{x}) \equiv -\gamma_0\gamma_5\psi_d(-\widetilde{x}), \end{aligned} \tag{78}$$

For the $d_{L(R)}$ fermion, the mass term also takes the usual form:

$$\mathcal{L}_m = \frac{ig}{4}\phi(x)\overline{\psi}_d(x)\psi_d(x), \quad \overline{\psi}_d(x) = \psi_d^\dagger(x)\bar{\gamma}_0 = \psi_d^\dagger(x)\gamma_5 \tag{79}$$

Since ψ_c, ψ_f and ψ_d transform differently under the $\overline{C}PT$ symmetries and one can not transform them from one to the other with continuous proper orthochronous Lorentz transformation, they can be regarded as three independent fields in quantum field theory.(We note that any continuous proper orthochronous Lorentz transformation will not change the definition of Majorana spinon basis.)

The three generations of neutrino fields described by $c_{L(R)}$, $f_{L(R)}$ and $d_{L(R)}$ fermions can also be identified by their different $\overline{C}PT$ transformation laws in momentum space, see Appendix A for details.

VII. NEUTRINO MASS MIXING MATRIX

A. Seesaw mechanism

It is well known that a Majorana mass term of the form $m\overline{\psi}(x)\psi(x)$ is prohibited for left-handed light neutrinos since it breaks the electric-weak gauge symmetry, and that is why the SM predicts zero neutrino mass. A nice way to fix this problem is to assume the existence of three generations of heavy sterile neutrinos, and masses of the left-handed light neutrinos can be induced by the see-saw mechanism[20–22]. The total mass matrix reads:

$$M_{total} = \begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix}, \tag{80}$$

where m_D is the 3 by 3 Dirac mass matrix and M is the 3 by 3 Majorana mass matrix of right handed sterile neutrinos.(We note that the left-handed neutrinos have a zero mass.) If we assume that m_D is on the electric-weak symmetry breaking energy scale($250GeV$) and M is on the grand unification theory(GUT) energy scale($10^{15}GeV$), a mass on the energy scale of $0.1eV$ can be induced for the left-handed light neutrino. With a proper choice of basis, m_D can be chosen as a diagonal matrix. Here we further assume m_D to be uniform, with the form $m_D = diag(m, m, m)$. The particular reason why we choose such a form is that the three generations of left-handed/right-handed neutrinos can be regarded as three distinguishable resonating states out of four Majorana zero modes at cutoff scale, as having been discussed in last section. Under such an assumption, the mass mixing pattern of the left-handed light neutrino is uniquely determined by the mass mixing pattern of the right-handed heavy sterile neutrinos. In principle, the mass mixing in charged lepton sector should also be taken into account for the experimentally observed left-handed light neutrino mixing pattern, however, as the charged lepton has a huge mass hierarchy, the contribution should be small and negligible within LO approximation. The mass matrix M_{total} can be complex in the presence of CP violation, but this effect is observed to be small and therefore is negligible within LO approximation.

In section V, we propose that the origin of the right-handed neutrino mass can be understood as the spontaneously breaking of the $\mathbb{Z}_2$ charge conjugation gauge symmetry. In the following we will apply the same idea to derive the entire right-handed neutrino mass matrix M.

B. Mixing pattern and predictions of neutrino mass

Firstly, according to the $\mathbb{Z}_2$ gauge (minimal coupling) principle, we can write down the most general $\overline{C}PT$ invariant mass term for three generations of right-handed neutrinos. We have:

$$\begin{aligned}
\mathcal{L}_m &= \frac{ig}{4}\phi(x)\left[\overline{\psi}_f(x)\psi_f(x) + \overline{\psi}_d(x)\psi_d(x) + \overline{\psi}_c(x)\gamma_5\psi_c(x)\right] \\
&+ \frac{ig'}{4}\phi(x)\left[\overline{\psi}_d(x)(1+\gamma_0\gamma_5)(1+\gamma_5)\psi_c(x) + \overline{\psi}_c(x)(1+\gamma_5)(1-\gamma_0\gamma_5)\psi_d(x)\right] \\
&+ \frac{ig'}{4}\phi(x)\left[\overline{\psi}_f(x)(1+\gamma_5)\psi_c(x) + \overline{\psi}_c(x)(1+\gamma_5)\psi_f(x)\right] \\
&+ \frac{ig'}{4}\phi(x)\left[\overline{\psi}_f(x)(1-\gamma_0\gamma_5)\psi_d(x) + \overline{\psi}_d(x)(1+\gamma_0\gamma_5)\psi_f(x)\right] \qquad (81)
\end{aligned}$$

Here we use the same coupling g for all the diagonal mass terms and g' for all the off-diagonal mass terms. Again, this is because the three generations of right-handed neutrinos are the three resonating states out of the *same* four Majorana zero modes at cutoff scale. The above argument can also be incorporated into traditional quantum field theory language(in the absence of cutoff physics) by imposing an additional $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ flavor symmetry to constraint the coupling constant, see Appendix B for details. We note that for $\psi_c(x)$ and $\psi_f(x)$, the boost generators are defined by $S_{0i} = \frac{1}{4}[\gamma_0, \gamma_i]$ while for $\psi_d(x)$, the boost generator is defined by $\bar{S}_{0i} = \frac{1}{4}[\bar{\gamma}_0, \gamma_i] = \frac{1}{4}[\gamma_5, \gamma_i]$. Such an interesting twist makes the above mass term invariant under the Lorentz transformation, despite the existence of $(1 \pm \gamma_0\gamma_5)$ term which does not seem to be invariant under the Lorentz boost.

In the extended SM, three generations of right-handed neutrinos are described by three copies of the same Majorana field. Let us redefine $\psi_d(x)$ by:

$$\psi'_d(x) = R^{-1}\psi_d(x) = \frac{1}{\sqrt{2}}(1-\gamma_0\gamma_5) \qquad (82)$$

The corresponding $\bar{\gamma}_0$ and $\bar{\gamma}_{1,2,3}$ will change back to γ_0 and $\gamma_{1,2,3}$. It is easy to see that $\psi'_d(x)$, $\psi'_c(x) \equiv \frac{1+\gamma_5}{\sqrt{2}}\psi_c(x)$ and $\psi_f(x)$ transform in the same way under the $\overline{C}PT$ symmetries, therefore, they can be interpreted as the three generations of right-handed sterile neutrinos in the extended SM. In terms of $\psi'_d(x)$, $\psi'_c(x)$ and $\psi_f(x)$, the $\overline{C}PT$ invariant mass term takes the following form:

$$\begin{aligned}
\mathcal{L}_m &= \frac{ig}{4}\phi(x)\left[\overline{\psi}_f(x)\psi_f(x) + \overline{\psi'}_d(x)\psi'_d(x) + \overline{\psi'}_c(x)\psi'_c(x)\right] + \frac{2ig'}{4}\phi(x)\left[\overline{\psi'}_d(x)\psi'_c(x) + \overline{\psi'}_c(x)\psi'_d(x)\right] \\
&+ \frac{\sqrt{2}ig'}{4}\phi(x)\left[\overline{\psi}_f(x)\psi'_c(x) + \overline{\psi'}_c(x)\psi_f(x)\right] + \frac{\sqrt{2}ig'}{4}\phi(x)\left[\overline{\psi}_f(x){\psi'}_d(x) + \overline{\psi'}_d(x)\psi_f(x)\right] \qquad (83)
\end{aligned}$$

with $\overline{\psi}'_c(x) = (\psi'_c)^\dagger(x)\gamma_0$ and $\overline{\psi}'_d(x) = (\psi'_d)^\dagger(x)\gamma_0$.

We see that the mass mixing pattern has already been fixed, regardless of the relative strength of g and g'. The mass matrix can be diagonalized by(the basis is ordered as ψ_f, ψ'_c, ψ'_d and $\frac{\phi_0}{4}$ is set to be 1):

$$M = \begin{pmatrix} g & \sqrt{2}g' & \sqrt{2}g' \\ \sqrt{2}g' & g & 2g' \\ \sqrt{2}g' & 2g' & g \end{pmatrix} = V^\dagger \begin{pmatrix} (1-\sqrt{5})g' + g & 0 & 0 \\ 0 & (1+\sqrt{5})g' + g & 0 \\ 0 & 0 & -2g' + g \end{pmatrix} V \quad (84)$$

with

$$V^\dagger = \begin{pmatrix} \sqrt{\frac{5+\sqrt{5}}{10}} & \sqrt{\frac{5-\sqrt{5}}{10}} & 0 \\ -\sqrt{\frac{5-\sqrt{5}}{20}} & \sqrt{\frac{5+\sqrt{5}}{20}} & -\frac{1}{\sqrt{2}} \\ -\sqrt{\frac{5-\sqrt{5}}{20}} & \sqrt{\frac{5+\sqrt{5}}{20}} & \frac{1}{\sqrt{2}} \end{pmatrix} \simeq \begin{pmatrix} 0.85 & 0.53 & 0 \\ -0.37 & 0.6 & -0.71 \\ -0.37 & 0.6 & 0.71 \end{pmatrix} \quad (85)$$

In terms of mixing angle, we have:

$$\theta_{23} = -45^\circ; \quad \theta_{13} = 0; \quad \theta_{12} = 31.7^\circ (\tan^2\theta_{12} = \frac{\sqrt{5}-1}{2}) \quad (86)$$

We note that the physical masses of the mass egienstates are the absolute value of Eq.(84), with $M_1 = |(1-\sqrt{5})g' + g|$, $M_2 = |(1+\sqrt{5})g' + g|$ and $M_3 = |-2g' + g|$, and the $\pm$ sign in front of θ_{23} is just a gauge choice of the basis.

Finally, due to the same reason that the three generations of right-handed neutrinos are the three resonating states out of the *same* four Majorana zero modes at cutoff scale, we further argue that the diagonal Yukawa coupling must have the same strength as the off-diagonal coupling with $|g| = |g'|$(see appendix B for detail discussions). According to the seesaw mechanism, the mass mixing matrix for left-handed light neutrino takes the same form as Eq.(85)(in the limit $m_D \ll M$), however, the mass hierarchy is reversed. The solution with $g = -g'$ implies $M_1 = M_2 = \sqrt{5}g$ and $M_3 = 3g$, which leads to $m_1/m_3 = m_2/m_3 = 3/\sqrt{5}$(here m_1, m_2 and m_3 are eigen masses of the left-handed light neutrinos) and can match the current experimental observations.(If we assume the small mass splitting Δm_{12} is negligible within LO approximation.) However, The solution $g = g'$ leads to $M_1 = (\sqrt{5}-2)g < M_3 = g < M_2 = (\sqrt{5}+2)g$ and contradicts to the current experimental results with either $m_1 \simeq m_2 < m_3$ or $m_1 \simeq m_2 > m_3$. Therefore, here we make the choice with $g = -g'$. Based on the current experimental data $\Delta m^2_{23} \simeq 2.5 \times 10^{-3} eV^2$, we obtain $m_1 = m_2 \simeq 0.075 eV$ and $m_3 \simeq 0.054 eV$.

C. Symmetry properties of neutrino mass mixing matrix

Now let us examine the symmetry of the derived mass mixing matrix. Although the mixing angle derived above is consistent with the GR pattern, the symmetry group is different from that in Ref.[37, 38], and it contains three $\mathbb{Z}_2$ generators U, S and R, defined by:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2} & -\frac{(\sqrt{5}+1)}{2} & \frac{(\sqrt{5}-1)}{2} \\ -\sqrt{2} & \frac{(\sqrt{5}-1)}{2} & -\frac{(\sqrt{5}+1)}{2} \end{pmatrix}; R = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i & i \\ -i & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -i & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{87}$$

They satisfy:

$$U^T M U = M; \quad S^T M S = M; \quad R^T M R = M, \tag{88}$$

and

$$\begin{aligned} U^2 &= 1; \quad S^2 = 1; \quad R^2 = 1, \\ US &= SU; \quad UR = RU; \quad SR = -URS, \end{aligned} \tag{89}$$

U is the center of the above symmetry algebra since it commutes with both S and R. As a $\mathbb{Z}_2$ generator, U can have eigenvalue ± 1. In the subspace with $U = -1$, S and R commute with each other, which leads to a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ group symmetry. While in the subspace with $U = 1$, S and R anticommute with each other, which leads to a $\mathbb{Z}_2$ Heisenberg group symmetry. We note that U and S are the two $\mathbb{Z}_2$ generators of the GR pattern[37, 38] characterized by the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ Klein symmetry and apply to generic g, g' in Eq.(84), while R is a new generator which arises from the special relation $g = -g'$.

D. The effect of CP violation

Before conclusion, we discuss the effect of CP violation for the neutrino mass mixing matrix. Recently, the DaYa-Bay's experiment has reported a non zero $\theta_{13} \simeq 8.8°$[16]. From our point of view, the experimentally observed (not very small) θ_{13} has already implied the presence of CP violation! This is because the GR pattern we derived has a zero θ_{13} within LO approximation, and if we ignore the charged lepton contribution for θ_{13} due to its huge mass hierarchy(This assumption is reasonable since in the CKM quark mass mixing matrix, θ_{13} is significantly small due to its huge mass hierarchy.), the experimentally obseved θ_{13}

must come from CP violation. On the other hand, our theory predicts $m_1 = m_2$ within LO approximation, therefore the experimentally observed small mass splitting Δm_{12} is also contributed by CP violation. Interestingly, the current experiment results point to the relation $|\Delta m_{12}/\Delta m_{23}| \sim \theta_{13}/\theta_{23}$. Our theory suggests that such a relation should not be a coincidence, and it actually indicates that the nonzero Δm_{12} and θ_{13} have a common origin – the CP violation. However, the mechanism of CP violation in lepton sector is not clear, and in our framework, the topological Berry phase[30] of Majorana zero modes could be a possible source of CP violation. We will leave a detailed study of CP violation physics in our future publications.

VIII. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper, we start with a simple 1D TSC model protected by $T^2 = -1$ time reversal symmetry and show that the pair of time reversal protected Majorana zero modes on each end carry a $T^4 = -1$ representation of time reversal symmetry and 1/4 spin. Then we generalize the $T^4 = -1$ fractionalized representation for a pair of Majorana zero modes into a $P^4 = -1$ parity symmetry and a $\overline{C}^4 = -1$ nontrivial charge conjugation symmetry as well. We also construct explicit condensed matter models and show that the proliferating of Majorana zero modes will lead to a relativistic dispersion and an $SU(2)$ spin.

These interesting observations from condensed matter systems motivate us to interpret a Majorana fermion as four Majorana zero modes(or a Lorentz spinon zero mode) and revisit its $\overline{C}PT$ symmetries. Surprisingly, we find that the $\overline{C}PT$ symmetries for a Majorana fermion made up of four Majorana zero modes satisfy a super algebra. The $\overline{C}PT$ super algebra for a Majorana fermion can be generalized into quantum filed theory as well. We further point out that the nontrivial charge conjugation symmetry $\overline{C}$ can be promoted to a $\mathbb{Z}_2$ gauge symmetry and its spontaneously breaking leads to the origin of (right-handed) neutrino mass. The $\mathbb{Z}_2$ gauge symmetry indicates the existence of the fifth force in our universe, which is possible to be detected in future LHC experiment. Indeed, the seesaw mechanism scenario requires such a fifth force. This is because the right-handed sterile neutrino does not carry electric-weak charge, therefore if we assume all the coupling terms arise from (gauge) interactions, there should be *no* coupling term between left-handed neutrinos and right-handed neutrinos. However, in the seesaw mechanism, there is a coupling term in the form of $L\widetilde{\phi}\nu_R$(the Dirac mass term with L as the lepton doublets, ϕ as the Higgs field and ν_R as the right-handed neutrino field). Although it is an "allowed" term by gauge invariance, it is not a "natural" term since there is no interaction between $L\widetilde{\phi}$ and ν_R. In the presence of the $\mathbb{Z}_2$ gauge force, such a term becomes nature since ν_L and ν_R can carry opposite half-$\mathbb{Z}_2$ charge. Here the concept of half-$\mathbb{Z}_2$ charge arises from the transformation Eq.(67), where the $d_{L(R)}$ fermion operator takes eigen value $\mp i$ under charge conjugation symmetry, which is indeed a $\mathbb{Z}_4$ charge. The reason why a fermion can carry a half-$\mathbb{Z}_2$(or $\mathbb{Z}_4$) charge is again due to the group extension of the nontrivial charge conjugation symmetry $\overline{C}$ over the fermion parity symmetry that makes the total symmetry group to be $\mathbb{Z}_4$. The half-$\mathbb{Z}_2$ charge assignment of a single fermion is also consistent with the fact that the mass term(a fermion bilinear) carries $\mathbb{Z}_2$ charge one.

These new concepts even provide us a natural way to understand the origin of three generations of neutrinos, as there are three inequivalent ways to form a pair of complex fermions(a particle and an anti-particle) out of four Majorana zero modes, characterized by the $T^4 = -1$, $(TP)^4 = -1$ and $(T\overline{C})^4 = -1$ fractionalized symmetries that the complex fermions carry. This argument requires that a Majorana fermion is not a point like particle and has an internal structure at cutoff scale. In the semiclassical limit, together with the $\mathbb{Z}_2$ gauge (minimal coupling) principle, we are able to uniquely determine the $\overline{C}PT$ invariant mass term and compute the neutrino mass mixing matrix with *no* fitting parameters within LO approximation(without CP violation and charged lepton contribution). We obtain $\theta_{12} = 31.7°, \theta_{23} = 45°$ and $\theta_{13} = 0°$(the golden ratio pattern), which is intrinsically close to the current experimental results. We further predict an exact mass ratio for the three mass eigenstates with $m_1/m_3 = m_2/m_3 = 3/\sqrt{5}$.

For future works, we would like to point out several interesting directions along this line of thinking: (a) *The quark CKM mass mixing matrix.* Since a Dirac fermion can be decomposed into a pair of Majorana fermions, the Majorana zero modes scenario will be applicable for the Dirac fermion as well. As a result, the origin of three generations of quarks and charged leptons can be understood in the same way. It is even possible to use similar terminology to compute the quark CKM mass mixing matrix. However, a crucial difference in the quark CKM mass mixing matrix is the mass hierarchy problem, which leads to a significant suppressing for its mixing angles. It is important to understand the origin of quark mass hierarchy. (b) *The cutoff problem.* To resolve the cutoff problem, the topological defects description for fundamental particles is very promising, and it has already been shown that a dual SM can be constructed from $SU(5)$ monopole[63]. Together with the Majorana zero modes idea proposed in this paper, it is possible to explain the origin of isospin, fractionalized charge and three colors of quarks. Of course, a more challenging and deep way to deal with the cutoff problem is to develop a mathematical framework for quantum field theory in discrete space-time. If a fundamental theory has extremely strong quantum fluctuation at cutoff scale, the discrete structure would become crucial. In a recent work[59, 60], topological non-linear sigma model in discrete space-time is proposed. Since gauge field can emerge from non-linear sigma model approaching quantum criticality[64, 65], it is possible to derive the SM from a discrete non-linear sigma model. (c) *Hidden super algebra for the SM.* From experimental point of view, to avoid fine-tuning, a super algebraic structure of the SM is demanded. Recent experimental results on the Higgs mass near $126GeV$[66, 67] point to a relation $M_{\text{Higgs}} \simeq (M_u + M_d + M_c + M_s + M_t + M_b)/\sqrt{2}$(the Higgs boson mass is intrinsically close to the summation of six quark masses divided by $\sqrt{2}$). We also notice $M_t \simeq M_W + M_Z$(top quark mass is intrinsically close to the summation of W and Z boson masses) by pass. If the above two relations are not coincident, they must be strong indications that SM might satisfy a hidden super algebra. We note that these interesting mass relations are merely among the known fermions and bosons in the SM, therefore they can not be explained by any traditional super-symmetry. Nevertheless, the Majorana zero modes might provide us a natural way to understand these relations. (d)*Super-extension of space-time structure.* The conjectured internal structure of a Majorana fermion in Fig. 6 might imply the presence of super-coordinates at cutoff scale. The discovery of Majorana zero modes brings us the novel concept of half degree of freedom[24, 25], but from a classical point of view, half degree of freedom contradicts the locality principle and there is no way to define half degree of freedom per spacial point. Thus the internal structure in Fig. 6 with four Majorana modes located on the vertices of a tetrahedra seems to be meaningless. Nevertheless, if we assume a cutoff theory contains not only the classical space-time coordinates but also fermionic coordinates, and a single Majorana mode lives on a fermionic coordinate, the locality problem can be resolved. Ref.[61] proposed a nice way to incorporate super-coordinates in the discrete topological non-linear sigma model, hence it has the potential to describe a quantum field theory with super-coordinates at cutoff. The presence of super-coordinates might automatically imply a super algebraic structure of the SM. Finally, such kinds of discrete topological non-linear sigma models are background independent and might provide us a new route towards super (quantum) gravity.

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Appendix A: $\overline{C}PT$ symmetries in momentum space

In this section, we will use a momentum space picture to derive the three generations of neutrinos. First, let us examine the $\overline{C}PT$ symmetry transformation of the Fourier modes $\gamma_\sigma(\mathbf{k}) = \frac{1}{\sqrt{V}}\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\gamma_\sigma(\mathbf{x})$ and $\gamma'_\sigma(\mathbf{k}) = \frac{1}{\sqrt{V}}\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\gamma'_\sigma(\mathbf{x})$. It is straightforward to derive:

$$\begin{aligned}\overline{C}\gamma_\uparrow(\mathbf{k})\overline{C}^{-1} &= -\gamma'_\downarrow(\mathbf{k}); \quad \overline{C}\gamma_\downarrow(\mathbf{k})\overline{C}^{-1} = -\gamma'_\uparrow(\mathbf{k}); \\ \overline{C}\gamma'_\uparrow(\mathbf{k})\overline{C}^{-1} &= \gamma_\downarrow(\mathbf{k}); \quad \overline{C}\gamma'_\downarrow(\mathbf{k})\overline{C}^{-1} = \gamma_\uparrow(\mathbf{k}),\end{aligned} \tag{A1}$$

$$\begin{aligned}P\gamma_\uparrow(\mathbf{k})P^{-1} &= -\gamma'_\uparrow(-\mathbf{k}); \quad P\gamma_\downarrow(\mathbf{k})P^{-1} = \gamma'_\downarrow(-\mathbf{k}); \\ P\gamma'_\uparrow(\mathbf{k})P^{-1} &= \gamma_\uparrow(-\mathbf{k}); \quad P\gamma'_\downarrow(\mathbf{k})P^{-1} = -\gamma_\downarrow(-\mathbf{k}),\end{aligned} \tag{A2}$$

$$\begin{aligned}T\gamma_\uparrow(\mathbf{k})T^{-1} &= -\gamma_\downarrow(-\mathbf{k}); \quad T\gamma_\downarrow(\mathbf{k})T^{-1} = \gamma_\uparrow(-\mathbf{k}); \\ T\gamma'_\uparrow(\mathbf{k})T^{-1} &= -\gamma'_\downarrow(-\mathbf{k}); \quad T\gamma'_\downarrow(\mathbf{k})T^{-1} = \gamma'_\uparrow(-\mathbf{k}),\end{aligned} \tag{A3}$$

We can apply the similar argument to the emergence of three generations of Majorana fermions for their Fourier modes in momentum space as well.

$$\begin{aligned}d_L(\mathbf{k}) &= \gamma_\uparrow(\mathbf{k}) - i\gamma'_\downarrow(\mathbf{k}); \quad d_R(\mathbf{k}) = \gamma'_\uparrow(\mathbf{k}) - i\gamma_\downarrow(\mathbf{k}) \\ c_L(\mathbf{k}) &= \gamma_\uparrow(\mathbf{k}) + i\gamma_\downarrow(\mathbf{k}); \quad c_R(\mathbf{k}) = \gamma'_\uparrow(\mathbf{k}) - i\gamma'_\downarrow(\mathbf{k}) \\ f_L(\mathbf{k}) &= \gamma_\uparrow(\mathbf{k}) + i\gamma'_\uparrow(\mathbf{k}); \quad f_R(\mathbf{k}) = \gamma_\downarrow(\mathbf{k}) + i\gamma'_\downarrow(\mathbf{k})\end{aligned} \tag{A4}$$

They carry the projective representation of T, TP and $T\overline{C}$ symmetries:

$$\begin{aligned}(TP)d_L(\mathbf{k})(TP)^{-1} &= -id_L^\dagger(-\mathbf{k}); \quad (TP)d_R(\mathbf{k})(TP)^{-1} = id_R^\dagger(-\mathbf{k}) \\ Tc_L(\mathbf{k})T^{-1} &= -ic_L^\dagger(\mathbf{k}); \quad Tc_R(\mathbf{k})T^{-1} = ic_R^\dagger(\mathbf{k}) \\ (T\overline{C})f_L(\mathbf{k})(T\overline{C})^{-1} &= -if_L^\dagger(\mathbf{k}); \quad (T\overline{C})f_R(\mathbf{k})(T\overline{C})^{-1} = if_R^\dagger(\mathbf{k}),\end{aligned} \tag{A5}$$

with $(TP)^2 = -1$,$T^4 = -1$ and $(T\overline{C})^4 = -1$ symmetry.

Their corresponding Hamiltonians of massless Majorana Lagrangian Eq.(60) has the following form in momentum space, e.g., for $d_{L(R)}$ femion:

$$\mathcal{H}_d = \frac{1}{4}\sum_{\mathbf{k}} \psi^\dagger(\mathbf{k})\bar{\gamma}_0\bar{\gamma}_i k_i \psi(\mathbf{k}), \tag{A6}$$

where $\psi(\mathbf{k})$ is the Fourier mode of $\psi(\mathbf{x})$, defined as $\psi(\mathbf{k}) = \frac{1}{\sqrt{V}}\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\psi(\mathbf{x})$. It is straightforward to verify that $\psi^\dagger(\mathbf{k}) = \psi^t(-\mathbf{k})$. If we assume the chiral basis has a spin polarization in the y-direction, we can fix the momentum to be $\mathbf{k} = (0, k, 0)$. Thus, we obtain:

$$\mathcal{H}_d = \frac{1}{4}\sum_k \left[k\gamma_\uparrow(-k)\gamma_\uparrow(k) - k\gamma_\downarrow(-k)\gamma_\downarrow(k) - k\gamma'_\uparrow(-k)\gamma'_\uparrow(k) + k\gamma'_\downarrow(-k)\gamma'_\downarrow(k)\right] \tag{A7}$$

In terms of the chiral fermion fields $d_L(\mathbf{k})$ and $d_R(\mathbf{k})$, we have:

$$\mathcal{H}_d = \frac{1}{2}\sum_k \left[kd_L^\dagger(k)d_L(k) - kd_R^\dagger(k)d_R(k)\right] \tag{A8}$$

For any given momentum $\mathbf{k}$, we can define its positive energy mode as a left-handed particle and the negative energy mode as a right-handed anti-particle.

For $c_{L(R)}$ and $f_{L(R)}$, their Hamiltonian in momentum space read:

$$\mathcal{H}_{c(f)} = \frac{1}{4}\sum_{\mathbf{k}} \psi^\dagger(\mathbf{k})\gamma_0\gamma_i k_i \psi(\mathbf{k}), \tag{A9}$$

If we assume the chiral basis has a spin polarization in the z-direction, we can fix the momentum to be $\mathbf{k} = (0, 0, k)$ In terms of the $c_{L(R)}$ and $f_{L(R)}$ fermion operators, we have:

$$\begin{aligned}\mathcal{H}_d &= \frac{1}{2}\sum_k \left[kc_L^\dagger(k)c_L^\dagger(-k) - kc_R^\dagger(k)c_R^\dagger(-k) + h.c.\right]; \\ \mathcal{H}_f &= \frac{1}{2}\sum_k \left[kf_L^\dagger(k)f_L^\dagger(-k) - kf_R^\dagger(k)f_R^\dagger(-k) + h.c.\right],\end{aligned} \tag{A10}$$

In the Nambu basis, we obtain:

$$\mathcal{H}_c = \frac{1}{2}\sum_k \left[\, c_L^\dagger(k)+c_R^\dagger(k),\ c_L(-k)-c_R(-k) \,\right] \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \begin{bmatrix} c_L(k)+c_R(k) \\ c_L^\dagger(-k)-c_R^\dagger(-k) \end{bmatrix},$$

$$\mathcal{H}_f = \frac{1}{2}\sum_k \left[\, f_L^\dagger(k)-f_R^\dagger(k),\ f_L(-k)+f_R(-k) \,\right] \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \begin{bmatrix} f_L(k)-f_R(k) \\ f_L^\dagger(-k)+f_R^\dagger(-k) \end{bmatrix}, \tag{A11}$$

After diagonalizing the above two Hamiltonians, we can again define a positive mode corresponding to the left-handed particle and a negative energy mode corresponding to the right-handed anti-particle.

Appendix B: The $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ flavor symmetry and beyond

In this section, we provide a symmetry argument for the choice of Yukawa couplings in Eq.(81). Let us start with the diagonal term and assume there are three independent couplings g_f, g_d and g_c.

$$\mathcal{L}_{m-d} = \frac{i}{4}\phi(x)\left[g_f\overline{\psi}_f(x)\psi_f(x) + g_d\overline{\psi}_d(x)\psi_d(x) + g_c\overline{\psi}_c(x)\gamma_5\psi_c(x)\right] \tag{B1}$$

According to the definitions of $\psi_f(x)$ and $\psi_c(x)$, they are related to each other by a $\mathbb{Z}_2$ symmetry transformation $\psi_c(x) = S_1\psi_f(x)$, where:

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{B2}$$

Let us rewrite the diagonal mass term as:

$$\begin{aligned}
\mathcal{L}_{m-d} &= \frac{i}{4}\phi(x)\left[g_f\overline{\psi}_f(x)S_1^{-1}S_1\psi_f(x) + g_d\overline{\psi}_d(x)\psi_d(x) + g_c\overline{\psi}_c(x)\gamma_5 S_1 S_1^{-1}\psi_c(x)\right] \\
&= \frac{i}{4}\phi(x)\left[g_f\psi_f(x)^\dagger S_1^{-1}S_1\gamma_0 S_1^{-1}S_1\psi_f(x) + g_d\overline{\psi}_d(x)\psi_d(x) + g_c\psi_c(x)^\dagger S_1 S_1^{-1}\gamma_0\gamma_5 S_1 S_1^{-1}\psi_c(x)\right] \\
&= \frac{i}{4}\phi(x)\left[g_f\psi_f(x)^\dagger S_1^{-1}\gamma_0\gamma_5 S_1\psi_f(x) + g_d\overline{\psi}_d(x)\psi_d(x) + g_c\psi_c(x)^\dagger S_1\gamma_0 S_1^{-1}\psi_c(x)\right] \\
&= \frac{i}{4}\phi(x)\left[g_f\psi_c(x)^\dagger\gamma_0\gamma_5\psi_c(x) + g_d\overline{\psi}_d(x)\psi_d(x) + g_c\psi_f(x)^\dagger\gamma_0\psi_f(x)\right] \\
&= \frac{i}{4}\phi(x)\left[g_f\overline{\psi}_c(x)\gamma_0\gamma_5\psi_c(x) + g_d\overline{\psi}_d(x)\psi_d(x) + g_c\overline{\psi}_f(x)\psi_f(x)\right]
\end{aligned} \tag{B3}$$

Comparing Eq.(B1) and Eq.(B3), we obtain $g_c = g_f$.

Similarly, $\psi_f(x)$ and $\psi_d(x)$ are also related by another $\mathbb{Z}_2$ symmetry transformation $\psi_d(x) = S_2 \psi_f(x)$ with:

$$S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{B4}$$

Again, we can rewrite the diagonal mass term as:

$$\begin{aligned} \mathcal{L}_{m-d} &= \frac{i}{4}\phi(x)\left[g_f \overline{\psi}_f(x) S_2^{-1} S_2 \psi_f(x) + g_d \overline{\psi}_d(x) S_2 S_2^{-1} \psi_d(x) + g_c \overline{\psi}_c(x)\gamma_5 \psi_c(x)\right] \\ &= \frac{i}{4}\phi(x)\left[g_f \psi_f(x)^\dagger \gamma_0 S_2^{-1} S_2 \psi_f(x) + g_d \psi_d(x)^\dagger \gamma_5 S_2 S_2^{-1} \psi_d(x) + g_c \overline{\psi}_c(x)\gamma_5 \psi_c(x)\right] \\ &= \frac{i}{4}\phi(x)\left[g_f \psi_f(x)^\dagger S_2^{-1} S_2 \gamma_0 S_2^{-1} S_2 \psi_f(x) + g_d \psi_d(x)^\dagger S_2 S_2^{-1} \gamma_5 S_2 S_2^{-1} \psi_d(x) + g_c \overline{\psi}_c(x)\gamma_5 \psi_c(x)\right] \\ &= \frac{i}{4}\phi(x)\left[g_f \psi_f(x)^\dagger S_2^{-1} \gamma_5 S_2 \psi_f(x) + g_d \psi_d(x)^\dagger S_2 \gamma_0 S_2^{-1} \psi_d(x) + g_c \overline{\psi}_c(x)\gamma_5 \psi_c(x)\right] \\ &= \frac{i}{4}\phi(x)\left[g_f \psi_d(x)^\dagger \gamma_5 \psi_d(x) + g_d \psi_f(x)^\dagger \gamma_0 \psi_f(x) + g_c \overline{\psi}_c(x)\gamma_5 \psi_c(x)\right] \\ &= \frac{i}{4}\phi(x)\left[g_f \overline{\psi}_d(x)\gamma_5 \psi_d(x) + g_d \overline{\psi}_f(x)\gamma_0 \psi_f(x) + g_c \overline{\psi}_c(x)\gamma_5 \psi_c(x)\right] \end{aligned} \tag{B5}$$

Comparing Eq.(B1) and Eq.(B5), we obtain $g_d = g_f$. Finally, we have $g_c = g_d = g_f = g$. Now we see that in traditional quantum field theory language, the choice of diagonal Yukawa couplings in Eq.(81) can be achieved by imposing the above two $\mathbb{Z}_2$ symmetries, which leads to a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ flavor symmetry.

Nevertheless, traditional quantum field theory can not tell us why there are three generations of neutrinos and where the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ flavor symmetry comes from. To understand these mysteries, the internal structure proposed in this paper – a Majorana fermion is made up of four Majorana zero modes plays a crucial role. At cutoff scale, all the mass terms should be regarded as interactions between the scalar particle ϕ and the four Majorana modes $\gamma_\uparrow, \gamma_\downarrow, \gamma'_\uparrow$ and $\gamma'_\downarrow$. For example, all the three terms in Eq.(B1) can be expressed as:

$$\begin{aligned} \frac{ig_f}{4}\phi(x)\overline{\psi}_f(x)\psi_f(x) &= \frac{ig_f}{2}\phi(x)\left[\gamma_\uparrow(x)\gamma'_\uparrow(x) - \gamma_\downarrow(x)\gamma'_\downarrow(x)\right]; \\ \frac{ig_d}{4}\phi(x)\overline{\psi}_d(x)\psi_d(x) &= \frac{ig_d}{2}\phi(x)\left[\gamma_\uparrow(x)\gamma'_\uparrow(x) - \gamma_\downarrow(x)\gamma'_\downarrow(x)\right]; \\ \frac{ig_c}{4}\phi(x)\overline{\psi}_c(x)\gamma_5\psi_c(x) &= \frac{ig_c}{2}\phi(x)\left[\gamma_\uparrow(x)\gamma'_\uparrow(x) - \gamma_\downarrow(x)\gamma'_\downarrow(x)\right], \end{aligned} \tag{B6}$$

The above expression implies that the three mass terms are indeed the same term at cutoff. In terms of traditional quantum field theory language, we can attribute the existence of three generations of neutrinos to the three different (local) ways of making a pair of complex fermions out of four Majorana zero modes. Therefore, the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ flavor symmetry is indeed a gauge symmetry from our perspective and we obtain $g_c = g_d = g_f = g$. However, at this point, one may confuse that if the three mass terms are the same, why we observe three generations of neutrinos rather than one. This is because discrete gauge theory can have a deconfinement phase in 3D where the three generations of neutrinos becomes well defined at low energy.

The same argument also apply to the off-diagonal mass term:

$$\begin{aligned}
\mathcal{L}_{m-od} &= \frac{ig_{cd}}{4}\phi(x)\left[\overline{\psi}_d(x)(1+\gamma_0\gamma_5)(1+\gamma_5)\psi_c(x) + \overline{\psi}_c(x)(1+\gamma_5)(1-\gamma_0\gamma_5)\psi_d(x)\right] \\
&+ \frac{ig_{cf}}{4}\phi(x)\left[\overline{\psi}_f(x)(1+\gamma_5)\psi_c(x) + \overline{\psi}_c(x)(1+\gamma_5)\psi_f(x)\right] \\
&+ \frac{ig_{df}}{4}\phi(x)\left[\overline{\psi}_f(x)(1-\gamma_0\gamma_5)\psi_d(x) + \overline{\psi}_d(x)(1+\gamma_0\gamma_5)\psi_f(x)\right], \qquad \text{(B7)}
\end{aligned}$$

which can be expressed as:

$$\begin{aligned}
&\frac{ig_{cd}}{4}\phi(x)\left[\overline{\psi}_d(x)(1+\gamma_0\gamma_5)(1+\gamma_5)\psi_c(x) + \overline{\psi}_c(x)(1+\gamma_5)(1-\gamma_0\gamma_5)\psi_d(x)\right] \\
= &\frac{ig_{cd}}{2}\phi(x)\left[\gamma_\uparrow(x)\gamma'_\uparrow(x) - \gamma_\downarrow(x)\gamma'_\downarrow(x)\right]; \qquad \text{(B8)}
\end{aligned}$$

$$\begin{aligned}
&\frac{ig_{cf}}{4}\phi(x)\left[\overline{\psi}_f(x)(1+\gamma_5)\psi_c(x) + \overline{\psi}_c(x)(1+\gamma_5)\psi_f(x)\right] \\
= &\frac{ig_{cf}}{2}\phi(x)\left[\gamma_\uparrow(x)\gamma'_\uparrow(x) - \gamma_\downarrow(x)\gamma'_\downarrow(x)\right]; \qquad \text{(B9)}
\end{aligned}$$

$$\begin{aligned}
&\frac{ig_{df}}{4}\phi(x)\left[\overline{\psi}_f(x)(1-\gamma_0\gamma_5)\psi_d(x) + \overline{\psi}_d(x)(1+\gamma_0\gamma_5)\psi_f(x)\right] \\
= &\frac{ig_{df}}{2}\phi(x)\left[\gamma_\uparrow(x)\gamma'_\uparrow(x) - \gamma_\downarrow(x)\gamma'_\downarrow(x)\right], \qquad \text{(B10)}
\end{aligned}$$

Thus we can derive $g_{cd} = g_{cf} = g_{df} = g'$. Finally, by comparing the diagonal and off-diagonal mass terms, we can further derive $|g| = |g'|$. Here the relative sign of g and g' can not be fixed because this relation is not a consequence of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ flavor symmetry(We note that flavor symmetry can *not* relate diagonal and off-diagonal mass terms).

In conclusion, all the above results come from a single principle – the three generations of neutrinos/anti-neutrinos are the three resonating states out of the *same* four Majorana zero modes at cutoff scale. We conjecture that the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ flavor gauge symmetry proposed here is also crucial for understanding the charged lepton and quark mass hierarchy problem, which might originate from the spontaneously breaking of such a flavor gauge symmetry.

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