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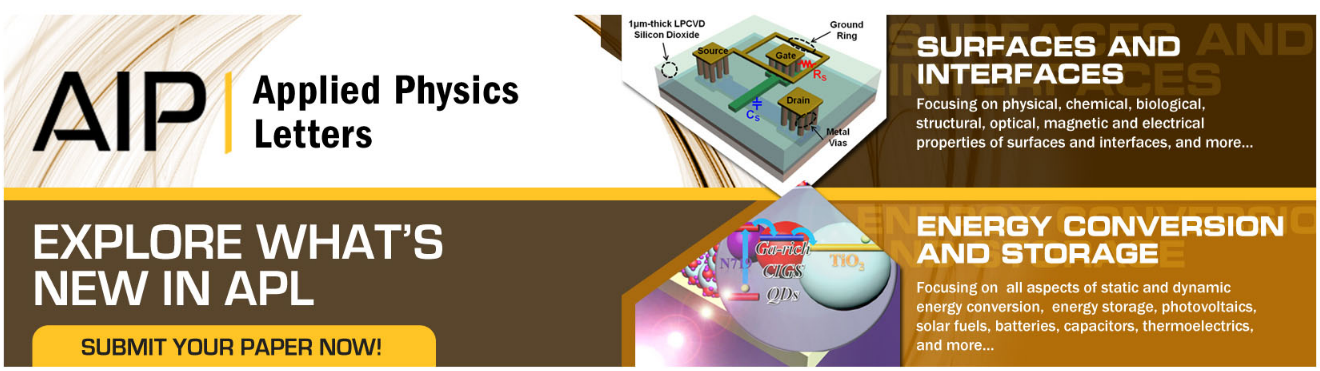

PHYSICS OF PLASMAS **20**, 082113 (2013)

Circular polarization of obliquely propagating whistler wave magnetic field

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The circular polarization of the magnetic field of obliquely propagating whistler waves is derived using a basis set associated with the wave partial differential equation. The wave energy is mainly magnetic and the wave propagation consists of this magnetic energy sloshing back and forth between two orthogonal components of magnetic field in quadrature. The wave electric field energy is small compared to the magnetic field energy. © *2013 AIP Publishing LLC*.
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I. INTRODUCTION

The whistler wave[1,2] is a cold magnetized plasma wave with dispersion relation

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{pe}^2/\omega^2}{1 - \frac{|\omega_{ce}|}{\omega}\cos\theta} \tag{1}$$

that is important in many different contexts with notable examples including chorus in Earth's magnetosphere,[3–5] collisionless magnetic reconnection,[6–8] the helicon plasma source,[9,10] and magnetic vortices.[2,11] The textbook derivation of whistler waves shows that the wave electric field is right-hand circularly polarized when the propagation wavevector is parallel to the background magnetic field.

In the limit that ω is slightly smaller than $|\omega_{ce}|\cos\theta$, the denominator of the second term is small and negative so that c^2k^2/ω^2 becomes very large; in this situation the wave is quasi-electrostatic and because the dispersion is $\omega \simeq |\omega_{ce}|\cos\theta$ the wave has a resonance cone character.

On the other hand, in the limit where ω is much smaller than $|\omega_{ce}|\cos\theta$ while the second term in Eq. (1) is still much larger than the first term (i.e., the "1" term), Eq. (1) reduces to

$$\omega = \frac{|\omega_{ce}|c^2k^2\cos\theta}{\omega_{pe}^2}, \tag{2}$$

which has the interesting feature of having no mass dependence. Magnetospheric measurements and helicon plasma sources are typically in the regime described by Eq. (2), i.e., are not in the resonance cone regime. We will restrict attention to the regime where the whistler wave is described by Eq. (2), i.e., we are assuming $\omega \ll |\omega_{ce}|\cos\theta$ and are not considering the resonance cone regime.

Tsurutani *et al.*[12] made the surprising observation that the magnetic field component of magnetospheric chorus, a form of whistler emission characterized by Eq. (2), is circularly polarized even when the wave vector is not parallel to the background magnetic field, i.e., is oblique. Verkhoglyadova *et al.*[13] have shown by means of rotation matrices that the wave magnetic field is indeed circularly polarized in this Eq. (2) regime even when the wave vector is oblique but the wave electric field vector is only circularly polarized when the wave vector is parallel to the background magnetic field. The purpose of this paper is to show that this result can be obtained in a somewhat more direct manner by examining solutions of the vector partial differential equation that underlies the dispersion relation given by Eq. (2). We will show that while the entire wave electric field is not circularly polarized, the components of the electric field transverse to the direction of propagation are circularly polarized. We will also show that so long as $\cos\theta$ is finite, the wave energy is predominantly magnetic and the wave involves energy sloshing back and forth between two orthogonal magnetic field components.

II. WHISTLER WAVE CURRENT AS A CONSEQUENCE OF ELECTRON E×B DRIFT

We consider waves propagating in a uniform background magnetic field $\mathbf{B} = B\hat{z}$. The whistler frequency regime is $\omega_{pi}, \omega_{ci} \ll \omega \ll \omega_{ce}, \omega_{pe}$ so ions may be considered stationary and the electrons undergo $\mathbf{E}\times\mathbf{B}$ drift. The whistler dispersion relation results when the entire current is assumed to result from this electron $\mathbf{E}\times\mathbf{B}$ drift so both displacement current and electron inertia are negligible. This corresponds to neglecting the "1" in Eq. (1) and the lack of dependence on mass in Eq. (2).

In this regime where ions are stationary and electron inertia can be dropped, the linearized electron equation of motion reduces to

$$\tilde{\mathbf{E}} + \tilde{\mathbf{u}}_e \times \mathbf{B} = 0, \tag{3}$$

where tildes denote first order (wave) variables. Because ions are stationary, the entire plasma wave current is given by

$$\tilde{\mathbf{J}} = nq_e\tilde{\mathbf{u}}_e. \tag{4}$$

We now show that the whistler wave regime characterized by Eq. (2) results when the above two equations are inserted into Faraday's law and Ampere's law.

Inserting Eq. (3) into Faraday's law gives

$$\frac{\partial\tilde{\mathbf{B}}}{\partial t} = \nabla\times(\tilde{\mathbf{u}}_e\times\mathbf{B}) \tag{5}$$

and substituting for $\tilde{\mathbf{u}}_e$ using Eq. (4) gives

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \frac{1}{nq_e}\nabla \times (\tilde{\mathbf{J}} \times \mathbf{B}). \tag{6}$$

Expanding the right hand side gives

$$\nabla \times (\tilde{\mathbf{J}} \times \mathbf{B}) = \tilde{\mathbf{J}}\nabla \cdot \mathbf{B} + \mathbf{B}\cdot\nabla\tilde{\mathbf{J}} - \mathbf{B}\nabla\cdot\tilde{\mathbf{J}} - \tilde{\mathbf{J}}\cdot\nabla\mathbf{B}. \tag{7}$$

Since $\mathbf{B}$ is spatially uniform and since $\nabla \cdot \tilde{\mathbf{J}} = 0$ in accordance with dropping displacement current all that is left is the second term in Eq. (7) so

$$\nabla \times (\tilde{\mathbf{J}} \times \mathbf{B}) = \mathbf{B}\cdot\nabla\tilde{\mathbf{J}} = B\frac{\partial \tilde{\mathbf{J}}}{\partial z} = \frac{B}{\mu_0}\frac{\partial}{\partial z}\nabla \times \tilde{\mathbf{B}}. \tag{8}$$

Thus, Eq. (6) becomes

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \frac{1}{nq_e}\frac{B}{\mu_0}\frac{\partial}{\partial z}\nabla \times \tilde{\mathbf{B}}, \tag{9}$$

which is the fundamental partial differential equation for whistler waves. The linear relation between $\tilde{\mathbf{B}}$ and $\nabla \times \tilde{\mathbf{B}}$ shows that whistler waves involve magnetic helicity, can have a vortex structure,[2] and have a mathematical similarity to force-free magnetic fields.[14]

The whistler wave dispersion relation results from operating on Eq. (9) with $B/(nq_e\mu_0)\,\partial/\partial z\nabla\times$ to obtain

$$\frac{\partial}{\partial t}\frac{1}{nq_e}\frac{B}{\mu_0}\frac{\partial}{\partial z}\nabla \times \tilde{\mathbf{B}} = \left(\frac{1}{nq_e}\frac{B}{\mu_0}\right)^2\frac{\partial^2}{\partial z^2}\nabla \times \nabla \times \tilde{\mathbf{B}}. \tag{10}$$

On using Eq. (9) to substitute in the left hand side this reduces to

$$\frac{\partial^2 \tilde{\mathbf{B}}}{\partial t^2} = -\left(\frac{1}{nq_e}\frac{B}{\mu_0}\right)^2\frac{\partial^2}{\partial z^2}\nabla^2\tilde{\mathbf{B}}. \tag{11}$$

Near a localized source the wave structure will be more complicated than a plane wave (e.g., the vortex structure discussed in Ref. 2), but far from the source a plane wave should be a good approximation. Using the plane wave dependence $\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$, Eq. (11) becomes

$$\omega^2 = \left(\frac{1}{nq_e}\frac{B}{\mu_0}\right)^2 k_z^2 k^2. \tag{12}$$

Noting that $B/(nq_e\mu_0) = \omega_{ce}c^2/\omega_{pe}^2$ the square root of Eq. (12) is

$$\omega = \frac{|\omega_{ce}|}{\omega_{pe}^2}c^2 k_z k, \tag{13}$$

where ω, k_z, and k are positive; thus Eq. (2) is recovered thereby proving that this limit of whistler waves results from the wave current coming from electron $\mathbf{E}\times\mathbf{B}$ drift.

III. NATURAL COORDINATE SYSTEM AND CIRCULAR POLARIZATION

In order to investigate circular polarization, we note that ω_{ce} is negative, use $B/(nq_e\mu_0) = \omega_{ce}c^2/\omega_{pe}^2$, and write Eq. (9) as

$$\omega\tilde{\mathbf{B}} = \frac{|\omega_{ce}|}{\omega_{pe}^2}c^2 k_z \nabla \times \tilde{\mathbf{B}}. \tag{14}$$

Intuitively, the linear relation between $\tilde{\mathbf{B}}$ and $\nabla \times \tilde{\mathbf{B}}$ suggests that the wave has a helical structure. However, because $\mathbf{k}\cdot\tilde{\mathbf{B}} = 0$, the wave magnetic field has no component along $\mathbf{k}$ and so it can be conjectured that the helix collapses to being a circle centered on $\mathbf{k}$, i.e., the wave magnetic field is conjectured to be circularly polarized about $\mathbf{k}$. This line of reasoning would not apply to the wave electric field since, unlike for the magnetic field, there is no guarantee that $\mathbf{k}\cdot\tilde{\mathbf{E}} = 0$.

Using a mathematical formalism given in Ref. 15 for a quite different context, we claim that the general plane wave solution to Eq. (14) is of the form

$$\tilde{\mathbf{B}} = k\nabla\chi \times \hat{z} + \nabla \times (\nabla\chi \times \hat{z}), \tag{15}$$

where $\chi \sim \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$, and ω is given by Eq. (13). To prove this assertion, consider

$$\begin{aligned}\nabla \times \tilde{\mathbf{B}} &= k\nabla \times (\nabla\chi \times \hat{z}) + \nabla \times \nabla \times (\nabla\chi \times \hat{z})\\ &= k\nabla \times (\nabla\chi \times \hat{z}) + \nabla\nabla\cdot(\nabla\chi \times \hat{z}) - \nabla^2(\nabla\chi \times \hat{z})\\ &= k\nabla \times (\nabla\chi \times \hat{z}) + k^2(\nabla\chi \times \hat{z})\\ &= k[\nabla \times (\nabla\chi \times \hat{z}) + k\nabla\chi \times \hat{z}]\\ &= k\tilde{\mathbf{B}}.\end{aligned} \tag{16}$$

Thus, Eq. (14) reduces to

$$\omega\tilde{\mathbf{B}} = \frac{|\omega_{ce}|}{\omega_{pe}^2}c^2 k_z k\tilde{\mathbf{B}}, \tag{17}$$

which becomes Eq. (13).

We now assume without loss of generality that $\mathbf{k} = k_x\hat{x} + k_z\hat{z}$ so Eq. (15) can be written as

$$\begin{aligned}\tilde{\mathbf{B}} &= ki(k_x\hat{x} + k_z\hat{z})\chi \times \hat{z} + i(k_x\hat{x} + k_z\hat{z}) \times (i(k_x\hat{x} + k_z\hat{z})\chi \times \hat{z})\\ &= -ikk_x\chi\hat{y} + i(k_x\hat{x} + k_z\hat{z}) \times (-ik_x\chi\hat{y})\\ &= -ikk_x\chi\hat{y} + (k_x\hat{z} - k_z\hat{x})k_x\chi.\end{aligned} \tag{18}$$

We note that this satisfies $\mathbf{k}\cdot\tilde{\mathbf{B}} = 0$. We now define

$$\tilde{\mathbf{B}}_1 = (k_x\hat{z} - k_z\hat{x})k_x\chi \tag{19}$$

and

$$\tilde{\mathbf{B}}_2 = -ikk_x\chi\hat{y}, \tag{20}$$

so

$$\tilde{\mathbf{B}} = \tilde{\mathbf{B}}_1 + \tilde{\mathbf{B}}_2. \tag{21}$$

We note that $\tilde{\mathbf{B}}_1\cdot\tilde{\mathbf{B}}_2 = 0$ so $\tilde{\mathbf{B}}$ is composed of two mutually orthogonal components, each orthogonal to $\mathbf{k}$. Let us define the unit vector

$$\hat{e}_1 = \frac{k_z\hat{x} - k_x\hat{z}}{\sqrt{k_x^2 + k_z^2}} = \frac{k_z\hat{x} - k_x\hat{z}}{k}, \tag{22}$$

so

$$\tilde{\mathbf{B}}_1 = -kk_x\chi\hat{e}_1. \tag{23}$$

It is seen that $\{\hat{e}_1, \hat{y}, \hat{k}\}$ form a right-handed Cartesian basis set since

$$\begin{aligned}
\hat{e}_1 \times \hat{y} &= \left(\frac{k_z\hat{x} - k_x\hat{z}}{k}\right) \times \hat{y} = \frac{k_z\hat{z} + k_x\hat{x}}{k} = \hat{k} \\
\hat{y} \times \hat{k} &= \hat{y} \times \left(\frac{k_x\hat{x} + k_z\hat{z}}{k}\right) = \frac{-k_x\hat{z} + k_z\hat{x}}{k} = \hat{e}_1 \\
\hat{k} \times \hat{e}_1 &= \left(\frac{k_x\hat{x} + k_z\hat{z}}{k}\right) \times \left(\frac{k_z\hat{x} - k_x\hat{z}}{k}\right) = \hat{y}.
\end{aligned} \tag{24}$$

We then define $\hat{e}_2 = \hat{y}$ and $\hat{e}_3 = \hat{k}$ so

$$\tilde{\mathbf{B}}_1 = -kk_x\chi\hat{e}_1, \tag{25}$$

$$\tilde{\mathbf{B}}_2 = -ikk_x\chi\hat{e}_2. \tag{26}$$

Thus $\tilde{\mathbf{B}} \cdot \hat{e}_3 = 0$, the wave propagates along $\hat{e}_3$, and

$$\frac{\tilde{B}_2}{\tilde{B}_1} = i \tag{27}$$

so

$$\tilde{\mathbf{B}} = \tilde{B}_1(\hat{e}_1 + i\hat{e}_2) \tag{28}$$

which gives our main result: the wave magnetic field is circularly polarized even though $\hat{e}_3$ is oblique with respect to $\hat{z}$. The circular polarization is thus respect to the $\hat{e}_3$ direction, i.e., with respect to the direction of propagation. This result, first obtained by Verkhoglyadova *et al.*,[13] is important because exact parallel propagation is a singular situation that would never occur in reality and so circular polarization would be an asymptotic limit rather than a physically observable property. The circular polarization of the magnetic field for oblique propagation means that real whistler waves can be identified by the magnetic field circular polarization even though the wave is propagating obliquely. This property should be useful in non-magnetospheric contexts such as fusion plasmas, magnetic reconnection, and helicon devices. In particular, it should be useful for establishing the polarization of magnetic antennas for launching oblique whistler waves.

The wave electric field of an obliquely propagating whistler wave is

$$\tilde{\mathbf{E}} = -\frac{1}{nq_e}\tilde{\mathbf{J}} \times \mathbf{B} = -\frac{1}{\mu_0 nq_e}k\tilde{\mathbf{B}} \times \mathbf{B} = \frac{kB}{\mu_0 nq_e}\hat{z} \times \tilde{\mathbf{B}}. \tag{29}$$

Unlike the wave magnetic field, the wave electric field has an $\hat{e}_3$ component for oblique propagation since

$$\begin{aligned}
\tilde{\mathbf{E}} \cdot \hat{e}_3 &= \frac{kB}{\mu_0 nq_e}\hat{z} \times \tilde{\mathbf{B}} \cdot \hat{e}_3 \\
&= \frac{kB\tilde{B}_1}{\mu_0 nq_e}\hat{z} \cdot (\hat{e}_1 + i\hat{e}_2) \times \hat{e}_3 \\
&= \frac{kB\tilde{B}_1}{\mu_0 nq_e}\hat{z} \cdot (-\hat{e}_2 + i\hat{e}_1) \\
&= -\frac{ik_x B\tilde{B}_1}{\mu_0 nq_e}.
\end{aligned} \tag{30}$$

Thus, as shown in Verkhoglyadova *et al.*,[13] the electric field does not lie in the $\hat{e}_1$, $\hat{e}_2$ plane if $k_x \neq 0$ and so the electric field is not circularly polarized with respect to the propagation vector **k**. However direct calculation shows that

$$\frac{\tilde{E}_2}{\tilde{E}_1} = \frac{\hat{z} \times \tilde{\mathbf{B}} \cdot \hat{e}_2}{\hat{z} \times \tilde{\mathbf{B}} \cdot \hat{e}_1} = \frac{\hat{z} \cdot \tilde{\mathbf{B}} \times \hat{e}_2}{\hat{z} \cdot \tilde{\mathbf{B}} \times \hat{e}_1} = \frac{\hat{z} \cdot \tilde{\mathbf{B}}_1 \times \hat{e}_2}{\hat{z} \cdot \tilde{\mathbf{B}}_2 \times \hat{e}_1} = \frac{\hat{z} \cdot \hat{e}_1 \times \hat{e}_2}{\hat{z} \cdot i\hat{e}_2 \times \hat{e}_1} = i, \tag{31}$$

so the $\hat{e}_1$, $\hat{e}_2$ components of the electric field wave electric field are circularly polarized. Thus, while the entire electric field is in general not circularly polarized, the components orthogonal to **k** are. In the special case where **k** is parallel to $\hat{z}$ so $k_x = 0$, Eq. (30) shows that $\mathbf{k} \cdot \mathbf{E}$ vanishes so the electric field would be circularly polarized. This is in contrast to the wave magnetic field which has no $\hat{e}_3$ component so the entire wave magnetic field is always circularly polarized. Hence, as observed by Tsurutani *et al.*,[12] measurements of the time dependence of the direction of the local magnetic field vector will give a rotation of the field vector that can be used to determine the direction of **k**, whereas measurements of the time dependence of the direction of the local electric field vector cannot be so used.

Textbooks conventionally discuss waves using the wave electric field as the fundamental quantity, not the wave magnetic field. Thus, if as demonstrated here, there exists a special symmetry property associated with the wave magnetic field but not the wave electric field, conventional textbook treatments will fail to discern this uniquely magnetic property.

IV. WAVE ENERGY CONTENT

The time averaged wave electric field energy is

$$\begin{aligned}
W_E &= \frac{1}{4}\varepsilon_0\tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}^* \\
&= \frac{1}{4}\left(\frac{kB}{\mu_0 nq_e}\right)^2 \varepsilon_0\hat{z} \times \tilde{\mathbf{B}} \cdot \hat{z} \times \tilde{\mathbf{B}}^* \\
&= \frac{\varepsilon_0}{4}\left(\frac{kB}{\mu_0 nq_e}\right)^2 \left(|\tilde{B}|^2 - |\hat{z} \cdot \tilde{\mathbf{B}}|^2\right) \\
&= \frac{\varepsilon_0}{4}\left(\frac{kB}{\mu_0 nq_e}\right)^2 \tilde{B}_1^2\left(|(\hat{e}_1 + i\hat{e}_2)|^2 - |\hat{z} \cdot (\hat{e}_1 + i\hat{e}_2)|^2\right) \\
&= \frac{\varepsilon_0}{4}\left(\frac{kB}{\mu_0 nq_e}\right)^2 \tilde{B}_1^2\left(2 - \frac{k_x^2}{k^2}\right).
\end{aligned} \tag{32}$$

The time averaged wave magnetic field energy is

$$W_B = \frac{\tilde{\mathbf{B}} \cdot \tilde{\mathbf{B}}^*}{4\mu_0} = \frac{\tilde{B}_1^2}{4\mu_0}|(\hat{e}_1 + i\hat{e}_2)|^2 = \frac{\tilde{B}_1^2}{2\mu_0} \tag{33}$$

so the ratio of electric field to magnetic field energy is

$$\begin{aligned}\frac{W_E}{W_B} &= \frac{\mu_0\varepsilon_0}{2}\left(\frac{kB}{\mu_0 n q_e}\right)^2\left(2-\frac{k_x^2}{k^2}\right)\\ &= \frac{\mu_0\varepsilon_0}{2}k^2\left(\frac{\omega_{ce}c^2}{\omega_{pe}^2}\right)^2\left(2-\frac{k_x^2}{k^2}\right)\\ &= \frac{\mu_0\varepsilon_0}{2}k^2\left(\frac{\omega}{k_z k}\right)^2\left(2-\frac{k_x^2}{k^2}\right)\\ &= \frac{\omega^2}{k_z^2c^2}\left(1-\frac{k_x^2}{2k^2}\right).\end{aligned} \tag{34}$$

Thus to the extent that $c^2k_z^2/\omega^2 \gg 1$ which is true if $c^2k^2/\omega^2 \gg 1$ and $\cos\theta$ is not infinitesimal, the electric field energy content is negligible. The wave has no particle kinetic energy content because electron inertia is assumed zero and ions are assumed stationary. There is no thermal energy content because the plasma is considered cold.

V. WAVE PROPAGATION AS A SLOSHING BACK AND FORTH BETWEEN TWO COMPONENTS OF MAGNETIC ENERGY

Section IV showed that if the wave propagation is not excessively oblique, most of the wave energy is magnetic and there is no particle kinetic energy. Thus, unlike electrostatic waves, the wave does not involve sloshing of energy back and forth between electric field energy and particle kinetic energy. Instead, as will now be shown, the wave can be considered as consisting of energy sloshing back and forth between $|\tilde{B}_1|^2$ and $|\tilde{B}_2|^2$, i.e., between two components of magnetic field energy.

Since

$$\begin{aligned}\nabla\times\tilde{\mathbf{B}}_1 &= k\tilde{\mathbf{B}}_2\\ \nabla\times\tilde{\mathbf{B}}_2 &= k\tilde{\mathbf{B}}_1,\end{aligned} \tag{35}$$

it is seen that Eq. (9) can be decomposed into the two coupled equations

$$\begin{aligned}\frac{\partial\tilde{\mathbf{B}}_1}{\partial t} &= \frac{\omega_{ce}}{\omega_{pe}^2}c^2\frac{\partial}{\partial z}\nabla\times\tilde{\mathbf{B}}_2\\ \frac{\partial\tilde{\mathbf{B}}_2}{\partial t} &= \frac{\omega_{ce}}{\omega_{pe}^2}c^2\frac{\partial}{\partial z}\nabla\times\tilde{\mathbf{B}}_1\end{aligned} \tag{36}$$

thus demonstrating that the wave involves a sloshing back and forth between the two orthogonal magnetic fields $\tilde{\mathbf{B}}_1$ and $\tilde{\mathbf{B}}_2$.

VI. POYNTING FLUX AND GROUP VELOCITY

Using Eqs. (13), (21), and (29), and the unit vectors $\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$, the Poynting vector is found to be

$$\begin{aligned}\mathbf{S} &= \frac{1}{2\mu_0}\mathrm{Re}(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*)\\ &= \frac{1}{2\mu_0}\frac{kB}{\mu_0 n q_e}\mathrm{Re}(\tilde{\mathbf{B}}\hat{z}\cdot\tilde{\mathbf{B}}^* - \hat{z}|\tilde{B}|^2)\\ &= \frac{1}{2\mu_0}\frac{kB|\tilde{B}_1|^2}{\mu_0 n q_e}\mathrm{Re}[(\hat{e}_1+i\hat{e}_2)\hat{z}\cdot(\hat{e}_1-i\hat{e}_2)\\ &\quad -\hat{z}(\hat{e}_1+i\hat{e}_2)\cdot(\hat{e}_1-i\hat{e}_2)]\\ &= \frac{c^2}{\omega_{pe}^2}\frac{|\omega_{ce}|\,|\tilde{B}_1|^2}{2\mu_0}(k_z\hat{k}+k\hat{z}).\end{aligned} \tag{37}$$

The group velocity is defined as the velocity with which wave energy is transported. Using Eq. (13) the group velocity is

$$\begin{aligned}\frac{\partial\omega}{\partial\mathbf{k}} &= \frac{|\omega_{ce}|}{\omega_{pe}^2}c^2\frac{\partial}{\partial\mathbf{k}}(k_z k)\\ &= \frac{|\omega_{ce}|}{\omega_{pe}^2}c^2\left(k\frac{\partial k_z}{\partial\mathbf{k}}+k_z\frac{\partial k}{\partial\mathbf{k}}\right)\\ &= \frac{|\omega_{ce}|}{\omega_{pe}^2}c^2(k\hat{z}+k_z\hat{k}).\end{aligned} \tag{38}$$

The wave energy flux $\mathbf{\Gamma}$ as determined by the group velocity and Eq. (33) is

$$\mathbf{\Gamma} = W_B\frac{\partial\omega}{\partial\mathbf{k}} = \frac{|\tilde{B}_1|^2}{2\mu_0}\frac{|\omega_{ce}|}{\omega_{pe}^2}c^2(k\hat{z}+k_z\hat{k}) \tag{39}$$

showing that $\mathbf{\Gamma} = \mathbf{S}$, thus demonstrating that to the extent that $c^2k_z^2/\omega^2 \gg 1$, the Poynting flux constitutes the entire wave energy flux, the wave energy is almost entirely magnetic, and the wave dynamics consists of energy sloshing back and forth between $\tilde{\mathbf{B}}_1$ and $\tilde{\mathbf{B}}_2$.

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