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A WAKE MODEL FOR FREE-STREAMLINE FLOW THEORY

PART I. FULLY AND PARTIALLY DEVELOPED WAKE FLOWS AND CAVITY FLOWS PAST AN OBLIQUE FLAT PLATE

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A wake model for the free-streamline theory is proposed to treat the two-dimensional flow past an obstacle with a wake or cavity formation. In this model the wake flow is approximately described in the large by an equivalent potential flow such that along the wake boundary the pressure first assumes a prescribed constant under-pressure in a region downstream of the separation points (called the near-wake) and then increases continuously from this under-pressure to the given free stream value in an infinite wake strip of finite width (the far-wake). The boundary of the wake trailing a lifting body is allowed to change its slope and curvature at finite distances from the body and is required to be parallel to the main stream only asymptotically at downstream infinity. The pressure variation along the far-wake takes place in such a way that the upper and lower boundaries of the far-wake form a branch slit of undetermined shape in the hodograph plane. One advantage of this wake model is that it provides a rather smooth continuous transition of the hydrodynamic forces from the fully developed wake flow to the fully wetted flow as the wake disappears. When applied to the wake flow past an inclined flat plate, this model yields the exact solution in a closed form for the whole range of the wake under-pressure coefficient. The separated flow over a slightly cambered plate can be calculated by a perturbation theory based on this exact solution.

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1. Introduction

For the physical flow of a real fluid past a bluff body, experimental

observations indicate that the flow generally separates from certain points of separation on the obstacle, resulting in a wake formation, or, in the case of the cavitating flow of a liquid medium, a vapor-gas cavity occupying a near-wake region. Striding across the separated streamline there is the so-called free shear layer which is known experimentally to be thin and usually quite steady within a certain distance downstream of the separation point. In the case of one-phase flows (such as air), the flow velocities vary rapidly across the shear layer compared with their changes along the streamlines. In the case of cavity flows, there also exist free streamlines which form the cavity boundary and thereby partition flows of two different densities. The vorticity in these shear layers is fed partly from the boundary layer in front of the separation point and partly from the reversed flow inside the wake. In view of the fact that the boundary layer over the body and the separated shear layer are thin, the flow exterior to this region may be considered irrotational as a first-order approximation, and the boundary of this potential flow may be approximated by the body surface and the free streamline. For this reason this part of the wake will be called the "near-wake", or the "free streamline range". The pressure in such a region is in general approximately constant, which will be called the wake under-pressure, or the cavity pressure in case of cavity flows.

In the further downstream, however, the shear layer gradually becomes broader as the vorticity diffuses and at the same time non-uniformity of the pressure distribution across the layer increases. As a result, these shear layers become unstable and do not continue smoothly far downstream, but roll up to form vortices or become directly the region of turbulent mixing. In the case of one-phase flows, the vortices formed are rather orderly within a certain range of the Reynolds number (e.g., for a circular cylinder, this range of the Reynolds number, based on the diameter, may span from 40 to 10^5 or more). The vortices are shed alternately on each side of the body with a characteristic frequency to form Karman vortex street. These vortices mix and diffuse rapidly and are eventually dissipated in the wake. In the case of cavitating flows, rather regular vortex wakes behind the cavity are usually observed also. In the consideration of the stability of the cavity boundary, the analysis of the Helmholtz instability of the slip-streams shows that small perturbations in them grow exponentially. The

rate of growth is roughly proportional to $\sqrt{\rho'/\rho}$ where ρ' is the smaller of the densities on the two sides of the slipstream (see, e.g., Birkhoff and Zarantonello (1957)). This provides the reason why the cavity slipstream generally appears less unstable than the free shear layers in one-phase wake flows. After the cavity closes, however, the flow is rather similar in nature to the ordinary wake flows. In such a range, the shape of the free streamline cannot be determined definitely and (with a constant upstream velocity) the wake flow is only stationary in the mean. This part of the wake will be called the "far-wake", or the "mixing range". In between the near and far wakes there may exist a transition region in which the separated free streamlines from the two sides of the body reattach to each other. Along the far wake the mean pressure increases gradually from the wake underpressure (or the cavity pressure) and finally recovers the main stream pressure at the downstream infinity.

As pointed out above, it may be expected on physical grounds that the flow outside the obstacle and the near-wake region may be approximated with good accuracy by a potential flow. Only when the attempt is made to extend this approximate potential flow to large distances from the body (including the far-wake) do the various wake flow and cavity flow models arise, such as the Riabouchinsky model, (Riabouchinsky, 1920) the re-entrant jet model, (see,e.g., Kreisel, 1946; Gilbarg and Serrin, 1950) and the Roshko wake model (Roshko, 1954, 1955). Some physical significances of these models have been discussed by Wu (1956). In each of these flow models an artifice of some sort is invariably introduced to admit the under-pressure coefficient as a free parameter, to account for the essential feature of the wake flow that the very complicated process of dissipation always takes place due to the viscous effects in the wake, and to replace the actual wake flow of the real fluid by a simplified device within the framework of an equivalent potential flow. The validity of these flow models will therefore have to be justified by their agreement with experimental observations of the actual flow field near the body as well as the hydrodynamic forces and moments acting on the body.

The problem of the steady, fully cavitated plane flow past a lifting hydrofoil of arbitrary shape has been treated by Wu (1956), adopting the Roshko model (for its relative mathematical simplicity) and using a

generalized Levi-Civita method for curved barriers in cavity flows. In the application of the general theory, two specific examples were carried out for the inclined flat plate and circular arc hydrofoils. The results in both cases are found in very good agreement with the experimental results of Parkin (1956) for the case when the cavity is longer than the hydrofoil. This theory, however, is unable to predict the conditions under which the cavity terminates in front of the trailing edge (the so-called partial cavity flow), and also fails to provide a correction to the hydrodynamic forces for such case. Furthermore, when this theory was later applied to hydrofoils of arbitrary shape the details of the numerical procedure was found to be quite complicated and the numerical instability was observed by this author on occasions. The Roshko model has also been applied by Mimura (1958) to treat the wake flow past an oblique flat plate, yielding results in good agreement with the experiments of Fage and Johansen (1927) for incidence angle α greater than 30°. Some deviation appears, however, for $\alpha = 15^{\circ}$ when the constant pressure near-wake becomes shorter than the plate. The drawbacks observed in these applications of the Roshko model promoted the present investigation which is aimed to remove these difficulties and to achieve further simplifications of the unnecessary details of the physical flow.

In order to develop a theory for the wake flow or cavity flow in which the region of constant pressure may have an arbitrary length, a plane wake flow model is proposed here with the following physical assumptions:
(i) The entire separated wake region is postulated to be bounded by two smooth free streamlines, each of which consists of two different parts. The first part covers the near-wake region which starts from the separation point down to a certain point which is determined from the theory. The pressure along this part assumes a given constant value, p_c, the wake under-pressure or the cavity pressure. The value of p_c will be assumed to be less than the free stream pressure, p_∞, throughout this work. Along the remaining part of the free streamline the pressure is assumed to change continuously from p_c to p_∞ at the downstream infinity.
(ii) The only kinematic condition on the free streamlines in the <u>physical plane</u> is that they become asymptotically parallel to the main flow at infinity.
(iii) The flow outside the wake is assumed inviscid and irrotational.

(iv) The images of the variable-pressure parts of the two free streamlines in the velocity plane (or the hodograph plane) are assumed to form a branch slit of undetermined shape. This will be referred to as the "hodograph-slit condition". Although this assumed flow configuration becomes too over-simplified and hence invalid in the far-wake, it is to be expected that the rough approximation of the far-wake will bear no predominant influence on the actual flow field near the body.

This wake model will now be applied to treat the wake (or cavity) flow past an oblique flat plate. The wake flow will be called fully developed (or fully cavitating in case of cavity flows) if the region of the constant pressure near-wake extends beyond the trailing edge of the plate, and will be called partially developed (or partially cavitating) if the near-wake region terminates in front of the trailing edge. For brevity these two flow regimes will also be called the full wake flow and the partial wake flow. The latter case will be treated separately in part B, in which case the constant pressure part on the lower free streamline disappears. Furthermore, these analyses may be utilized to treat the wake flow past a plate with a small camber; the resultant flow may be considered as a small perturbation with respect to the basic (nonlinear) flow past the flat plate. The treatment of this last problem, however, will be postponed to a future paper.

It remains to state that the present free streamline theory is applicable to both the wake flows in one-phase medium (such as in air) and the cavity flows in water since the present theoretical result is found in good agreement with the experimental observations of Fage and Johansen, which deals with the wake flow in air, and Parkin's experiments, which are concerned with cavity flows in water. However, it may be conjectured that in the unsteady flow case there would exist a marked difference between the one-phase wake flow and the cavity flow past the same obstacle at equal values of the wake under-pressure coefficient and the corresponding cavitation number. One argument in support of this contention is that the inertial effects due to the fluid matter inside the one-phase wake and that inside the vapor-gas cavity in a liquid have different specific densities and hence bear different dynamic influences on the corresponding flow in unsteady motion.

A. FULLY DEVELOPED WAKE FLOWS AND CAVITY FLOWS

2. Analysis of the Flow Field

Adopting the present wake model, we consider specifically the steady, irrotational plane flow of an incompressible fluid, with free stream velocity U and pressure p_∞, impinging on an oblique flat plate such that the flow is separated from the leading edge A and trailing edge B, forming two free streamlines ACI and $BC'I$ which are assumed to become asymptotically parallel to the main stream at the downstream infinity I (see Figure 1). The shape of ACI and $BC'I$ in the physical plane is otherwise unknown á priori. We adopt a Cartesian coordinate system (x,y), with the x-axis lying along the plate AB and the origin at the leading edge A. The flow outside the wake is assumed inviscid and irrotational, hence there exists a velocity potential ϕ. As usual, we introduce the complex variable $z = x + iy$, the complex potential $f(z) = \phi + i\psi$, and the complex velocity

$$w(z) = df/dz = u - iv = q e^{-i\theta} \tag{1}$$

where u, v are the x, y-components of the velocity, $q = |w|$, and θ is the inclination of the velocity vector to the x-axis. Let α be the angle of attack of the plate, then

$$w = w_o = U e^{-i\alpha} \quad \text{at} \quad z = \infty \ . \tag{2}$$

The kinematic and dynamic conditions on the free streamlines ACI and $BC'I$ are imposed as follows. We assume that the pressure

$$p = p_c \leqslant p_\infty \qquad \text{along } AC \text{ and } BC' \ , \tag{3}$$

and that p varies continuously and monotonically from p_c to p_∞ along CI and $C'I$. From (3) and the Bernoulli equation it follows that

$$q = \text{const.} = q_c \qquad \text{along } AC \text{ and } BC' \ , \tag{3a}$$

so that the Bernoulli equation of the external flow may be written

$$p + \frac{1}{2}\rho q^2 = p_\infty + \frac{1}{2}\rho U^2 = p_c + \frac{1}{2}\rho q_c^2 \ . \tag{4}$$

Since the points C and C' are located on the two branches of the same stagnation streamline $\psi = 0$, and since they are the end points of a constant pressure region, we obviously have

$$\psi_C = \psi_{C'} = 0 \ , \quad q_C = q_{C'} \ . \tag{5a}$$

For the complete determination of the points C and C', we no introduce the assumptions that the potential ϕ and the flow inclination θ have respectively the same values at C and C':

$$\phi_C = \phi_{C'} \ , \quad \theta_C = \theta_{C'} \ . \tag{5b}$$

Equations (5a) and (5b) can be combined as

$$f_C = f_{C'} \ , \quad w_C = w_{C'} \ . \tag{6}$$

In the far-wake region bounded by streamlines CI and C'I, the flow is assumed to be dissipated in such a way that p and w on the boundary CI and C'I change monotonically from p_c and w_C, eventually recovering their main stream values p_∞ and w_o at the downstream infinity. The images of the free streamlines CI and C'I, on which $\psi = 0$, is further assumed to form a branch slit (of undetermined shape) in the hodograph w-plane so that the flow field in the hodograph plane will be simply connected and simply covered. This condition is called the <u>hodograph-slit condition</u>. The locations of C and C' in the physical plane are otherwise unknown a priori, and must be determined as a part of the problem. The postulated configuration of the fully developed wake flow (or cavity flow) requires that $\mathrm{Re}(z_{C'} - z_B) \geqslant 0$; otherwise the wake flow becomes partially developed. Aside from this phenomenological description of the free streamlines, the details of the flow within the wake (presumably viscous and rotational) are otherwise immaterial in connection with the exterior flow and hence will not be pursued further in the present work.

It may be pointed out here that in the previous treatment of similar flow problems by Wu (1956) and Mimura (1958), the two conditions in (5b) were replaced by $\theta_C = \alpha$ and $\theta_{C'} = \alpha$, and the hodograph-slit condition was reduced to the special form that CI and C'I become straight lines parallel to the main flow. The reasons for adopting the present conditions

are first, this model gives a reasonably good description of the flow outside the wake in comparison of the actual flow visualization; second, use of these conditions provides a rather smooth transition to the partial wake flow; and lastly, the present flow model results in a simplified analysis for the problem. The validity of the present theory of course may only be justified by its agreement with the experimental results.

For simplicity, both the plate length ℓ and the constant speed q_c on the cavity wall will be normalized to unity so that from (4) we have

$$q_c = 1 \quad , \qquad U = (1 + \sigma)^{-1/2} \tag{7a}$$

where

$$\sigma = (p_\infty - p_c) \; / \; (\tfrac{1}{2} \rho U^2) \quad . \tag{7b}$$

The dimensionless parameter σ is usually called the wake under-pressure coefficient, or the cavitation number for cavity flows; this parameter characterizes the wake flow. In fact, the different flow regimes of the fully and partially developed wake flows can also be indicated by different ranges of σ.

Several streamlines ψ = const. in the w-plane are illustrated in Figure 1. Under the normalization $q_c = 1$ and the hodograph-slit condition, the bounding streamline $\psi = 0$ forms the boundary of the semi-circle of unit radius in the lower half w-plane and the slit CIC'; the entire flow is mapped onto the interior of the simply-covered semi-circle. It is convenient to introduce the parametric ζ-plane, defined by

$$\zeta = \frac{1}{2} (w^{-1} + w) \tag{8a}$$

with the inverse transformation

$$w = \zeta - (\zeta^2 - 1)^{1/2} \tag{8b}$$

for which a branch cut is made along the ζ-axis between $\zeta = -1$ and 1, the branch is chosen such that $(\zeta^2 - 1)^{1/2} \rightarrow \zeta$, and hence $w \rightarrow 0$, as $\zeta \rightarrow \infty$. By the transformation (8) the entire flow is mapped onto the upper half ζ-plane, with the point $w_o = Ue^{-i\alpha}$ mapped into

$$\zeta_o = \frac{1}{2}\,(w_o^{-1} + w_o) = \frac{1}{2}\,(\frac{1}{U}\,e^{i\alpha} + U\,e^{-i\alpha}) \quad . \tag{9}$$

Since $\psi = 0$ on the entire real ζ-axis, the complex potential $f(\zeta)$ can be continued analytically into the lower half ζ-plane by

$$f(\bar{\zeta}) = \overline{f(\zeta)} \quad . \tag{10}$$

From the asymptotic behavior of the streamlines ψ = const. passing through the point I, it is evident that $f(\zeta)$ must have a simple pole at $\zeta = \zeta_o$. The function $f(\zeta)$ is regular everywhere else in the upper half ζ-plane; in particular, $f = 0$ at $\zeta = \infty$. Let the complex constant k be the residue of $f(\zeta)$ at ζ_o, and consider the function $g(\zeta) = f(\zeta) - k(\zeta - \zeta_o)^{-1} - \bar{k}\,(\zeta - \bar{\zeta}_o)^{-1}$. By virtue of (10) it is obvious that $g(\bar{\zeta}) = \overline{g(\zeta)}$ whereby $g(\zeta)$ is continued analytically into the entire ζ-plane. But $g(\zeta)$ is seen to be bounded everywhere in the ζ-plane and hence must be a constant by Liouville's theorem. This constant is zero since $g = 0$ at $\zeta = \infty$. Therefore we have

$$f = \frac{k}{\zeta - \zeta_o} + \frac{\bar{k}}{\zeta - \bar{\zeta}_o} \quad .$$

Since a small circle (counterclockwise) around D in the f-plane is mapped into a large semi-circle (clockwise) in the ζ-plane, it follows that $f \longrightarrow 0$ like ζ^{-2} as $|\zeta| \longrightarrow \infty$. To ensure this we must have $k + \bar{k} = 0$, or k is purely imaginary. Hence the above equation may be written

$$F = \frac{A}{4}\,\frac{1}{(\zeta - \zeta_o)(\zeta - \bar{\zeta}_o)} \tag{11}$$

where A is a real constant. Since $\psi = 0$ on the lines CI and C'I, we must have $\arg(\zeta - \zeta_o) + \arg(\zeta - \bar{\zeta}_o) = 0$ for ζ lying on these lines; hence CI and C'I are straight lines parallel to the imaginary ζ-axis. In particular, at C and C', we find

$$\zeta_C = \zeta_{C'} = \mathrm{Re}\,\zeta_o = \frac{1}{2}\,(U^{-1} + U)\cos\alpha \quad .$$

We shall now assign the range of U for the full wake flow (or the fully cavitating flow) by the condition* that the point C' is located down-

*For further discussion of this condition, see the remarks following equation (17).

stream of the trailing edge B, that is, $\zeta_{C'} \leqslant \zeta_B = 1$. The corresponding range of U is determined from this condition and the above equation as

$$U_1 \leqslant U \leqslant 1 \ , \qquad U_1 = (1 - \sin\alpha)/\cos\alpha = \cos\alpha/(1 + \sin\alpha) \tag{12a}$$

in which the upper limit $U \leqslant 1$ follows from the physical condition $\sigma \geqslant 0$ (see Equation 7a). From (7a) and (12a) we find for the full wake flow

$$0 \leqslant \sigma \leqslant \sigma_1 \ , \quad \sigma_1 = U_1^{-2} - 1 = 2\tan\alpha\cot\left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \ . \tag{12b}$$

Note that $\sigma_1 \sim 2\alpha$ as $\alpha \to 0$, and $\sigma_1 \sim 4(\alpha - \pi/2)^{-2}$ as $\alpha \to \pi/2$. Thus for a given α, $0 < \alpha < \pi/2$, U has a lower limit $U_1(\alpha)$ and σ has an upper limit $\sigma_1(\alpha)$ for the fully developed wake flow. Let $W = e^{-i\gamma}$ at C and C', then from (8),

$$\cos\gamma = \zeta_C = \frac{1}{2}(U^{-1} + U)\cos\alpha \ . \tag{13}$$

This equation asserts that $0 \leqslant \gamma \leqslant \alpha$ for U lying in the range (12a). Thus the flow inclination γ at C and C' is always less than its free stream value α; they are equal only when $U = 1$.

Combining (8), (9) and (11), we obtain

$$f = \frac{Aw^2}{(w - w_o)(w - \bar{w}_o)(w - 1/w_o)(w - 1/\bar{w}_o)} \tag{14}$$

where w_o is given by (2). We see here that the present model yields the complex potential as a one-valued analytic function of w in a closed form. The above solution can actually be written down directly by the following argument. As previously explained, f has a simple pole at $w = w_o$; this is the first factor in the denominator of (14). The second factor $(w - \bar{w}_o)$ is the reflection of the first pole into the real w-axis so that $\psi = 0$ on the real w-axis. The remaining factors may be written as $(w^{-1} - w_o)(w^{-1} - \bar{w}_o)$, which are the reflection of the first two factors into the unit circle $w\bar{w} = 1$ so that $\psi = 0$ on $|w| = 1$. Furthermore, f has only one zero in the entire flow and that $f = O(w^2)$ as $w \to 0$ is obvious from the local analytical behavior of f near $w = 0$. This argument leads exactly to (14). It is convenient, however, to refer on occasions to the variable ζ and the parametric solution (8) and (11).

The physical plane z is determined by integration

$$z(w) = \int_{-1}^{w} \frac{1}{w} \frac{df}{dw} \, dw = \left[\frac{f}{w}\right]_{-1}^{w} + \int_{-1}^{w} \frac{f}{w^2} \, dw \tag{15a}$$

which gives

$$z(w) + a = \frac{f(w)}{w} + iB\left\{\left(\frac{1}{\bar{w}_o} - \bar{w}_o\right)\left[\frac{1}{w_o}\log(w-w_o) - w_o \log\left(w - \frac{1}{w_o}\right)\right]\right.$$

$$\left. - \left(\frac{1}{w_o} - w_o\right)\left[\frac{1}{\bar{w}_o}\log(w-\bar{w}_o) - \bar{w}_o \log\left(w - \frac{1}{\bar{w}_o}\right)\right]\right\} \tag{15b}$$

where the constant B is given by

$$A/B = 2(U^{-1}-U)\sin\alpha\,(w_o - 1/w_o)(\bar{w}_o - 1/\bar{w}_o)$$

$$= 2(U^{-1}-U)\sin\alpha\left[(U^{-1}+U)^2 - (2\cos\alpha)^2\right] . \tag{15c}$$

The real constant a in (15b) is equal to the value of the right hand side at $w = -1$ so that $z = 0$ at the point A. This result shows that $z(w)$ has a simple pole and a logarithmic singularity at the points w_o, $\bar{w}_o$, $1/w_o$, $1/\bar{w}_o$. The logarithmic singularities of $z(w)$ are admissible since the flow does not cover the entire z-plane due to the infinitely long wake. In order that $z(w)$ will be single-valued in the flow region, two branch cuts are introduced in the w-plane, one from w_o along IC to $1/\bar{w}_o$, the other being the reflection of the first cut into the real axis. (Since $w = \infty$ is also a logarithmic branch point of $z(w)$, a third branch cut is necessary between $w = \infty$ and any one of $\bar{w}_o$, $1/w_o$, $1/\bar{w}_o$. But this cut can be made entirely outside the flow region in the w-plane.) Now since the plate has unit length, $z(1) - z(-1) = 1$. The result of this calculation gives

$$A = \left[(U^{-1}+U)^2 - (2\cos\alpha)^2\right] \Big/ K , \tag{16a}$$

$$K = 2\,\frac{(U^{-1}+U)^2 + (2\cos\alpha)^2}{(U^{-1}+U)^2 - (2\cos\alpha)^2} + \frac{\pi(U^{-1}+U)}{2\sin\alpha} + \frac{(U^{-1}+U)^2 - (2\cos\alpha)^2}{(U^{-1}-U)\sin\alpha}\tan^{-1}\left(\frac{U^{-1}-U}{2\sin\alpha}\right) \tag{16b}$$

This relation determines the coefficient A, and therefore completes the solution. It is noted that A and B are positive real constants.

When the point w moves along the cut from C' to I and back to C, the function $\log(w - w_o)$ increases by $2\pi i$ while the other functions in (15b) recover their original values. Hence

$$z_C - z_{C'} = (2\pi B/w_o)(\bar{w}_o - 1/\bar{w}_o) = -2\pi B(U^{-2} - \cos 2\alpha - i \sin 2\alpha), \quad (17a)$$

which shows that $x_C < x_{C'}$, that is, the projection of the point C on the plate is always upstream of C'. Consequently, as the cavitation number increases (U decreases), the point C will pass over head the trailing edge B before the point C' will reach B at $U = U_1$ (see equation 12). It will be seen later that in order to have a smooth transition to the partial wake flow regime treated below, we should adopt for the range of fully developed wake flow, instead of (12), the condition $\mathrm{Re}\, z_C \geqslant \mathrm{Re}\, z_B$, and then consider a different flow regime after x_C becomes equal to x_B and before the point C' reaches the trailing edge B. However, it is found that the hydrodynamic forces in the present case given below is continuous even for $U < U_1$, although the flow configuration for $U < U_1$ is no longer the full wake flow under consideration. From the numerical results it will be shown that the transition to the partial wake flow case can be interpolated very smoothly, especially when the incidence angle α is small. Therefore for the practical purpose, condition (12) may be used as the approximate range of U in the full wake flow regime. The final result is given by (27c).

The transverse distance between the points C and C' in the direction normal to the main flow is

$$h = \mathrm{Im}\left[(z_C - z_{C'})e^{-i\alpha}\right] = 2\pi B(1 + U^{-2}) \sin\alpha \ . \quad (17b)$$

This quantity may be compared with the lateral spacing of the Karman vortex street behind the oblique plate.

3. Lift and Drag

With q_c normalized to unity, the pressure difference $(p - p_c)$ may be written, from (4), as

$$p - p_c = \frac{1}{2}\rho(1 - q^2) = \frac{1}{2}\rho(1 - w\bar{w}) \quad . \tag{18}$$

Let the x- and y-component of the hydrodynamic force acting on the plate be denoted by X and Y, then

$$X + iY = i\int_A^B (p-p_c)\,dz = \frac{1}{2}\,i\rho\int_{CABC'} (1-w\bar{w})dz \tag{19}$$

where in the last step the contour of integration is extended to CABC' since $w\bar{w} = 1$ on AC and BC'. The first term of the last integral is simply

$$X_1 + iY_1 = \frac{1}{2}\,i\rho(z_{C'}-z_C) = -(i\pi\rho B/w_o)(\bar{w}_o - 1/\bar{w}_o)$$

by using (17a). The complex conjugate of the second term of the integral in (19) is

$$X_2 - iY_2 = \frac{1}{2}\,i\rho\int_{CABC'} w\bar{w}\,\frac{d\bar{z}}{d\bar{f}}\,d\bar{f} = \frac{1}{2}\,i\rho\int_{CABC'} w\,df = \frac{1}{2}\,i\rho\oint f\,dw$$

since f is purely real on CABC'. The last step is obtained by integration by parts and by making use of condition (6); the corresponding contour in the w-plane is counterclockwise around the semi unit circle. Now the integrand is analytic, regular everywhere inside the contour except at the simple pole $w = w_o$. Hence, by applying the theorem of residues,

$$X_2 - iY_2 = i\pi\rho B w_o(\bar{w}_o - 1/\bar{w}_o) \quad .$$

Combining $X_1 + iY_1$ and $X_2 + iY_2$ to obtain $X + iY$, we find

$$X = 0 \quad , \qquad Y = \pi\rho B\,(U^{-2} - U^2) \quad . \tag{20}$$

Therefore the hydrodynamic force acts normal to the plate, of magnitude Y.

The normal force coefficient is defined, as usual, by $C_N = Y/(\frac{1}{2}\rho U^2 \ell)$ where ℓ is the plate length (which is set to unity presently). Hence from (20), (15c) and (16), we obtain

$$C_N = \pi(U^{-1} + U) \,/\, (KU^2 \sin\alpha) \quad , \tag{21}$$

where K is given by (16b). The lift and drag coefficients are of course

$$C_L = C_N \cos\alpha \quad , \qquad\qquad C_D = C_N \sin\alpha \quad . \tag{22}$$

The coefficients C_L and C_D are plotted versus the under-pressure coefficient σ for several values of α, as shown in Figures 2 and 3. In Figures 4 and 5 the coefficients C_L and C_D are also plotted versus σ on a log scale in order to incorporate with the values of C_L and C_D in the partial wake flow case. Fortunately there are several experimental results available for comparison with the theory. The experimental measurements of C_L and C_D for a cavitated flat plate in a high speed water tunnel reported by Parkin (1956) are shown in Figures 2 and 3. Also included in these figures are the experimental data of C_L and C_D for a cavitating flat plate in a free jet water tunnel presented by Silberman (1959). The present theory over predicts slightly these data at small cavitation numbers, but the general trend of agreement between the theory and experiments may be considered to be good. In Figure 4 several values of C_L derived from the experiments made by Fage and Johansen (1927) for the wake flow of air past a flat plate are included; these data were obtained without wall effect corrections. A comparison shows that the present theory is in excellent agreement with these experimental results.

In the limit as $U \rightarrow 1$ (or $\sigma \rightarrow 0$), we find from (21) that

$$C_N = 2\pi \sin\alpha \,/\, (4 + \pi \sin\alpha)$$

which is the familiar classical result of Kirchhoff for the infinite cavity flow past an inclined laminar. When the plate is set normal to the flow, $\alpha = \pi/2$, we have by symmetry that $\theta_C = \theta_{C'} = \pi/2$ which are the conditions adopted by Roshko (1954) in proposing his model. For $\alpha = \pi/2$, we deduce from (21) that

$$C_D = C_N = \frac{\pi}{2} \left\{ \frac{U^3}{1+U^2} + \frac{U^2}{1-U^2} \left[\frac{\pi}{2} - (1+U^2) \tan^{-1} U \right] \right\}^{-1} \tag{23}$$

which is the result of Roshko (1954).

When the point C' approaches B, so that the region of constant pressure is limited to only the space above the plate, one derives for $U = U_1$ (see Equation (12)) and for α small the result:

$$C_L \sim C_N \sim \pi\alpha \quad . \tag{24}$$

Therefore, as the full wake flow is at the transition to the partial wake flow, the lift coefficient on the plate held at a small incidence angle is approximately half the aerodynamic value $2\pi\alpha$.

4. Pressure Distribution and the Free Streamline Configuration

The pressure distribution on the wetted and separated sides of the plate is readily determined from (4) and (15), which provide a parametric representation of $p(x, y)$. On the separated side, $p = p_c$, hence

$$C_p \equiv (p - p_\infty) / (\tfrac{1}{2}\rho U^2) = -\sigma \quad . \tag{25a}$$

On the wetted side, w is real, $-1 < w < 1$, hence from (4) and (7),

$$C_p = 1 - (1 + \sigma)\, w^2 \qquad \text{for} \quad -1 < w < 1 \;; \tag{25b}$$

and from (15), for $-1 < w < 1$,

$$x(w) = \frac{AU^2}{(1+U^2+2U\cos\alpha)^2} + \frac{AU^2 w}{(w^2+U^2-2wU\cos\alpha)(1+w^2U^2-2wU\cos\alpha)}$$

$$+ AU^2 \int_{-1}^{w} (w^2+U^2-2wU\cos\alpha)^{-1}(1+w^2U^2-2wU\cos\alpha)^{-1}\, dw \quad . \tag{26}$$

The above parametric solution $C_p(w)$ and $x(w)$ is shown in Figure 6 for $\alpha = 29.85^\circ$, 49.85°, 69.85°, 90° with the respective under-pressure coefficient $\sigma = 0.924$, 1.230, 1.360, and 1.380. The corresponding experimental results were obtained by Fage and Johansen (1927); the mean "base pressure coefficient" σ_{exp} were observed experimentally for the wake flow in air. However, no correction due to the tunnel wall effect was made for these data. A comparison shows that the present theory and the experi-

ments are in excellent agreement.

The shapes of the free streamlines AC and BC' can be determined from (15) as follows. Along AC, $w = e^{-i\theta}$, $\gamma \leqslant \theta \leqslant \pi$, γ being given by (13), we have

$$z - z_A = \frac{AU^2}{(1+U^2+2U\cos\alpha)^2} + \frac{AU^2 e^{i\theta}}{\left[1+U^2-2U\cos(\theta-\alpha)\right]\left[1+U^2-2U\cos(\theta+\alpha)\right]}$$

$$+ i\,AU^2 \int_\theta^\pi e^{i\theta}\left[1+U^2-2U\cos(\theta-\alpha)\right]^{-1}\left[1+U^2-2U\cos(\theta+\alpha)\right]^{-1} d\theta \,. \quad (27a)$$

Along BC', $w = e^{-i\theta}$, $0 \leqslant \theta \leqslant \gamma$, we obtain

$$z - z_B = \frac{AU^2 e^{i\theta}}{\left[1+U^2-2U\cos(\theta-\alpha)\right]\left[1+U^2-2U\cos(\theta+\alpha)\right]} - \frac{AU^2}{(1+U^2-2U\cos\alpha)^2}$$

$$- i\,AU^2 \int_0^\theta e^{i\theta}\left[1+U^2-2U\cos(\theta-\alpha)\right]^{-1}\left[1+U^2-2U\cos(\theta+\alpha)\right]^{-1} d\theta \,. \quad (27b)$$

The shapes of the free streamlines AC and BC' can be calculated from these equations for given α and σ.

In particular, the transverse distance h between the points C and C' in the direction normal to the main flow, given by (17b), can be expressed by using (15c), (16) and (21) as

$$\frac{h}{\ell} = \frac{1}{\sigma} C_N \sin\alpha = \frac{C_D}{\sigma} \quad (27c)$$

where the plate length ℓ is restored for completeness. This simple result can also be derived by a momentum consideration applied to the far-wake ICC'I. This value of h/ℓ is plotted versus σ in Figure 7 for several values of α. Also shown in Figure 7 are a few values of h/ℓ calculated by Fage and Johansen (1927), using Kármán's stability relation $h = 0.281a$ and the measured values of the vortex spacing "a". The present theoretical result compares favorably well with such estimates, although this flow model is not expected to reproduce any details of the far-wake flow. In the actual

measurements of h/ℓ, however, Fage and Johansen reported that h/ℓ increases towards the downstream as the vortices diffuse.

B. PARTIALLY DEVELOPED WAKE FLOWS AND CAVITY FLOWS

5. The Flow Model; Analysis of the Flow Field

As described in Section 1, the partially developed wake flow is defined by the configuration that the near-wake of constant pressure p_c covers only a part of the suction side of the lifting plate, which starts from the leading edge A and terminates at a certain point C to the upstream of the trailing edge B (see Figure 8). The pressure in the wake further downstream increases continuously and recovers at downstream infinity its upstream value p_∞. In order to describe this type of flow with a good approximation and to render the flow exterior to the wake subject to a simple analysis, the following model is proposed.

The stagnation streamline ID is split into two branches, one of them follows along the lower surface to the trailing edge B and then forms a free streamline BI, extending to the downstream infinity I. The other branch separates from the leading edge A to form another free streamline ACB'I such that $p = p_c$ on AC, with $x_C \leq x_B$, and that p increases monotonically along CB'I. The streamline CB' is assumed to be parallel to the plate. The point B' is defined by

$$\psi_B = \psi_{B'} = 0 \quad \text{and} \quad \mathrm{Re}(z_{B'} - z_B) = 0 \ . \tag{28a}$$

Furthermore, we assume that the complex velocity w takes the same value at B and B',

$$w_{B'} = w_B \ (= u_T \text{ say}) \ . \tag{28b}$$

This condition implies that the velocity at B' is parallel to the plate, and that the pressure at B' and B are equal. Physically it is conceivable that, if the wake at the trailing edge is narrow, the pressure cannot vary appreciably across the wake. This condition will be imposed even if the wake may be moderately thick. However, for $\alpha > 45^\circ$, the condition

(28b) would be expected to lose gradually it physical significance since the non-uniformity of the pressure distribution across the wake and over the suction side of the flat plate will extend over such a wide region that the flow outside the wake can no longer be approximated by this simple description. After all, from the physical point of view, the partial wake flow would be of interest only for small and moderate values of α. For this reason, our present treatment will be limited to the range $0 < \alpha < 45^{\circ}$.

Conditions (28a, b) define the point B' and replace the assumptions (5a, b) for the case of full wake flows. The streamlines B'I and BI will again be assumed to form a slit in the hodograph plane (the hodograph-slit condition). As in the previous case, the plate length ℓ and the constant speed q_c on AC are again both normalized to unity. Since p increases monotonically along CB'I, the pressure p_T at B and B' is greater than p_c, and hence obviously $0 < u_T < 1$ where u_T is the value of w at the trailing edge B and B'.

It should be noted that, if conditions (28a, b) are to be fulfilled, a circulation around the wake must be introduced, and consequently the potential f will not have the same value at B and B'. In fact, we must have $f_{B'} - f_B = \Gamma$ where Γ is the circulation around BDACB'. The existence of a circulation is an essential feature of the partial wake flow, as compared with the previous case; without the circulation the transition to the fully wetted flow would be greatly impaired.

Under the normalization $q_c = 1$ and the hodograph-slit condition, the flow is mapped conformally into the interior of a simply-covered semi-circle of unit radius in the lower half w-plane as illustrated in Figure 8. The illustration is self-evident and needs no further elaboration.

By the conformal transformation (8) the entire flow is mapped into the upper half ζ-plane, with the point $w_o = U e^{-i\alpha}$ mapped to ζ_o given by (9); the boundary of the semi-circle in the w-plane is mapped into the entire real ζ-axis. From the configuration of the streamlines near $\zeta = \zeta_o$, it is again evident that f must have a simple pole at $\zeta = \zeta_o$. Furthermore, in order to satisfy (28a, b) a vortex must be introduced at $\zeta = \zeta_o$. From these singular properties of f and the property (10)

it follows that the solution $f(\zeta)$ must be of the form

$$2\pi f(\zeta) = \frac{Q}{\zeta - \zeta_o} + \frac{\bar{Q}}{\zeta - \bar{\zeta}_o} - i\Gamma \log \frac{\zeta - \zeta_o}{\zeta - \bar{\zeta}_o} \tag{29}$$

where Q (complex in general) is the strength of the simple pole and Γ (real) is the circulation about $\zeta = \zeta_o$. From the local conformal behavior of $f(\zeta)$ near $\zeta = \infty$, as was explained for (11), we must again require that $f = O(\zeta^{-2})$ as $\zeta \to \infty$. Expanding the right hand side of (29) for large ζ and equating the coefficient of ζ^{-1} equal to zero, we obtain the following relationship between Q and Γ :

$$Q + \bar{Q} = -i\Gamma(\zeta_o - \bar{\zeta}_o) = (U^{-1} - U)\Gamma \sin\alpha \ , \tag{30}$$

where use has been made of (9). Let the value of ζ at B and B' be ζ_T, then from (8) the value of w at B and B' is

$$u_T = \zeta_T - (\zeta_T^2 - 1)^{1/2} \ . \tag{31}$$

Obviously we must have $\zeta_T > 1$ so that $0 < u_T < 1$ for the partial wake flow. The streamlines BI and B'I form a slit which is perpendicular to the real ζ-axis at ζ_T. Then, since a small semi-circle around B in the f-plane is mapped into a small quarter-circle around ζ_T in the ζ-plane, it follows that $df/d\zeta = 0$ at $\zeta = \zeta_T$. From this condition and (30) we obtain

$$\zeta_T = \frac{Q\bar{\zeta}_o - \bar{Q}\zeta_o}{Q - \bar{Q}} = \frac{1}{2}(\zeta_o + \bar{\zeta}_o) + \frac{1}{2}\frac{Q+\bar{Q}}{Q-\bar{Q}}(\bar{\zeta}_o - \zeta_o)$$

$$= \frac{1}{2}\left(\frac{1}{U} + U\right)\cos\alpha + \frac{1}{2i}\frac{Q+\bar{Q}}{Q-\bar{Q}}\left(\frac{1}{U} - U\right)\sin\alpha \ . \tag{32}$$

The physical plane z is determined by the integration

$$z = \int_{-1}^{w} \frac{1}{w}\frac{df}{dw}\, dw \ . \tag{33a}$$

Carrying out the integration and making use of (30), we obtain

$$2\pi(z+a) = \frac{2Q}{(w-w_o)(w-1/w_o)} + \frac{2\bar{Q}}{(w-\bar{w}_o)(w-1/\bar{w}_o)} + \frac{b-i\Gamma}{w_o}\log(w-w_o)$$

$$- w_o(b+i\Gamma)\log(w-\frac{1}{w_o}) + \frac{\bar{b}+i\Gamma}{\bar{w}_o}\log(w-\bar{w}_o) - \bar{w}_o(\bar{b}-i\Gamma)\log(w-\frac{1}{\bar{w}_o}) \quad , \tag{33b}$$

where

$$b = -2Q\Big/\left(\frac{1}{w_o} - w_o\right) = -2Q\Big/\left(\frac{1}{U}\, e^{i\alpha} - U e^{-i\alpha}\right) , \tag{33c}$$

and the real constant $(2\pi a)$ is equal to the value of the right hand side of (33b) at $w = -1$ so that $z = 0$ at the point A. The function $z(w)$ has a simple pole and a logarithmic singularity at the points w_o, $\bar{w}_o$, $1/w_o$, $1/\bar{w}_o$. In order that $z(w)$ be single valued in the flow field, two branch cuts are introduced in the w-plane, one from w_o along IB and its image path (reflected into the real w-axis) to $w = \bar{w}_o$, the other being the image of the first cut into the unit circle $w\bar{w} = 1$. As the point w traces along the cut from B, around the point w_o and ends up at B', the function $\log(w-w_o)$ increases by $2\pi i$, whereas the other functions in (33b) are unaltered. Hence from (33)

$$z_{B'} - z_B = i(b - i\Gamma)/w_o \quad .$$

But condition (28a) requires that $(z_{B'}-z_B)$ be purely imaginary, say

$$z_{B'} - z_B = i\beta\Gamma \tag{34}$$

where β is a real constant. By comparison we have

$$b = (\beta w_o + i)\,\Gamma \quad . \tag{35}$$

From (30), (33c) and (35) we can solve for β, Q and b, giving

$$\beta = 2U \sin\alpha/(1-U^2\cos 2\alpha) \quad , \tag{36}$$

$$Q = \frac{1}{2}\Gamma\left\{\left(\frac{1}{U} - U\right)\sin\alpha - i\cos\alpha\left(\frac{1}{U} - U + 2\beta U^2\sin\alpha\right)\right\} \quad . \tag{37}$$

Substituting this equation in (32), we obtain

$$\zeta_T = \frac{1}{2}\left(\frac{1}{U} + U\right)\cos\alpha + \frac{(1-U^2)^2\sin^2\alpha}{2U\cos\alpha(1-U^2+2\beta U^3\sin\alpha)} \tag{38}$$

which is determinate for given U and α. Now application of the condition $\zeta_T > 1$ to (38) for the partial wake flow will lead to a permissible range of U for each α, say $0 \leq U < U_p(\alpha)$ such that $\zeta_T(U_p, \alpha) = 1$. However, it can be verified that $U_p(\alpha)$ is approximately equal to $U_1(\alpha)$ defined by (12) for moderate and small values of α. In fact, it is readily shown that $\zeta_T(U_1, \alpha) = 1 + \frac{1}{2}\alpha^3 + 0(\alpha^4)$ as $\alpha \to 0$. Therefore, the difference between U_1 and U_p will not be pursued further, and $0 < U < U_1$ will be used as the approximate range of U for the partial wake flow.

Upon substitution of (35)-(37) into (33), we obtain

$$\begin{aligned} 2\pi(z+a)/\Gamma = {} & \frac{2U(1-U^2)\sin\alpha\left[1+w^2-2wU\cos\alpha(1-\beta U\sin\alpha)\right]}{(U^2+w^2-2wU\cos\alpha)(1+w^2U^2-2wU\cos\alpha)} \\ & + \beta\log\left[(w-w_o)(w-\bar{w}_o)\right] - w_o(\beta w_o+2i)\log(w-1/w_o) \\ & - \bar{w}_o(\beta\bar{w}_o-2i)\log(w-1/\bar{w}_o) \quad . \end{aligned} \tag{39}$$

Finally the circulation strength Γ is determined by the scale of the plate length such that $z_B - z_A = z(u_T) - z(-1) = 1$. The result of this calculation yields

$$\Gamma = \pi/\Lambda\,(U, \alpha) \quad , \tag{40a}$$

$$\begin{aligned} \Lambda = {} & U(1-U^2)\sin\alpha\left\{\frac{1+u_T^2-2u_TU\cos\alpha(1-\beta U\sin\alpha)}{(U^2+u_T^2-2u_TU\cos\alpha)(1+u_T^2U^2-2u_TU\cos\alpha)}\right. \\ & \left. - 2\,\frac{1+U\cos\alpha(1-\beta U\sin\alpha)}{(1+U^2+2U\cos\alpha)^2}\right\} + \frac{1}{2}\beta\log\frac{u_T^2+U^2-2u_TU\cos\alpha}{1+u_T^2U^2-2u_TU\cos\alpha} \\ & + 2U\cos\alpha(1-\beta U\sin\alpha)\tan^{-1}\frac{U(1+u_T)\sin\alpha}{1-u_TU^2+U(1-u_T)\cos\alpha} \quad , \end{aligned} \tag{40b}$$

in which β is given by (36) and u_T by (31) and (38). This equation determines the circulation Γ in terms of U and α.

It is of interest to note the limiting case of the fully wetted flow.* From (36), (38) and (31) we deduce immediately that as $U \to 0$,

$$\beta \sim 2U \sin\alpha \ , \quad \zeta_T \sim (1+U^2 \cos 2\alpha)/(2U\cos\alpha) \ , \quad u_T \sim U\cos\alpha \ ; \quad (41a)$$

and hence from (40)

$$\Gamma \sim \pi U \sin\alpha \left\{ 1-(U\sin\alpha)^2 \log (U\sin\alpha)^2 + O(U^2) \right\} \quad \text{as} \quad U \to 0 \ . \quad (41b)$$

Furthermore, it is seen from (34) that $z_{B'} - z_B \to 0$ like U^2 as $U \to 0$, and from (39) that $z_C = z(1) \to 0$ as $U \to 0$. Thus as $U \to 0$ (or rather $U/q_c \to 0$ as $q_c \to \infty$ for fixed U), the constant pressure region vanishes and the thickness of the wake reduces to zero, the flow thereby becomes fully wetted. The results that $u_T = U\cos\alpha$, and $\Gamma/(\text{chord}) = \pi U \sin\alpha$ are of course both well-known in the airfoil theory.

6. Lift and Drag in the Partial Wake Flow

The calculation of the hydrodynamic forces on the inclined plate in a partial wake flow is less straightforward than in the full wake flow case, since if the forces are to be determined by integration of the pressure difference across the plate, the pressure on the suction side of the plate is now subject to certain arbitrary interpretation. However, in view of the physical significance of the condition (28b), we shall assume that the hydrodynamic force acting on the plate is equal to that on the closed body B'CADBB', with its base BB' exposed to a uniform base pressure p_T. This assumption enables us to calculate the force directly from the exterior potential flow without considering the viscous flow of the real fluid within the wake. The force so determined may be conjectured to include the effects due to the cavity formation near the leading edge and the equivalent dissipation in this potential flow model.

For the present purpose the Bernoulli equation may be written

$$p + \frac{1}{2}\rho w \bar{w} = p_T + \frac{1}{2}\rho u_T^2 \qquad (42)$$

* The fully wetted flow past a flat plate can physically be realized only when the leading edge is sufficiently round.

where u_T is given by (31) and (38). According to the assumption stated above, the hydrodynamic force acting on the plate is given by

$$X + iY = i \oint_{\mathcal{C}} (p - p_T) dz = \frac{1}{2} i \rho \oint_{\mathcal{C}} (u_T^2 - w\bar{w}) dz \tag{43}$$

where the contour $\mathcal{C}$ denotes the path B'CAB. This is the pressure integral on the closed body B'CABB' since $p = p_T$ on BB'. The first term of this integral is simply

$$X_1 + iY_1 = \frac{1}{2} i \rho u_T^2 \oint_{\mathcal{C}} dz = \frac{1}{2} i \rho u_T^2 (z_B - z_{B'}) = \frac{1}{2} \rho u_T^2 \beta \Gamma$$

by using (34). The complex conjugate of the second term in (43) is

$$X_2 - iY_2 = \frac{1}{2} i \rho \oint_{\mathcal{C}} w\bar{w} \frac{d\bar{z}}{d\bar{f}} df = -\frac{1}{2} i \rho \oint_{\mathcal{C}_w} w \frac{df}{dw} dw$$

where the contour $\mathcal{C}_w$ is counterclockwise around the semi-unit circle in the w-plane. Now from the previous solution (29), (8), it is seen that the above integrand $w\, df/dw$ is an analytic function of w, whose only singularity within the contour $\mathcal{C}_w$ is a double pole at $w = w_o$, at which the residue is found to be $-(b + i\Gamma)w_o/2\pi$. Hence, by the theorem of residues,

$$X_2 - iY_2 = -\frac{1}{2} \rho (b + i\Gamma) w_o = -\rho \Gamma w_o (i + \frac{1}{2} \beta w_o)$$

where use has been made of (35). Combining $X_1 + iY_1$ and $X_2 + iY_2$ to obtain $X + iY$, we find

$$X + iY = \rho U \Gamma e^{i\alpha} \left[i + \frac{1}{2} \beta U \left(\frac{u_T^2}{U^2} e^{-i\alpha} - e^{i\alpha} \right) \right] . \tag{44}$$

It is noted from the above result that the force component parallel to the plate, X, generally does not vanish in the partial wake flow. In particular when $U \to 0$, use of the limiting values (41a, b) in (44) yields

$$X = -\rho U \Gamma \sin\alpha$$

which is known as the leading edge suction in the airfoil theory. That the tangential force component X is in general not zero perhaps cannot be explained entirely within the framework of the potential theory. Partly this is because the approximated mechanism of dissipation takes place over a portion of the plate. In the real physical case, the flow pattern is of course very complex.

Finally, resolving the force into the lift L and drag D, we obtain

$$D + i L = (X + i Y) e^{-i\alpha} = \rho U \Gamma \left[i + \frac{1}{2} \beta U \left(\frac{u_T^2}{U^2} e^{-i\alpha} - e^{+i\alpha} \right) \right] , \qquad (45)$$

and hence

$$C_L = \frac{L}{\frac{1}{2} \rho U^2 (\text{chord})} = \frac{2}{U} \Gamma \left[1 - \frac{1}{2} \beta U \left(1 + \frac{u_T^2}{U^2} \right) \sin\alpha \right] , \qquad (46)$$

$$C_D = \frac{D}{\frac{1}{2} \rho U^2 (\text{chord})} = \beta \Gamma \left(\frac{u_T^2}{U^2} - 1 \right) \cos\alpha , \qquad (47)$$

where β is given by (36), u_T by (31) and (38), and Γ by (40). Near the transition between the full wake and partial wake flows, we let $U = U_1$ (see Equation 12) and consider the small values of α (since this partial wake flow model becomes inadequate for $\alpha > \pi/4$). For such a case we derive from (12), (36), (31), (38) that

$$U_1 \sim 1 - \alpha + \frac{1}{2} \alpha^2 , \quad \beta \sim 1 - \alpha + \frac{3}{2} \alpha^2 , \quad u_T \sim 1 - \alpha^{3/2} - \frac{3}{4} \alpha^{5/2} ; \qquad (48)$$

and from (40), $\Gamma \sim \frac{1}{2} \pi\alpha$. Substituting these values in (46), (47), we obtain

$$C_L \sim \pi\alpha , \quad C_D \sim \alpha C_L , \quad \text{for} \quad U = U_1 , \quad \alpha \ll 1 . \qquad (49)$$

On the other hand, in the limiting case of the fully wetted flow, $U \rightarrow 0$, we substitute (41) in (46) and (47), giving

$$C_L \sim 2\pi \sin\alpha \left\{ 1 - (U \sin\alpha)^2 \log (U \sin\alpha)^2 + O(U^2) \right\} , \qquad (50a)$$

$$C_D \sim - 2\pi U^2 \sin^4\alpha \cos\alpha . \qquad (50b)$$

Equation (49) coincides with (24) and (22) which are the upper limits of C_L and C_D in the full wake flow for $\alpha \ll 1$; this indicates that the transition from the full wake to partial wake flow is smooth for small and moderate values of incidence angle α. Equation (50a) shows that for small U, C_L is slightly greater than the classical aerodynamic value $2\pi \sin\alpha$ and eventually tends to $2\pi \sin\alpha$ as $U \rightarrow 0$. This is known as the leading edge bubble effect which produces a small positive camber over the original flat plate airfoil. For small U, (50b) shows that C_D attains a negative value, which is very small for small α and is of smaller order than the classical leading edge suction. This rather unfavorable result may be attributed to the over-simplification of this partial wake flow model.

Equations (46) and (47) are plotted in Figures 2-5 for a range of α from 2° to 40°. For moderate values of α, the result shows that the transition from the full wake to partial wake flow becomes increasingly less smooth with increasing α; a smooth curve in the transition region is appropriately faired-in with dashed lines. Furthermore, the drag has been found to become negative (but small in magnitude) beyond a certain range of the cavitation number σ; this part of the curve is shown by the dotted lines. In spite of these rough approximations, the present wake flow model is seen able to account for the salient features of the wake flow, as the incorporated experimental results clearly indicate.

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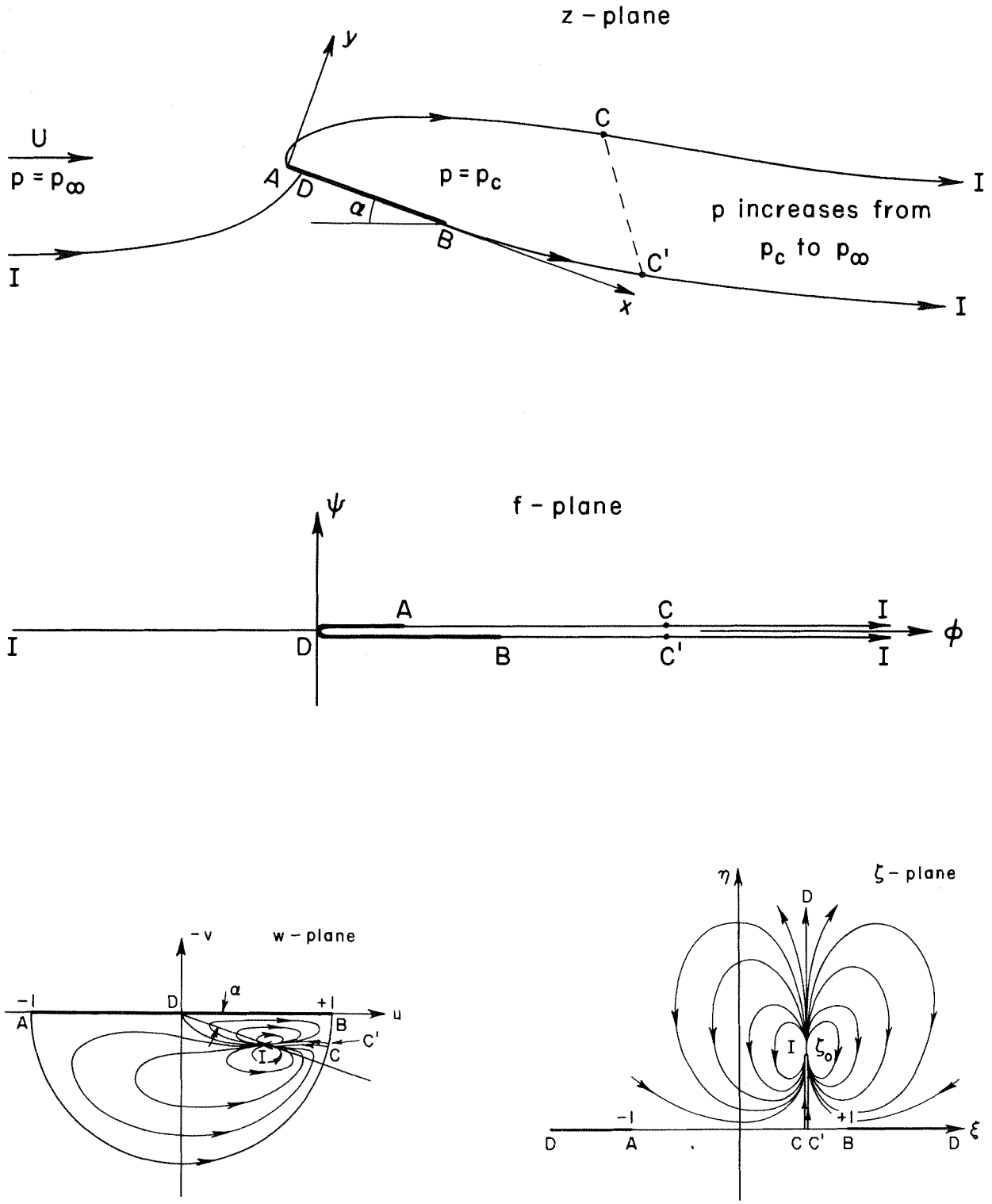

Figure 1. The free streamline model for the fully developed wake flow past an oblique flat plate and its conformal mapping planes.

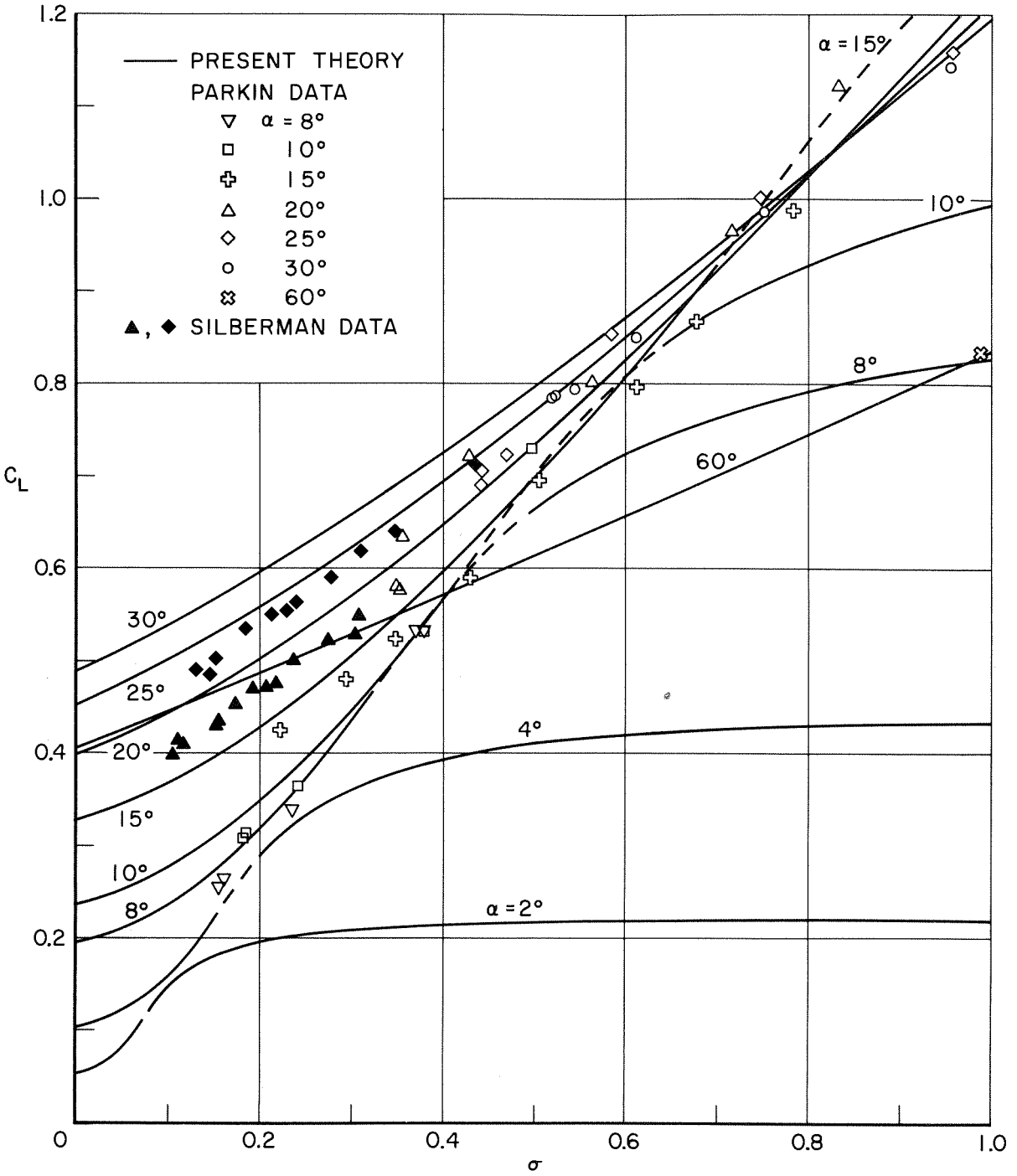

Figure 2. Variation of C_L with the wake underpressure coefficient σ (or the cavitation number) for the flat plate. Parkin's experiments were performed in a high speed closed water tunnel, whereas Silberman's experiments in a free jet water tunnel, both data being reproduced here with σ equal to the cavitation number based on the measured cavity pressure and without the correction of the tunnel boundary effect.

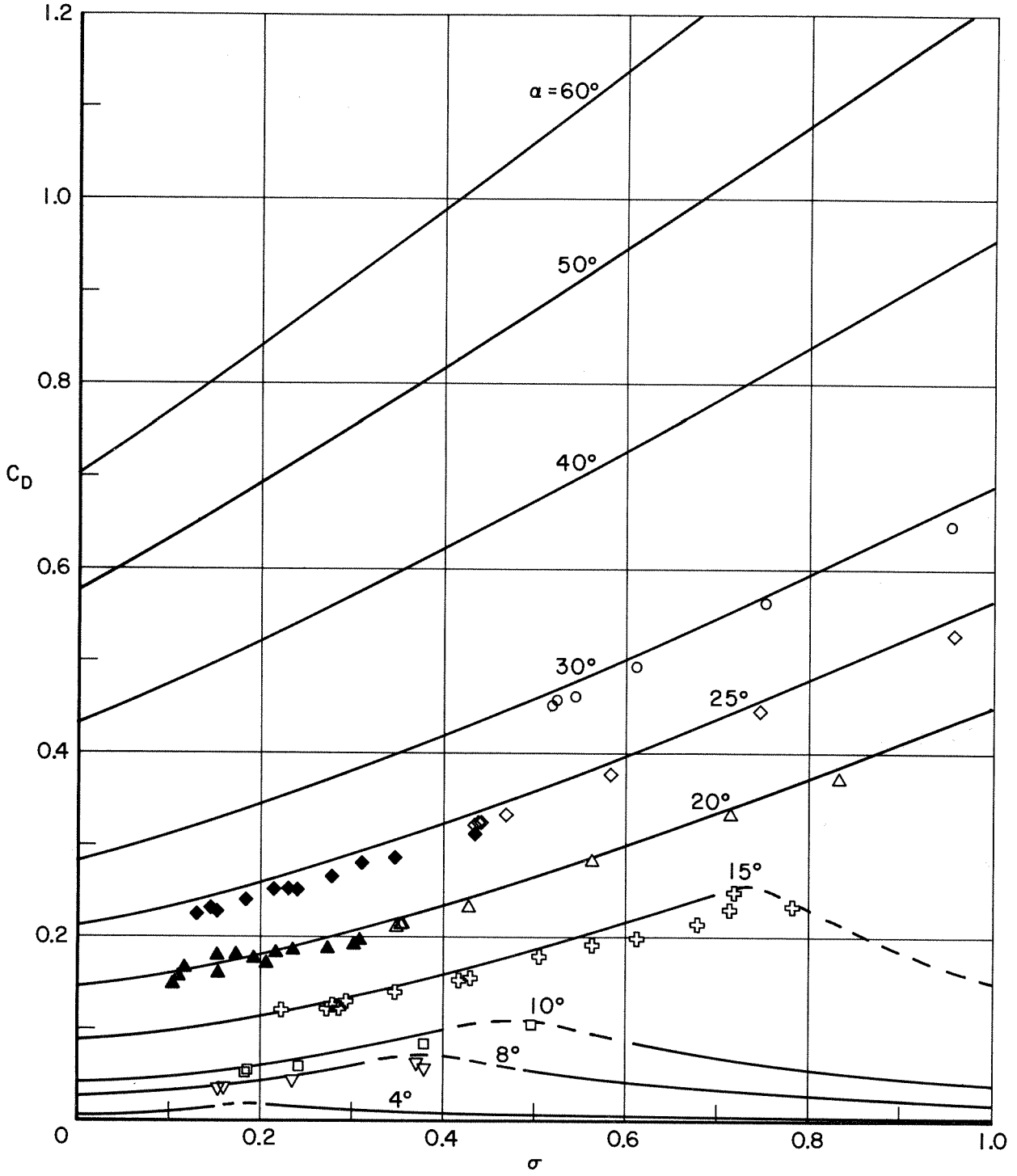

Figure 3. Variation of C_D with σ for the flat plate (same legend as Figure 2).

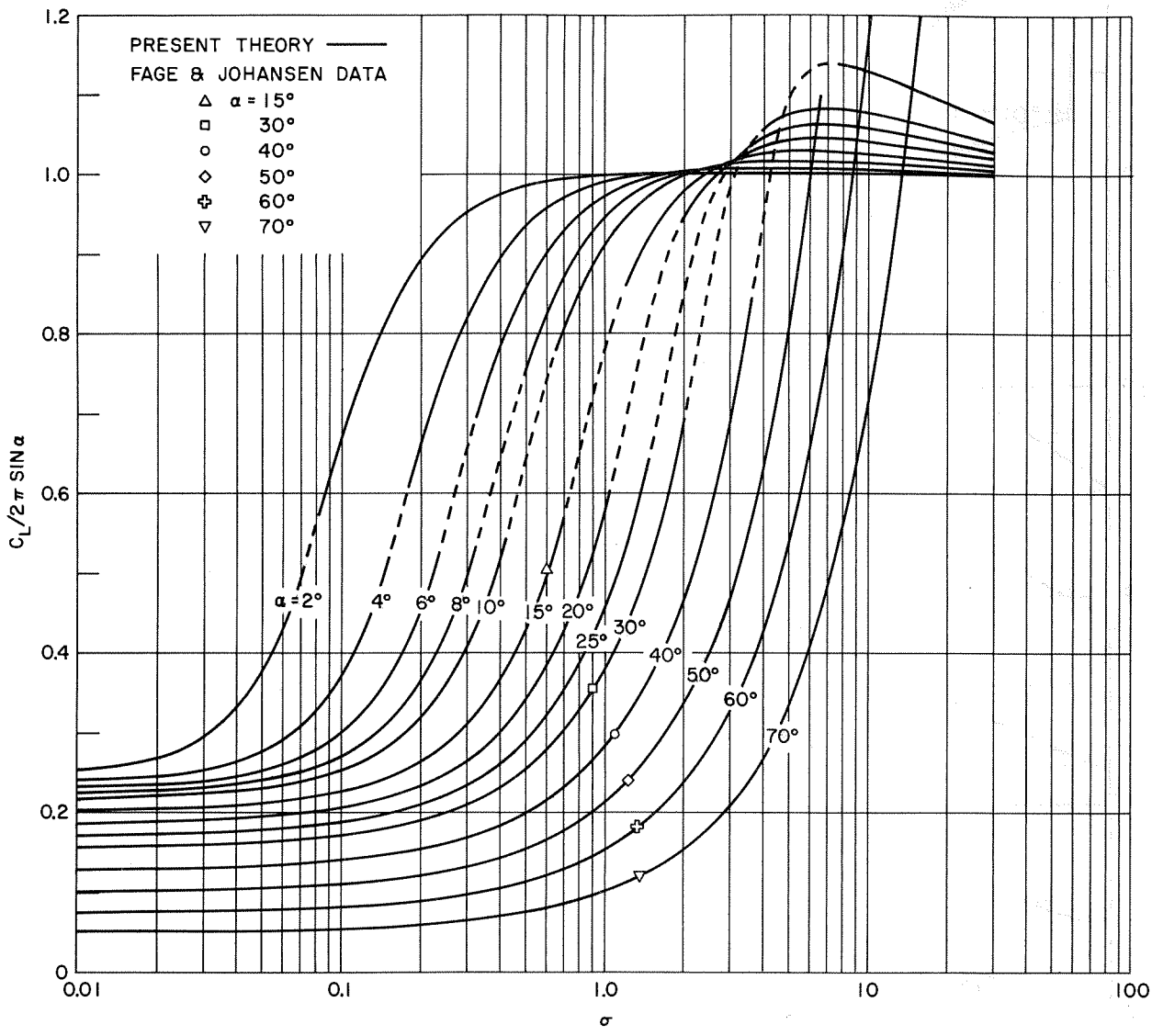

Figure 4. Variation of C_L over an extended range of σ to cover both the fully and partially developed wake flows past the flat plate, the transition between these two regimes of wake flows being appropriately faired-in with dashed lines. The experiments of Fage and Johansen were carried out in a wind tunnel, the results being reproduced with the wake underpressure coefficient σ based on the measured constant base pressure and without the correction of the tunnel wall effect.

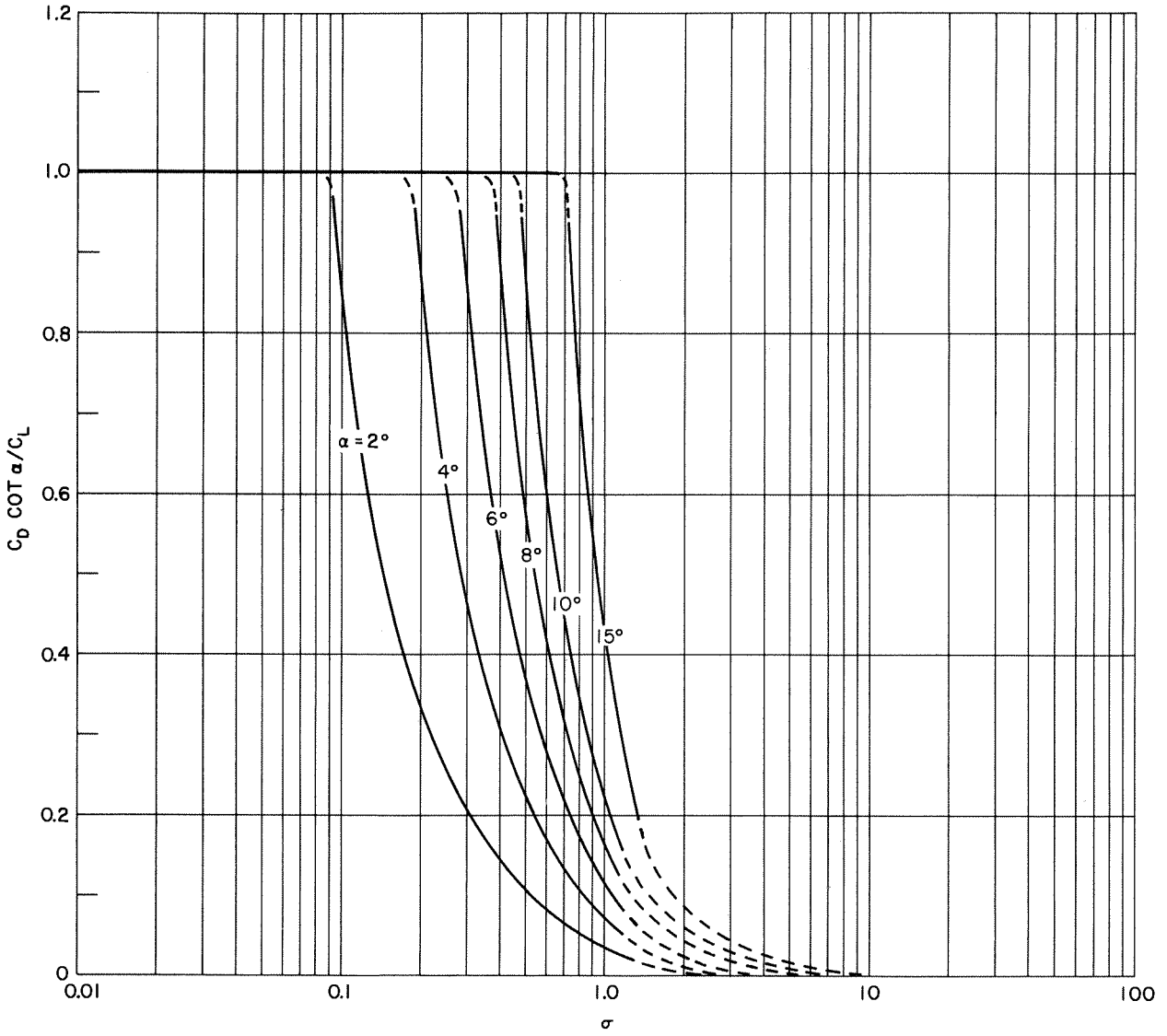

Figure 5. Variation of C_D over an extended range of σ for the flat plate. The dashed lines for $\sigma > 1$ are not the theoretical results, but are shown to indicate what would be expected on physical grounds.

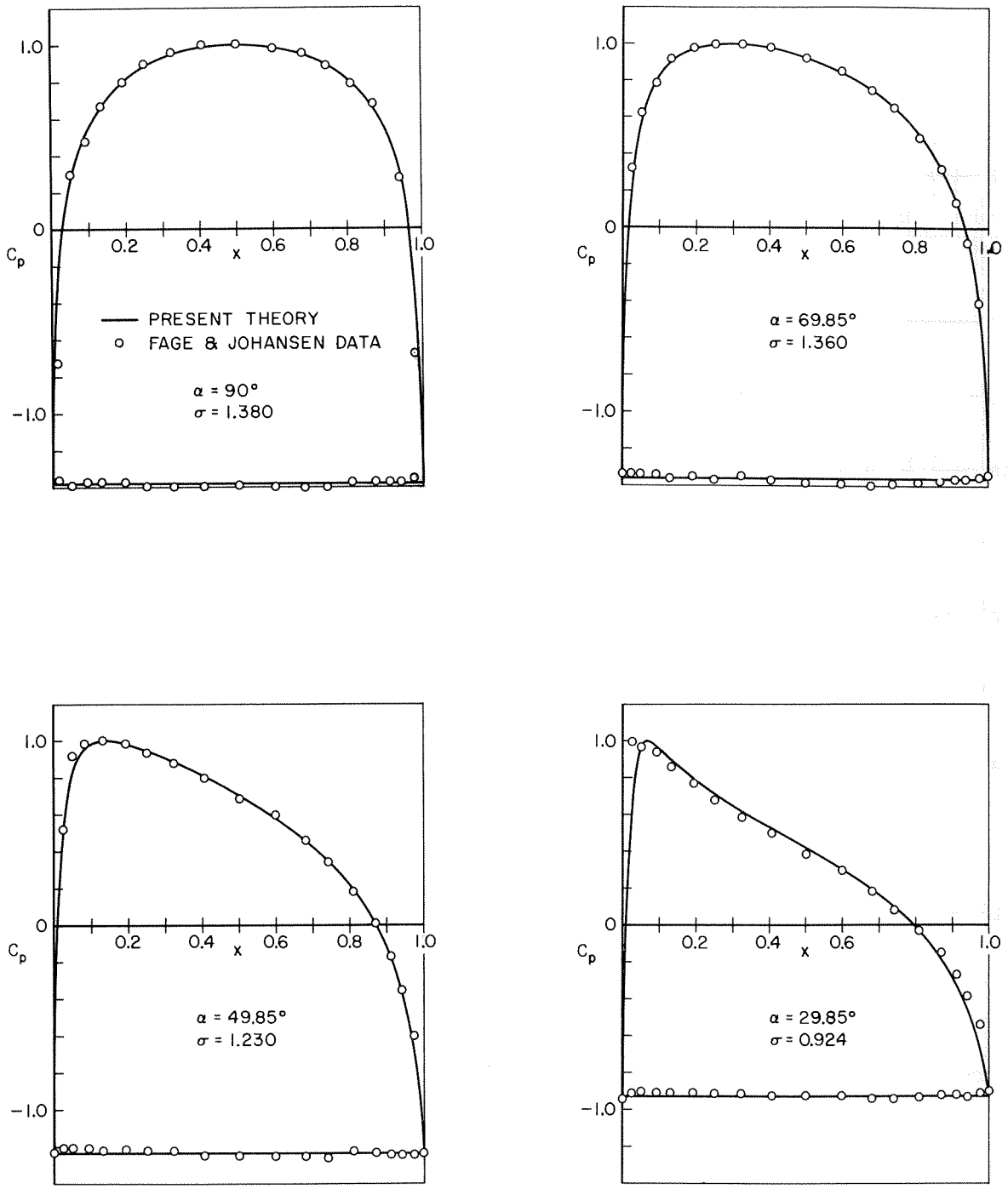

Figure 6. Pressure distributions on the wetted and separated sides of the oblique plate. The experimental results of Fage and Johansen were obtained by reading graphs in their paper since tabulated data were not available.

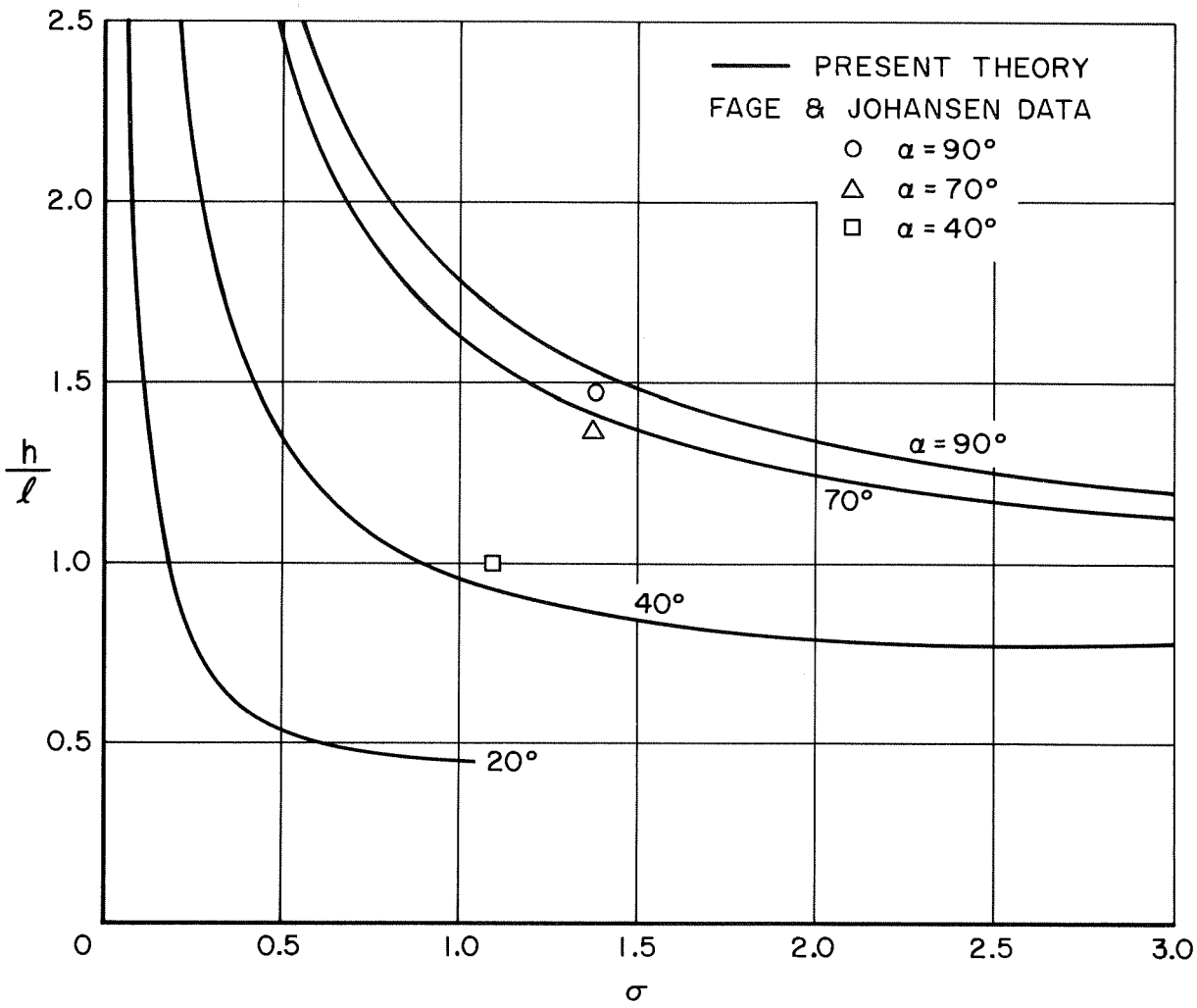

Figure 7. Variation of the asymptotic width of the wake with σ.

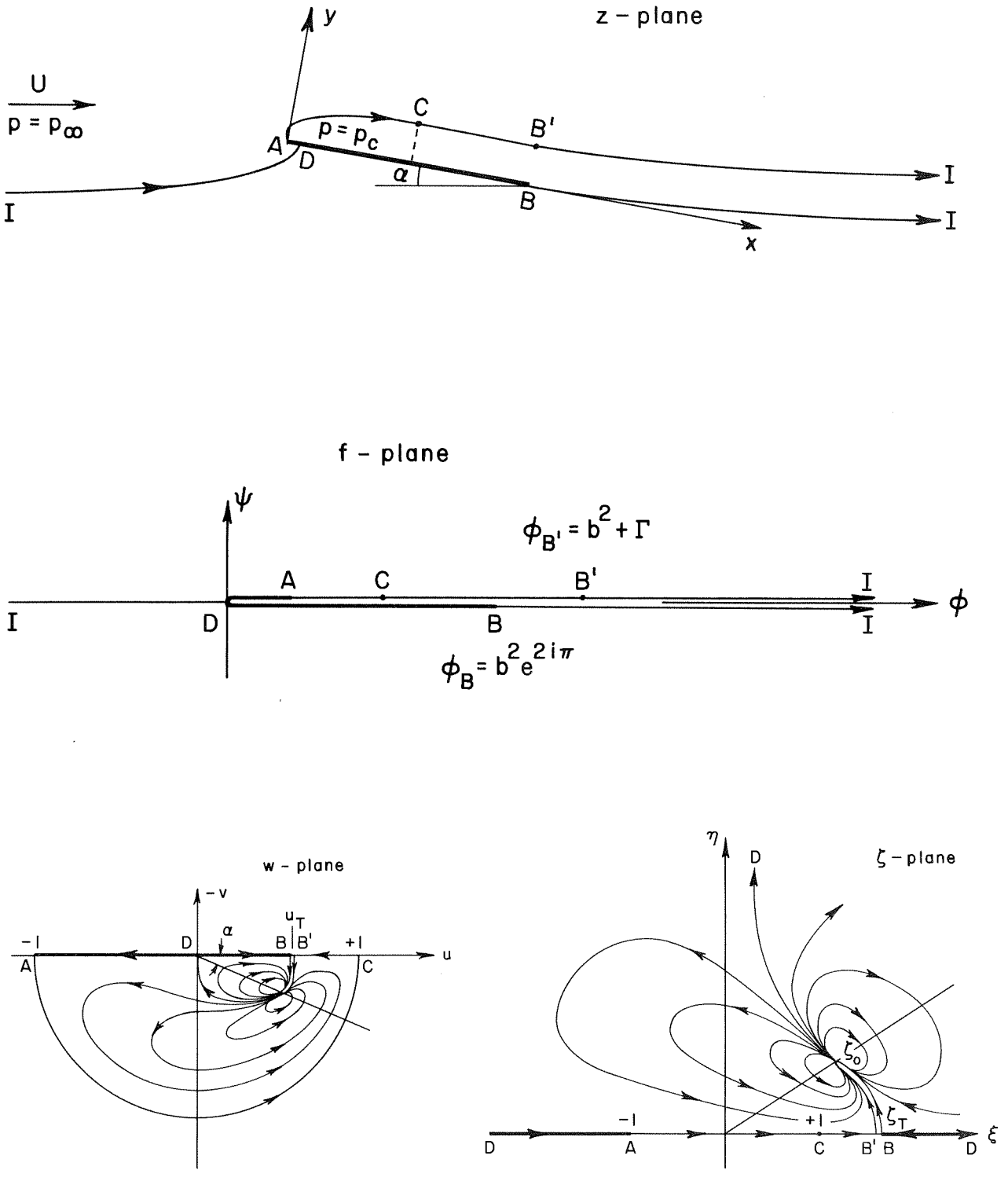

Figure 8. The free streamline model for the partially developed wake flow past an oblique flat plate and its conformal mapping planes.

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