THE BIRMAN–MURAKAMI–WENZL ALGEBRAS OF TYPE D_n

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ABSTRACT. The Birman–Murakami–Wenzl algebra (BMW algebra) of type D_n is shown to be semisimple and free of rank $(2^n+1)n!!-(2^{n-1}+1)n!$ over a specified commutative ring R, where $n!!=1\cdot 3\cdots(2n-1)$. We also show it is a cellular algebra over suitable ring extensions of R. The Brauer algebra of type D_n is the image of an R-equivariant homomorphism and is also semisimple and free of the same rank, but over the ring $\mathbb{Z}[\delta^{\pm 1}]$. A rewrite system for the Brauer algebra is used in bounding the rank of the BMW algebra above. As a consequence of our results, the generalized Temperley–Lieb algebra of type D_n is a subalgebra of the BMW algebra of the same type.

KEYWORDS: associative algebra, Birman–Murakami–Wenzl algebra, BMW algebra, Brauer algebra, cellular algebra, Coxeter group, generalized Temperley–Lieb algebra, root system, semisimple algebra, word problem in semigroups

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1. INTRODUCTION

In [2], Birman and Wenzl, and independently in [24], Murakami, defined algebras indexed by the natural numbers which play a role in both the representation theory of quantum groups and knot theory. They were given by generators and relations. In [23], Morton and Wasserman gave them a description in terms of tangles. These are the Birman–Murakami–Wenzl algebras (usually abbreviated to BMW algebras) for the Coxeter system of type A_n. They behave nicely with respect to restriction to the algebras generated by subsets of the generators. For instance, the BMW algebras of a restricted type embed naturally into the bigger ones. This is similar to the fact that in Weyl groups subgroups generated by subsets of the standard reflections are themselves Weyl groups. The Hecke algebra of type A_n is a natural quotient of the Birman–Murakami–Wenzl algebra of type A_n and the Temperley–Lieb algebra, conceived originally for statistics (cf. [26]), is a natural subalgebra. Inspired by the beauty of these results, the existence of Temperley–Lieb algebras of other types ([11, 15, 17, 18]) and the existence of a faithful linear representation of the braid group ([9, 10]), the authors defined analogues for other simply laced Coxeter diagrams and found some of their properties in [5]. The faithful linear representations of the braid group were shown first by Bigelow in [1] and Krammer in [20]. They used a representation introduced by Lawrence in [21].

In this paper we consider the algebras when the Coxeter diagram is of type D_n. We prove the conjecture stated in [5, Section 7.1], which is Theorem 1.1. Here, $n!!=1\cdot 3\cdots(2n-1)$. We work over the quotient ring R of $\mathbb{Z}[\delta,\delta^{-1},l,l^{-1},m]$ by

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the ideal generated by $m(1-\delta)-(l-l^{-1})$ instead of the field $\mathbb{Q}(l,\delta)$ in which it embeds (see Lemma 3.7).

Theorem 1.1. *The BMW algebra of type* D_n *over* R *is free of rank*

$$(2^n+1)n!!-(2^{n-1}+1)n!.$$

When tensored with $\mathbb{Q}(l,\delta)$*, it is semisimple.*

The result produces linear representations of the Artin group of type D_n similar to the representations of the braid group on n strands which arose from the BMW algebra of type A_{n-1}. These include the faithful representations related to the Lawrence–Krammer representations occurring in [9] as well as the representations occurring in [5]. Furthermore, specific information about the representations is given in terms of sets of orthogonal roots and irreducible representations of Weyl groups of type D_r for certain r (cf. Remark 7.5).
These sets of orthogonal roots are also used in the description of a cellular basis, whose elements are determined by pairs of such root sets and a Weyl group element. This leads, for suitable extensions of the coefficient ring R, to cellularity of the BMW algebra $\mathrm{B}(\mathrm{D}_n)$ in the sense of [16, Definition 1.1]. For $\mathrm{B}(\mathrm{A}_n)$, this result is known thanks to [28].

Theorem 1.2. *The BMW algebra of type* D_n *is cellular if the coefficient ring* R *is extended to an integral domain containing an inverse to* 2*.*

As a consequence of the work we are able to show the Temperley–Lieb algebra of type D_n as defined in [11, 15, 18] is a natural subalgebra.

Corollary 1.3. *The generalized Temperley–Lieb algebra of type* D_n *is a natural subalgebra of both the Brauer algebra and the BMW algebra of type* D_n *over the rings* $Z[\delta,\delta^{-1}]$ *and* R*, respectively.*

The current work completes the proof that there is an isomorphism from the BMW algebra to the algebra of tangles having a pole of order 2 studied in [7]. For each element of the cellular basis, the two corresponding root sets determine the set of horizontal strands at the top and bottom, respectively, and the corresponding Weyl group element determines the vertical strands of the tangle. The isomorphism is discussed at the end of this paper.
Putting together [23], Theorem 1.1 and the main theorem of [8], we have reached a complete description of the BMW algebras of spherical simply laced type.

2. Overview

We proceed as follows. First, in Section 3, we introduce the BMW algebra $\mathrm{B}(M)$ over R for M of type A_n ($n\geq 1$), D_n ($n\geq 4$), or E_n ($n=6,7,8$), which we denote ADE. Then the Brauer algebra, $\mathrm{Br}(M)$, of the same type over $\mathbb{Z}[\delta^{\pm 1}]$ is obtained from $\mathrm{B}(M)$ by specializing m to 0 and l to 1. This algebra was defined in [4] where it was shown to be free over R of rank $(2^n+1)n!!-(2^{n-1}+1)n!$ in case $M=\mathrm{D}_n$. The modding out of m and $l-1$ gives a surjective R-equivariant map $\mu:\mathrm{B}(M)\mapsto \mathrm{Br}(M)$.
The Brauer algebra $\mathrm{Br}(M)$ is given in terms of generators e_i, r_i for i running over the nodes of M, and relations determined by M (cf. Definition 3.3). The subalgebra of $\mathrm{Br}(M)$ generated by the r_i is the group algebra over $\mathbb{Z}[\delta^{\pm 1}]$ of $W(M)$, the Coxeter group of type M.

The specialization enables us to pass from monomials in $\mathrm{B}(\mathrm{D}_n)$ to monomials in $\mathrm{Br}(\mathrm{D}_n)$. We will use this observation to find a basis of monomials for $\mathrm{B}(\mathrm{D}_n)$ from a similar basis in $\mathrm{Br}(\mathrm{D}_n)$.

In Section 4 we summarize results from [4] and [6] which show how the monomials of $\mathrm{Br}(M)$ determine sets of mutually orthogonal roots, which in the case $M = \mathrm{A}_{n-1}$ are directly related to tops and bottoms of the well-known Brauer diagrams. The monomials, including powers of δ, form a monoid inside $\mathrm{Br}(M)$, denoted $\mathrm{BrM}(M)$ (see Definition 3.3).

In Sections 5 and 6 we use the following strategy to produce a basis of $\mathrm{B}(\mathrm{D}_n)$ from elements of $\mathrm{BrM}(\mathrm{D}_n)$. A word $\underline{a}$ in the generators of the Brauer monoid $\mathrm{BrM}(M)$ is said to be of *height* t if the number of generators r_i occurring in it is equal to t. We say that $\underline{a}$ is *reducible* to another word $\underline{b}$ if $\underline{b}$ can be obtained from $\underline{a}$ by a finite sequence of specified rewrite rules (listed in Table 2) that do not increase the height. This process will be called a *reduction*. The significance of such a reduction is that the word $\underline{a}$ also corresponds to a unique monomial in the BMW algebra and that a parallel reduction (with rules listed in Table 1) can be carried out in the BMW algebra in the sense that the monomial in $\mathrm{B}(\mathrm{D}_n)$ corresponding to $\underline{a}$ can be rewritten as a linear combination of monomials all of which are represented by words of height less than or equal to the height of $\underline{a}$, with equality occurring for at most one term (see Proposition 3.5(ii)). We exhibit a finite set of reduced words to which each word reduces; see Corollary 6.12. This will lead to a set T of reduced words such that every word in the generators of $\mathrm{B}(\mathrm{D}_n)$ can be reduced to an element of T up to multiples by powers of δ. The above argument will give that, when viewed as elements of $\mathrm{B}(\mathrm{D}_n)$, the set T is a spanning set of $\mathrm{B}(\mathrm{D}_n)$.

In Section 7 we prove our main result by constructing a suitable set T of monomials corresponding to specific triples consisting of pairs of sets of mutually orthogonal roots and a Weyl group element. We also prove Corollary 1.3 by showing that the generalized Temperley–Lieb algebra of type D_n, embeds in $\mathrm{B}(\mathrm{D}_n)$ and in $\mathrm{Br}(\mathrm{D}_n)$.

In Section 8 we show that if the ring of coefficients is extended to an integral domain containing 2^{-1}, the algebra $\mathrm{B}(\mathrm{D}_n)$ is cellular in the sense of [16, Definition 1.1]. In our proof, we need the ring extension in order to invoke [14, Theorem 1.1] where cellularity of the Hecke algebras of type D_n is proved for such rings of coefficients. This Hecke algebra is a natural quotient of $\mathrm{B}(\mathrm{D}_n)$ and the Hecke algebras of type D_{n-2t} occur as subalgebras with different idempotents as identities in the analysis.

We have applied the above results in [7], where a tangle algebra $\mathbf{KT}(\mathrm{D}_n)$ over R on n strands was introduced. This algebra was shown to be a homomorphic image of the BMW algebra $\mathrm{B}(\mathrm{D}_n)$ of type D_n and Theorem 1.1 gives that $\mathbf{KT}(\mathrm{D}_n)$ is an isomorphic image of it.

Part of the work reported here grew out of the PhD. thesis of one of us, [13]. The other two authors wish to acknowledge Caltech and Technische Universiteit Eindhoven for enabling mutual visits.

3. BMW and Brauer Algebras

The BMW algebras of type A_n $(n \geq 1)$, D_n $(n \geq 4)$, and E_n $(n = 6, 7, 8)$ have been discussed extensively in [5]. We assume that M is a Coxeter diagram which is one of these (in particular, it has no multiple bonds). Our main results will only concern M of type A_{n-1} and D_n. The BMW algebra of type M is defined over the ring $R = \mathbb{Z}[l^{\pm 1}, m, \delta^{\pm 1}]/(m(\delta - 1) - (l^{-1} - l))$.

Definition 3.1. The BMW algebra $\mathrm{B}(M)$ of type M is the free algebra over R given by generators g_i, e_i with i running over the nodes of the diagram M, subject to the relations in the BMW Relations Table 1 where $i \sim j$ denotes adjacency of two nodes i and j.

	for i
(RSrr)	$g_i^2 = 1 - m(g_i - l^{-1}e_i)$
(RSer)	$e_i g_i = l^{-1} e_i$
(RSre)	$g_i e_i = l^{-1} e_i$
(HSee)	$e_i^2 = \delta e_i$
	for $i \not\sim j$
(HCrr)	$g_i g_j = g_j g_i$
(HCer)	$e_i g_j = g_j e_i$
(HCee)	$e_i e_j = e_j e_i$
	for $i \sim j$
(HNrrr)	$g_i g_j g_i = g_j g_i g_j$
(HNrer)	$g_j e_i g_j = g_i e_j g_i + m(e_j g_i - e_i g_j + g_i e_j - g_j e_i)$ $+m^2(e_j - e_i)$
(RNrre)	$g_j g_i e_j = e_i e_j$
(RNerr)	$e_i g_j g_i = e_i e_j$
(HNree)	$g_j e_i e_j = g_i e_j + m(e_j - e_i e_j)$
(RNere)	$e_i g_j e_i = l e_i$
(HNeer)	$e_j e_i g_j = e_j g_i + m(e_j - e_j e_i)$
(HNeee)	$e_i e_j e_i = e_i$
	for $i \sim j \sim k$
(HTeere)	$e_j e_i g_k e_j = e_j g_i e_k e_j$
(RTerre)	$e_j g_i g_k e_j = e_j e_i e_k e_j + m(e_j e_i g_k e_j - l e_j)$

TABLE 1. BMW Relations

Remark 3.2. The set of relations given is redundant. In fact, the relations (HNrer), (HNree), (HNeer), (HNeee), (HTeere), and (RTerre) follow from the others, as we will explain. Moreover, if $\mathrm{B}(M)$ is tensored with a ring in which m is invertible, they all follow from (RSrr), (RSre), (HCrr), (HNrrr), and (RNere). This is shown in [5] where these were labeled (D1), (R1), (B1), (B2), and (R2), respectively.
We will prove the stated redundancies, starting with (HNeee). By (RNerr), (RSre), and (RNere), respectively, we have

$$e_i e_j e_i \quad = \quad e_i g_j g_i e_i = l^{-1} e_i g_j e_i = e_i.$$

For (HNeer) we multiply (RNerr) from the right by g_j, apply (RSrr) to the right hand side, and, for the final equality, (RNerr), and (RNere):

$$\begin{aligned} e_j e_i g_j &= e_j g_i g_j^2 = e_j g_i (1 - m g_j + m l^{-1} e_j) = e_j g_i - m e_j g_i g_j + m l^{-1} e_j g_i e_j \\ &= e_j g_i - m e_j e_i + m e_j. \end{aligned}$$

(HNree) is derived in a similar way. The equation (HNrer) is dealt with in [5, Proposition 2.3] by use of the relations we have obtained. For (HTeere), we use

(RNerr) and (RNrre), respectively:

$$e_je_ig_ke_j = e_jg_ig_jg_ke_j = e_jg_ie_ke_j.$$

Recall here that $i \not\sim k$ because the diagram M has no triangles. For (RTerre) write $e_jg_ig_ke_j = e_jg_ig_jg_j^{-1}g_ke_j$ and use the expression $g_j^{-1} = g_j + m - me_j$ which follows from (RSrr) and is given in [5, Proposition 2.1].

Definitions 3.3. Let M be a graph of type ADE. We define the Brauer monoid $\mathrm{BrM}(M)$ to be the monoid generated by the elements r_i and e_i $(i \in M)$ and δ subject to the relations in the Brauer Relations Table 2. The Brauer algebra of type M is the monoid algebra $\mathbb{Z}[\mathrm{BrM}(M)]$.

label	relation	label	relation
(δ)	δ is central	(δ^{-1})	$\delta\delta^{-1} = 1$
	for i		
(RSrr)	$r_i^2 = 1$	(RSer)	$e_ir_i = e_i$
(RSre)	$r_ie_i = e_i$	(HSee)	$e_i^2 = \delta e_i$
	for $i \not\sim j$		
(HCrr)	$r_ir_j = r_jr_i$	(HCer)	$e_ir_j = r_je_i$
(HCee)	$e_ie_j = e_je_i$		
	for $i \sim j$		
(HNrrr)	$r_ir_jr_i = r_jr_ir_j$	(HNrer)	$r_je_ir_j = r_ie_jr_i$
(RNrre)	$r_jr_ie_j = e_ie_j$	(RNerr)	$e_ir_jr_i = e_ie_j$
(HNree)	$r_je_ie_j = r_ie_j$	(RNere)	$e_ir_je_i = e_i$
(HNeer)	$e_je_ir_j = e_jr_i$	(HNeee)	$e_ie_je_i = e_i$
	for $i \sim j \sim k$		
(HTeere)	$e_je_ir_ke_j = e_jr_ie_ke_j$	(RTerre)	$e_jr_ir_ke_j = e_je_ie_ke_j$

Table 2. Brauer Relations

The r_i in $\mathrm{BrM}(M)$ generate a subgroup of the Brauer monoid that we denote W. This is a Coxeter group of type M as the r_i satisfy the required relations and, after factoring out the ideal of $\mathrm{Br}(M)$ generated by the e_i, we obtain the group algebra of W over $\mathbb{Z}[\delta^{\pm 1}]$.

We consider the Brauer algebra of type M as an algebra over $\mathbb{Z}[\delta^{\pm 1}]$. Here δ is in the center of $\mathrm{BrM}(M)$ and we identify this δ with the δ in $Z[\delta^{\pm 1}]$. Since the other defining relations of the Brauer monoid are the defining relations of the corresponding BMW algebra $\mathrm{B}(M)$ modulo the ideal $(l-1, m)$ generated by $l-1$ and m, the Brauer algebra $\mathrm{Br}(M)$ can be identified with $\mathrm{B}(M) \otimes_R R/(l-1, m)$. The corresponding equivariant map $a \mapsto a \otimes 1$ will be denoted by μ.

Just as for $\mathrm{B}(M)$, some of the relations in the Brauer Relations Table 2 are redundant; see [4, Lemma 3.1]. We will need to rewrite words in the generators r_i and e_i, with $\delta^{\pm 1}$ viewed as coefficients. This necessitates the extra relations that are displayed in Table 2.

Definitions 3.4. By F_n we denote the monoid that is the central product of the free monoid on the symbols r_i, e_i $(i = 1, \dots, n)$ with the infinite cyclic group generated

by δ. Its elements will be called *words*. There is a surjective homomorphism of monoids $\pi : F_n \to \mathrm{BrM}(M)$ mapping the symbols r_i, e_i, and δ to the corresponding elements of $\mathrm{BrM}(M)$. The monomial in $\mathrm{B}(M)$ corresponding to $\underline{a} \in F_n$, obtained by replacing r_i by g_i and leaving e_i and δ as before, will be denoted $\rho(\underline{a})$, so $\mu(\rho(\underline{a})) = \pi(\underline{a})$. A word $\underline{a} \in F_n$ is said to be of *height* t if the number of r_i occurring in it is equal to t; we denote this number t by $\mathrm{ht}(\underline{a})$.
We say that $\underline{a}$ is *reducible* to another word $\underline{b}$, or that $\underline{b}$ is a *reduction* of $\underline{a}$, if $\underline{b}$ can be obtained by a finite sequence of specified rewrites, listed in the Brauer Relations Table 2, starting from $\underline{a}$, that do not increase the height. We call a word in F_n *reduced* if it cannot be further reduced to a word of smaller height. We have labeled the relations in the tables above with R or H according to whether the rewrite from left to right strictly lowers the height or not. If the number stays the same, we call it H for *homogeneous*. Our rewrite system will be the set of all rewrites in the Brauer Relations Table 2 in either direction in the homogeneous case when an H appears in its label and from left to right only in case an R occurs in its label. We write $\underline{a} \rightsquigarrow \underline{b}$ if $\underline{a}$ can be reduced to $\underline{b}$; for example (RNerr) gives $e_2r_3e_2 \rightsquigarrow e_2$ if $2 \sim 3$. If the height does not decrease during a reduction, we sometimes use the term *homogeneous reduction* and write $\underline{a} \leftrightsquigarrow \underline{b}$; for example, (HNeee) gives $e_2 \leftrightsquigarrow e_2e_3e_2$ if $2 \sim 3$. If it does decrease, we also speak of a *strict reduction*.
Homogeneous reduction induces a congruence relation on F_n, to which we will refer as *homogeneous equivalence*. We denote its set of equivalence classes by $F_n/\leftrightsquigarrow$. The congruence property turns it into a monoid.

The reductions in F_n are important because they have a meaning for both the Brauer algebra and the corresponding BMW algebra. For each of the relations in the Brauer Relations Table 2, there is a corresponding relation in the BMW Relations Table 1. In Section 7, the following proposition will be used to find a basis of $\mathrm{B}(\mathrm{D}_n)$ that has the same size as a basis of $\mathrm{Br}(\mathrm{D}_n)$.

Proposition 3.5. *Suppose $\underline{a} \rightsquigarrow \underline{b}$ with $\underline{a}$, $\underline{b} \in F_n$.*
(i) $\pi(\underline{a}) = \pi(\underline{b})$ *in* $\mathrm{BrM}(M)$.
(ii) There are a finite number of $\lambda_{\underline{c}} \in R$ such that, in $\mathrm{B}(M)$,

$$\rho(\underline{a}) = \rho(\underline{b}) + \sum_{\underline{c} \in F_n,\, \mathrm{ht}(\underline{c}) < \mathrm{ht}(\underline{a})} m\lambda_{\underline{c}}\rho(\underline{c}).$$

Proof. (i). For each reduction step of the sequence of relations, the word evaluated in $\mathrm{BrM}(M)$ is the same because the relations are satisfied in $\mathrm{BrM}(M)$ by definition. This means $\pi(\underline{a}) = \pi(\underline{b})$, proving (i).

(ii). The expressions in the BMW Relations Table 1 all have one term on each side whose coefficient is not a multiple of m. These terms are the same as in the Brauer Relations Table 2 with g_i instead of r_i. Indeed, if $l = 1$ and the terms with coefficient m are ignored, the tables are the same. Each reduction step in $\underline{a} \rightsquigarrow \underline{b}$, replaces the term on the left with the corresponding one on the right side of the equality in the table plus terms that are multiples of m and have strictly smaller height. The end result is $\rho(\underline{b})$ plus terms that are multiples of m, whose height has been reduced at least once. As $\underline{a} \rightsquigarrow \underline{b}$ involves only a finite sequence of specific rewrites from Table 2, only a finite number of substitutions from Table 1 has been applied, and so only a finite number of summands occurs at the right hand side of the equality in (ii). $\square$

Notation 3.6. For $x_1, \dots, x_q \in \{r_1, \dots, r_n, e_1, \dots, e_n, \delta^{\pm 1}\}$, we write $(x_1 \cdots x_q)^{\mathrm{op}} = x_q \cdots x_1$, thus defining an opposition map on F_n. This notation is compatible with the maps π and ρ when $\cdot^{\mathrm{op}}$ on $\mathrm{B}(M)$ and $\mathrm{Br}(M)$ is interpreted as the anti-involution of [5, Remark 2.1(i)] and [4, Remark 5.7], respectively.

To end this section, we discuss properties of R which show how to relate some properties of sets of monomials in $\mathrm{B}(\mathrm{D}_n)$ to corresponding ones in $\mathrm{Br}(\mathrm{D}_n)$ using the maps π and ρ.

Lemma 3.7. *The ring R embeds in $\mathbb{Q}(\delta)[l^{\pm 1}]$ and also in $\mathbb{Q}(l, \delta)$.*

Proof. Let $D = \mathbb{Z}[l^{\pm 1}, \delta^{\pm 1}]$, which is a unique factorization domain, and let F be its field of fractions. Put $s(m) = (1-\delta)m - (l - l^{-1})$. Notice $s(m)$ is primitive and so irreducible in $F[m]$ by Gauss' Lemma. Hence $R = D[m]/(s(m))$ is an integral domain. Its field of fractions is $\mathbb{Q}(l, \delta)$. Finally, $\mathbb{Q}(\delta)[l^{\pm 1}]$ is a subring of $\mathbb{Q}(l, \delta)$ containing both D and m, as the latter is equal to $(l - l^{-1})/(1-\delta)$ modulo $s(m)$, and so also contains R. $\square$

The following lemma will give a lower bound for the rank of $\mathrm{B}(\mathrm{D}_n)$.

Lemma 3.8. *Suppose that T is a finite set of monomials in F_n whose images $(\pi(t))_{t \in T}$ are linearly independent in $\mathrm{Br}(\mathrm{D}_n)$. Then $(\rho(t))_{t \in T}$ are linearly independent in $\mathrm{B}(\mathrm{D}_n)$.*

Proof. Suppose that $\sum_{t \in T} \lambda_t \rho(t)$ with $\lambda_t \in R$ is a non-trivial linear combination that is equal to 0 in $\mathrm{B}(\mathrm{D}_n)$. Then the same non-trivial linear relation holds over the principal ideal domain $\mathbb{Q}(\delta)[l^{\pm 1}]$ into which R embeds according to Lemma 3.7. Rescale the coefficients by a suitable power of $l-1$ to guarantee $\lambda_s \notin (l-1)\mathbb{Q}(\delta)[l^{\pm 1}]$ for some $s \in T$. Now $\mu(\lambda_s) \neq 0$ and $\pi(t) = \mu(\rho(t))$ for $t \in T$ (cf. Definitions 3.4), so $\sum_{t \in T} \mu(\lambda_t)\pi(t)$ is a non-trivial linear combination in $\mathrm{Br}(\mathrm{D}_n)$, that is equal to 0, contradicting the linear independence assumption on $\pi(t)_{t \in T}$. $\square$

The following result will yield the right upper bound on the rank of $\mathrm{B}(\mathrm{D}_n)$.

Proposition 3.9. *Let M be of type* ADE *and let T be a set of words in F_n such that the $(\pi(t))_{t \in T}$ is a basis of $\mathrm{Br}(M)$. If each word in F_n can be reduced to an element of $\delta^{\mathbb{Z}}T$, then $\rho(T)$ is a basis of $\mathrm{B}(M)$ and each element of T is a reduced word.*

Proof. Assume that each word in F_n can be reduced to an element of $\delta^{\mathbb{Z}}T$. We first prove that $\rho(T)$ is a linear spanning set of $\mathrm{B}(M)$. If not, there is a word $\underline{a}$ in F_n such that $\rho(\underline{a})$ is not in the linear span of $\rho(T)$. Pick one of smallest height. Then, by assumption, $\underline{a} \rightsquigarrow \underline{b}$ for some $\underline{b} \in \delta^{\mathbb{Z}}T$. Proposition 3.5(ii) implies that $\rho(\underline{a}) - \rho(\underline{b})$ is a linear combination of monomials in $\mathrm{B}(M)$ of height lower than $s = \mathrm{ht}(\underline{a})$. If $s = 0$, this means $\rho(\underline{a}) = \rho(\underline{b}) \in \delta^{\mathbb{Z}}\rho(T)$. Otherwise $s > 0$ and we may assume, using induction on height, that monomials in $\mathrm{B}(M)$ of height lower than s are all in the linear span of the elements in $\rho(T)$ of height lower than s. Then the right

hand side in the expression of $\rho(\underline{a}) - \rho(\underline{b})$ as a linear combination of monomials of lower height is in the linear span of $\rho(T)$. Consequently, $\rho(\underline{a})$ is in the same linear span, a contradiction. We have shown that $\mathrm{B}(M)$ is spanned by $\rho(T)$.
It now follows from Lemma 3.8 that $(\rho(t))_{t\in T}$ is a basis of $\mathrm{B}(\mathrm{D}_n)$. If $\underline{a} \in T$ is not reduced, then there is $\underline{b} \in F_n$ with $\underline{a} \rightsquigarrow \underline{b}$ and $\mathrm{ht}(\underline{a}) > \mathrm{ht}(\underline{b})$. After applying the assumption to $\underline{b}$, we may assume $\underline{b} \in \delta^{\mathbb{Z}}T$ and still $\mathrm{ht}(\underline{a}) > \mathrm{ht}(\underline{b})$. In view of the hypothesis, $\pi(\underline{a})$ and $\delta^i\pi(\underline{b})$ for some $i \in \mathbb{Z}$ are distinct members of a basis of $\mathrm{Br}(\mathrm{D}_n)$, so $\pi(\underline{a})$ and $\pi(\underline{b})$ are linearly independent. On the other hand, by Proposition 3.5(i), $\pi(\underline{a}) = \pi(\underline{b})$, a contradiction. $\square$

4. Admissible Sets and the Function Monoid

Let $n \in \mathbb{N}$, $n \geq 4$. We summarize some of the results of [4] and [6] about admissible sets with a special focus on type $M = \mathrm{D}_n$. These are particular sets of mutually orthogonal positive roots. The results will be used to monitor the reduction of words in F_n. We will fix a root system Φ for W and a set of simple roots $\alpha_1, \ldots, \alpha_n$ with indices for $M = \mathrm{D}_n$ as indicated in the Dynkin diagram of Figure 1.

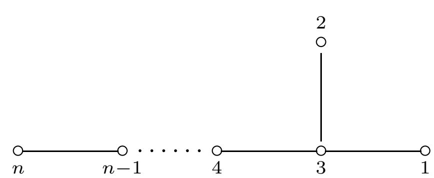


Figure 1. The diagram of type D_n with node labels

In terms of the standard orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$ of $\mathbb{R}^n$, these simple roots are $\alpha_1 = \varepsilon_1 + \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_1$, $\alpha_3 = \varepsilon_3 - \varepsilon_2$, ..., $\alpha_n = \varepsilon_n - \varepsilon_{n-1}$. Accordingly, we will write $\Phi^+ = (\mathbb{Z}_{\geq 0}\alpha_1 + \mathbb{Z}_{\geq 0}\alpha_2 + \cdots + \mathbb{Z}_{\geq 0}\alpha_n) \cap \Phi$ where $\mathbb{Z}_{\geq 0}$ are the non-negative integers. The elements of Φ^+ are called the *positive roots* of Φ (or simply D_n); they are of the form $\varepsilon_j - \varepsilon_i$ and $\varepsilon_i + \varepsilon_j$ for $n \geq j > i \geq 1$. Recall $\Phi = \Phi^+ \cup (-\Phi^+)$. The reflection in $\mathbb{R}^n$ with root β is denoted r_β. The map $r_i \mapsto r_{\alpha_i}$ $(i = 1, \ldots, n)$ extends to an isomorphism from W to a reflection subgroup of the orthogonal group on $\mathbb{R}^n$. We often identify W with this reflection group by means of the isomorphism.
There are some standard properties of the root systems we are using which we mention here for convenience. All roots have square norm 2. The inner products are all ± 2, ± 1, or 0. If $(\beta, \gamma) = 1$, then $\beta - \gamma$ is a root, and if $(\beta, \gamma) = -1$, then $\beta + \gamma$ is a root, equal to $r_\beta\gamma$. Further if $(\beta, \gamma) = 0$, then $\beta \pm \gamma$ is never a root. We often encounter the situation in which $(\beta, \alpha_i) = 0$, $i \sim j$, and $\beta - \alpha_j$ is a root. Then $(\beta - \alpha_j, \alpha_i) = 1$ and so $\beta - \alpha_j - \alpha_i = r_i r_j \beta$ is also a root.

Remark 4.1. There are two notions of height. The first is $\mathrm{ht}(\underline{a})$ for $\underline{a}$ an element of F_n (cf. Definitions 3.4). The second is the more standard notion of height of a positive root β. This is $\sum \lambda_i$ for the root $\sum \lambda_i\alpha_i$ where the α_i are the simple roots. We also denote this $\mathrm{ht}(\beta)$ and trust no confusion will arise.

In order to recognize the elements of the ideal in $\mathrm{BrM}(\mathrm{D}_n)$ generated by e_1e_2, see Definition 6.3, we will need the notion of orthogonal mates.

Definitions 4.2. For $\beta = \varepsilon_i - \varepsilon_j$ a root in the root system Φ of type D_n embedded in $\mathbb{R}^n$ as indicated above, its *orthogonal mate* is defined to be $\beta^* = \varepsilon_i + \varepsilon_j$ and, vice versa, the *orthogonal mate* of β^* is $\beta^{**} = \beta$. Furthermore, we write r^*_β for r_{β^*}, the reflection whose root is the orthogonal mate of β. For the simple roots α_i we also write r^*_i instead of $r^*_{\alpha_i}$.
If $n > 4$, the roots orthogonal to β form a subsystem of Φ of type $\mathrm{A}_1\mathrm{D}_{n-2}$ and β^* is the unique positive root in the A_1 component of this subsystem. If $n = 4$, the choice of orthogonal mate essentially depends on the choice of an orthogonal pair of simple roots.
There are several equivalent definitions of admissible sets as outlined in [6, Proposition 2.3]. For our purposes we may define a set B of mutually orthogonal positive roots to be *admissible* if and only if, when $\alpha_1, \alpha_2, \alpha_3 \in B$ and there exists a root α for which $(\alpha_i, \alpha) = \pm 1$ for all i, then $r_\alpha r_{\alpha_1} r_{\alpha_2} r_{\alpha_3}\alpha$ or $-r_\alpha r_{\alpha_1} r_{\alpha_2} r_{\alpha_3}\alpha$ is also in B. Given any set, B, of mutually orthogonal positive roots, a straightforward exercise shows there is a unique smallest admissible set containing B. This set is called the *admissible closure* of B, notation B^{cl}; see [4, Definition 2.2].
By $\mathcal{A}$ we denote the collection of all admissible sets (including the empty set). This set has a natural W-action given by

$$wB = \Phi^+ \cap \{\pm w\beta \mid \beta \in B\}$$

for $w \in W$. A representative of each W-orbit in $\mathcal{A}$ is given in [4, Table 3]; this is a corrected version of a similar table in [6]. We will need these only for types A_n and D_n, which, for the convenience of the reader, are summarized in Lemma 4.4 and Table 3. The meaning of M_Y and S_Y in Table 3 will become clear later (in Proposition 5.8).

Notation 4.3. By $\mathcal{Y}$ we denote the collection of the following sets of nodes of D_n.

$$\begin{aligned} Y(t) &= \{n, n-2, \ldots, n-2t+2\}, \text{ for } t \in [0, \lfloor n/2 \rfloor] \\ Y^*(t) &= \{n, n-2, \ldots, n-2t+4, 1, 2\}, \text{ for } t \in [1, \lfloor n/2 \rfloor] \\ Y'(n/2) &= \{1, 4, 6, 8, \ldots, n\} \text{ if } n \text{ is even.} \end{aligned}$$

For $Y \in \mathcal{Y}$, the set B_Y is the admissible closure of the set of roots α_j for $j \in Y$. For $Y = Y^*(t)$, this implies that B_Y is the set of roots α_j and α^*_j for $j \in Y$. For $Y = Y(t)$ or $Y'(n/2)$ however, no orthogonal mates occur and so B_Y is the set of roots α_j for $j \in Y$.

Lemma 4.4. *Each W-orbit in $\mathcal{A}$ has a unique representative B_Y for $Y \in \mathcal{Y}$.*

For instance, if $n = 4$, there are three orbits of admissible sets of size 2, with representatives B_Y, where $Y = Y(2)$, $Y'(2)$, and $Y^*(1)$, respectively.
Notice that, if B is an admissible set containing a root as well as its orthogonal mate, then it is a union of roots together with their orthogonal mates.

The following proposition is proved in [4, Theorem 3.6]; the fact that e_iB as described below is well defined is shown in [4, Lemma 3.3(v)].

Proposition 4.5. *Let M be of type* ADE. *The action of W on $\mathcal{A}$ extends to an action of the Brauer monoid* $\mathrm{Br}(M)$ *determined by the following rules for the*

M	$\lvert B_Y \rvert$	M_Y	Y	$S_Y = \{x\hat{e}_Y \mid x \text{ as below}\}$
A_n	t	A_{n-2t}	$Y(t)$	$r_1, \ldots, r_{n-2t}$
D_n	t	$\mathrm{A}_1\mathrm{D}_{n-2t}$	$Y(t)$	$\underline{z}_n^*, r_1, \ldots, r_{n-2t}$
D_n	0	D_n	$\emptyset$	$r_1, \ldots, r_n$
D_n (n odd)	$(n-1)/2$	A_1	$Y((n-1)/2)$	$\underline{z}_n^*$
D_n (n even)	$n/2$	A_1	$Y(n/2)$	r_1
D_n (n even)	$n/2$	A_1	$Y'(n/2)$	r_2
D_n	$2t$	A_{n-2t-1}	$Y^*(t)$	$r_4, \ldots, r_{n-2t+2}$

TABLE 3. Cocliques $Y \in \mathcal{Y}$ of M and sizes of the admissible sets B_Y for types A_n and D_n. The index t in the line for A_n satisfies $0 \le t \le \lfloor n/2 \rfloor$. The index t in the first line for D_n satisfies $0 < t < \lfloor (n-1)/2 \rfloor$. In the last line it satisfies $0 < t \le \lfloor n/2 \rfloor$.

generators e_i, where $i \in M$ and $B \in \mathcal{A}$.

$$\begin{aligned} \delta B &= B, \\ e_i B &= \begin{cases} B & \text{if } \alpha_i \in B, \\ (B \cup \{\alpha_i\})^{\mathrm{cl}} & \text{if } \alpha_i \perp B, \\ r_\beta r_i B & \text{if } \beta \in B \setminus \alpha_i^\perp. \end{cases} \end{aligned}$$

For this action, if $a \in \mathrm{BrM}(M)$ and B, $C \in \mathcal{A}$ satisfy $B \subseteq C$, then $aB \subseteq aC$.

When considering words $\underline{a}$ of F_n and $B \in \mathcal{A}$, we will often write $\underline{a}B$ instead of $\pi(\underline{a})B$ and $B\underline{a}$ instead of $\pi(\underline{a})^{\mathrm{op}}B$. The latter defines right actions of F_n and $\mathrm{BrM}(M)$ on $\mathcal{A}$.

Remark 4.6. Let $M = \mathrm{D}_n$ and suppose that $\underline{a}$ is a word in F_n. Then, by Lemma 4.4, $\underline{a}(\emptyset) \in WB_Y$ for some $Y \in \mathcal{Y}$. In [4, Proposition 4.9], it is shown that, up to powers of δ, the elements $\pi(\underline{a})$ of $\mathrm{BrM}(\mathrm{D}_n)$ are in bijective correspondence with triples (B, B', z) with $\underline{a}(\emptyset)$ and $\underline{a}^{\mathrm{op}}(\emptyset)$ in the same W-orbit WB_Y for $Y \in \mathcal{Y}$ and $z \in W(M_Y)$, where M_Y is the Coxeter type corresponding to Y as specified in column 3 of Table 3. We will prove a counterpart of this result. In fact, we will prove a stronger statement (Corollary 6.12) about rewrites of $\underline{a}$ in F_n rather than equality for $\pi(\underline{a})$ in $\mathrm{BrM}(\mathrm{D}_n)$. In general, the expressions appearing in [loc. cit.] are not reduced, which makes them unsuitable for rewrite purposes. By way of example, we mention that, for $e_n r_n^*$, we will find an expression of height 1 (namely $\underline{z}_n^*$ as in Lemma 5.6) where r_n^* (see Definition 4.2) has large height.

5. Elementary properties of the Brauer monoid

In this section we prepare for the reduction of words in F_n for the Brauer monoid, $\mathrm{BrM}(\mathrm{D}_n)$, of type D_n. The purpose of this and the next section is to show that, up to homogeneous equivalence, each element of F_n has a unique reduced word. This goal is achieved in Corollary 6.12.

We will use the action of Proposition 4.5. Let $\underline{a} \in F_n$.
Our immediate goal will be to show that $\underline{a}$ can be rewritten to a reduced word that is uniquely determined by $\pi(\underline{a})$ up to homogeneous equivalence, so the reduced

word will be a unique element of $F_n/\leftrightsquigarrow$. In fact, we shall be working with words in F_n but often think of them as representing classes in $F_n/\leftrightsquigarrow$.
Later, in Sections 7 and 8, we will use words in F_n to represent monomials in $\mathrm{B}(\mathrm{D}_n)$. Before we continue we introduce some notation.

Notation 5.1. Suppose that k and i are two nodes of D_n. Let $i = i_1, i_2, \ldots, i_r = k$ be the geodesic path from i to k in D_n. Then we set $e_{i,k} = e_{i_1}e_{i_2}\cdots e_{i_r}$, which we interpret as an element of F_n. Notice the first factor is e_i and the last is e_k. In particular, for $i < k$ and $k \geq 3$, we have $e_{i,k} = e_ie_{i+1}\cdots e_k$ unless $i = 1$ in which case it is $e_1e_3e_4\cdots e_k$. Also $e_{1,2} = e_1e_3e_2$ is a special case.
Let β be a positive root. If $\beta = \sum \lambda_i\alpha_i$ we call the *support* of β the set of nodes i for which $\lambda_i \neq 0$; it is denoted $\mathrm{Supp}(\beta)$. As in [5], we will write, if k is a node of the diagram, $\mathrm{Proj}(k, \beta)$ for the node of D_n in $\mathrm{Supp}(\beta)$ nearest to k. There is a unique one as the support is a connected set of nodes in the Dynkin diagram D_n, which is a tree.

Definition 5.2. If $k \in \mathrm{Supp}(\beta)$, then, as follows directly from [5, Proposition 3.2], there is a unique Weyl group element $a_{\beta,k}$ of smallest length that maps $\{\alpha_k\}$ to $\{\beta\}$ in the action of Proposition 4.5 (so $a_{\beta,k}\{\alpha_k\} = \{\beta\}$). Its height, as a monomial of $\mathrm{Br}(\mathrm{D}_n)$, is equal to $\mathrm{ht}(\beta)-1$. The opposite element $a^{\mathrm{op}}_{\beta,k}$ maps $\{\beta\}$ to $\{\alpha_k\}$. We will often view $a_{\beta,k}$ as an element of F_n in the guise of a shortest expression for $a_{\beta,k}$ as a product of simple reflections. Since any two such expressions are homogeneously equivalent, they represent the same element of $F_n/\leftrightsquigarrow$, which suffices for our purpose of reductions.
We extend the definition of $a_{\beta,k}$ to the case where $k \notin \mathrm{Supp}(\beta)$. For β a positive root with $k \notin \mathrm{Supp}(\beta)$ and k' the node next to k on the geodesic path from k to $j = \mathrm{Proj}(k, \beta)$, we set $a_{\beta,k} = a_{\beta,j}e_{j,k'}$ in F_n.

We will be mainly concerned with the case $k = n$.

Lemma 5.3. *The elements $a_{\beta,n}$ satisfy the following properties.*

(i) If $j \leq n-1$, then $a_{\alpha_j,n}e_n = e_{j,n}$.
(ii) If j is a node of D_n such that $\beta - \alpha_j$ is a root, then $a_{\beta,n}e_n \leftrightsquigarrow r_ja_{\beta-\alpha_j,n}e_n$.
(iii) $\mathrm{ht}(a_{\beta,n}) = \mathrm{ht}(\beta) - 1$.

Proof. (i). Clearly n is not in the support of α_j and so $a_{\alpha_j,n} = e_{j,n-1}$. The required equality follows from multiplication by e_n on the right.
(ii). We first consider the case where $n \in \mathrm{Supp}(\beta)$. In this case $a_{\beta,n}$ is any word of shortest length which takes α_n to β. Its length is $\mathrm{ht}(\beta) - 1$. As mentioned above and in [5, Proposition 2.3] it is a unique up to homogeneous equivalence. If $j \neq n$, then $a_{\beta-\alpha_j,\alpha_n}$ is a word of shortest length taking α_n to $\beta - \alpha_j$ and so $r_ja_{\beta-\alpha_j,\alpha_n}$ is a word of shortest length taking α_n to β, proving that $a_{\beta,n} \leftrightsquigarrow r_ja_{\beta-\alpha_j,n}$, and so $a_{\beta,n}e_n \leftrightsquigarrow r_ja_{\beta-\alpha_j,n}e_n$.
If $j = n$, then $\beta-\alpha_n$ is a root. Because of the structure of the roots of D_n, this means the coefficient in β of both α_n and α_{n-1} as a linear combination of simple roots is 1 and so $\beta-\alpha_n$ has $n-1$ in its support but not n. In particular, $a_{\beta-\alpha_n,n-1}$ is a word of height $\mathrm{ht}(\beta)-1$ taking α_{n-1} to $\beta-\alpha_n$. As n is not in the support of $\beta-\alpha_n$ but $n-1$ is, $a_{\beta-\alpha_n,n-1}$ is a word in r_i with $i \leq n-1$ and further as the coefficient of α_{n-1} in $\beta-\alpha_n$ is just 1, all the r_i occurring in a reduced word for $a_{\beta-\alpha_n,n-1}$ have $i \leq n-2$. In particular r_n and $a_{\beta-\alpha_n,n-1}$ commute. Also, $a_{\beta-\alpha_n,n} = a_{\beta-\alpha_n,n-1}e_{n-1}$

by definition. Now $r_n a_{\beta-\alpha_n,n} e_n = r_n a_{\beta-\alpha_n,n-1} e_{n-1} e_n \leftrightsquigarrow a_{\beta-\alpha_n,n-1} r_n e_{n-1} e_n$. By (HNree) $r_n e_{n-1} e_n \leftrightsquigarrow r_{n-1} e_n$. In terms of the action of Proposition 4.5, this implies $a_{\beta-\alpha_n,n-1} r_{n-1}\{\alpha_n\} = a_{\beta-\alpha_n,n-1}\{\alpha_{n-1}+\alpha_n\}$. Recall $a_{\beta-\alpha_n,n-1}$ is a shortest word in $r_1,\dots,r_{n-1}$ taking $\{\alpha_{n-1}\}$ to $\{\beta-\alpha_n\}$ and so is a word of shortest length taking $\{\alpha_{n-1}+\alpha_n\}$ to $\{\beta\}$ as $a_{\beta-\alpha_n,n-1}$ fixes $\{\alpha_n\}$. Now $a_{\beta-\alpha_n,n-1} r_{n-1}$ is a shortest word taking $\{\alpha_n\}$ to $\{\beta\}$ and so $a_{\beta-\alpha_n,n-1} r_{n-1} \leftrightsquigarrow a_{\beta,n}$. This gives $r_n a_{\beta-\alpha_n,n} e_n \leftrightsquigarrow a_{\beta-\alpha_n,n-1} r_{n-1} e_n \leftrightsquigarrow a_{\beta,n} e_n$.
If $n \notin \mathrm{Supp}\,(\beta)$, let $i = \mathrm{Proj}(\beta, n)$. If $i > 3$, the argument above applies directly with i instead of n and $j \le i$, giving $r_j a_{\beta-\alpha_j,i} e_i \leftrightsquigarrow a_{\beta,i} e_i$. The assertion now follows from right multiplication by $e_{i+1,n}$. For $i = 3$, the root β is $\alpha_3+\alpha_2$ or $\alpha_3+\alpha_1$ and the arguments are similar. Notice i cannot be 1 or 2 as $\beta-\alpha_j$ is a root.

(iii). This is direct from (ii) and the definition of $a_{\beta,n}$. □

Remark 5.4. As the proof uses the relation (HNree) which is not binomial in the BMW algebra, two homogeneously equivalent words of (ii) do not necessarily have the same image under ρ in the BMW algebra. Indeed, if $j = n$ and $\beta = \alpha_{n-1}+\alpha_n$, then $a_{\beta,n} = r_{n-1}$ and $a_{\beta-\alpha_n,n} = e_{n-1}$, so $\rho(a_{\beta,n} e_n) = g_{n-1} e_n$ is distinct from $\rho(r_n a_{\beta-\alpha_n,n} e_n) = g_n e_{n-1} e_n$. As indicated in Proposition 3.5, the two expressions are equal up to sums of monomials of lower height (with coefficients in the ideal generated by m).

We have denoted words in F_n by underlined symbols like $\underline{a}$. In the remainder of the paper we will need to reduce words which have specific r_i or e_i in them. It is notationally awkward to have long strings underlined, and so we will dispense with this for words including such r_i and e_i. For example we write $\underline{a} r_i r_j e_i \rightsquigarrow \underline{a} e_j e_i$ rather than $\underline{a r_i r_j e_i} \rightsquigarrow \underline{a e_j e_i}$. We continue to underline general elements of F_n as $\underline{a}$.
Let M be a Coxeter diagram with n nodes. The *Matsumoto–Tits rewrite rules* of type M on $s_1,\dots,s_k$ are the following rewrite rules in the free monoid on $s_1,\dots,s_n$.

$$\begin{aligned} s_i s_i &\rightsquigarrow 1 \\ s_i s_j &\rightsquigarrow s_j s_i \ \text{ if } \ i \not\sim j \\ s_i s_j s_i &\rightsquigarrow s_j s_i s_j \ \text{ if } \ i \sim j \end{aligned}$$

Note that the second and the third rule are homogeneous.

Lemma 5.5. *Let M be a Coxeter diagram with n nodes. Then any two reduced words with respect to the Matsumoto–Tits rewrite rules of type M on $s_1,\dots,s_n$ are homogeneously equivalent, that is, can be rewritten into each other by means of a series of the second and the third rewrite rules.*

Proof. The result can be found in [27] and is independently proved in [22]. A more general version is found in [3]. □

As a first application, note that, for the subgroup W of $\mathrm{BrM}(\mathrm{D}_n)$, the rewrite rules with $r_1,\dots,r_n$ instead of $s_1,\dots,s_k$ coincide with (RSrr), (HCrr), and (HNrrr) of Table 2. Therefore, each element of W corresponds to a unique reduced word of F_n up to homogeneous equivalence. In other words, the equivalence classes in F_n

of reduced words over $\{r_1,\ldots,r_n\}$ correspond bijectively with the elements of the Coxeter group W. This implies that, for each reduced word $\underline{a}\in F_n$ all of whose symbols are in $\{r_1,\ldots,r_n\}$, its homogeneous equivalence class is uniquely determined by $\pi(\underline{a})$. In Proposition 5.8, we will generalize this application to recognize Coxeter groups of type M_Y for each $Y\in\mathcal{Y}$, using words $\underline{s}_i$ in F_n to be specified in Notation 5.7.

A slightly less general statement holds for $\mathrm{B}(\mathrm{D}_n)$ instead of $\mathrm{BrM}(\mathrm{D}_n)$. As of the above-mentioned rewrite rules, (HCrr) and (HNrrr) are binomial in Table 1 as well, for each reduced $\underline{a}\in F_n$ all of whose symbols are in $\{r_1,\ldots,r_n\}$, its homogeneous equivalence class is uniquely determined by $\rho(\underline{a})$ as well. In Proposition 8.4, we will generalize this application, using the same words $\underline{s}_i$ as above in F_n to recognize subquotients of $\mathrm{B}(\mathrm{D}_n)$ isomorphic to Hecke algebras of type M_Y for $Y\in\mathcal{Y}$.

Observe that F_{n-1} is a submonoid of F_n.

Lemma 5.6. *Let $\underline{z}_n^*=e_{n,2}r_1e_{3,n}$ for $n\geq 3$, $\underline{z}_1^*=r_2e_1$, and $\underline{z}_2^*=r_1e_2$ all of these viewed as words in F_n. Then $\underline{z}_n^*$ has height 1 and occurs in the following reductions for $n\geq 3$.*

(i) $r_n^*e_n\rightsquigarrow\underline{z}_n^*$ *and* $e_nr_n^*\rightsquigarrow\underline{z}_n^*$.
(ii) $\underline{z}_n^*\leftrightsquigarrow e_{n,3}r_2e_1e_{3,n}$.
(iii) *For $n\geq 4$ and $i\in\{1,\ldots,n-2\}$,* $e_i\underline{z}_n^*\leftrightsquigarrow\underline{z}_i^*e_n\leftrightsquigarrow e_n\underline{z}_i^*$ *and* $r_i\underline{z}_n^*\leftrightsquigarrow\underline{z}_n^*r_i$.
(iv) $\underline{z}_n^*e_{n-2}\leftrightsquigarrow e_n\underline{z}_{n-2}^*$ *and* $e_n\underline{z}_n^*\leftrightsquigarrow\underline{z}_n^*e_n\leftrightsquigarrow\delta\underline{z}_n^*$.
(v) $\underline{z}_n^*\underline{z}_n^*\rightsquigarrow\delta e_n$.

For n equal to 1 or 2, statements (i) and (v) also hold.

Proof. Assume first $n\geq 3$. By definition, there is only one factor r_i in $\underline{z}_n^*$ and so its height is at most 1.

To see that it does not have height zero we use the representation $\rho_{W\{\alpha_n\}}$ of [4, Theorem 3.6]. In particular we are considering $Y(1)=\{\alpha_n\}$. Consider the action of $\underline{z}_n^*$ in the notation of [loc. cit.] on the 1-space spanned by the vector $\xi_{\{\alpha_n\}}$. Indeed $e_{2,n}\xi_{\{\alpha_n\}}=\delta\xi_{\{\alpha_2\}}$ and then $r_1\xi_{\{\alpha_2\}}=\xi_{\{\alpha_2\}}h_{1,\alpha_2}$. It follows from [loc. cit.] that h_{1,α_2} is one of the generators of $M_{\{n\}}$ which in this case is the Weyl group of the diagram of type $\mathrm{A}_1\mathrm{D}_{n-2}$. Now act by $e_{3,n}$ to see $\underline{z}_n^*\xi_{\{\alpha_n\}}=\delta\xi_{\{\alpha_n\}}h_{1,\alpha_2}$. If $\underline{z}_n^*$ could be reduced it would have height 0 and the action on $\xi_{\{\alpha_n\}}$ would either be 0 or would be $\xi_{\{\beta\}}\delta^k$ for β a root and for some k a contradiction. This means $\underline{z}_n^*$ has height 1.

(i). Let $w_{2,n}=r_3r_2r_4r_3r_5r_4\cdots r_{n-1}r_{n-2}r_nr_{n-1}$ be as in [5, Lemma 3.1] and set $w_{n,2}=w_{2,n}^{\mathrm{op}}$. Then $r_n^*=w_{n,2}r_1w_{2,n}$ where r_n^* was defined in Definition 4.2. In order to show the required reductions, we use repeatedly the reducing relation (RNrre), that is, $r_jr_ie_j\rightsquigarrow e_ie_j$ for $i\sim j$. In particular, $w_{n,2}e_n\rightsquigarrow e_{2,n}$. Now $r_1e_{2,n}\rightsquigarrow e_2r_1e_{3,n}$ and $r_n^*e_n=w_{n,2}r_1w_{2,n}e_n\rightsquigarrow e_{n,2}r_1e_{3,n}=\underline{z}_n^*$. A similar computation shows that $e_nr_n^*\rightsquigarrow\underline{z}_n^*$.

(ii). This statement holds because of $e_3e_2r_1e_3\leftrightsquigarrow e_3r_2e_1e_3$, which is immediate from the defining relation (HTeere).

(iii). For $i\in\{2,\ldots,n-2\}$, by the definition of $e_{k,n}$, (HCee), and (HNeee),

$$\begin{aligned}
e_i\underline{z}_n^* &\leftrightsquigarrow e_{n,i+2}e_ie_{i+1}e_{i,2}r_1e_{3,n}\leftrightsquigarrow e_{n,i+2}e_{i,2}r_1e_{3,n}\\
&\leftrightsquigarrow e_{n,i+2}e_{i,2}r_1e_{3,i}e_{i+1}e_{i+2,n}\leftrightsquigarrow e_{i,2}r_1e_{3,i}e_{n,i+2}e_{i+1}e_{i+2,n}\\
&\leftrightsquigarrow \underline{z}_i^*e_n\leftrightsquigarrow e_n\underline{z}_i^*.
\end{aligned}$$

By (HCer), (HNree), and (HNeer),

$$\begin{array}{rcl} r_i\underline{z}_n^* & \leftrightsquigarrow & e_{n,i+2}r_ie_{i+1}e_{i,2}r_1e_{3,n} \leftrightsquigarrow e_{n,i+2}r_{i+1}e_ie_{i-1,2}r_1e_{3,n} \\ & \leftrightsquigarrow & e_{n,i+2}e_{i+1}r_{i+2}e_{i,2}r_1e_3e_{4,n} \leftrightsquigarrow e_{n,2}r_1e_{3,i}r_{i+2}e_{i+1}e_{i+2,n} \\ & \leftrightsquigarrow & e_{n,2}r_1e_{3,i}r_{i+1}e_{i+2,n} \leftrightsquigarrow e_{n,2}r_1e_{3,i}e_{i+1}r_ie_{i+2,n} \\ & \leftrightsquigarrow & \underline{z}_n^*r_i. \end{array}$$

The case $i = 1$ is notationally different but can be done the same way as $i = 2$.

(iv). In view of the palindromic nature of the word $\underline{z}_i^*$ and the fact, proved in (iii), that $\underline{z}_i^*$ and e_n commute homogeneously, we see that e_i and $\underline{z}_n^*$ commute homogeneously. Applying this with $i = n-2$ gives $\underline{z}_{n-2}^*e_n \leftrightsquigarrow e_n\underline{z}_{n-2}^*$. The second chain of homogeneous equivalences is a direct consequence of (RSee).

(v). By (RSee), (HCer), (HNeee), and (RSrr),

$$\begin{array}{rcl} \underline{z}_3^*\underline{z}_3^* & = & e_3e_2r_1e_3e_3e_2r_1e_3 \rightsquigarrow \delta e_3r_1e_2e_3e_2r_1e_3 \rightsquigarrow \delta e_3r_1e_2r_1e_3 \\ & \rightsquigarrow & \delta e_3r_1r_1e_2e_3 \rightsquigarrow \delta e_3e_2e_3 \rightsquigarrow \delta e_3. \end{array}$$

Also, by (HSee), (HCer), (HNeee), and (RSrr),

$$\begin{array}{rcl} \underline{z}_n^*\underline{z}_n^* & = & e_{n,4}e_3r_1e_2e_3e_{4,n}e_{n,4}e_3r_1e_2e_3e_{4,n} \rightsquigarrow \delta e_{n,4}e_3r_1e_2e_3e_2r_1e_3e_{n,4} \\ & \rightsquigarrow & \delta e_{n,4}e_3e_{4,n} \rightsquigarrow \delta e_n. \end{array}$$

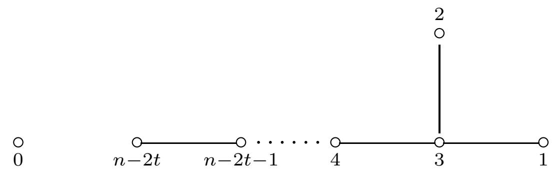
The cases $n = 1$ and $n = 2$ can be done separately. □

Notation 5.7. Let M be of type ADE. For any coclique Y of M, we write $e_Y = \prod_{y\in Y} e_y$ and $\hat{e}_Y = \delta^{-|Y|}\prod_{y\in Y} e_y$. All factors commute, so we need not care about the order in which they occur. For instance $Y(0) = \emptyset$ and $\hat{e}_{Y(0)} = 1$, whereas $Y(1) = \{\alpha_n\}$ and $\hat{e}_{Y(1)} = \hat{e}_n$.
We distinguish the following elements of F_n according to the different possibilities for $Y \in \mathcal{Y}$. We need $\underline{z}_n^*$ as in Lemma 5.6 and $\underline{e}_n^* = e_{n,2}e_1e_{3,n}$, which is the height zero analog of $\underline{z}_n^*$. The elements $\underline{s}_i$ and $\underline{f}_i$ will play roles reminiscent of r_i and e_i.

$$\begin{array}{rcl} Y = Y(t) \qquad (t>0) & : & \underline{s}_0 = \underline{z}_n^*\delta^{-1}\hat{e}_Y,\ \underline{s}_i = r_i\hat{e}_Y, \\ & & \underline{f}_0 = \underline{e}_n^*\delta^{-1}\hat{e}_Y,\ \underline{f}_i = e_i\hat{e}_Y \quad (1 \le i \le n-2t) \\ Y = Y(0) & : & \underline{s}_i = r_i,\ \underline{f}_i = e_i \quad (1 \le i \le n) \\ Y = Y(\frac{n-1}{2}) \qquad (n \text{ odd}) & : & \underline{s}_0 = \underline{z}_n^*\delta^{-1}\hat{e}_Y,\ \underline{f}_0 = \underline{e}_n^*\delta^{-1}\hat{e}_Y \\ Y = Y(\frac{n}{2}) \qquad (n \text{ even}) & : & \underline{s}_1 = r_1^*\delta^{-1}\hat{e}_Y,\ \underline{f}_1 = \underline{\hat{e}}_2\hat{e}_Y \\ Y = Y'(\frac{n}{2}) \qquad (n \text{ even}) & : & \underline{s}_1 = r_2^*\delta^{-1}\hat{e}_Y,\ \underline{f}_1 = \underline{\hat{e}}_1\hat{e}_Y \\ Y = Y^*(t) \qquad (t>0) & : & \underline{s}_i = r_{i+3}\hat{e}_Y, \\ & & \underline{f}_i = \underline{e}_{i+3}\delta^{-1}\hat{e}_Y \quad (1 \le i \le n-2t-1) \end{array}$$

Let $Y \in \mathcal{Y}$. The indices i of $\underline{s}_i$ and $\underline{f}_i$ occurring in Notation 5.7 are attached to the diagram M_Y in such a way that 0 (if it occurs) corresponds to the isolated component A_1 of M_Y and the other component (of type A or D) is labeled as usual for A and as indicated in Figure 1 for D. For instance, in case $Y = Y(t)$ with

$0 < t < (n-1)/2$, the diagram $M_Y = \mathrm{A}_1\mathrm{D}_{n-2t}$ is labeled as follows.



In the proposition below we establish the Matsumoto–Tits rewrite rules for the Coxeter group of type M_Y with generators $\pi(\underline{s}_i)$ as in (5.7) and identity $\hat{e}_Y$. The $\underline{f}_i$ will be studied in the next section.

Proposition 5.8. *Let $n \geq 4$ and $Y \in \mathcal{Y}$. The words $\underline{s}_i$ in F_n, for i a node of M_Y, have height* 1 *and satisfy the following properties.*

(i) With respect to the rewrite system of Table 2 in F_n, the words $\underline{s}_i$ satisfy the Matsumoto–Tits rewrite rules of type M_Y with identity element $\hat{e}_Y$. That is, they satisfy $\hat{e}_Y\hat{e}_Y \leftrightsquigarrow \hat{e}_Y$, $\underline{s}_i\hat{e}_Y \leftrightsquigarrow \underline{s}_i$, $\hat{e}_Y\underline{s}_i \leftrightsquigarrow \underline{s}_i$, $\underline{s}_i\underline{s}_i \rightsquigarrow \hat{e}_Y$, $\underline{s}_i\underline{s}_j \leftrightsquigarrow \underline{s}_j\underline{s}_i$ if $i \not\sim j$, and $\underline{s}_i\underline{s}_j\underline{s}_i \leftrightsquigarrow \underline{s}_j\underline{s}_i\underline{s}_j$ if $i \sim j$, where i and j are nodes of M_Y.

(ii) The elements $\pi(\underline{s}_i)$, for i running through the nodes of M_Y, generate a Coxeter group of type M_Y in $\mathrm{BrM}(\mathrm{D}_n)$ with identity element $\pi(\hat{e}_Y)$.

(iii) For $Y \in \mathcal{Y}$, denote U_Y the set of words in $F_n\hat{e}_Y$ that are minimal expressions in the $\underline{s}_i$ (where i runs over the nodes of M_Y) for elements of the Coxeter group of (ii). Then the restriction of π to U_Y induces a bijection from the set of homogeneous equivalence classes in U_Y onto this Coxeter group.

Proof. Recall $\mathrm{ht}(e_Y) = 0$. By Lemma 5.6(ii), $\mathrm{ht}(\underline{z}_n^*) = 1$, so $\mathrm{ht}(\overline{s}_0) = 1$, and, clearly, $\mathrm{ht}(\underline{s}_i) = 1$ for $i > 0$.

(i). We verify the individual rewrite rules in the case where $Y = Y(t)$ and leave the other cases to the reader (as they are similar or easier). Those involving $\hat{e}_{Y(t)}$ at the left hand side are straightforward applications of the rules (HSee), (HCee), (HCre), and (HNeee).

$\underline{s}_i\underline{s}_i \rightsquigarrow \hat{e}_{Y(t)}$. By Lemma 5.6(iv) we see $e_n\underline{z}_n^* \leftrightsquigarrow \underline{z}_n^*e_n$ and $\underline{z}_n^*e_j \leftrightsquigarrow e_j\underline{z}_n^*$ for $j \leq n-2$ and so $\underline{z}_n^*e_{Y(t)} \leftrightsquigarrow e_{Y(t)}\underline{z}_n^*$. Hence $\underline{z}_n^*$ commutes homogeneously with $e_{Y(t)}$ and so in the definition of $\overline{s}_0$ it does not matter on which side $e_{Y(t)}$ occurs. In particular, using Lemma 5.6(v), we find $\underline{s}_0\underline{s}_0 \leftrightsquigarrow \underline{z}_n^*\underline{z}_n^*e_{Y(t)}e_{Y(t)}\delta^{-2-2t} \rightsquigarrow \delta e_n e_{Y(t)}\delta^{-2-t} \rightsquigarrow e_{Y(t)}\delta^{-t}$, which is the identity element of $\pi(U_{Y(t)})$. This settles the case $i = 0$. For $i > 0$, the assertion $\underline{s}_i\underline{s}_i \rightsquigarrow \hat{e}_{Y(t)}$ follows directly from the fact that $e_{Y(t)}$ and r_i commute and (HSrr).

$\underline{s}_i\underline{s}_j \rightsquigarrow \underline{s}_j\underline{s}_i$ if $i \not\sim j$. For $i = 0$ and $j > 0$, this follows from Lemma 5.6(iii). For $i > 0$ and $j > 0$, it is immediate from (HCrr).

$\underline{s}_i\underline{s}_j\underline{s}_i \rightsquigarrow \underline{s}_j\underline{s}_i\underline{s}_j$ if $i \sim j$. Here we must have $i, j > 0$. Now it is immediate from (HNrrr).

(ii). The fact that the $\pi(\underline{s}_i)$ $(0 \leq i \leq n-2t)$ generate a quotient of the Coxeter group of type $M_{Y(t)}$ is immediate from (i) and the fact that a rewrite rule $x \rightsquigarrow y$ in F_n implies $\pi(x) = \pi(y)$. Therefore, it suffices to show that there is a surjective homomorphism from the group generated by the $\pi(\underline{s}_i)$ onto $W(M_{Y(t)})$. This follows from [4, Lemma 1.3].

(iii). This is immediate from (ii) and Lemma 5.5. □

6. Reduction in the Brauer monoid

In this section we continue to discuss reductions of words in F_n. The main purpose is to show that each word in F_n can be reduced to a particular form described in Theorem 6.11. We first study the product of a generator and a word $a_{\beta,n}e_n$, which is in reduced form. Here $a_{\beta,n}$ is given by Definition 5.2. In the action of Proposition 4.5, the element $\pi(a_{\beta,n}e_n)$ maps $\emptyset$ to $\{\beta\}$, so after left multiplication by e_i it will map $\emptyset$ to $\{\alpha_i\}$ or (in case $\alpha_i \perp \beta$) to $\{\alpha_i, \beta\}$, and, after left multiplication with r_i, it will map $\emptyset$ to $\{r_i\beta\}$. The lemma below will find corresponding reduced words. In order to control the kernel of this action, we need a little more notation.

Notation 6.1. For $Y \in \mathcal{Y}$, let Z_Y be the subsemigroup of F_n generated by all $\delta^j e_Y$ for all $j \in \mathbb{Z}$, and $\underline{s}_i$ and $\underline{f}_i$ for all nodes i of M_Y as in Notation 5.7. We also write Z_n instead of $Z_{Y(1)}$. The subsemigroup $Z_\emptyset$ coincides with F_n.

Lemma 6.2. *Let $M = \mathrm{D}_n$, let $i \in \{1, \ldots, n\}$, and let $\beta \in \Phi^+$. Then the word $e_i a_{\beta,n} e_n$ reduces to a word in $a_{\beta',n}Z_n$, where β' is a positive root with $\mathrm{ht}(\beta') \leq \mathrm{ht}(\beta)$. Also, $r_i a_{\beta,n} e_n$ can be reduced to a word in $a_{\beta',n}Z_n$, where $\{\beta'\} = r_i\{\beta\}$. Moreover, if $\underline{a} \in F_n$, then $\underline{a}e_n$ can be reduced to a word in $a_{\beta',n}Z_n$, where β' is a positive root with $\mathrm{ht}(\beta') \leq \mathrm{ht}(\underline{a})$ and $\beta' \in \underline{a}\{\alpha_n\}$.*

Proof. We proceed by induction on $\mathrm{ht}(\beta)$. If $\mathrm{ht}(\beta) = 1$ we have $\beta = \alpha_j$ for some node j of D_n. By Lemma 5.3, $a_{\beta,n}e_n \rightsquigarrow e_{j,n}$.
Consider first $e_i a_{\beta,n} e_n$. By the above, $e_i a_{\beta,n} e_n \rightsquigarrow e_i e_{j,n}$. If $\{i, j\} = \{1, 2\}$, then $e_i e_{j,n} = e_1 e_{2,n} \rightsquigarrow \delta^{-1} e_2 e_{3,n} e_n^* = \delta^{-1} a_{\alpha_2,n} e_n e_n^* = a_{\alpha_2,n}\underline{f}_0 \in a_{\alpha_2,n}Z_n$. By symmetry of the diagram, the case $j = 1$ can be replaced by $j = 2$ and handled in a similar way, so assume $j \geq 2$. If $i < j$, then e_i can be commuted to the right and be absorbed into Z_n as $\underline{f}_i$. If $i = j - 1$, then we may assume $i \geq 2$ as we already handled the case $\{i, j\} = \{1, 2\}$, and so $e_i a_{\beta,n} e_n \rightsquigarrow e_i e_{j,n} = e_{i,n} = a_{\alpha_i,n} e_n$. If $i = j$ we obtain $e_i a_{\beta,n} e_n \rightsquigarrow \delta e_{i,n}$. If $i = j + 1$ we can use $e_i e_j e_i \leftrightsquigarrow e_i$ to derive $e_i a_{\beta,n} e_n \rightsquigarrow e_{i,n}$. Otherwise $i \not\sim j$ and $i \neq j$; commute the e_i past terms in $e_{j,n}$ to obtain $e_i a_{\beta,n} e_n \rightsquigarrow e_j e_{j+1} \cdots e_i e_{i-1} e_i \cdots e_n$. Now use $e_i e_{i-1} e_i \leftrightsquigarrow e_i$ and commute the preceding terms e_k to the right and absorb them into Z_n as products of $\underline{f}_k$. In each of these cases $e_i a_{\beta,n} e_n \rightsquigarrow a_{\beta',n}\underline{z}$ for some $\underline{z} \in Z_n$ and some $\beta' \in \{\alpha_i, \alpha_j\} = e_i\{\alpha_j\}$, as required.
We now consider $r_i a_{\beta,n} e_n$ with $\beta = \alpha_j$ for some node j, where $r_i a_{\beta,n} e_n \rightsquigarrow r_i e_{j,n}$. There are two special cases which we handle directly, viz., $j = 2$ with $i = 1$ and $j = 1$ with $i = 2$. For the first we have $r_1 e_{2,n} \rightsquigarrow e_2 r_1 e_{2,n} \delta^{-1}$. Notice $e_2 \rightsquigarrow e_{2,n} e_{n,2} \delta^{-1}$ and so $r_1 e_{2,n} \rightsquigarrow e_2 r_1 e_{3,n} \rightsquigarrow e_{2,n} e_{n,2} r_1 e_{3,n} \delta^{-1} = e_{2,n} \underline{z}_n^* \delta^{-1} = a_{\alpha_2,n} \underline{z}_n^* \delta^{-1}$ and we are done as $\underline{z}_n^* \delta^{-1} = \underline{s}_0 \in Z_n$. The other case is similar. Assume, therefore, that these special cases do not occur. If $i = j-1$, we have $r_i e_{j,n-1} e_n = a_{r_i\beta} e_n$ and we are done. If $i < j-1$, then r_i commutes homogeneously through to give $e_{j,n} r_i$, unless we have $i = 1$ and $j = 3$, a case that can be treated as $i = 2$ and $j = 3$, which is done below; observe that the expression $e_{j,n} r_i$ is equal to $e_{j,n}\underline{s_i} = a_{a_j,n} e_n \underline{s_i}$ and satisfies all the requirements. If $i = j$, use $r_i e_i \rightsquigarrow e_i$ to see that $r_i e_{j,n-1} e_n \rightsquigarrow e_{i,n} \in a_{\beta,n} Z_n$. As above if $i = j + 1$, then by (HNree), $r_{j+1} e_j e_{j+1} e_{j+2,n} \rightsquigarrow r_j e_{j+1} e_{j+2,n} = r_j e_{j+1,n} = a_{\alpha_j + \alpha_{j+1},n} e_n$. This is what is required as here $\beta = \alpha_j$ and $r_{j+1}\alpha_j = \alpha_j + \alpha_{j+1}$. Otherwise, $i > j+1$ and $r_i e_{j,n} \rightsquigarrow e_{j,i-2} r_i e_{i-1} e_{i,n} \rightsquigarrow e_{j,i-2} r_{i-1} e_{i,n}$. Now if $j < i-2$ we see $e_{j,i-2} = e_{j,i-3} e_{i-2}$, and we use $e_{i-2} r_{i-1} \rightsquigarrow e_{i-2} e_{i-1} r_{i-2}$ to find $e_{j,i-1} r_{i-2} e_{i,n}$ and, commuting r_{i-2} homogeneously to the right, we obtain the required form.

We may suppose then that β has height greater than 1 and so there is a node j for which $\beta-\alpha_j$ is a root. Throughout this part of the proof we use Lemma 5.3 when $\beta-\alpha_j$ is a root to see that up to homogeneous equivalence $a_{\beta,n}e_n \leftrightsquigarrow r_j a_{\beta-\alpha_j}e_n$. Again, consider first $e_i a_{\beta,n}e_n$. Choose $j=i$ if possible. If so, we use $e_i r_i \rightsquigarrow e_i$ to obtain $e_i r_i a_{\beta-\alpha_i,n}e_n \rightsquigarrow e_i a_{\beta-\alpha_i,n}e_n$. The resulting word has lower height than $e_i r_i a_{\beta-\alpha_i,n}e_n$ and we use induction to finish. Suppose $i \not\sim j$ and $i \neq j$. Then $e_i r_j a_{\beta-\alpha_j,n}e_n \rightsquigarrow r_j e_i a_{\beta-\alpha_j,n}e_n$. Now apply the induction hypothesis to $e_i a_{\beta-\alpha_j,n}e_n$ so $e_i a_{\beta-\alpha_j,n}e_n \rightsquigarrow a_{\beta',n}e_n\underline{z}$ where $\underline{z}\in Z_n$ and $\mathrm{ht}(\beta')<\mathrm{ht}(\beta)$. In view of this inequality, induction applies to the statement involving $r_j a_{\beta',n}e_n$. Acting by r_j could raise the height at most one, still leaving $\mathrm{ht}(r_j\beta')\le\mathrm{ht}(\beta)$ as needed. Suppose $i\sim j$. We know that (α_i,β) is not 1 as we have chosen $j=i$ if possible above. This means either $(\beta,\alpha_i)=0$ or $(\beta,\alpha_i)=-1$. Suppose first $(\beta,\alpha_i)=0$. Then $(\beta-\alpha_j,\alpha_i)=1$ and so $\beta-\alpha_j-\alpha_i$ is a root and $a_{\beta,n}e_n = r_j r_i a_{\beta-a_j-\alpha_i,n}e_n$; now $e_i r_j r_i a_{\beta-\alpha_j-\alpha_i,n}e_n \rightsquigarrow e_i e_j a_{\beta-\alpha_j-\alpha_i,n}e_n$, and we can finish by induction to get the result as the height of the root $\beta-\alpha_j-\alpha_i$ is at most $\mathrm{ht}(\beta)-2$. Suppose now $(\beta,\alpha_i)=-1$. Then $e_i a_{\beta,n}e_n \rightsquigarrow e_i r_j a_{\beta-\alpha_j,n}e_n \rightsquigarrow e_i e_j r_i a_{\beta-\alpha_j,n}e_n$. Notice $(\beta-\alpha_j,\alpha_i)=-1+1=0$ and so $r_i(\beta-\alpha_j)=\beta-\alpha_j$, from which we derive $e_i a_{\beta,n}e_n \rightsquigarrow e_i e_j r_i a_{\beta-\alpha_j,n}e_n \rightsquigarrow e_i e_j a_{\beta-\alpha_j,n}e_n\underline{z}$ for some $\underline{z}\in Z_n$ by the induction hypothesis for the action of r_i. Using the induction hypothesis twice more, we find $e_i a_{\beta,n}e_n \rightsquigarrow e_i a_{\beta',n}e_n\underline{z}' \rightsquigarrow a_{\beta''}e_n\underline{z}''$ for certain roots β' and β'' whose height is at most $\mathrm{ht}(\beta)-1$ and $\underline{z}'$, $\underline{z}''\in Z_n$. This ends the part of the proof involving left multiplication by e_i.

We now consider $r_i a_{\beta,n}e_n$ where $\mathrm{ht}(\beta)>1$. If $(\beta,\alpha_i)=-1$, then $r_i a_{\beta,n}e_n \leftrightsquigarrow a_{\beta+\alpha_i,n}e_n$ by Lemma 5.3 and we are done. Suppose $(\beta,\alpha_i)=1$. Then $\beta-\alpha_i$ is a root and $a_{\beta,n}e_n \leftrightsquigarrow r_i a_{\beta-\alpha_i,n}e_n$. Now use $r_i r_i a_{\beta-\alpha_i,n}e_n \rightsquigarrow a_{\beta-\alpha_i,n}e_n \leftrightsquigarrow a_{r_i\beta,n}e_n$ to finish.

Therefore, we can assume $(\beta,\alpha_i)=0$. There is a node j for which $\beta-\alpha_j$ is a root and so $r_i a_{\beta,n}e_n \leftrightsquigarrow r_i r_j a_{\beta-\alpha_j,n}e_n$ by Lemma 5.3. The arguments here are similar to the ones at the beginning of this proof when $\mathrm{ht}\,\beta>1$. In particular, if $i\not\sim j$ and $i\neq j$ this reduces to $r_j r_i a_{\beta-\alpha_j,n}e_n$ and we use induction for r_i acting in the case $(\alpha_i,\beta-\alpha_j)=0$.

The only remaining case is $i\sim j$ and still $(\beta,\alpha_i)=0$. Here $\beta-\alpha_i-\alpha_j$ is a root orthogonal to α_j and $a_{\beta,n}e_n \leftrightsquigarrow r_j r_i a_{\beta-\alpha_j-\alpha_i,n}e_n$ by Lemma 5.3. We consider $r_i r_j r_i a_{\beta-\alpha_j-\alpha_i,n}e_n$ and so use the homogeneous relation $r_i r_j r_i \leftrightsquigarrow r_j r_i r_j$, the induction hypothesis and $(\alpha_j,\beta-\alpha_j-\alpha_i)=0$ to derive $r_i r_j r_i a_{\beta-\alpha_j-\alpha_i,n}e_n \rightsquigarrow r_j r_i a_{\beta-\alpha_j-\alpha_i,n}e_n\underline{z} \rightsquigarrow a_{\beta,n}e_n\underline{z}'$ with $\underline{z}$, $\underline{z}'\in Z_n$, as required. This proves all but the last part of the lemma.

As for the last statement, without loss of generality, we may assume that $\underline{a}e_n$ is reduced. We argue by induction on the length of $\underline{a}$. Whenever $\underline{a}$ is equal to $a_{\beta,n}$, there is nothing to show. In particular, we may assume that $\underline{a}$ has positive length; say it starts with e_i or r_i. By induction, we have $\underline{a}e_n \rightsquigarrow e_i a_{\beta,n}e_n\underline{z}$ or $\underline{a}e_n \rightsquigarrow r_i a_{\beta,n}e_n\underline{z}$ with $\mathrm{ht}(\beta)\le\mathrm{ht}(\underline{a})$ for some $\underline{z}\in Z_n$. The proof now follows from the second statement in view of $\beta'\in a_{\beta',n}\{\alpha_n\}$, which is clear from the definition of $a_{\beta',n}$. □

We now return to the sets $Y\in\mathcal{Y}$ and use Z_Y of Notation 6.1 to reduce words of the form $\underline{a}e_Y$. Sometimes we come across $e_n e_n^*$, which is homogeneously equivalent to $e_{n,2}e_1e_2e_{2,n}$ up to powers of δ. In that case, we usually invoke Proposition 6.4

below to reduce the word further. In the other cases, we have $Y = Y(t)$ for some t or $Y = Y'(n/2)$.

Notation 6.3. By Θ we denote the ideal of $\mathrm{Br}(\mathrm{D}_n)$ generated by e_1e_2. For U any subring of $\mathrm{Br}(\mathrm{D}_n)$, we also write Ue_1e_2U for the set of all linear combinations of expressions of the form ue_1e_2v with $u, v \in U$. So $\Theta = \mathrm{Br}(\mathrm{D}_n)e_1e_2\mathrm{Br}(\mathrm{D}_n)$.

Note that $Z_{Y^*(t)}$ is contained in Θ for each $t \in [1, \lfloor n/2 \rfloor]$.

Proposition 6.4. *Let Q be the subalgebra of $\mathrm{Br}(\mathrm{D}_n)$ generated by all r_i and e_i for $i > 1$. Then Q is isomorphic to $\mathrm{Br}(\mathrm{A}_{n-1})$ and satisfies the following properties.*

(i) In F_n any word containing $\mathrm{e}_1\mathrm{e}_2$ can be reduced to a word of the form $\underline{u}\mathrm{e}_1\mathrm{e}_2\underline{v}$ where $\underline{u}$ and $\underline{v}$ are words in r_i and e_i for $i > 1$, so $\pi(\underline{u}), \pi(\underline{v}) \in Q$.

(ii) The ideal Θ coincides with Qe_1e_2Q. It is isomorphic to the ideal in Q generated by any e_i $(i > 1)$. An explicit height preserving isomorphism is determined by $u\hat{e}_1e_2v \mapsto ue_2v$ for $u, v \in Q$.

Proof. The isomorphism of Q with $\mathrm{Br}(\mathrm{A}_{n-1})$ follows from the determination of $\mathrm{Br}(\mathrm{D}_n)$ in [4].

(i). This can be shown along the lines of the last paragraph of [5, Section 7.1].

(ii). Let $u, v, u', v' \in Q$. By considerations in the Brauer algebra of type A_{n-1} there is a monomial h in the submonoid of $\mathrm{BrM}(\mathrm{D}_n)$ generated by e_i, r_i for $i \geq 4$, such that $e_2vu'e_2 = e_2h$. We then have $u\hat{e}_1e_2vu'\hat{e}_1e_2v' = u\hat{e}_1e_2vu'e_2\hat{e}_1v' = u\hat{e}_1e_2h\hat{e}_1v' = ue_2h\hat{e}_1^2v' = u\hat{e}_1e_2hv'$, and the same multiplication worked out for $ue_2vu'e_2v'$ shows it is equal to ue_2hv', which proves that the indicated map preserves products. The rank of domain and range is

$$\sum_{t=1}^{\lfloor n/2 \rfloor} \left(\frac{n!}{2^t t!(n-2t)!} \right)^2 (n-2t)!$$

by [4, Lemma 1.3], and so the map is an isomorphism. $\square$

As a consequence, the reduction rules for words mapping into Θ all follow from reductions in $\mathrm{BrM}(\mathrm{A}_{n-1})$ (applied to elements of the ideal generated by one of the e_j). We will be using these observations several times below.

The word $e_{Y(t)}$ commutes homogeneously with the elements $\underline{z}_n^*$ and r_i, e_i $(i = 1, \ldots, n-2t)$, so, up to homogeneous equivalence, it does not matter on which side $e_{Y(t)}$ is located in these expressions for elements of $Z_{Y(t)}$.

Lemma 6.5. *Fix $t \in \{1, \ldots, \lfloor n/2 \rfloor\}$. Consider a word $\underline{a}$ for which $\underline{a}B_{Y(t)}$ is in the same W-orbit as $B_{Y(t)}$ and a word $\underline{b}$ for which $\underline{b}B_{Y^*(t)}$ is in the same W-orbit as $B_{Y^*(t)}$. Then $\underline{a}e_{Y(t)}$ and $\underline{b}e_{Y^*(t)}$ each reduce to a word of the form*

$$a_{\beta_n,n}a_{\beta_{n-2},n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}\underline{z} \tag{1}$$

with $\beta_{n-2k} \in \Phi^+$ for $0 \leq k \leq t-1$ such that β_{n-2k} has support in D_{n-2k} for each k, and $\underline{z} \in Z_{Y(t)}$ in the first case and $\underline{z} \in Z_{Y^*(t)}$ in the second case. Also, $\beta_n \in \underline{a}e_n(\emptyset)$ and $\{\beta_n, \beta_n^*\} \in \underline{b}e_n(\emptyset)$. The same applies to $Y'(n/2)$ instead of $Y(n/2)$ if n is even.

Proof. Consider first the case of $\underline{a}$. The statement that $\beta_n \in \underline{a}e_n(\emptyset)$ is straightforward from the definition and the fact that the terms distinct from $a_{\beta_n,n}$ do not move $\alpha_n = e_n(\emptyset)$.
Notice that $a_{\beta_n,n}a_{\beta_{n-2},n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}e_{Y(t)}$ is homogeneously equivalent to $a_{\beta_n,n}e_na_{\beta_{n-2},n-2}e_{n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}e_{n-2t+2}$. By Lemma 6.2, $\underline{a}e_n$ can be reduced to $a_{\beta_n,n}e_n\underline{z}_n$ for some $\beta_n \in \Phi^+$ and $\underline{z}_n \in Z_n$. In particular, up to homogeneous equivalence, cf. Lemma 5.6(iii), we may assume $\underline{z}_n = \underline{a}'$ or $\underline{z}_n = \underline{z}_n^*\underline{a'}$ for some $\underline{a}' \in F_{n-2}$. We denote this as $(\underline{z}_n^*)^\varepsilon a'$ where we set $\varepsilon = 0$ if it is $\underline{a}'$ and $\varepsilon = 1$ if it is $\underline{z}_n^*\underline{a'}$; so $\varepsilon \in \{0,1\}$.
If $t = 1$, we are done by Lemma 6.2. Therefore, we may assume $t > 1$. By induction on n, we find

$$\underline{a}'e_{Y(t)\setminus\{n\}} \rightsquigarrow a_{\beta_{n-2},n-2}e_{n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}e_{n-2t+2}\underline{z}_{n-2t+2}$$

for some $\underline{z}_{n-2t+2} \in Z_{Y(t)\setminus\{e_n\}}$. As $\underline{a} \in F_{n-2}$, the terms of $\pi(\underline{a}'e_{Y(t)\setminus\{n\}})$ never include r_n, r_{n-1}, e_n, or e_{n-1}, and so the support of β_{n-2} is in D_{n-2}.
Now, by Lemma 5.6, for $\varepsilon \in \{0,1\}$, thanks to $\underline{z}_n^*e_{n-2} \leftrightsquigarrow e_n\underline{z}_{n-2}^*\hat{e}_{n-2}$, we have, up to powers of δ

$$\begin{aligned}
\underline{a}e_{Y(t)} &\rightsquigarrow a_{\beta_n,n}e_n(\underline{z}_n^*)^\varepsilon\underline{a}'e_{Y(t)\setminus\{n\}}\\
&\rightsquigarrow a_{\beta_n,n}e_n(\underline{z}_n^*)^\varepsilon a_{\beta_{n-2},n-2}e_{n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}e_{n-2t+2}\underline{z}_{n-2t+2}\\
&\rightsquigarrow a_{\beta_n,n}e_na_{\beta_{n-2},n-2}e_{n-2}(\underline{z}_{n-2}^*)^\varepsilon\cdots a_{\beta_{n-2t+2},n-2t+2}e_{n-2t+2}\underline{z}_{n-2t+2}\\
&\rightsquigarrow a_{\beta_n,n}e_na_{\beta_{n-2},n-2}e_{n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}e_{n-2t+2}(\underline{z}_{n-2t+2}^*)^\varepsilon\underline{z}_{n-2t+2}\\
&\rightsquigarrow a_{\beta_n,n}a_{\beta_{n-2},n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}\underline{z}
\end{aligned}$$

with $\underline{z} = e_ne_{n-2}\cdots e_{n-2t+2}(\underline{z}_{n-2t+2}^*)^\varepsilon\underline{z}_{n-2t+2} \in Z_{Y(t)}$ and β_{n-2k} has support in $\{\alpha_1,\ldots,a_{n-2k}\}$ for each k, as required.
Notice that $\underline{z}_{n-2t+2}^*$ homogeneously commutes with elements of $Z_{Y(t)}$ by Lemma 5.6, (RSer) and (RSre), and the fact that $\underline{z}_{n-2t+2}^*$ starts and ends with e_{n-2t+2}.
The case $\underline{b}e_{Y^*(t)}$ runs along the same lines and is simpler in view of Proposition 6.4. □

The special case $t = 1$ gives the following corollary.

Corollary 6.6. *If $\underline{a}\{\alpha_n\} = \{\beta\}$ or $\underline{b}\{\alpha_n,\alpha_n^*\} = \{\beta,\beta^*\}$, then $\underline{a}e_n \rightsquigarrow a_{\beta,n}e_n\underline{z}$ with $\underline{z} \in Z_n$ or $\underline{b}e_n \rightsquigarrow a_{\beta,n}e_n\underline{z}$ with $\underline{z} \in Z_{\{\alpha_n,\alpha_n^*\}}$.*

Remark 6.7. If B is in the W-orbit of $B_{Y(t)}$ with $\beta \in B$, then $Ba_{\beta,n}e_n$ is in the same orbit. This is clear for the terms from W in $a_{\beta,n}$ and also for the terms e_j which map α_{j-1} to α_j, which is r_jr_{j-1}. The same is true for r_3r_1 moving α_1 to α_3. Now $Ba_{\beta,n}e_n$ contains α_n plus roots all in the subsystem of type D_{n-2}. The term β_{n-2} is one of these, which can be associated to one of the roots of B other than β. In this way, an order of the roots of B gives the terms β_{n-2s} which occur. The same is true for $Y'(n/2)$ instead of $Y(t)$ if n is even. A similar result is true for the case of $Y^*(t)$; here $Ba_{\beta,n}e_n$ contains α_n, α_n^* as well as roots in D_{n-2} together with their orthogonal mates.

We will consider the different ways to write $\underline{a}e_Y$ in this reduced form. The case of $t = 2$ will suffice to argue the general case. If $n \geq 5$, there are two possibilities, $Y(2)$ and $Y^*(2)$. As mentioned before, if $n = 4$, there is one more, for $Y = Y'(2)$.
For $Y = Y(2) = \{n, n-2\}$, we consider words of the form $a_{\beta_n,n}a_{\beta_{n-2},n-2}\underline{z}$ where $\underline{z} \in Z_{Y(2)}$. We need a lemma that involves words in F_n mapping $\{\alpha_n,\alpha_{n-2}\}$ to

$\{\beta, \gamma\}$ in $\mathcal{A}$ and the ways to reduce them. Similarly for $Y = Y^*(2)$ we consider words mapping $\{\alpha_n, \alpha_n^*, \alpha_{n-2}, \alpha_{n-2}^*\}$ to $\{\beta, \beta^*, \gamma, \gamma^*\}$.

Lemma 6.8. *Suppose that* $\underline{a} \in F_n$ *satisfies* $\underline{a}\{\alpha_n, \alpha_{n-2}\} = \{\beta, \gamma\}$ *and* $\underline{b} \in F_n$ *satisfies* $\underline{b}\{\alpha_n, \alpha_n^*, \alpha_{n-2}, \alpha_{n-2}^*\} = \{\beta, \beta^*, \gamma, \gamma^*\}$ *. Then of the two possible reductions of* $\underline{a}e_{Y(2)}$ *and* $\underline{b}e_{Y^*(2)}$ *as in Lemma 6.5, at least one can be reduced to the other, that is, for some* $\underline{z} \in Z_{Y(2)}$ *or* $\underline{z} \in Z_{Y^*(2)}$*, respectively, we have*

$$\begin{array}{ll} \textit{either} & a_{\beta,n}a_{\beta_{n-2},n-2}e_ne_{n-2} \rightsquigarrow a_{\gamma,n}a_{\gamma_{n-2},n-2}e_ne_{n-2}\underline{z} \\ \textit{or} & a_{\gamma,n}a_{\gamma_{n-2},n-2}e_ne_{n-2} \rightsquigarrow a_{\beta,n}a_{\beta_{n-2},n-2}e_ne_{n-2}\underline{z}. \end{array}$$

For $n = 4$ *and* $\{\beta, \gamma\} \in WB_{Y'(2)}$*, the same statement holds with* $Y'(2)$ *instead of* $Y(2)$ *and* $\{\alpha_4, \alpha_1\}$ *instead of* $\{\alpha_4, \alpha_2\}$*.*

Proof. We deal with $\underline{a}$ first. Suppose first that either β or γ has n in its support. Without loss of generality, we assume $n \in \mathrm{Supp}(\beta)$. Then $\pi(a_{\beta,n}) \in W$ as n is in the support of β. By (HNeee),

$$\begin{aligned} a_{\beta,n}e_na_{\beta_{n-2},n-2}e_{n-2} &\leftrightsquigarrow a_{\beta,n}a_{\beta_{n-2},n-2}e_{n-2}e_n \\ &\leftrightsquigarrow a_{\beta,n}a_{\beta_{n-2},n-2}e_{n-2}e_{n-1}e_{n-2}e_n \\ &\leftrightsquigarrow a_{\beta,n}a_{\beta_{n-2},n}e_{n-2}e_{n-1}e_ne_{n-2} \\ &\leftrightsquigarrow a_{\beta,n}a_{\beta_{n-2},n}e_ne_{n-2} \end{aligned}$$

As $\pi(\alpha_{\beta,n})$ is in the Weyl group, $a_{\beta,n}\{\beta_{n-2}\}$ is a single root. As $a_{\beta,n}a_{\beta_{n-2},n-2}\underline{z}\{\alpha_n, \alpha_{n-2}\} = \{\beta, \gamma\}$ we also have $\underline{z}\{\alpha_n, \alpha_{n-2}\} = \{\alpha_n, \alpha_{n-2}\}$. Now $a_{\beta_{n-2},n-2}\{\alpha_{n-2}, \alpha_n\} = \{\beta_{n-2}, \alpha_n\}$ and $a_{\beta,n}\{\beta_{n-2}, \alpha_n\} = \{\gamma, \beta\}$. This means $a_{\beta,n}\{\beta_{n-2}\} = \{\gamma\}$. Also $a_{\beta_{n-2},n}\{\alpha_n\} = \{\beta_{n-2}\}$ and so $a_{\beta,n}a_{\beta_{n-2},n}\{\alpha_n\} = \{\gamma\}$.
Now Corollary 6.6 and Lemma 5.6 give

$$a_{\beta,n}a_{\beta_{n-2},n}e_ne_{n-2} \rightsquigarrow a_{\gamma,n}e_n\underline{z}'e_{n-2} \rightsquigarrow a_{\gamma,n}e_na_{\gamma_{n-2},n-2}e_{n-2}\underline{z}$$

for some $\underline{z} \in Z_{Y(2)}$ and $\underline{z}' \in Z_n$. In particular the lemma holds in this case.
Suppose then that n is in the support of neither β nor γ. We argue by induction on n. The two reductions of $\underline{a}e_ne_{n-2}$ are $a_{\beta,n-1}e_{n-1}e_na_{\beta_{n-2},n-2}e_{n-2}$ and $a_{\gamma,n-1}e_{n-1}e_na_{\gamma_{n-2},n-2}e_{n-2}$ up to right multiples by elements of $Z_{Y(2)}$.
We know both β_{n-2} and γ_{n-2} do not have n or $n-1$ in their support. We will argue that neither has $n-2$ in the support either. Then $a_{\beta_{n-2},n-2}e_{n-2} \rightsquigarrow a_{\beta_{n-2},n-3}e_{n-3}e_{n-2}$ and similarly for $a_{\gamma_{n-2},n-2}e_{n-2}$.
Notice $\{\beta, \gamma\}a_{\beta,n-1} = \{\gamma', \alpha_{n-1}\}$ and $\{\beta, \gamma\}a_{\beta,n-1}e_n = \{\beta_{n-2}, \alpha_n\}$. Now γ' is a root lying in the subsystem of type $\gamma' \in D_{n-1}$ and so does not have n in its support. Moreover, $(\alpha_{n-1}, \gamma') = 0$ and $\gamma' \neq \alpha_{n-1}^*$ as $\{\beta, \gamma\}$ is in the W-orbit of $B_{Y(2)}$. This means γ' does not have $n-2$ in its support either or $(\gamma', \alpha_{n-2}) = -1$, in which case, by definition, $\gamma' = \beta_{n-2}$ and this root does not have $n-2$ in its support as claimed.
The two reductions of $\underline{a}e_ne_{n-2}$ are now $a_{\beta,n-1}e_{n-1}e_na_{\beta_{n-2},n-3}e_{n-3}e_{n-2}$ and $a_{\gamma,n-1}e_{n-1}e_na_{\gamma_{n-2},n-3}e_{n-3}e_{n-2}$ up to right multiples by elements of $Z_{Y(2)}$.
Now both $a_{\beta,n-1}e_{n-1}a_{\beta_{n-2},n-3}e_{n-3}$ and $a_{\gamma,n-1}e_{n-1}a_{\gamma_{n-2},n-3}e_{n-3}$ belong to F_{n-1}. By induction on n, one can be reduced to the other—up to a right factor from $Z_{\{\alpha_{n-1},\alpha_{n-3}\}}$, say

$$a_{\beta,n-1}e_{n-1}a_{\beta_{n-2},n-3}e_{n-3} \quad \rightsquigarrow \quad a_{\gamma,n-1}e_{n-1}a_{\gamma_{n-2},n-3}e_{n-3}\underline{z}' \tag{2}$$

for $\underline{z}' \in Z_{\{\alpha_{n-1},\alpha_{n-3}\}}$. Due to (HNeee) and the definition of Z_{n-1} we have $\underline{z}^*_{n-1}e_{n-2} \leftrightsquigarrow e_{n-1}\underline{z}^*_{n-2}$. Terms in $\underline{z}'$ generated by e_i or r_i with $i < n-3$ are in $Z_{Y(2)}$. If there is $\underline{z}^*_{n-3}$, then, using Lemma 5.6(iv), we can replace it with $\underline{z}^*_{n-1}$. If $\underline{z}'$ is a product of generators with index less than $n-4$ we have $\underline{z}'e_ne_{n-2} \leftrightsquigarrow e_ne_{n-2}\underline{z}'$ with $\underline{z}' \in Z_{Y(2)}$. In case $\underline{z}' = \underline{z}^*_{n-1}$, we find $\underline{z}^*_{n-1}e_ne_{n-2} \rightsquigarrow \underline{z}^*_{n-1}e_{n-2}e_n \rightsquigarrow e_{n-1}\underline{z}^*_{n-2}e_{n-2}e_n\delta^{-1} \rightsquigarrow e_{n-1}e_{n-2}e_n\delta^{-1}\underline{z}^*_{n-2}$, as $e_{n-2}] \rightsquigarrow e^2_{n-2}\delta^{-1}$.
Now multiplication by e_ne_{n-2} on both sides of the reduction (2) and application of Lemma 5.6(iv) gives

$$\begin{aligned} a_{\beta,n}e_na_{\beta_{n-2},n-2}e_{n-2} &\rightsquigarrow a_{\gamma,n-1}e_{n-1}a_{\gamma_{n-2},n-3}e_{n-3}\underline{z}'e_{n-2}e_n \\ &\rightsquigarrow a_{\gamma,n-1}e_{n-1}a_{\gamma_{n-2},n-3}e_{n-3}e_{n-2}e_n\underline{z} \\ &\rightsquigarrow a_{\gamma,n}a_{\gamma_{n-2},n-2}\underline{z} \end{aligned}$$

for $\underline{z} \in Z_n$, as required. Here, $\underline{z}$ is the same as $\underline{z}'$ unless $\underline{z}'$ has a factor $\underline{z}^*_{n-1}$ or $\underline{z}^*_{n-3}$, in which case we can take it to be $\underline{z}^*_{n-1}$ by Lemma 5.6. In the case with $\underline{z}^*_{n-1}$ occurring, the extra e_{n-1} commutes to the left. If $\underline{z}' = \underline{z}''\underline{z}^*_{n-1}$, then $\underline{z}$ reduces to $\underline{z}''\underline{z}^*_{n-2}$.
Next we deal with $\underline{b}$. In this case again, using Proposition 6.4, we can proceed as above. The words $\underline{b}e_{Y^*(2)}$ can be taken to belong to the subalgebra Q of type A_{n-1} and so the reduction is simpler. Notice here $\{\beta,\beta^*,\gamma,\gamma^*\}a_{\beta,n}e_n = \{\gamma',\gamma'^*,\alpha_n,\alpha_n^*\}$ with support of γ' in D_{n-2}. □

This case, for just two roots, extends to admissible sets of arbitrary size by the next lemma.

Lemma 6.9. *Let $\underline{a}$ be a word in F_n and choose $Y \in \mathcal{Y}$ such that $\underline{a}(\emptyset) \in WB_Y$. Let $t \in \{0,\ldots,\lfloor n/2\rfloor\}$ be such that $Y \in \{Y(t), Y'(t), Y^*(t)\}$. Then there are positive roots β_{n-2k} for $k = 0,\ldots,t-1$ such that β_{n-2k} has support in D_{n-2k} for each k and $\underline{a}\hat{e}_Y$ can be reduced to an element of $\underline{a}'Z_Y$ where*

$$\underline{a}' = a_{\beta_n,n}a_{\beta_{n-2},n-2}\cdots a_{\beta_{n-2t+2},n-2t+2}$$

and every word reduced from $\underline{a}\hat{e}_Y$ as in Lemma 6.5 can also be reduced to a word in $\underline{a}'Z_Y$.

Proof. Set $B = \underline{a}B_Y$. By Lemma 6.5 there is a unique reduction up to right multiplication by elements of Z_Y for each ordering of the elements of B. We use Lemma 6.8 to see that the order of, say the first two, does not matter, in the sense that one reduction can be reduced to another. Continuing this way with α_{n-2} and α_{n-4}, we see that the words as in Lemma 6.5 for all orders of the roots of B can be reduced to a particular one. This proves the lemma. □

Notation 6.10. The lemma allows us to define $a_{B,n}$, for $B \in WB_Y$, as the unique word $\underline{a}'e_Y \in F_n$ up to homogeneous equivalence and powers of δ determined by Lemma 6.9 with $B = \underline{a}'B_Y = \underline{a}'(\emptyset)$. When $t = 0$, we take $a_{B,n}$ to be the identity of the Brauer algebra.

The sets U_Y were introduced in Proposition 5.8(iii).

Theorem 6.11. *Each $\underline{a} \in F_n$ can be reduced to a word of the form $a_{B,n}\underline{z}a^{\mathrm{op}}_{B',n}\delta^k$ where k is an integer, B, $B' \in WB_Y$, and $\underline{z} \in U_Y$ for some $Y \in \mathcal{Y}$.*

Proof. Put $B = \underline{a}(\emptyset)$ and $B' = \underline{a}^{\rm op}(\emptyset)$. It follows from Lemma 6.5 that the two sets belong to the same W-orbit inside $\mathcal{A}$, namely the one containing B_Y. It suffices to prove the statement of the theorem for Z_Y instead of U_Y because $\underline{a}(\emptyset) = B$ and, by Proposition 4.5, the presence of $e_Y e_i$ in $\underline{z}$ for some i non-adjacent to all members of Y in M would imply that $a_{B,n}\underline{z}a^{\rm op}_{B',n}(\emptyset)$ contains $a_{B,n}(B_Y \cup \{\alpha_i\})$, a set of size greater than $|B|$; however $\underline{a}(\emptyset) = B$ has size $|B|$, a contradiction.
Consider the case $Y = Y(t)$ and suppose $B \in WB_{Y(t)}$. If $B = \emptyset$, then $\underline{a}$ does not contain any occurrences of e_i as $e_i(\emptyset)$ contains α_i (cf. the last assertion of Proposition 4.5). This means that $\underline{a}$ is a product of r_i and the Matsumoto–Tits rewrite rules for W suffice for the validity of the theorem in this case, with $t = 0$ and $Y = Y(0) = \emptyset$.
Therefore, we may assume that $B \neq \emptyset$, so there is an index i such that e_i occurs in $\underline{a}$. If $i \neq n$, then by homogeneous equivalence, we can replace e_i by $e_{i,n}e_{n-1,i}$. Thus $\underline{a} = \underline{b}e_n\underline{c}$ for certain $\underline{b}$, $\underline{c} \in F_n$. By Lemma 6.9 applied to both $\underline{b}$ and $\underline{c}^{\rm op}$, we can reduce $\underline{a}$ to $a_{\beta,n}\underline{z}_n a^{\rm op}_{\beta',n}$ for some $\beta \in B$, $\beta' \in B'$ and $\underline{z}_n \in Z_n$. Then, by an argument as in the proof of Lemma 6.5, $\underline{z}_n = e_n\underline{a}'(\underline{z}^*_n)^\varepsilon$ with $\underline{a}' \in F_{n-2}$ and $\varepsilon \in \{0, 1\}$. This deals with the case where $t = 1$.
Suppose $t > 1$. By induction on n, the word $\underline{a}'$ reduces to $a_{D,n-2}\underline{z}'a^{\rm op}_{D',n-2}$ for some $\underline{z}' \in Z_{X''}$, where $X'' = \{\alpha_{n-2}, \ldots, \alpha_{n-2t+2}\}$ and D and D' are admissible sets in the root system of type D_{n-2} with support in $\{1, \ldots, n-2\}$. Due to Lemma 5.6 (ii), (iv),

$$e_n e_{n-2}\underline{z}^*_n \leftrightsquigarrow e_n e_n \underline{z}^*_{n-2} \leftrightsquigarrow \delta e_n \underline{z}^*_{n-2} \leftrightsquigarrow e_n e_{n-2}\underline{z}^*_{n-2}.$$

By induction on t, this gives $e_{Y(t)}\underline{z}^*_n \leftrightsquigarrow e_{Y(t)}\underline{z}^*_{n-2t+2}$, which is the same as $e_n e_{X''}\underline{z}^*_n \leftrightsquigarrow e_n e_{X''}\underline{z}^*_{n-2t+2}$. So, by Lemmas 6.9 and 5.6 parts (iii) and (iv),

$$\begin{aligned}
\underline{a} &\rightsquigarrow a_{\beta,n}\underline{z}_n a^{\rm op}_{\beta',n} \leftrightsquigarrow a_{\beta,n}e_n\underline{a}'(\underline{z}^*_n)^\varepsilon a^{\rm op}_{\beta',n} \rightsquigarrow a_{\beta,n}e_n a_{D,n-2}\underline{z}'a^{\rm op}_{D',n-2}(\underline{z}^*_n)^\varepsilon a^{\rm op}_{\beta',n}\\
&\rightsquigarrow a_{\beta,n}a_{D,n-2}e_n\underline{z}'(\underline{z}^*_n)^\varepsilon a^{\rm op}_{D',n-2}a^{\rm op}_{\beta',n}\\
&\rightsquigarrow a_{\beta,n}a_{D,n-2}e_n\underline{z}'(\underline{z}^*_{n-2t+2})^\varepsilon a^{\rm op}_{D',n-2}a^{\rm op}_{\beta',n} \rightsquigarrow a_{B,n} z a^{\rm op}_{B',n}
\end{aligned}$$

for some $\underline{z} \in Z_{Y(t)}$. This handles the case $Y = Y(t)$.
If $B \in WB_{Y'(n/2)}$, the same arguments apply. Finally, if $B \in WB_{Y^*(t)}$, then, due to Proposition 6.4, the same arguments apply to Q with the root system of type A_{n-1} having support in $\{2, \ldots, n\}$. □

The following corollary extends Lemma 5.5 to the the Brauer monoid.

Corollary 6.12. *For each $\underline{a} \in F_n$, all reduced elements of F_n reducible from $\underline{a}$ are homogeneously equivalent to an element of the form $a_{B,n}\underline{z}a^{\rm op}_{B',n}\delta^k$ with B and B' in WB_Y for some $Y \in \mathcal{Y}$ and $\underline{z} \in U_Y$. Here the elements B and B' are uniquely determined by $B = \underline{a}(\emptyset)$ and $B' = \underline{a}^{\rm op}(\emptyset)$, respectively.*

Proof. The form is immediate from the theorem. Uniqueness up to homogeneous equivalence follows from Lemma 6.9 for $a_{B,n}$ and $a_{B',n}$ and from the Matsumoto–Tits' rewrite rules for $\underline{z} \in U_Y$, as stated in Proposition 5.8. □

As a consequence of Corollary 6.12, all reduced words that are reductions from $\underline{a} \in F_n$ in $\mathrm{Br}(\mathrm{D}_n)$ are homogeneously equivalent.
The proof of Corollary 6.12 has implications for the ordinary Brauer algebra of type A_{n-1} which we can take to be generated by r_i, e_i for $2 \leq i \leq n$. Here there are no

r_i^*, and $U_{Y(t)}$ consists of the reduced words on $\underline{s}_i$ for $2 \leq i \leq n-2t$, while W is generated by the r_i for $2 \leq i \leq n$ and is isomorphic to the symmetric group of n points.

Corollary 6.13. *Let* $\mathrm{Br}(\mathrm{A}_{n-1})$ *be the Brauer algebra of type* A_{n-1}*. Let* π *be the map from* F_{n-1} *to* $\mathrm{Br}(\mathrm{A}_{n-1})$ *taking* r_i *or* e_i *to the element in* $\mathrm{Br}(\mathrm{A}_{n-1})$ *with the same label. For each* $\underline{a} \in F_{n-1}$*, all reduced words in* F_n *reducible from* $\underline{a}$ *are homogeneously equivalent to an element of the form* $a_{B,n-1}\underline{z}a^{\mathrm{op}}_{B',n-1}\delta^k$ *with* B *and* B' *in* $WB_{Y(t)}$ *for some* $t \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ *and* $\underline{z} \in U_{Y(t)}$*. Here the elements* B *and* B' *are uniquely determined by* $B = \underline{a}(\emptyset)$ *and* $B' = \underline{a}^{\mathrm{op}}(\emptyset)$*, respectively. Also,* $U_{Y(t)}$ *is the Weyl group of type* A_{n-1-2t}*.*

We chose the index $n-1$ here so there will be no confusion between these coefficients and the ones in Corollary 6.12.

7. Proof of Theorem 1.1 and Corollary 1.3

In this section we prove Theorem 1.1 and Corollary 1.3.

Proof of Theorem 1.1. By Lemma 3.8, the rank of $\mathrm{B}(\mathrm{D}_n)$ is at least $\dim(\mathrm{Br}(\mathrm{D}_n))$, which by [4, Theorem 1.1] equals $(2^n + 1)n!! - (2^{n-1} + 1)n!$.

Now let T be the set of elements $a_{B,n}\underline{z}a^{\mathrm{op}}_{B',n}$ in F_n as in Corollary 6.12. Then the elements of T correspond to triples $(B, B', \underline{z})$ where B and B' are in the W-orbit in $\mathcal{A}$ containing B_Y for some $Y \in \mathcal{Y}$ and $\underline{z} \in U_Y$. Now T is a finite set and, by Corollary 6.12, every $\underline{a} \in F_n$ reduces to an element of T up to a power of δ.
For the remainder of the proof of the first statement of Theorem 1.1, we note that, by [4, Proposition 4.9 and the proof of Theorem 1.1], $(\pi(t))_{t \in T}$ is a basis of $\mathrm{Br}(\mathrm{D}_n)$. Now Proposition 3.9 applies, so $(\rho(t))_{t\in T}$ is a basis of $\mathrm{B}(\mathrm{D}_n)$. This shows that $\mathrm{B}(\mathrm{D}_n)$ is free of rank as claimed in Theorem 1.1.
To show that $\mathrm{B}(\mathrm{D}_n)$ tensored over $\mathbb{Q}(l, \delta)$ is semisimple we use the surjective equivariant map $\mu : \mathrm{B}(\mathrm{D}_n) \otimes_R \mathbb{Q}(\delta)[l^{\pm 1}] \to \mathrm{Br}(\mathrm{D}_n)$ over $\mathbb{Q}(\delta)$; cf. Definitions 3.3. We know its image $\mathrm{Br}(\mathrm{D}_n)$ is semisimple by [4, Corollary 5.6] and so has no nilpotent left ideals. Suppose $\mathrm{B}(\mathrm{D}_n) \otimes_R \mathbb{Q}(\delta, l)$ has a nontrivial nilpotent ideal. Take a nonzero element of it expressed in the basis we have found. Multiply the element by a suitable polynomial in l so that all coefficients are in $\mathbb{Q}(\delta)[l^{\pm 1}]$. As in the proof of Lemma 3.8, rescale the coefficients by a power of $l-1$ so that all coefficients remain in $\mathbb{Q}(\delta)[l^{\pm 1}]$ but some coefficient λ_s lies outside $(l-1)\mathbb{Q}(\delta)[l^{\pm 1}]$. The result is a nonzero nilpotent element in $\mathrm{B}(\mathrm{D}_n) \otimes \mathbb{Q}(\delta)[l^{\pm 1}]$ with $\mu(\lambda_s) \neq 0$, so its image under π is a nonzero nilpotent element of $\mathrm{Br}(\mathrm{D}_n)$. Furthermore, any multiple is nilpotent both in $\mathrm{B}(\mathrm{D}_n) \otimes \mathbb{Q}(\delta, l)$ and in $\mathrm{Br}(\mathrm{D}_n)$ and so generates a nontrivial nilpotent ideal of $\mathrm{Br}(\mathrm{D}_n)$, a contradiction. This completes the proof of Theorem 1.1.

Although we did not need the statement for the proof of the main theorem, it may be worthy of mention that, by Proposition 3.9, each word in T as above is reduced. We will need the elements $\rho(a_{B,n})$ for the words $a_{B,n}$ in F_n introduced in Notation 6.10. These words were defined up to homogeneous equivalence. Since different elements from the homogeneous class of $a_{B,n}$ may give different elements in $\mathrm{B}(\mathrm{D}_n)$, see Remark 5.4, we need to select a particular element in each class.

Notation 7.1. Let $Y \in \mathcal{Y}$. For each $B \in WB_Y$, we take $a_{B,n}$ to be a specific word in F_n from its homogeneous equivalence class in F_n and write $b_{B,n} = \rho(a_{B,n})$ for its image in $\mathrm{B}(\mathrm{D}_n)$ under ρ.

Corollary 7.2. *For $n \geq 4$, the elements $b_{B,n}\rho(\underline{z})b_{B',n}$ for $(B, B', \underline{z}) \in \bigcup_{Y\in\mathcal{Y}} WB_Y \times WB_Y \times U_Y$ are a basis of* $\mathrm{B}(\mathrm{D}_n)$.

Remark 7.3. Let Θ' be the ideal of $\mathrm{B}(\mathrm{D}_n)$ generated by e_1e_2. We can also choose a basis of Θ' of the form $b_{B,n-1}\rho(\underline{z})b_{B',n-1}$ where the $b_{B,n-1}$ are chosen as in Notation 7.1 using Corollary 6.13 for $\mathrm{Br}(\mathrm{A}_{n-1})$.

Remark 7.4. A consequence of Theorem 1.1 is that natural subalgebras generated by $\{g_i, e_i \mid i \in K\}$ for K a set of nodes of M have the usual desired subalgebra structure, that is, are naturally isomorphic to the BMW algebra whose type is the restriction of M to K. In particular, the subalgebra generated by $\{g_i, e_i \mid 2 \leq i \leq n\}$ is the full $\mathrm{B}(\mathrm{A}_{n-1})$ rather than a proper homomorphic image. The same applies to the algebra generated by all g_i, e_i for $i \leq n-1$ which is $\mathrm{B}(\mathrm{D}_{n-1})$ and not a proper image.

Proof of Corollary 1.3. The generalized Temperley–Lieb algebra of type D_n has been studied in [11, 15, 18]. The elements e_i either in $\mathrm{B}(\mathrm{D}_n)$ or in $\mathrm{Br}(\mathrm{D}_n)$ commute for $i \not\sim j$ by (HCee). For $i \sim j$, we have $e_ie_je_i = e_i$ by (HNeee). Also, $e_i^2 = \delta e_i$ by (HSee). The free algebra on $e_1, \ldots, e_n$ with this presentation over $\mathbb{Z}[\delta^{\pm 1}]$ is called the (generalized) Temperley–Lieb algebra of type D_n over $\mathbb{Z}[\delta^{\pm 1}]$; we will denote it by $\mathbf{TL}(\mathrm{D}_n)$. The subalgebra generated by $e_1, \ldots, e_n$ in $\mathrm{Br}(\mathrm{D}_n)$ is a homomorphic image of $\mathbf{TL}(\mathrm{D}_n)$; the subalgebra of $\mathrm{B}(\mathrm{D}_n)$ generated by these elements is a homomorphic image of $\mathbf{TL}(\mathrm{D}_n) \otimes_{\mathbb{Z}[\delta^{\pm 1}]} R$. The words in F_n corresponding to generators for these subalgebras consist solely of the symbols $e_1, \ldots, e_n, \delta$ and so are of height 0.

In [18, Theorem 4.2 and Lemma 6.5], a description of a generating set for the Temperley–Lieb algebra is given in terms of decorated diagrams with some restrictions. In [7] diagrams such as these were introduced for the full algebra $\mathrm{Br}(\mathrm{D}_n)$. In particular, in [7, Theorem 1.1] it is shown there is an isomorphism, ν, from $\mathrm{Br}(\mathrm{D}_n)$ to the span of the diagrams as a basis over $Z[\delta, \delta^{-1}]$. In [18, Lemmas 6.5 and 6.6] it is shown that the specific images $\nu(e_i)$ generate the full Temperley–Lieb algebra and so the e_i in $\mathrm{Br}_n(\mathrm{D}_n)$ generate the full Temperley–Lieb algebra. The actual multiplication of $\nu(e_i)$ with $\nu(e_j)$ in [7, Section 4] has a coefficient ξ which sometimes appears. However, by results in [7] the coefficient ξ does not appear for words of height 0 and so does not appear here.

Now apply Proposition 3.8 to see that the algebra generated by $\rho(e_i)$ is the full Temperley–Lieb algebra over R.

This completes the proof of Corollary 1.3. It follows from this that the subalgebra of $\mathrm{Br}(\mathrm{D}_n)$ generated by e_j $(j \geq 2)$ is isomorphic to the Temperley–Lieb algebra of type A_{n-1}.

In [8], it is shown that the Temperley–Lieb monomials are the terms $a_{B,n}$ for B of height 0, where the concept of height 0 for B is given in Section 9.

Remark 7.5. By use of μ and the Tits Deformation Theorem, see [3, IV.2, exercise 26] or [25, Lemma 85], it can be shown that the irreducible degrees associated to $\mathrm{B}(\mathrm{D}_n)$ are the same as for $\mathrm{Br}(\mathrm{D}_n)$. This can also be shown by use of Theorem 6.11 for representations with Θ' in the kernel as in [4] and for the others from the connection of Θ' to $\mathrm{B}(\mathrm{A}_{n-1})$ as in the proof of Theorem 1.1.

8. Cellularity

Let S be a commutative algebra over R. In this section we prove Theorem 1.2, which states that $\mathrm{B}(\mathrm{D}_n)\otimes_R S$ is cellular in the sense of Graham–Lehrer [16, Definition 1.1] if S contains an inverse to 2. We recall the definition from [16].

Definition 8.1. An associative algebra A over a commutative ring S is *cellular* if there is a quadruple $(\Lambda, T, C, *)$ satisfying the following three conditions.

(C1) Λ is a finite partially ordered set. Associated to each $\lambda\in\Lambda$, there is a finite set $T(\lambda)$. Also, C is an injective map
$$C:\quad \coprod_{\lambda\in\Lambda} T(\lambda)\times T(\lambda)\to A$$
whose image is an S-basis of A.

(C2) The map $*: A\to A$ is an S-linear anti-involution such that $C(x,y)^* = C(y,x)$ whenever $x,y\in T(\lambda)$ for some $\lambda\in\Lambda$.

(C3) If $\lambda\in\Lambda$ and $x,y\in T(\lambda)$, then, for any element $a\in A$,
$$aC(x,y)\equiv \sum_{u\in T(\lambda)} r_a(u,x)C(u,y) \mod A_{<\lambda},$$
where $r_a(u,x)\in S$ is independent of y and where $A_{<\lambda}$ is the S-submodule of A spanned by $\{C(x',y')\mid x',y'\in T(\mu) \text{ for } \mu<\lambda\}$.

Such a quadruple $(\Lambda, T, C, *)$ is called a *cell datum* for A.

Now let S be an integral domain containing R as a subring with $2^{-1}\in S$. We introduce a quadruple $(\Lambda, T, C, *)$ and prove that it is a cell datum for $A=\mathrm{B}(\mathrm{D}_n)\otimes_R S$. The map $*$ on A will be the opposition map $\cdot^{\mathrm{op}}$ of Notation 3.6. Before describing the other three components of the quadruple $(\Lambda, T, C, *)$, we will relate the subalgebras of A generated by monomials corresponding to the elements of U_Y, defined in Proposition 5.8(iii), to Hecke algebras. For this purpose we need a version of Proposition 5.8 that applies to A rather than $\mathrm{BrM}(\mathrm{D}_n)$. This requires a version of Lemma 5.6 for $\mathrm{B}(\mathrm{D}_n)$ rather than $F_n/\leftrightsquigarrow$, with $\rightsquigarrow$ replaced by equality in $\mathrm{B}(\mathrm{D}_n)$. Here, as in Remark 7.3, we let Θ' be the ideal of $\mathrm{B}(\mathrm{D}_n)$ generated by e_1e_2.

Lemma 8.2. *For $n\geq 3$, the monomials $\hat{z}_n^* = \rho(\underline{z}_n^*)$ in $\mathrm{B}(\mathrm{D}_n)$ satisfy the following equations, where $g_n^*=\rho(r_n^*)$. Here we are considering e_i and g_i to be in $\mathrm{B}(\mathrm{D}_n)$, namely $g_i=\rho(r_i)$ and $e_i=\rho(e_i)$ where $r_i\in F_n$ and the e_i within the parentheses is also in F_n.*

(i) $g_n^*e_n = \hat{z}_n^* = e_ng_n^*$.
(ii) $\hat{z}_n^* = e_{n,3}g_2e_1e_{3,n}$.
(iii) For $n\geq 4$ and $i\in\{1,\ldots,n-2\}$, $e_i\hat{z}_n^* = \hat{z}_i^*e_n = e_n\hat{z}_i^*$ *and* $g_i\hat{z}_n^* = \hat{z}_n^*g_i$.
(iv) $\hat{z}_n^*e_{n-2} = e_n\hat{z}_{n-2}^*$ *and* $e_n\hat{z}_n^* = \hat{z}_n^*e_n = \delta\hat{z}_n^*$.
(v) $(\hat{z}_n^*)^2\in \delta e_n - m\delta\hat{z}_n^* + \Theta'$.

For n equal to 1 or 2, both (i) and (v) hold.

Proof. Many of the proofs are the same as for Lemma 5.6. Differences occur when the relations are not monomial as extra terms occur with coefficients divisible by m.

(i). The proof is similar to the one of Lemma 5.6(i); note that the relations (RNrre) for $\mathrm{B}(\mathrm{D}_n)$ are also binomial.

(ii). Again, the only relation used in the proof of Lemma 5.6(ii) is (HTeere), which is binomial for $\mathrm{B}(\mathrm{D}_n)$.

(iii). Let $i \in \{2, \ldots, n-2\}$. The relation $e_i\hat{z}_n^* = e_n\hat{z}_i^*$ can be derived from the definition of $e_{k,n}$, and the binomial relations (HCee) and (HNeee), as in the proof of Lemma 5.6.

The proof of $g_i\hat{z}_n^* = \hat{z}_n^* g_i$ is a bit more involved. By (HCer), (RNrre), (RSrr) and (RNerr),

$$\begin{aligned}
g_i\hat{z}_n^* &= e_{n,i+2}g_ie_{i+1,2}g_1e_{3,n} = e_{n,i+2}g_i^2g_{i+1}e_{i,2}g_1e_{3,n}\\
&= e_{n,i+2}g_{i+1}e_{i,2}g_1e_{3,n} - me_{n,i+2}g_ig_{i+1}e_{i,2}g_1e_{3,n}\\
&\qquad\qquad +ml^{-1}e_{n,i+2}e_ig_{i+1}e_{i,2}g_1e_{3,n}\\
&= e_{n,i+2}e_{i+1}g_{i+2}^{-1}e_{i,2}g_1e_{3,n} - me_{n,2}g_1e_{3,n} + me_{n,i+2}e_{i,2}g_1e_{3,n}\\
&= e_{n,i+1}g_{i+2}^{-1}e_{i,2}g_1e_{3,n} - m\hat{z}_n^* + me_{i,2}g_1e_{n,i+2}e_{3,n}\\
&= e_{n,2}g_{i+2}^{-1}g_1e_{3,n} - m\hat{z}_n^* + me_{i,2}g_1e_{n,i}e_n\\
&= e_{n,2}g_{i+2}^{-1}g_1e_{3,n} - m\hat{z}_n^* + m\hat{z}_i^*e_n.
\end{aligned}$$

Since each of the three summands is invariant under opposition, (observe that $(\hat{z}_n^*)^{\mathrm{op}} = \hat{z}_n^*$ follows from (i)), so is $g_i\hat{z}_n^*$. This shows $g_i\hat{z}_n^* = (g_i\hat{z}_n^*)^{\mathrm{op}} = (\hat{z}_n^*)^{\mathrm{op}}g_i = \hat{z}_n^*g_i$.

The case $i = 1$ is notationally different but can be done the same way as $i = 2$.

(iv). By (iii) with $i = n-2$ we have $e_{n-2}\hat{z}_n^* = \hat{z}_{n-2}^*e_n = e_n\hat{z}_{n-2}^*$. Taking images under $\cdot^{\mathrm{op}}$ and using opposition invariance of $\hat{z}_n^*$, we find $\hat{z}_n^*e_{n-2} = (e_{n-2}\hat{z}_n^*)^{\mathrm{op}} = (\hat{z}_{n-2}^*e_n)^{\mathrm{op}} = e_n\hat{z}_{n-2}^*$, as required for the first equation. The second chain of equations is a direct consequence of (RSee).

(v).

For $n \geq 3$, by (HSee), (HCer), (HNeee), and (RSrr),

$$\begin{aligned}
(\hat{z}_n^*)^2 &= e_{n,3}g_1e_2e_{3,n}e_{n,3}g_1e_2e_{3,n} = e_{n,3}g_1e_2e_3g_1e_2e_{3,n}\delta\\
&= \delta e_{n,3}g_1e_2e_3e_2g_1e_{3,n} = \delta e_{n,3}g_1e_2g_1e_{3,n} = \delta e_{n,3}g_1^2e_2e_{3,n}\\
&= \delta e_{n,3}(1 - mg_1 + ml^{-1}e_1)e_{2,n}\\
&= \delta e_{n,3}e_{2,n} - \delta me_{n,3}g_1e_{2,n} + \delta ml^{-1}e_{n,3}e_1e_2e_{3,n}\\
&\in \delta e_n - \delta m\hat{z}_n^* + \Theta'.
\end{aligned}$$

The cases $n = 1$ and $n = 2$ are easily proved. □

Definition 8.3. For $t \in \{0, \ldots, \lfloor n/2\rfloor\}$, we write J_{t+1} to denote the ideal of A generated by Θ', $e_{Y(t')}$ for all $t' > t$, and $e_{Y'(n/2)}$ if n is even and $n > 2t$. In particular, J_1 is the ideal generated by all e_i and, for n even with $t = n/2$, the ideal J_{t+1} coincides with Θ'.

Recall the words $\underline{s}_i$ $(0 \leq i \leq n-2t)$ and $\hat{e}_{Y(t)} = e_{Y(t)}\delta^{-t}$ given in Notation 5.7.

Proposition 8.4. *Let $t \in \{0, \ldots, \lfloor n/2\rfloor\}$. The monomials $\hat{s}_i = \rho(\underline{s}_i)$, for i a node of $M_{Y(t)}$, satisfy the following relations.*

(i) The element $\rho(\hat{e}_{Y(t)})$ acts as an identity element on the $\hat{s}_i$, that is, $\rho(\hat{e}_{Y(t)})\hat{s}_i = \hat{s}_i$ and $\hat{s}_i\rho(\hat{e}_{Y(t)}) = \hat{s}_i$, while $\rho(\hat{e}_{Y(t)})^2 = \rho(\hat{e}_{Y(t)})$. Moreover, the $\hat{s}_i$ satisfy the braid relations (HCrr) and (HNrrr) of Table 1 with g_i replaced by $\hat{s}_i$ and 1 by $\rho(\hat{e}_{Y(t)})$.

(ii) Each monomial $\hat{s}_i$ satisfies the quadratic Hecke algebra relation modulo the ideal J_{t+1}, that is, $\hat{s}_i^2 + m\hat{s}_i - \rho(\hat{e}_{Y(t)}) \in J_{t+1}$.

If n is even and $t = n/2$, the corresponding statement holds for $Y'(n/2)$ replacing $Y(n/2)$.

Proof. Here again g_i and e_i are considered in $\mathrm{B}(\mathrm{D}_n)$.

(i). The relations involving $\rho(\hat{e}_{Y(t)})$ are easily derived from Lemma 8.2. Note the resemblance with the proof of Proposition 5.8.

Use of (RSrr), (HCrr), and (HNrrr) gives the relations not involving $\hat{s}_0$. It remains to verify the commuting of $\hat{s}_0$ with $\hat{s}_i$ for $i \in \{1, \ldots, n-2t\}$. By Lemma 8.2(iii) $g_i\hat{z}_n^* = \hat{z}_n^* g_i$. This gives

$$\hat{s}_0\hat{s}_i \quad = \quad z_n^*\delta^{-1}\hat{e}_{Y(t)}g_i\hat{e}_{Y(t)} = z_n^* g_i\delta^{-1}\hat{e}_{Y(t)} = g_i\hat{e}_{Y(t)}z_n^*\delta^{-1}\hat{e}_{Y(t)} = \hat{s}_i\hat{s}_0.$$

(ii). For $i \in \{1, \ldots, n-2t\}$, we have $\hat{s}_i^2 = g_i\hat{e}_{Y(t)}g_i\hat{e}_{Y(t)} = g_i^2\hat{e}_{Y(t)} = (1 - mg_i + ml^{-1}e_i)\hat{e}_{Y(t)} = \hat{e}_{Y(t)} - m\hat{s}_i + ml^{-1}e_{Y(t)\cup\{i\}}\delta^{-1-t}$. Here $e_{Y(t)\cup\{i\}}$ is in J_{t+1}, so $\hat{s}_i^2 + m\hat{s}_i - \hat{e}_{Y(t)} \in J_{t+1}$.

As for $\hat{s}_0^2$, by Lemma 8.2(iv), $\hat{z}_n^* e_{Y(t)} = e_{Y(t)}\hat{z}_{n-2t+2}^*$ and $e_{Y(t)}\hat{z}_n^* = \hat{z}_{n-2t+2}^* e_{Y(t)}$, so, by Lemma 8.2(iii),(iv),(v), and in view of $\Theta' \subseteq J_{t+1}$, we have

$$\begin{aligned}
\hat{s}_0^2 &= \hat{z}_n^* e_{Y(t)}\hat{z}_n^* e_{Y(t)}\delta^{-2t-2} = \hat{z}_n^* e_{Y(t)}e_{Y(t)}\hat{z}_n^* e_{Y(t)}\delta^{-3t-2} \\
&= e_{Y(t)}(\hat{z}_{n-2t+2}^*)^2 e_{Y(t)}^2\delta^{-3t-2} \\
&\in e_{Y(t)}(e_{n-2t+2} - m\hat{z}_{n-2t+2}^*)e_{Y(t)}\delta^{-2t-1} + J_{t+1} \\
&= e_{Y(t)}e_{n-2t+2}e_{Y(t)}\delta^{-2t-1} - me_{Y(t)}\hat{z}_{n-2t+2}^* e_{Y(t)}\delta^{-2t-1} + J_{t+1} \\
&= \rho(\hat{e}_{Y(t)}) - m\hat{z}_n^* e_{Y(t)}e_{Y(t)}\delta^{-2t-1} + J_{t+1} \\
&= \rho(\hat{e}_{Y(t)}) - m\hat{z}_n^* e_{Y(t)}\delta^{-t-1} + J_{t+1} \\
&= \rho(\hat{e}_{Y(t)}) - m\hat{s}_0 + J_{t+1}.
\end{aligned}$$

□

We will next exploit the elements $b_{B,n}$ of Notation 7.1. Recall from Proposition 5.8(iii) the definition of $U_{Y(t)}$.

Notation 8.5. Let H_Y be the linear span of $\rho(U_Y)$.

Corollary 8.6. *For $t \in \{0, \ldots, \lfloor n/2 \rfloor\}$, the linear subspace $H_{Y(t)}$ of A satisfies the following properties.*

(i) The linear subspace $H_{Y(t)} + J_{t+1}$ is a subalgebra of A whose quotient algebra mod J_{t+1} is isomorphic to the Hecke algebra of type $M_{Y(t)}$. Moreover, $\rho(U_{Y(t)})$ is a basis of $H_{Y(t)}$.

(ii) For each $i \in \{1, \ldots, n\}$ and $B \in WB_{Y(t)}$, we have $g_i b_{B,n} e_{Y(t)} \in b_{r_iB,n}H_{Y(t)} + J_{t+1}$ and $e_i b_{B,n} e_{Y(t)} \in b_{B'',n}H_{Y(t)} + J_{t+1}$ for some $B'' \in WB_{Y(t)}$.

(iii) The linear subspace $H_{Y(t)}$ is invariant under opposition.

If n is even, the similarly defined linear span $H_{Y'(n/2)}$ equals $S\hat{e}_{Y'(n/2)}$ and satisfies the same properties.

Proof. If $\underline{a} \in U_{Y(t)}$ is a minimal expression in the $\underline{s}_i$ of the element $\pi(\underline{a}) \in W(M_{Y(t)})$, then, as a consequence of Lemma 5.5 and the relations established in Proposition 8.4(i), $\rho(\underline{a})$ depends only on $\pi(\underline{a})$ and not on the choice of the minimal expression.

(i). By the above and Proposition 8.4(ii), the spanning set $\rho(U_{Y(t)})$ of $H_{Y(t)}$ has size at most $|W(M_{Y(t)})|$. Due to Corollary 7.2 there is no collapse, so the spanning set has size equal to $|W(M_{Y(t)})|$ and is a basis of $H_{Y(t)}$. By Proposition 8.4, the linear subspace $H_{Y(t)} + J_{t+1}$ is closed under multiplication and satisfies the Hecke algebra defining relations mod J_{t+1} on the generators $\hat{s}_i$ $(0 \leq i \leq n-2t)$. In particular, $(H_{Y(t)} + J_{t+1})/J_{t+1}$ is a quotient of the Hecke algebra of type $M_{Y(t)}$. But, its rank is equal to $|W(M_{Y(t)})|$, which is the Hecke algebra dimension, and so $(H_{Y(t)} + J_{t+1})/J_{t+1}$ is isomorphic to the Hecke algebra of type $M_{Y(t)}$.

(ii). In view of Corollary 7.2 and Proposition 4.5, $b_{B,n}H_{Y,t}b_{B',n} + J_{t+1}$ is the linear span of J_{t+1} and all monomials x in $\mathrm{B}(\mathrm{D}_n)$ such that $\mu(x)(\emptyset) = B$ and $\mu(x)^{\mathrm{op}}(\emptyset) = B'$. But $x = g_i b_{B,n} e_{Y,t}$ satisfies $\mu(x)(\emptyset) = r_i B$ and $\mu(x)^{\mathrm{op}}(\emptyset) = Y(t)$, so $g_i b_{B,n} e_{Y,t} \in b_{r_iB,n}H_{Y,t} + J_{t+1}$.
Similarly, $\mu(e_i b_{B,n} e_{Y(t)})(\emptyset) = \mu(e_i b_{B,n})B_{Y(t)} = e_i B$ always contains a member B'', say, of WB, and $\mu(e_i b_{B,n} e_{Y(t)})^{\mathrm{op}}(\emptyset) = e_{Y(t)}(\pi(a_{B,n}))^{\mathrm{op}}\{\alpha_i\}$ contains $B_{Y(t)}$, so $e_i b_{B,n} e_{Y(t)} \in b_{B'',n}H_{Y(t)} + J_{t+1}$. Here if $\alpha_i \perp B$ the expression is in J_{t+1}.

(iii). It is readily verified that each $\hat{s}_i$ is fixed under opposition. As the opposite of a minimal expression in the $\hat{s}_i$ is again a minimal expression, $\rho(U_{Y(t)})$ is invariant under opposition. Hence, so is $H_{Y(t)}$. □

We now give the cell datum for $A = \mathrm{B}(\mathrm{D}_n)_R \otimes S$. View A_{n-1} as the subdiagram of D_n on the nodes $2, \ldots, n$. As an algebra over S, the ideal Θ' of A generated by e_1e_2 is isomorphic to the ideal of $\mathrm{B}(\mathrm{A}_{n-1}) \otimes_R S$ generated by e_2; see Proposition 6.4. The ideal generated by e_2 is a cellular algebra as $\mathrm{B}(\mathrm{A}_{n-1})$ is cellular by [28, Theorem 3.11] and it inherits the cellular structure from that of $\mathrm{B}(\mathrm{A}_{n-1})$. In fact, it corresponds to the ideals with cell datum associated with partitions of $n-2t$ for $1 \leq t \leq \lfloor n/2 \rfloor$.
Let $(\Lambda_\theta, T_\theta, C_\theta, *_\theta)$ be the cell datum for Θ'. It is clear from [28, Theorem 3.11] that $*_\theta$ coincides with the restriction to Θ' of the map $\cdot^{\mathrm{op}}$. Moreover, the elements $g_1 - g_2$ and $e_1 - e_2$ are in the kernel of the action of A on Θ' by left multiplication, as well as by right multiplication.
For $0 \leq t \leq \lfloor n/2 \rfloor$ we let $(\Lambda_t, T_t, C_t, *_t)$ be the cell datum for the Hecke algebra $H_{Y(t)}$ mod J_{t+1} of type $M_{Y(t)}$ (see Corollary 8.6(i)) with $*_t$ the restriction to $H_{Y(t)}$ of $\cdot^{\mathrm{op}}$. If $n = 2t$, there is another copy needed which we denote $(\Lambda'_{n/2}, T'_{n/2}, C'_{n/2}, *'_{n/2})$; it corresponds to the admissible set $Y'(n/2)$. By [14], these cell data are known to exist if $\frac{1}{2} \in S$. We take the values of C_t to be in $H_{Y(t)}$.
The poset Λ is the disjoint union of Λ_θ together with the posets Λ_t of the cell data for the various Hecke algebras $H_{Y(t)}$ mod J_{t+1}, as well as $\Lambda'_{n/2}$ if n is even. We make Λ into a poset as follows. For a fixed t or θ it is already a poset, and we keep the same partial order. Furthermore, any element of Λ_t is greater than any element of Λ_s if $t < s$. In particular the elements of Λ_0 are greater than the elements of Λ_t for any $t \geq 1$. Moreover, if n is even, any element of $\Lambda'_{n/2}$ is smaller than any element of Λ_t for $t < n/2$. Finally, we decree that any element of Λ_θ is smaller than any element of Λ_t $(0 \leq t \leq n/2)$ or $\Lambda'_{n/2}$.

Let $t \in \{0, \ldots, \lfloor n/2 \rfloor\}$. For $\lambda \in \Lambda_t$, we set $T(\lambda) = WB_{Y(t)} \times T_t(\lambda)$ and, if n is even, for $\lambda \in \Lambda'_{n/2}$, we set $T(\lambda) = WB_{Y'(n/2)} \times T'_{n/2}(\lambda)$. For $\lambda \in \Lambda_\theta$, we set $T(\lambda) = T_\theta(\lambda)$. This determines T.

We define C as follows. For $t \in \{0, \ldots, \lfloor n/2 \rfloor\}$, $\lambda \in \Lambda_t$, and $(B, x), (B', y) \in T(\lambda)$, we have

$$C\big((B,x),(B',y)\big) = b_{B,n} C_t(x,y) b^{\mathrm{op}}_{B',n}.$$

Similarly on $\Lambda'_{n/2} \times \Lambda'_{n/2}$. For $\lambda \in \Lambda_\theta$, the map C on $T(\lambda) \times T(\lambda)$ is just C_θ.

Since we have already defined $*$ by the opposition map, this concludes the definition of $(\Lambda, T, C, *)$. We next verify the conditions (C1), (C2), (C3).

(C1). The map C has been chosen so that its image is an S-basis of Θ' (the image of C_θ), joint with the set of elements $b_{B,n} C_t(x,y) b^{\mathrm{op}}_{B',n}$ for $B, B' \in WB_{Y(t)}$ and $C_t(x,y)$ running through a basis of $H_{Y(t)}$, and $b_{B,n} C'_{n/2}(x,y) b^{\mathrm{op}}_{B',n}$ for $B, B' \in WB_{Y'(n/2)}$ and $C'_{n/2}(x,y)$ running through a basis of $H_{Y'(n/2)}$. By Corollary 7.2, this implies that the image of C is a basis of A. Injectivity of C follows from injectivity of C_θ, C_t $(0 \le t \le n/2)$, $C'_{n/2}$ if n is even, and Theorem 1.1, which guarantees that no collapses of dimensions of the individual parts occur.

(C2). Clearly, $*$ is an S-linear anti-involution. Let $t \in \{0, \ldots, \lfloor n/2 \rfloor\}$, $\lambda \in \Lambda_t$, and $(B, x), (B', y) \in T(\lambda)$. Then $(b_{B,n} C_t(x,y) b^{\mathrm{op}}_{B',n})^{\mathrm{op}} = b_{B',n} C_t(x,y)^{\mathrm{op}} b^{\mathrm{op}}_{B,n}$, so, in order to establish $\big(C((B,x),(B',y))\big)^* = C((B',y),(B,x))$, it suffices to verify that $C_t(x,y)^{\mathrm{op}}$ coincides with $C_t(y,x)$. Now $*_t$ on $H_Y(t)$ mod J_{t+1} coincides with opposition, so modulo J_{t+1} we have $C_t(x,y)^{\mathrm{op}} = C_t(x,y)^{*_t} = C_t(y,x)$ by the cellularity of $(\Lambda_t, T_t, C_t, *_t)$. On the other hand, as $H_{Y(t)}$ is invariant under opposition, see Corollary 8.6(iii), and contains the values of C_t, it contains $C_t(x,y)^{\mathrm{op}} - C_t(y,x)$, so $C_t(x,y)^{\mathrm{op}} - C_t(y,x) \in H_{Y(t)} \cap J_{t+1} = \{0\}$, whence $C_t(x,y)^{\mathrm{op}} = C_t(y,x)$, as required.
The case of $\lambda \in \Lambda'_{n/2}$ for n even is similar. If $\lambda \in \Lambda_\theta$ and $x, y \in T(\lambda)$, then $C(x,y)^* = C(y,x)$ is immediate from the cellularity of $(\Lambda_\theta, T_\theta, C_\theta, *_\theta)$.

(C3). Let $\lambda \in \Lambda_t$ and $(B,x), (B',y) \in T(\lambda)$. Fix $i \in \{1, \ldots, n\}$. It clearly suffices to prove the formulas for a running over the generators g_i and e_i of $\mathrm{B}(\mathrm{D}_n)$.
By choice of C_t, we have $C_t(x,y) \in H_{Y(t)}$, and, see Proposition 8.4(i), $C_t(x,y) = \rho(\hat{e}_{Y(t)}) C_t(x,y)$. According to Corollary 8.6(ii), there is $z_{B,i} \in H_{Y(t)}$, depending only on B and i, such that $g_i b_{B,n} \rho(\hat{e}_{Y(t)}) \in b_{g_i B,n} z_{B,i} + J_{t+1}$. As $(\Lambda_t, T_t, C_t, *_t)$ is a cell datum for $H_{Y(t)}$ mod J_{t+1}, for each $u \in T_t(\lambda)$, there are $\nu_i(u, B, x) \in S$, independent of B' and y, such that

$$z_{B,i} C_t(x,y) \quad \in \quad \sum_{u \in T_t(\lambda)} \nu_i(u,B,x) C_t(u,y) + (H_{Y(t)})_{<\lambda} + J_{t+1}.$$

Since both $(H_{Y(t)})_{<\lambda}$ and J_{t+1} are contained in $A_{<\lambda}$, we find

$$\begin{aligned} g_i C((B,x),(B',y)) &= g_i b_{B,n} \rho(\hat{e}_{Y(t)}) C_t(x,y) b^{\mathrm{op}}_{B',n} \\ &\in b_{r_i B,n} z_{B,i} C_t(x,y) b^{\mathrm{op}}_{B',n} + A_{<\lambda} \\ &= \sum_{u \in T_t(\lambda)} \nu_i(u,B,x) b_{r_i B,n} C_t(u,y) b^{\mathrm{op}}_{B',n} + A_{<\lambda} \\ &= \sum_{u \in T_t(\lambda)} \nu_i(u,B,x) C((r_i B,u),(B',y)) + A_{<\lambda}, \end{aligned}$$

as required.
Rewriting (RSrr) to $e_i = lm^{-1}(g_i^2 + mg_i - 1)$, we see that, if $m^{-1} \in S$, the proper behavior of the cell data under left multiplication by e_i is taken care of by the above formulae for g_i. A proof in full generality can be given that is similar to the above proof for g_i using Corollary 8.6(ii).
For $\lambda \in \Lambda_\theta$, the formulas are straight from those for Θ' as $g_1 a = g_2 a$ and $e_1 a = e_2 a$ for each $a \in \Theta'$.
This establishes that $(\Lambda, T, C, *)$ is a cell datum for A and so completes the proof of Theorem 1.2.
Alternatively, the information we have provided shows $\mathrm{B}(\mathrm{D}_n)$ is an iterated inflation of Hecke algebras of type D_n, D_{n-2t}, and A_{n-1-2s} for $s, t \geq 1$ and so are cellular by [19].

9. Discussion

The following consequence of Corollary 6.12 will be of use in [8]. It involves the products $A_{B,n} = a_{B,n}\hat{e}_Y$ where $B \in WB_Y$.

Theorem 9.1. *Let $Y \in \mathcal{Y}$. For each $B \in WB_Y$ there is, up to homogeneous equivalence and powers of δ, a unique word $A_{B,n}$ in $F_n\hat{e}_Y$ satisfying the following three properties for each node i of D_n.*

(i) $r_iA_{B,n} \rightsquigarrow A_{r_iB,n}h$ *for some* $h \in U_Y$. *Furthermore, if* $r_iB > B$, *then* h *is the identity* $\hat{e}_Y$ *of* U_Y.
(ii) *If* $|e_iB| = |B|$, *then* $e_iA_{B,n} \rightsquigarrow A_{e_iB,n}h$ *for some* $h \in \delta^{\mathbb{Z}}U_Y$ *and* $\mathrm{ht}(e_iB) \leq \mathrm{ht}(B)$.
(iii) *If* $|e_iB| > |B|$, *then* $e_iA_{B,n}$ *reduces to an element of* $\mathrm{BrM}(\mathrm{D}_n)e_U\mathrm{BrM}(\mathrm{D}_n)$ *for some set of nodes* U *strictly containing* Y.

In [6, Proposition 3.1] it is shown that there is a natural order on each W-orbit in $\mathcal{A}$, and in fact, [6, Corollary 3.6], each such orbit has a unique maximal element under this order. The ordering is also involved in a notion of height for elements of $\mathcal{A}$, denoted $\mathrm{ht}(B)$ for $B \in \mathcal{A}$, which satisfies $\mathrm{ht}(B) < \mathrm{ht}(C)$ whenever B and C are in the same W-orbit in $\mathcal{A}$ and satisfy $B < C$. Moreover, if $r_iB > B$, then $\mathrm{ht}(r_iB) = \mathrm{ht}(B) + 1$. There are certain minimal elements $Y \in \mathcal{A}$ (such as the sets B_Y for $Y \in \mathcal{Y}$ described above). Then $\mathrm{ht}(B)$ will the distance to the maximal element in the Hasse diagram of the part of the poset $WB = WB_Y$. In particular, $\mathrm{ht}(B_Y) = 0$. The word $A_{B,n}$ has height $\mathrm{ht}(B)$ and moves $\emptyset$ to B in the left action: $A_{B,n}(\emptyset) = B$.
The words $A_{B,n}$ are as given by Notation 6.10 using Lemma 6.9 and involve an ordering of the roots of B. Height considerations as above give an algorithm for choosing a representative for $A_{B,n}$. To begin, pick β_n to be a root of smallest height in B. Then β_{n-2} should be a root of smallest height in $a_{\beta_n,n}e_nB \setminus \{\alpha_n\}$ in case $B \in WB_{Y(t)}$. Continue at each step picking the next root β_{n-2s} as one of smallest height from the roots remaining. Similarly for other $Y \in \mathcal{Y}$.
An alternative proof of the results of this paper, using the methods of [8] is possible. The proof in that paper deals with the case $M = \mathrm{E}_n$ $(n = 6, 7, 8)$ and involves a search of a finite number of finite posets. The search can be avoided in the case $M = \mathrm{D}_n$ by using the specific structure of the root system and induction on n. In [8], the definition of $A_{B,n}$ (which is denoted a_B there) is given by the following algorithm which need not be the same as the one above.

Algorithm 9.2. Given $B \in \mathcal{A}$, determine a word $A_{B,n}$ of minimal height with $A_{B,n}(\emptyset) = B$.

(i) If the number of simple nodes in B is t, then $A_{B,n}$ is a product of e_i which moves these simple nodes to Y as described in [8].
(ii) If $r_k B < B$ for some node k, then $A_{B,n} = r_k A_{r_k B,n}$.
(iii) Otherwise, there are adjacent nodes j, k with $\alpha_j \in B$; then $A_{B,n} = e_j A_{e_k B,n}$ where $A_{e_k B,n}$ has been defined inductively. Here $e_k B = r_j r_k B$, $e_j e_k B = B$, and $\mathrm{ht}(e_k B) = \mathrm{ht}(B)$.

The main result of this paper concerns an upper bound for the BMW algebra, given by a presentation. A lower bound, as can be seen in the proof of the main theorem, is in [4]. In particular, the current results finish the proof of the main theorem in the paper [7] on the tangle algebra $\mathbf{KT}(\mathrm{D}_n)$, which gives a topological depiction of $\mathrm{B}(\mathrm{D}_n)$.
On the level of the Brauer algebra, for a monomial a, the admissible set $B = a(\emptyset)$ determines the connections of the horizontal strands at the top in the following way: if $\varepsilon_i - \varepsilon_j$ belongs to B, then there is a horizontal strand from top node i to top node j that does not go around the pole. If $\varepsilon_i + \varepsilon_j$ belongs to B, then there is a horizontal strand from top node i to top node j that goes around the pole. If $\varepsilon_i - \varepsilon_j$ and $\varepsilon_i + \varepsilon_j$ both belong to B, then Θ, the pair of loops going around the pole as defined in [7], belongs to the tangle. In addition, the paper [7] gives an alternative proof of the lower bound on the rank of $\mathrm{B}(\mathrm{D}_n)$.
Furthermore, the ideal Θ' in $\mathrm{B}(\mathrm{D}_n)$ has a nice interpretation in the tangle algebra $\mathbf{KT}(\mathrm{D}_n)$, where they are ordinary tangles with no loops around the pole and with coefficient Θ, as described in [7]. Here there must be at least one horizontal strand at the top and one at the bottom. These tangles span the ideal corresponding to $\mu^{-1}(Q)$, for Q as in Proposition 6.4. This ideal is easily seen to be the ideal in $\mathrm{B}(\mathrm{A}_{n-1})$ identified in Section 7.

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