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Kinetic-Theory Description of Conductive Heat Transfer from a Fine Wire

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The Maxwell moment method utilizing the two-sided Maxwellian distribution function is applied to the problem of conductive heat transfer between two concentric cylinders at rest. Analytical solutions are obtained for small temperature differences between the cylinders. The predicted heat transfer agrees very well with experiments performed on the heat loss from a fine wire by Bomelburg and Schäfer, Rating, and Eucken. Comparison with results given by Grad's thirteen-moment equations, and with those given by Fourier's "law" plus the Maxwell–Smoluchowski temperature-jump boundary condition, shows that the two-sided character of the distribution function is a crucial factor in problems involving surface curvature.

I. INTRODUCTION

AT present there is no general agreement concerning the connection between highly rarefied gas flows and gas dynamics as described by the Navier–Stokes equations. Some insight into the nature of the transition between these two regimes is provided by the work of Willis,[1] who employed the simplified Krook[2] model for the collision integral in the Maxwell–Boltzmann equation for the single-particle velocity distribution function. But the Krook model implies isotropic scattering, which is highly suspect when there are large mean velocity and mean temperature differences or surface curvature, especially in a rarefied gas. One would like to preserve the main features of the collision process, while retaining the ability to deal with nonlinear problems.

Following Maxwell,[3] some progress can be made toward this goal by converting the Maxwell–Boltzmann equation into an integral equation of transfer, or full-range moment equation, for any quantity Q that is a function only of the components of the particle velocity. Actually Maxwell utilized a special form of the distribution function. We may extend his method by employing an arbitrary distribution function, which is expressed in terms of a certain number of undetermined functions of space and time, selected in such a way that essential physical features of the problem are introduced.[4] Of course the proper number of moments (Q's) must be taken to ensure that a complete set of equations is obtained for these undetermined functions.

As a simple example of this approach, one of the present authors introduced the "two-sided Maxwellian," which is a natural generalization of the situation for free molecule flow.[4] In body coordinates all outwardly directed particle velocity vectors lying within the "cone of influence" (region 1 in Fig. 1) are described by the function $f = f_1$, where

$$f_1 = \frac{n_1(\mathbf{R}, t)}{[2\pi kT_1(\mathbf{R}, t)/m]^{\frac{3}{2}}} \exp\left\{-\frac{m[\boldsymbol{\xi} - \mathbf{u}_1(\mathbf{R}, t)]^2}{2kT_1(\mathbf{R}, t)}\right\}. \quad \text{(1a)}$$

In region 2 (all other $\boldsymbol{\xi}$),

$$f = f_2 = \frac{n_2(\mathbf{R}, t)}{[2\pi kT_2(\mathbf{R}, t)/m]^{\frac{3}{2}}} \cdot \exp\left\{-\frac{m[\boldsymbol{\xi} - \mathbf{u}_2(\mathbf{R}, t)]^2}{2kT_2(\mathbf{R}, t)}\right\}, \quad \text{(1b)}$$

where $n_1 \cdots \mathbf{u}_2$ are ten initially undetermined functions of $\mathbf{R}$ and t. One important advantage of the

[1] D. R. Willis, Phys. Fluids 5, 127 (1962).
[2] R. L. Bhatnager, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).
[3] J. C. Maxwell, Phil. Trans. Roy. Soc. **157**, 49 (1867); also, *Scientific Papers* (Dover Publications, Inc., New York, 1952), Vol. 2, p. 26.
[4] L. Lees, "A Kinetic Theory Description of Rarefied Gas Flows," GALCIT Hypersonic Research Project, Memorandum No. 51 (1959).

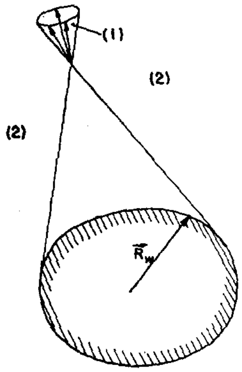

FIG. 1. Cone of influence for discontinuous distribution function.

"two-sided Maxwellian" is that the surface boundary conditions are easily incorporated into the analysis. For example, for completely diffuse reemission the re-emitted particles have a Maxwellian velocity distribution corresponding to the surface temperature T_w by definition, and the mean velocity of the re-emitted particles is identical with the local surface velocity. Thus [Eq. (1a)], $u_1(\mathbf{R}, t) = \mathbf{u}_w$, and $T_1(\mathbf{R}, t) = T_w$ when $\mathbf{R} = \mathbf{R}_w$. When there is no net mass transfer at the surface, an additional boundary condition must be satisfied which is similar to the usual free-molecule flow condition, except that now $\mathbf{u}_2 \neq \mathbf{u}_\infty$.[4]

Once the form of the distribution function is selected, the collision integral appearing in the Maxwell moment equations can be evaluated for any arbitrary law of force between the particles.[5] However, Maxwell's famous inverse fifth-power force law of repulsion provides an important simplification, because the collision integrals are evaluated once and for all in terms of simple combinations of stresses and heat fluxes.[4] In a previous paper[6] we applied this method to the problem of steady, plane compressible Couette flow in order to study the effects of large temperature differences and dissipation in the simplest possible geometry. In the present paper we want to study particularly the effect of surface curvature in a rarefied gas.

II. DESCRIPTION OF THE PROBLEM

The present problem deals with the conductive heat transfer from a metallic wire to a monatomic gas at rest. A fine wire is placed coaxially in a large cylindrical bell jar and is electrically heated. The wire temperature is known from its electrical resistance, while the heat input is found by measuring the current. At normal gas density, heat conduction from the wire is clearly independent of the gas pressure; while at very low gas density the heat loss is proportional to gas pressure. When the gas density is in the transition range, the relation between heat conduction and pressure is not as simple, but the two limiting regimes are joined smoothly (Fig. 2). This simple device has long been used by many investigators[7-11] to determine the thermal conductivity of gases and to study the phenomena of temperature jump and energy accommodation at the wire surface. Some authors have also approached the problem analytically, but they are all forced to introduce certain *ad hoc* assumptions, which restrict their results to small values of the ratio of the mean free path to the wire radius.

Up until very recently, experiments with such a heated wire furnished one of the few sets of data for the full range of gas densities from the free molecular regime to the continuum regime. Moreover, the present problem is fundamentally important because it contains the effect of both convex and concave surfaces. Along with plane Couette flow and shock-wave structure, this problem has received a good deal of attention in rarefied gas dynamics. Weber,[7] and Schäfer, Rating, and Eucken[8] subdivide the annulus into three parts: two free molecular heat-transfer regions near the solid surfaces $R_1 < R < R_1 + \Delta R_1$, $R_2 - \Delta R_2 < R < R_2$ (R_1, R_2 are the wire and bell jar radii, respectively); and a region between $R_1 + \Delta R_1$ and $R_2 - \Delta R_2$ where continuum heat conduction is assumed. The ar-

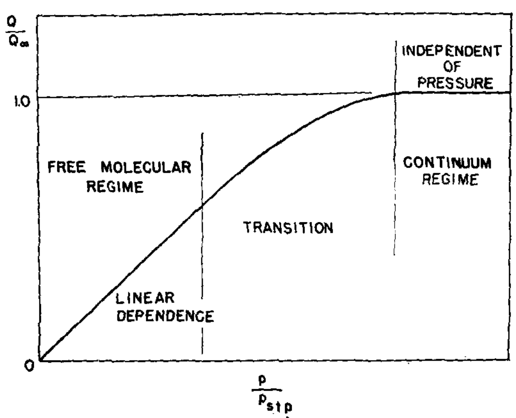

FIG. 2. Variation of heat transfer with gas pressure.

[5] M. L. Lavin and J. K. Haviland, Phys. Fluids **5**, 274 (1962).

[6] C. Y. Liu and L. Lees, *Advances in Applied Mechanics, Supplement 1, Rarefied Gas Dynamics* (Academic Press Inc., New York, 1961), pp. 391-428.

[7] S. Weber, Kgl. Danske Videnskab. Selskab. Mat.-Fys. Medd. **19**, 11 (1942).

[8] K. Schäfer, W. Rating, and A. Eucken, Ann. Physik **42**, 176 (1942).

[9] H. Gregory and C. T. Archer, Proc. Royal Soc. (London) **A110**, 91 (1926).

[10] B. G. Dickins, Proc. Royal Soc. (London) **A143**, 517 (1934).

[11] H. S. Gregory, Phil. Mag. **22**, 257 (1936).

bitrary quantities ΔR_1 and ΔR_2 are functions of the mean free path λ, and in Weber's case they are taken simply equal to $(15/8)\lambda$. In Schäfer, Rating, and Eucken's calculation, $\Delta R_1/\lambda$ and $\Delta R_2/\lambda$ are functions of λ/R_1 and λ/R_2, respectively. In their studies, the implication that λ is small in comparison with R_1 and R_2 has been made. The choice of ΔR_1 and ΔR_2 is clearly related to the temperature-jump boundary condition proposed by Smoluchowski (see D in Sec. V).[12] He suggested that for a small degree of rarefaction, the difference between the gas temperature and wall temperature at the solid surface is equal to $-(15/8)\lambda\ (dT/dn)_{\text{wall}}$, where $(dT/dn)_{\text{wall}}$ is the gas-temperature gradient normal to the wall. Application of Smoluchowski's relation to the present problem is discussed in D of Sec. V.

Gregory[9] and his followers have investigated this "hot-wire" method over the period of a decade. Their primary goal is accurate determination of the gaseous thermal conductivity as a function of temperature. Early developments were more along technical lines than analytical, such as keeping the wire temperature constant under different conditions, elimination of convective losses, etc. They merely used the usual Fourier result that the total heat transfer Q is $2\pi k_c l(T_{\text{I}} - T_{\text{II}})/\ln (R_2/R_1)$, in which k_c is the "classical" thermal conductivity of the gas,[13] l the length of the wire, and T_{I} and T_{II} are the temperatures of the wire and the bell jar, respectively. They allow k_c to decrease if the pressure decreases appreciably below atmospheric[9]; however, their original focal point is the temperature dependence but not the pressure effect.

Later, Dickins[10] adopted Gregory's apparatus to determine accommodation coefficients.[14] He corrected the heat transfer Q at low pressures by an amount Λ/R_1, so that

$$Q = \frac{2\pi k_c l(T_{\text{I}} - T_{\text{II}})}{\ln (R_2/R_1) + (\Lambda/R_1)},$$

in which $\Lambda = (15/8)\lambda(2 - a)/a$, and a is Knudsen's accommodation coefficient. As determined by Dickins' experiment "a" is about 0.9 for most gases except helium and hydrogen. The correction Λ is easily seen to be based on Smoluchowski's relation.[12]

Two years later, Gregory[11] generalized the same relation for polyatomic gases, but Λ then included a numerical factor which accounts for intermolecular forces and has to be determined by experimental data on viscosity and specific heat. At the same time, microscopic studies have also been made by Zener[15] and Devonshire[16] on the general aspect of solid–gas interchange of energy. They require experimental determination of certain constants related to intermolecular forces.

Welander in 1954[17] worked the problem anew but used a different constant for Λ, in which the factor $(2 - a)/a$ is replaced by $(2 - Ka)/a$.[18] The quantity K is found to be 0.827 by solving the "Krooked" Boltzmann equation,[19] in which the collision integral is taken to be $(8/15)(c/\lambda)(f - f_0)$, where f is the unknown velocity distribution function, f_0 is the local Maxwellian, and c the mean thermal velocity $c = (8kT/m\pi)^{\frac{1}{2}}$. Welander attempted to extend the validity of the $(2 - Ka)/a$ expression to the free molecular regime by allowing K to be a function of gas density. Under the assumption that $|(dT/dR)_{\text{wall}}(\lambda/T)|$ is small in comparison with unity and that the distribution function differs slightly from the *local* Maxwellian, he obtained an integral equation governing the K function, but he did not solve that equation. Instead he estimated K from experimental data given by Schäfer, Rating, and Eucken,[8] and found that K varied between 0.1 and 0.6. The fact that K depends only on pressure is rather obvious; yet Welander's result demonstrates very little beyond this point.

Though it might be difficult to record all the investigations of this "simple" problem since the first use of the apparatus by Schleiermacher[20] for determination of gaseous conductivity in 1888, yet it is clear that a thorough theoretical investigation of the problem is long overdue.

III. FORMULATION OF THE PROBLEM ACCORDING TO THE MAXWELL MOMENT METHOD

A. Distribution Function and Mean Quantities

We consider a wire of radius R_1 placed at the center of a concentric cylinder of radius R_2, with

[12] M. v. Smoluchowski, Akad. Wiss. Wien. **107**, 304 (1889); **108**, 5 (1899).

[13] It should be noted that the Fourier relation $q = -k_c\nabla T$ holds only at *normal* densities; thus the "classical" thermal conductivity k_c introduced here is merely for convenience. (See Sec. V: D.)

[14] The thermal accommodation coefficient "a" advanced by Knudsen [M. Knudsen, *Kinetic Theory of Gases* (Methuen and Company, Ltd., London, England, 1934)] is defined as $a = (E_g - E_r)/(E_g - E_w)$, where E_g is the energy transported to surface by incident molecules in equilibrium at the gas temperature, E_r is the actual energy carried away by molecules leaving the surface, and E_w the energy of re-emitted molecules in equilibrium at the wall temperature.

[15] C. Zener, Phys. Rev. **40**, 335 (1932).

[16] A. F. Devonshire, Proc. Roy. Soc. (London) **A158**, 269 (1937).

[17] P. Welander, Arkiv Fysik **7**, 555 (1954).

[18] Welander also used a different numerical factor $75\pi/128$ instead of 15/8; however, the quantative difference is negligible.

[19] P. Welander, Arkiv Fysik **7**, 507 (1954). Welander's paper is published at the same time as Krook's work.[2]

[20] A. Schleiermacher, Ann. physik. Chem. **34**, 623 (1888).

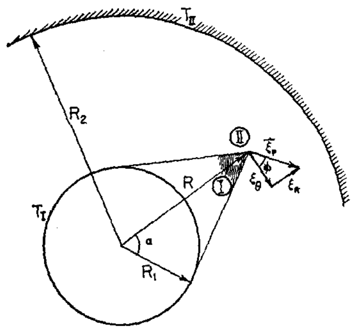

FIG. 3. Configuration in cylindrical coordinates.

$R_2 > R_1$ (Fig. 3). The wire is heated to a temperature T_{I}, while the outer cylinder is kept at temperature T_{II}. The annular region ($R_1 < R < R_2$) is filled with monatomic gas at an arbitrary density level, which is characterized by the mean free path λ evaluated at a convenient reference point (say $R = R_1$). If the wire is sufficiently long, end effects are negligible; thus the problem is axially symmetric and two dimensional.

As discussed in the Introduction, the simplest distribution function having a "two-sided" character and capable of giving a smooth transition between the highly rarefied gas regime and the continuum limit consists of two Maxwellians, each containing several parametric functions. All outwardly directed molecules with planar velocity vector $\boldsymbol{\xi}_p$,

$$\xi_p = (\xi_R^2 + \xi_\theta^2)^{\frac{1}{2}}, \qquad \phi = \tan^{-1}(\xi_R/\xi_\theta),$$

lying inside the wedge of influence (region 1 in Fig. 3) are characterized by one Maxwellian f_1, where

$$f = f_1 \quad \text{for} \quad \alpha < \phi < \pi - \alpha,$$

in which

$$\alpha = \cos^{-1}(R_1/R).$$

Then, all molecules with planar velocity ξ_p lying outside of region 1 are characterized by f_2, i.e.,

$$f = f_2 \quad \text{for} \quad \pi - \alpha < \phi < 2\pi + \alpha.$$

The requirement that f should be discontinuous on the sides of the "wedge of influence" is the most basic feature in the present scheme; its importance will be seen shortly.

In order to satisfy at least the three conservation equations and the heat-flux equation, one finds that four parametric functions specifying f_1 and f_2 are the absolute minimum. Thus we prescribe that

$$f_1 = n_1\left(\frac{m}{2\pi kT_1}\right)^{\frac{3}{2}} \exp\left[-\frac{m}{2kT_1}(\xi_P^2 + \xi_Z^2)\right];$$

likewise

$$f_2 = n_2\left(\frac{m}{2\pi kT_2}\right)^{\frac{3}{2}} \exp\left[-\frac{m}{2kT_2}(\xi_P^2 + \xi_Z^2)\right],$$

where $n_1(R)$, $T_1(R)$, $n_2(R)$, $T_2(R)$ are the four unknown functions of radial distance. Here the n's have the dimension of a number density, while the T's have the dimension of a temperature, and it must be stressed that each individual function has no explicit physical significance in general.

Knowing the distribution function f, one can evaluate all mean quantities Q by averaging over all velocity space,

$$\langle Q\rangle = \int Qf\,d\xi = \int_\alpha^{\pi-\alpha}\int_0^{+\infty}\int_{-\infty}^{+\infty} Qf_1\,d\xi_Z\,d\xi_P\,d\phi + \int_{\pi-\alpha}^{2\pi+\alpha}\int_0^{+\infty}\int_{-\infty}^{+\infty} Qf_2\,d\xi_Z\,d\xi_P\,d\phi.$$

For example, the mean density is

$$\rho = \rho(R) = \int mf\,d\xi = \frac{m}{2\pi}[n_1(\pi - 2\alpha) + n_2(\pi + 2\alpha)], \tag{2}$$

and the mean temperature is

$$T = T(R) = \frac{n_1T_1(\pi - 2\alpha) + n_2T_2(\pi + 2\alpha)}{n_1(\pi - 2\alpha) + n_2(\pi + 2\alpha)}. \tag{3}$$

Notice that the angular dependence appears directly, while T_1, T_2, n_1, n_2 will bring in a purely radial dependence. Expressions for radial velocity u_R, hydrostatic pressure p, normal stresses, and radial heat transfer q_R are listed below for later usage:

$$u_R = \left(\frac{2\pi k}{m}\right)^{\frac{1}{2}}(\cos\alpha)\frac{n_1(T_1)^{\frac{1}{2}} - n_2(T_2)^{\frac{1}{2}}}{n_1(\pi - 2\alpha) + n_2(\pi + 2\alpha)}; \tag{4}$$

$$p = (k/2\pi)[n_1T_1(\pi - 2\alpha) + n_2T_2(\pi + 2\alpha)] = -\tfrac{1}{3}(P_{RR} + P_{\theta\theta} + P_{ZZ}); \tag{5a}$$

$$\left.\begin{aligned} P_{RR} &= -\langle m\xi_R^2\rangle = -(k/2\pi)[n_1T_1(\pi - 2\alpha + \sin 2\alpha) + n_2T_2(\pi + 2\alpha - \sin 2\alpha)],\\ P_{\theta\theta} &= -\langle m\xi_\theta^2\rangle = -(k/2\pi)[n_1T_1(\pi - 2\alpha - \sin 2\alpha) + n_2T_2(\pi + 2\alpha + \sin 2\alpha)],\\ P_{ZZ} &= -\langle m\xi_Z^2\rangle = -(k/2\pi)[n_1T_1(\pi - 2\alpha) + n_2T_2(\pi + 2\alpha)];\end{aligned}\right\} \tag{5b}$$

$$q_R = -(2/\pi m)^{\frac{1}{2}}(\cos\alpha)[n_1(kT_1)^{\frac{3}{2}} - n_2(kT_2)^{\frac{3}{2}}]. \tag{6}$$

It should be pointed out here that the normal stresses in different directions are generally not the same; namely, $P_{RR} \neq P_{\theta\theta} \neq -p$.

B. Differential Equations and Boundary Conditions

In cylindrical coordinates the Maxwell integral equation of transfer is as follows[4]:

$$\frac{1}{R}\frac{\partial}{\partial R}\left[R\int f\xi_R Q\,d\xi\right]+\frac{1}{R}\frac{\partial}{\partial\theta}\int f\xi_\theta Q\,d\xi+\frac{\partial}{\partial Z}\int f\xi_Z Q\,d\xi-\int\frac{f}{R}\left(\xi_\theta^2\frac{\partial Q}{\partial\xi_R}-\xi_\theta\xi_R\frac{\partial Q}{\partial\xi_\theta}\right)d\xi=\Delta Q,\qquad(7)$$

where[4]

$$Q=Q(\xi_R,\xi_\theta,\xi_Z)=Q(\xi_P,\phi,\xi_Z),$$

$$\Delta Q=\iiiint(Q'-Q)ff_1V\,d\xi\,d\xi_1\,b\,db\,d\epsilon.$$

Because of two-dimensionality and axial-symmetry, Eq. (7) further reduces to

$$\frac{1}{R}\frac{d}{dR}\left(R\int f\xi_R Q\,d\xi\right)-\int\frac{f}{R}\left(\xi_\theta^2\frac{\partial Q}{\partial\xi_R}-\xi_\theta\xi_R\frac{\partial Q}{\partial\xi_\theta}\right)d\xi=\Delta Q.\qquad(8)$$

Setting $Q = m$, $m\xi_R$, $\frac{1}{2}m\xi^2$, respectively, we find $\Delta Q = 0$ because the mass, momentum, and energy are invariant during collisions, and we obtain the ordinary continuity, radial momentum, and energy equations. Since we are primarily interested in radial heat transfer, we take $Q_4 = \frac{1}{2}m\xi_R\xi^2$, which yields the heat-flux equation in which the collision integral ΔQ, for simplicity, is evaluated with Maxwell's inverse-fifth-power force law $F = (m_1m_2\tilde{K}/r^5)$, and is found to be proportional to the heat flux q_R.[4]

In order to bring out all pertinent parameters governing the problem we choose $n_{\rm I}$, $T_{\rm I}$, R_1 as the characteristic number density, temperature, and length, respectively. We also utilize the fact that the Maxwell mean free path evaluated at condition I is

$$\lambda_{\rm I}=(1/3A_2\rho_{\rm I})(\pi kT_{\rm I}/\tilde{K})^{\frac{1}{2}},$$

where $A_2 = 1.3682$ is the value of the scattering integral for Maxwell molecules,[3] and k is the Boltzmann constant. A_2 and $\tilde{K}$ are related to the classical coefficient of viscosity by the expression

$$\mu_c=kT/\tfrac{3}{2}A_2(2m\tilde{K})^{\frac{1}{2}}.$$

Denoting all normalized quantities by a bar superscript, the four differential equations governing the four unknown functions are as follows:

Continuity:

$$\bar{n}_1(\bar{T}_1)^{\frac{1}{2}}=\bar{n}_2(\bar{T}_2)^{\frac{1}{2}};\qquad(9a)$$

R Momentum:

$$(\sin 2\alpha-2\alpha)\frac{d}{d\bar{R}}(\bar{n}_1\bar{T}_1-\bar{n}_2\bar{T}_2)+\pi\frac{d}{d\bar{R}}(\bar{n}_1\bar{T}_1+\bar{n}_2\bar{T}_2)=0;\qquad(9b)$$

Energy:

$$\bar{n}_1\bar{T}_1^{\frac{3}{2}}-\bar{n}_2\bar{T}_2^{\frac{3}{2}}=\beta;\qquad(9c)$$

Heat Flux:

$$(\sin 2\alpha-2\alpha)\cdot\frac{d}{d\bar{R}}(\bar{n}_1\bar{T}_1^2-\bar{n}_2\bar{T}_2^2)+\pi\frac{d}{d\bar{R}}(\bar{n}_1\bar{T}_1^2+\bar{n}_2\bar{T}_2^2)+\frac{4}{15}\frac{R_1}{\lambda_{\rm I}}\frac{\beta}{\bar{R}}[\bar{n}_1(\pi-2\alpha)+\bar{n}_2(\pi+2\alpha)]=0,\qquad(9d)$$

in which β is the integration constant.

The normalized boundary conditions are at $\bar{R} = 1$,

$$\bar{T}_1=1,\qquad(10a)$$

$$\bar{n}_1=1;\qquad(10c)$$

at $\bar{R} = R_2/R_1$,

$$\bar{T}_2=T_{\rm II}/T_{\rm I}.\qquad(10b)$$

Since $P_{RR} \neq P_{\theta\theta} \neq -p$, the momentum equation (9b) does not imply $(dp/dR) = 0$. This observation is important, because a pressure gradient exists owing to heat conduction but not because of fluid flow. Also, the heat-flux equation (9d), relating q_R to two higher moments

$$\int mf\xi_P^2\sin^2\phi(\xi_P^2+\xi_Z^2)\,d\xi$$

and

$$\int mf\xi_P^2\cos^2\phi(\xi_P^2+\xi_Z^2)\,d\xi,$$

bears no resemblance to Fourier's "law" in general; in fact Eq. (9d) reduces to $q_R = -k_c\ (dT/dR)$ only if the local full-range Maxwellian is introduced into the left-hand side. In other words, Eq. (9d) would give Fourier's "law" to the first order if the Chapman–Enskog expansion procedure is employed.[4]

There are three parameters governing this problem: the rarefaction parameter $\lambda_{\rm I}/R_1$ of Eq. (9d), the temperature ratio $T_{\rm II}/T_{\rm I}$ appearing in the boundary condition, and the radius ratio R_2/R_1 describing the geometrical configuration. One can readily see that Eqs. (9) would all become algebraic at the free molecular limit, namely, $(\lambda_I/R_1) \to \infty$; thus, $n_1, \cdots, T_2$ would all have the constant values prescribed by the boundary conditions. Then the distribution function f would not only be discontinuous in velocity space, but also independent of space coordinates. Nevertheless, mean quantities [see Eqs. (2)–(6)] would still depend on $\bar{R}$ even in free molecular flow. It has been mentioned previously that the set of equations (9), reduces to the usual Fourier formulation if an expansion in λ_I/R_1 is employed. Of course the complete solutions to these equations will demonstrate these limiting characteristics.

IV. SOLUTIONS FOR SMALL TEMPERATURE DIFFERENCES BETWEEN CYLINDERS

In general, one can utilize Eqs. (9a), (9c) to express $\bar{n}_1, \bar{n}_2, \bar{T}_1, \bar{T}_2$ in favor of two unknown functions, as in reference 6. Then one has to integrate Eqs. (9b) and (9d) numerically for these two functions. For example, if we designate

$$\bar{n}_1\bar{T}_1 + \bar{n}_2\bar{T}_2 = G(\bar{R}), \qquad \bar{n}_1\bar{T}_1 - \bar{n}_2\bar{T}_2 = K(\bar{R}),$$

then by using Eqs. (9a) and (9c), we can express all four unknown functions by these two new functions G and K as

$$\bar{n}_1 = \frac{2}{\beta^2}\frac{KG}{(1/K + 1/G)}, \qquad \bar{n}_2 = \frac{2}{\beta^2}\frac{KG}{(1/K - 1/G)};$$

$$\bar{T}_1 = \frac{\beta^2}{4}\left(\frac{1}{K} + \frac{1}{G}\right)^2, \qquad \bar{T}_2 = \frac{\beta^2}{4}\left(\frac{1}{K} - \frac{1}{G}\right)^2.$$

Substituting into Eqs. (9b) and (9d), we obtain two governing equations

$$\frac{dG}{d\bar{R}} = \frac{2\alpha - \sin 2\alpha}{\pi}\frac{dK}{d\bar{R}},$$

$$\frac{dK}{d\bar{R}} = \frac{\left(\frac{32}{15}\frac{1}{\beta^3}\frac{R_1}{\lambda_I}\right)\left(\frac{\pi G - 2\alpha G}{1/K^2 - 1/G^2}\right)\frac{1}{\bar{R}}}{2(2\alpha - \sin 2\alpha)\left(\frac{1}{G^2} - \frac{1}{K^2}\right) + \pi\frac{G}{K^2} - \frac{K}{G^3}\frac{(\sin 2\alpha - 2\alpha)^2}{\pi}} = f_n(K, G, \bar{R}).$$

Boundary conditions can be converted easily into conditions for G and K. In actual numerical integration, interpolation would be more practical than an iteration scheme; namely, for a given R_1/λ_I, one may work with a spectrum of β's which leads to a spectrum of corresponding $\bar{T}_{II}$'s. Then for a prescribed $\bar{T}_{II}$, the corresponding β value at that particular density level can be found by interpolation.

By examining the situation more closely, one finds that the linear problem is in fact the most important. In all experiments previously performed, the wire is only slightly heated and its temperature never exceeds the temperature of the bell jar by more than 15%. With large temperature difference, pure conduction would be quite difficult to achieve. Thus, Eqs. (9) are linearized in order to acquire analytical solutions, to compare with experiments, and to study particularly the effect of surface curvature.

When the temperature ratio T_{II}/T_I departs little from unity, the four functions $\bar{n}_1, \bar{n}_2, \bar{T}_1, \bar{T}_2$ also depart from unity by an amount small compared with one. Symbolically, if

$$T_{II}/T_I = 1 - \epsilon, \quad \text{where} \quad \epsilon \ll 1,$$

then

$$\bar{n}_1 = 1 + N_1, \qquad \bar{n}_2 = 1 + N_2, \qquad \bar{T}_1 = 1 + t_1, \qquad \bar{T}_2 = 1 + t_2, \tag{11}$$

in which $N_1, N_2, t_1, t_2 \ll 1$.

Such a limiting process implies that each of the distribution functions f_1 and f_2 are slightly perturbed over a *constant* Maxwellian, but the distribution function is still discontinuous on the surface of the wedge of influence (Fig. 3). This procedure is intrinsically different from perturbation over a full-range *local* Maxwellian which usually is space dependent; the latter procedure follows practically the same line as the Chapman–Enskog scheme. Thus it should not be surprising to learn that the results so obtained would be useful only when the gas is slightly rarefied. In other words, the present linearization implies no restriction on the value of the rarefaction parameter. The scheme does imply that N_1, N_2, t_1, t_2 are of the same order of magnitude.

Introducing Eqs. (11) in Eqs. (9) one readily finds the set of governing equations for quantities N_1, N_2, t_1, t_2:

$$N_1 + \tfrac{1}{2}t_1 = N_2 + \tfrac{1}{2}t_2, \tag{12a}$$

$$(\sin 2\alpha - 2\alpha)(d/d\bar{R})(N_1 + t_1 - N_2 - t_2) + \pi(d/d\bar{R})(N_1 + t_1 + N_2 + t_2) = 0, \qquad (12b)$$

$$(N_1 - N_2) + \tfrac{3}{2}(t_1 - t_2) = \beta. \qquad (12c)$$

By using Eqs. (12a) and (12c), one obtains

$$t_1 - t_2 = \beta \quad \text{and} \quad N_2 - N_1 = \tfrac{1}{2}\beta.$$

Equation (12b) then yields

$$N_1 + t_1 + N_2 + t_2 = \text{const},$$

or

$$N_1 + t_1 = \text{const}.$$

The heat-flux equation (9d) becomes

$$(d/d\bar{R})(t_1 + t_2) + (8/15)(R_1/\lambda_{\rm I})(\beta/\bar{R}) = 0. \qquad (12d)$$

Integrating and applying the boundary conditions

$$t_1 = 0 \quad \text{at} \quad \bar{R} = 1, \qquad (13a)$$
$$N_1 = 0 \qquad (13b)$$
$$t_2 = -\epsilon \quad \text{at} \quad \bar{R} = R_2/R_1 = \bar{R}_2, \qquad (13c)$$

one obtains the following solutions:

$$\begin{aligned} N_1 &= (4/15)(R_1/\lambda_{\rm I})\beta \ln \bar{R}, \\ N_2 &= \beta[\tfrac{1}{2} + (4/15)(R_1/\lambda_{\rm I}) \ln \bar{R}], \\ t_1 &= -(4/15)(R_1/\lambda_{\rm I})\beta \ln \bar{R}, \\ t_2 &= -\beta[1 + (4/15)(R_1/\lambda_{\rm I}) \ln \bar{R}] \end{aligned} \qquad (14)$$

with

$$\beta = \frac{\epsilon}{1 + (4/15)(R_1/\lambda_{\rm I}) \ln (R_2/R_1)}.$$

Using subscript ∞ to denote quantities evaluated at the continuum limit, we then find the heat-transfer ratio as

$$\frac{Q}{Q_\infty} = \frac{q_R}{q_{R\infty}} = \frac{\beta}{\beta_\infty} = \frac{1}{1 + [(4/15)(R_1/\lambda_{\rm I}) \ln (R_2/R_1)]^{-1}}. \qquad (15)$$

Inserting the results of N_1, N_2, t_1, t_2 into Eq. (3), we obtain the temperature distribution

$$\frac{1 - \bar{T}}{1 - \bar{T}_{\rm II}} = \delta\left[\frac{1}{2} + \frac{1}{\pi}\cos^{-1}\left(\frac{R_1}{R}\right)\right] + (1 - \delta)\frac{\ln (R/R_1)}{\ln (R_2/R_1)}, \qquad (16)$$

with

$$\delta = [1 + (4/15)(R_1/\lambda_{\rm I}) \ln (R_2/R_1)]^{-1}.$$

Other mean quantities can be immediately written down from Eqs. (2), (4), (5), (6), and (14) in a similar fashion.

V. DISCUSSION

A. Heat Transfer and Comparison with Experiment

Knudsen's formula for heat loss from a surface of temperature $T_{\rm I}$ to a stream of incident molecules of temperature $T_{\rm II}$ at low pressure[21] has been generally accepted as a good one:

$$q_R|_{\text{at surface}} = \left(\frac{2k}{\pi m}\right)^{\frac{1}{2}} \frac{p_a}{(T_{\rm I})^{\frac{1}{2}}}(T_{\rm I} - T_{\rm II}), \qquad (17)$$

where a is Knudsen's thermal accommodation coefficient, which in the present study has been taken to be unity. Studies on accommodation coefficients, though quantitatively inconclusive, leave no doubt about the validity of Eq. (17). Dickins[10] observed the linear dependency of thermal conductivity on pressure below 5 cm Hg. Mann[22] also confirmed that "a" is independent of pressure within 2–3% accuracy up to 330 μ for an instrument with $R_2/R_1 = 1250$. Of course direct measurements on heat loss support this fact quantitatively (Sec. II).

Owing to its "two-sided" character the present formulation naturally brings out Eq. (17) as a limit for $R_1/\lambda_{\rm I} \rightarrow 0$, as can be seen from Eqs. (6) and (14). Here it shows clearly that if $p_{\rm I} \ll (1/30)(m/2\pi kT_{\rm I})^{\frac{1}{2}}(R_1/\lambda_{\rm I}) \ln (R_2/R_1)$, Knudsen's formula is quite applicable.

On the other hand, Eqs. (6) and (14) readily yield the Fourier result

$$q_R = k_c(T_{\rm I} - T_{\rm II})/R \ln (R_2/R_1),$$

as soon as we set $(R_1/\lambda_{\rm I}) \rightarrow \infty$ and utilize the relation $k_c = (15/4)\mu_c(k/m)$ for a monatomic gas.

Calculation of heat loss over the whole range of densities has also been done by Ai using Grad's thirteen-moment equations.[23] Ai obtains the Fourier heat-conduction relation over the whole range of densities and gives the result

$$\frac{Q}{Q_\infty} = \frac{1}{1 + [(8/15)(R_1/\lambda_{\rm I}) \ln (R_2/R_1)]^{-1}}, \qquad (18)$$

which yields a value twice as large as the actual heat loss at low pressures. As one can learn from Knudsen's formula [Eq. (17)] the heat loss at low

[21] E. H. Kennard, *Kinetic Theory of Gases* (McGraw-Hill Book Company, Inc., New York, 1938), 1st ed., pp. 311–327.
[22] W. B. Mann, Proc. Royal Soc. (London) **A146**, 776 (1934).
[23] D. K. Ai, "Cylindrical Couette Flow in a Rarefied Gas According to Grad's Equations," GALCIT Hypersonic Research Project, Memorandum No. 56 (1960).

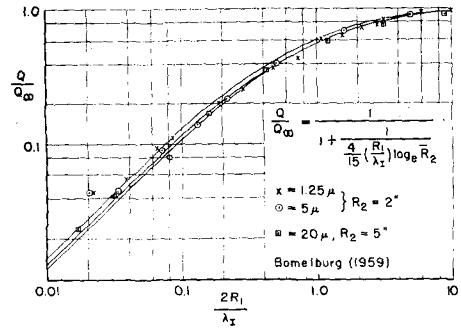

FIG. 4. Comparison with experiment (reference 25).

pressures is proportional to the difference between the gas temperature and the temperature of the solid wall. Grad's formulation lacks the "two-sided" angular effect, and always overestimates the temperature difference at the wall by a factor of 2 (see next section). The same factor is found when the Fourier relation is used in conjunction with the Maxwell-Smoluchowski temperature "jump" condition (Sec. V: D).

Numerous experiments have been performed using the heated fine wire to determine gaseous conductivity, or mostly thermal accommodation coefficients. Conductivity measurements are often made at normal density with different temperatures, while the determinations of accommodation coefficients are usually done at low pressures. Unfortunately, data obtained in the past years are utterly inconsistent. Values of accommodation coefficient for a given pressure differing from each other by one or two orders of magnitude are not surprising at all. Hartnett[24] in his survey report on accommodation coefficients attributes this discrepancy to three factors: (1) the properties of the solid surface which are usually unspecified greatly affect the result; (2) evaluation of the accommodation coefficient by Knudsen's formula [Eq. (17)] for free molecular flow is often unjustified, because the pressures are usually not low enough to ensure the free molecular limit; (3) use of an excessive radiation correction. Besides the inconsistency of these experiments, most publications give only the final accommodation coefficients; a backward deduction to the heat loss is not only dangerous but also impossible owing to lack of knowledge of some physical constants employed in their computations.

The most recent measurement designed solely to study conductive heat transfer is done by Bomelburg.[25] He uses Wollaston wire of diameters 1.25, 5, and 10 μ, and bell jars of diameters 4 in. and 10 in.[26] His results in the transition regime are reproduced in Fig. 4, in which the three curves represent calculations according to Eq. (15). It is understandable that at low pressures when $\lambda_I > 200\, R_1$, radiation and end losses become dominant; thus, conduction measurements at this range would be more difficult. But Fig. 4 shows that Bomelburg's experiment agrees with Eq. (15) fairly well.

Tracing back chronologically, we find the measurements by Schäfer, Rating, and Eucken in 1942.[8] They use a platinum wire of $R_1 = 0.00208$ cm in a tube of inner radius $R_2 = 0.294$ cm. Tests are run at 3.5°C with pressures ranging from 1 to 1/3000 atm, so the ratio $2R_1/\lambda_I$ covers a wide region from 0.1 to 1000. Heat-transfer results for argon and CO_2 are plotted in Fig. 5, in which the solid curve again represents Eq. (15). Points for argon all fall along the predicted curve with a maximum deviation of 10 percent at the lowest pressure point. The fact that the heat loss for CO_2 also obeys Eq. (15) is rather surprising, as the experiments of CO_2 went down to as low as 1/200 atm. One could expect that the factor 4/15 in Eq. (15), supposedly valid for a monatomic gas only, should be modified for a polyatomic gas. However, the general agreement is certainly not accidental.

Other experiments giving heat-loss data have all been performed with diatomic gases. Gregory and Archer's measurement[9] (1926) using air and hydrogen gave Q/Q_∞ values 20–30% lower than that predicted by Eq. (15). Knudsen's classical experiment[27] (1911) also using hydrogen at various pressure levels showed higher heat loss than expected. Fredlund[28] later correlated Knudsen's data with a formula exactly like Eq. (15). His correlation required a numerical factor approximately three times

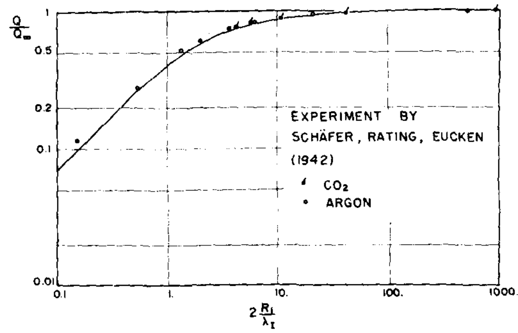

FIG. 5. Comparison with experiment (reference 8).

[24] J. P. Hartnett, *Advances in Applied Mechanics, Supplement 1, Rarefied Gas Dynamics*, (Academic Press Inc., New York, 1961), pp. 1–28.
[25] H. J. Bomelburg, Phys. Fluids **2**, 717 (1959).

[26] D. K. Ai (private communication).
[27] M. Knudsen, Ann. Physik **34**, 593 (1911).
[28] E. Fredlund, Ann. Physik **28**, 319 (1937).

larger than 4/15, which is qualitatively in the right direction for a polyatomic gas correction.

B. Temperature Distribution

The expression for the mean gas temperature [Eq. (16)] is rewritten here

$$\frac{1-\bar{T}}{1-\bar{T}_{\mathrm{II}}}=\delta\left[\frac{1}{2}+\frac{1}{\pi}\cos^{-1}\left(\frac{R_1}{R}\right)\right]+(1-\delta)\frac{\ln(R/R_1)}{\ln(R_2/R_1)}, \quad (16)$$

where

$$\delta=[1+(4/15)(R_1/\lambda_{\mathrm{I}})\ln(R_2/R_1)]^{-1}.$$

The significance of this parameter will be clear shortly. This expression shows that the temperature field is composed of two parts (see Fig. 6):

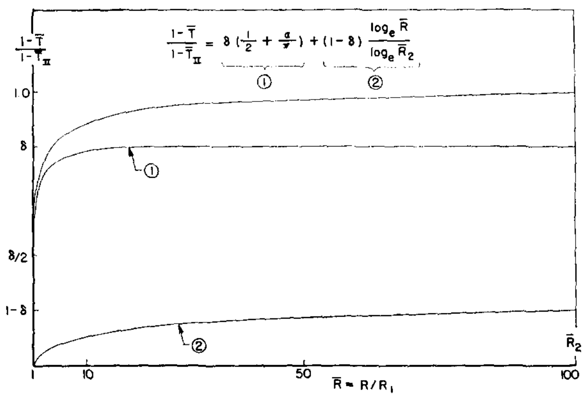

FIG. 6. Temperature distribution between two cylinders.

the first (part 1) is an angular part weighted by δ, the second (part 2) is logarithmic, and it is in turn weighted by the quantity $(1-\delta)$. Part 1 has the character of a free molecular temperature field, while the other part has the same character as the Fourier solution. One should note that $\cos^{-1}(R_1/R)$ becomes practically equal to $\frac{1}{2}\pi$ at about ten diameters from the center of the inner cylinder. Thus, at fairly low pressures, the physical presence of the wire has no influence on the temperature at a point several diameters away. As δ becomes smaller, the logarithmic part would penetrate deeper from R_2 towards R_1, and it finally dominates the whole temperature field when $\delta\to 0$. A sketch of the temperature distribution for various values of $2\lambda_{\mathrm{I}}/R_1$ is given in Fig. 7.

From another point of view, the temperature distribution can be interpreted as the composition of an "outer" solution and an "inner" solution. The "outer" solution describes the temperature field

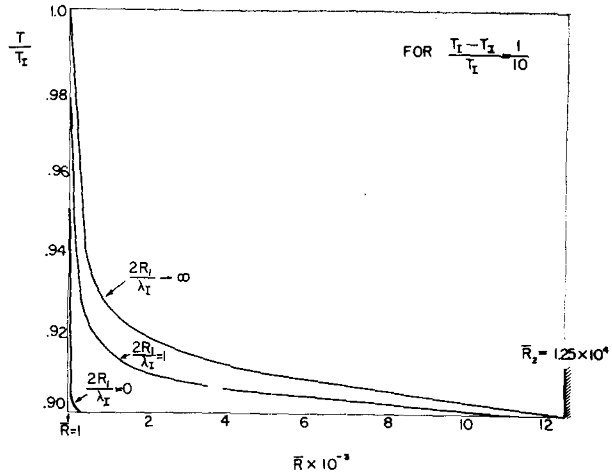

FIG. 7. Temperature distribution for various values of $2R_1/\lambda_{\mathrm{I}}$.

corresponding to the solution of the Fourier heat-conduction problem with a temperature "jump" at the inner wire when $R_2/R_1 \gg 1$, which is the only case in which this "splitting" makes sense. This field is given by the expression

$$\left(\frac{1-\bar{T}}{1-\bar{T}_{\mathrm{II}}}\right)_{\mathrm{outer}}=\delta\left[1+\frac{4}{15}\frac{R_1}{\lambda_{\mathrm{I}}}\ln\left(\frac{R_2}{R_1}\right)\right].$$

The inner solution is of the form

$$C[\tfrac{1}{2}+(1/\pi)\cos^{-1}(R_1/R)],$$

where the quantity in brackets is exactly the free-molecule solution, and C is an undetermined constant. Now, if one matches the inner and outer solutions by requiring that

$$\lim_{\bar{R}\to\infty}\left(\frac{1-\bar{T}}{1-\bar{T}_{\mathrm{II}}}\right)_{\mathrm{inner}}=\lim_{\bar{R}\to 1}\left(\frac{1-\bar{T}}{1-\bar{T}_{\mathrm{II}}}\right)_{\mathrm{outer}},$$

then $C=\delta$. Evidently Eq. (16) represents the full solution which is valid over the whole region.

The temperature-jump phenomenon is solely accounted for by the angular part (part 1), which contributes to $(1-\bar{T})/(1-\bar{T}_{\mathrm{II}})$ a difference of $\frac{1}{2}\delta$ at the surface of the wire [Eq. (16)]. At the free molecular limit the gas temperature equals the algebraic mean of T_{I} and T_{II} as one would expect in this linearized case. Grad's scheme employing only the logarithmic term gives a temperature profile[29]

$$\frac{1-\bar{T}}{1-\bar{T}_{\mathrm{II}}}=\delta+(1-\delta)\frac{\ln(R/R_1)}{\ln(R_2/R_1)}, \quad (19)$$

from which one finds the "temperature jump" at the wire surface to be δ, which differs from the present

[29] This expression though not given explicitly in reference 23 can be deduced from it easily.

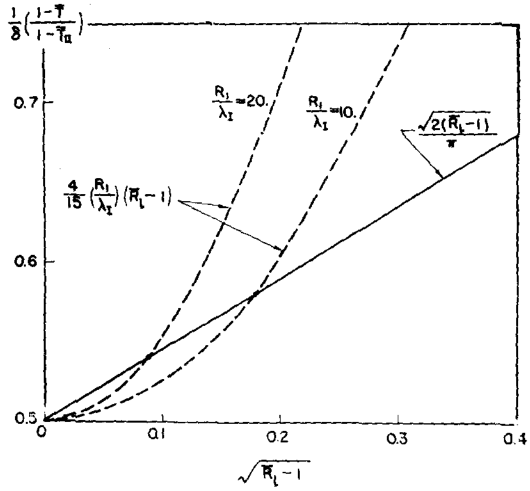

FIG. 8. Thickness of Knudsen layer.

result by a factor of 2, thus overestimating the heat loss by the same factor. Moreover, Eq. (19) yields a minimum (or maximum) somewhere in the annulus and is clearly physically unrealistic. The importance of the two-sided angular effect is fully shown here. On the other hand, at the wall of the outer cylinder the gas temperature is practically the same as the wall temperature T_{II} for any value of R_1/λ_1 if $R_2 > 20\, R_1$. In that case, as far as the outer cylinder is concerned, there is virtually no free molecular limit regardless of how large the mean free path becomes.

At the surface of the hot wire there exists a thin region usually known as a Knudsen layer at small degrees of rarefaction where the angular part (part 1, Fig. 6) effectively influences the temperature profile. This layer can be brought out by considering that

at $\bar{R}_l = 1 + \Delta$ with $\Delta \ll 1$,

$$\alpha \approx (2\,\Delta)^{\frac{1}{2}} = [2(\bar{R}_l - 1)]^{\frac{1}{2}},$$

and

$$\ln \bar{R}_l \approx \Delta = \bar{R}_l - 1,$$

so

$$\frac{1}{\delta}\left(\frac{1 - \bar{T}}{1 - \bar{T}_{II}}\right) \approx \frac{1}{2} + \frac{1}{\pi}\,[2(\bar{R}_l - 1)]^{\frac{1}{2}} + \frac{4}{15}\frac{R_1}{\lambda_I}\,(\bar{R}_l - 1). \qquad (20)$$

Figure 8 sketches the separate terms of Eq. (20). The layer is then defined by the region where the angular and logarithmic parts are of the same order of magnitude, namely,

$$(1/\pi)[2(\bar{R}_l - 1)]^{\frac{1}{2}} \approx (4/15)(R_1/\lambda_1)(\bar{R}_l - 1),$$

or

$$\bar{R}_l - 1 \approx 2.86\,(\lambda_I/R_1)^2,$$

when $(\lambda_I/R_1) \ll 1$. Thus the thickness of this Knudsen layer is proportional to the *square* of the Knudsen number.

A comparison between predicted and experimental temperature distributions has not been possible, because, up to the present time, the only temperature distribution measurements in rarefied gases were done by Lazareff,[30] and Mandell and West[31] between two parallel plates. Experiments designed to chart temperatures between two concentric cylinders have not yet been initiated.

C. "Free Molecular" Criterion

The minimum size of the mean free path required to ensure that free molecular conditions prevail in an experiment has always been a puzzling question. The choice between conditions like $\lambda_I \gg R_2 > R_1$ or $R_2 > \lambda_I \gg R_1$ is quite uncertain. The confusion can be totally avoided by considering the quantity δ. One realizes that δ is in fact the true criterion for free molecular flow; neither R_1/λ_I nor R_2/R_1 alone governs the situation, i.e., $\delta \to 1$ signifies the free molecular limit, while $\delta \to 0$ represents the continuum regime. For instance, with an apparatus of given R_2/R_1, $\delta \to 1$ can be reached by reducing the gas density or, at a given gas condition, one achieves free molecular flow by decreasing the R_2/R_1 ratio.

Referring to the definition of δ, one can now safely impose numerically that

$$(R_1/\lambda_I)\ln(R_2/R_1) \le 1/10,$$

or equivalently

$$\frac{\lambda_I}{R_2 - R_1} \ge \frac{10\ln(R_2/R_1)}{(R_2/R_1) - 1},$$

as the domain of free molecular flow, where Knudsen's formula [Eq. (17)] is valid within about 3%. On the other hand, the condition, say,

$$(4/15)(R_1/\lambda_1)\ln(R_2/R_1) \ge 20,$$

or

$$\frac{\lambda_I}{R_2 - R_1} \le \frac{1}{75}\frac{\ln(R_2/R_1)}{(R_2/R_1) - 1},$$

represents the continuum limit where the Fourier result will be correct within 5%. Figure 9 shows these

[30] P. Lazareff, Ann. Physik **37**, 233 (1912).
[31] W. Mandell and J. West, Proc. Phys. Soc. (London) **37**, 20 (1925).

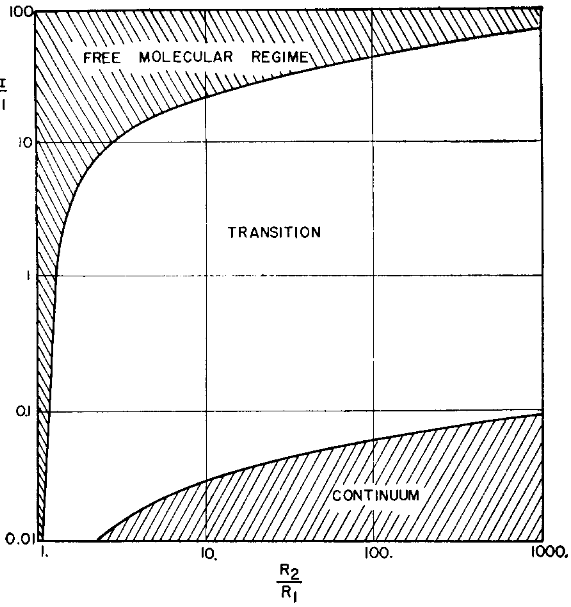

Fig. 9. Classification of density regimes.

two domains as well as the transition region for different values of R_2/R_1.

D. Fourier–Maxwell–Smoluchowski Formulation

It has been generally accepted that the Fourier relation with the Maxwell–Smoluchowski temperature-jump boundary condition would be fairly correct for gases of small degree of rarefaction. Its limit of validity for a problem involving curvatures has never been investigated. According to the relation

$$q_R = -k_c(dT/dR),$$

and the boundary conditions that

$$T_{\text{I}} - T(R_1) = -(15/8)\lambda_{\text{I}}(dT/dR)_{R=R_1},$$

and

$$T(R_2) - T_{\text{II}} = -(15/8)\lambda_{\text{I}}(dT/dR)_{R=R_2},$$

one would obtain an expression equivalent to Eq. (16) for the temperature distribution, namely,

$$\frac{1-\bar{T}}{1-\bar{T}_{\text{II}}} = \delta' + \left[1 - \left(1 + \frac{R_1}{R_2}\right)\delta'\right] \cdot \frac{\ln (R/R_1)}{\ln (R_2/R_1)}, \quad (21)$$

with

$$\delta' = \left[\left(1 + \frac{R_1}{R_2}\right) + \frac{8}{15}\frac{R_1}{\lambda_{\text{I}}}\ln\left(\frac{R_2}{R_1}\right)\right]^{-1}.$$

The ratio of heat transfer to the heat transfer in the limit $\lambda_I/R_1 \rightarrow 0$ is

$$\frac{Q}{Q_\infty} = \frac{q_R}{q_{R\infty}} = \left[1 + \frac{1 + (R_1/R_2)}{(8/15)(R_1/\lambda_{\text{I}})\ln (R_2/R_1)}\right]^{-1}. \quad (22)$$

In the case when $R_2 \approx R_1$, namely, the gap between two cylinders is small in comparison with R_1, the curvature effect then is not important and Eqs. (21) and (22) would be quite correct even at low pressures. This situation is not surprising, as we have learned from the linearized case

$$[(T_{\text{I}} - T_{\text{II}})/T_{\text{I}} \ll 1]$$

of the *plane* Couette flow problem.[6,32] There the Fourier–Maxwell–Smoluchowski result can be valid even when λ_{I} is large, at least with the minimum number of moments. However, the present type of apparatus normally has a large R_2/R_1 ratio; therefore, the heat transfer predicted by the Smoluchowski method [Eq. (22)] overshoots by a factor of 2 when $\lambda_{\text{I}} \geq R_1$, exactly as in Ai's result.[33] The present temperature profile [Eq. (21)] differs from Ai's [Eq. (19)] appreciably, because $\delta \neq \delta'$ in general. Now, one may confidently confirm a long time belief that, as far as gross quantities (like heat flux, total drag) are concerned, the Navier–Stokes–Fourier relations along with velocity-slip or temperature-jump boundary conditions would be fairly good for a *linearized problem in which all curvature effects can be considered negligible*, but

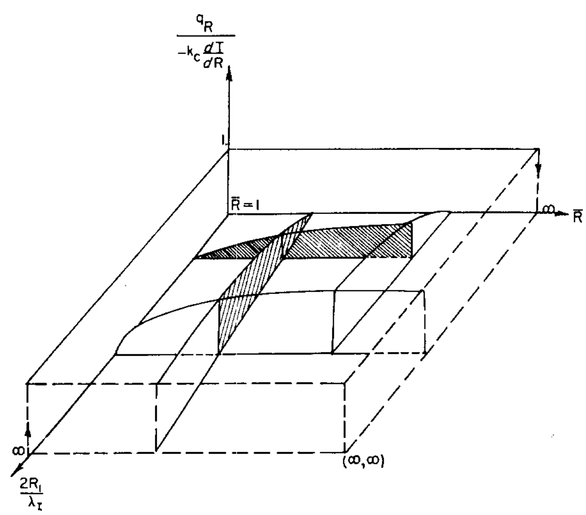

Fig. 10. Departure from Fourier's relation.

[32] H. T. Yang and L. Lees, "Plane Couette Flow at Low Mach Number According to the Kinetic Theory of Gases," GALCIT Hypersonic Research Project, Memorandum No. 36 (1957). Also, Proceedings of 5th Midwestern Conference on Fluid Mechanics (1957).

[33] Ai in his study has imposed a boundary condition $T(R_2) = T_{\text{II}}$ at all density levels, which would be true only if $R_2 > 20R_1$ (see B in Sec. V), so his solution contains an implicit assumption of $R_2/R_1 \gg 1$.

details (like velocity or temperature profile) so obtained would be open to doubt.

The domain of validity of the Fourier–Maxwell–Smoluchowski formulation for this "hot-wire" instrument now can be estimated from Eqs. (21) and (22) as

$$\lambda_I/R_1 \ll (8/15) \ln (R_2/R_1) \quad \text{if} \quad (R_2/R_1) \gg 1,$$

or numerically, say,

$$\lambda_I/R_1 < (8/150) \ln (R_2/R_1).$$

A sketch of the $q_R/[-k_c\ (dT/dR)]$ ratio (Fig. 10) based on Eqs. (6) and (14) shows that Fourier's "law" is valid either far from the center wire at any density, or everywhere at normal density. One interesting note is that the ratio falls to zero at the wire surface for finite λ_I; this behavior arises owing to the infinite temperature gradient resulting from differentiation of $\cos^{-1}(R_1/R)$. It is quite similar to the situation of having an infinite velocity gradient at the forward stagnation point of a cylinder in rarefied gas flow. The gradient becomes finite if the curvature becomes small and the cylinder is transformed to a flat disk.[34]

ACKNOWLEDGMENTS

The work discussed in this paper was carried out under the sponsorship and with the financial support of the U. S. Army Research Office and the Advanced Research Projects Agency, Contract No. DA-04-495-Ord-3231.

[34] H. Oguchi, in *Progress in Astronautics and Rocketry, Vol. 7, Hypersonic Flow Research*, edited by F. R. Riddell (Academic Press Inc., New York, 1962), pp. 13–36.