THE PHYSICS OF FLUIDS VOLUME 2, NUMBER 6 NOVEMBER-DECEMBER, 1959

Some Interior Problems of Hydromagnetics

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(Received June 22, 1959)

The static boundary problems of line currents and dipoles immersed in a perfectly conducting static fluid are considered first. The perturbing effect of moving fluid on the magnetostatic boundary about an isolated line current is then investigated. In this case, the initial circular boundary is distorted into an ellipse with major axis transverse to the direction of flow.

INTRODUCTION

It is of interest in both astronautics and geomagnetism to inquire into the mutual interaction between a magnetic field and a conducting stream. Previous investigators have considered the problem of a magnetic dipole introduced as a perturbation on an already existing uniform magnetic field in a conducting fluid.[1] We consider the limiting case of a perfectly conducting static fluid containing no initial field, but under an initial hydrostatic pressure. With these conditions a line current will produce a cylindrical cavity in the fluid, the radius of which will be given by

$$R = \mu I_0/2\pi P_\infty, \tag{1}$$

where I_0 is the current (cgs − emu units), μ the magnetic permeability, and P_∞ the fluid hydrostatic pressure. Equation (1) is self-evident from the symmetry of the situation. The hydrostatic pressure is balanced by the action of the interior magnetic field on a boundary current sheet in the conducting fluid. A magnetic dipole will result in a cavity whose boundary is less obvious. In the next section we will derive its shape.

STATIC DIPOLE CAVITY

The dipole cavity can be considered as the limit produced by two approaching line currents. When they are far enough apart, each will be surrounded by its own cylindrical cavity. This type of solution will cease when the two cylinders become tangent, producing a figure eight. Thereafter, the two cylinders will merge. The conditions to be satisfied along the entire exterior boundary are equality of hydrostatic and magnetic pressure,

$$(B_x^2 + B_y^2)/8\pi = P_\infty, \tag{2}$$

and tangency of the magnetic field,

$$dy/dx = B_y/B_x. \tag{3}$$

Of course, within the cavity, B_x and B_y will each be harmonic except at the sources ($\pm\, x_0$, 0) which approaches the origin in the limit.

To solve this problem it is convenient to consider (x, y) to be functions of (u, v) where

$$u = B_y/(B_x^2 + B_y^2)$$

and

$$v = -B_x/(B_x^2 + B_y^2).$$

Here x and y will be conjugate harmonic functions of u and v. From Eq. (2) the exterior cavity boundary will appear as a circle of radius $(8\pi P_\infty)^{-1}$ in the u-v plane. Equation (3) may be expressed as

$$-\frac{v}{u} = \frac{\dfrac{\partial x}{\partial u} + \dfrac{\partial x}{\partial v}\dfrac{dv}{du}}{\dfrac{\partial y}{\partial u} + \dfrac{\partial y}{\partial v}\dfrac{dv}{du}}. \tag{4}$$

Combining Eq. (4) with the Cauchy-Riemann conditions, and the fact that $dv/du = -u/v$ on the boundary, leads to

$$\partial x/\partial v = 0; \qquad \partial y/\partial u = 0. \tag{5}$$

These are the boundary conditions on the circle. The vertical line of contact ($0, \pm\, y_0$) will transform into a cut along the real axis of the uv plane.

Along the cut, $x = 0$ and $\partial y/\partial v = 0$; clearly there will be a square root singularity at the inner tip of the cut, where $x = y = 0$. The line current singularities will transform into the origin of the uv plane. The length of the cut will be determined by the strength of the line currents; for the dipole it extends into the origin and this case can be treated analytically.

To proceed, we expand x and y in power series about the origin as follows:

$$x = \sum_{n=1,3,\cdots}^{\infty} a_n r^{n/2} \sin\frac{n\theta}{2},$$

[1] M. G. S. el Mohandis, *Astrophys. J.* 129, 172 (1959).

$$y = \sum_{n=1,3,\cdots}^{\infty} a_n r^{n/2} \cos\frac{n\theta}{2},$$

where

$$r^2 = u^2 + v^2 \quad \text{and} \quad \theta = \tan^{-1} v/u.$$

In this form x and y automatically satisfy the slit conditions, and it remains only to satisfy conditions (5) on the circle boundary. This can be done by taking $a_3 = -a_1/3(8\pi P_\infty)^{\frac{1}{2}}$, and all the other coefficients equal to zero. The constant a_1 is, of course, determined by the dipole strength M. The coordinates of the cavity boundary may now be expressed parametrically in terms of θ as follows:

$$x = [M/(8\pi P_\infty)^{\frac{1}{2}}]^{\frac{1}{2}}\left[\sin\frac{\theta}{2} - \left(\frac{1}{3}\right)\sin\frac{3\theta}{2}\right],$$

$$y = [M/(8\pi P_\infty)^{\frac{1}{2}}]^{\frac{1}{2}}[\cos\theta/2 - (\tfrac{1}{3})\cos(3\theta/2)].$$

The boundary of the cavity surrounding the dipole is plotted in Fig. 1. It might be noted that the current density is constant at the interface.

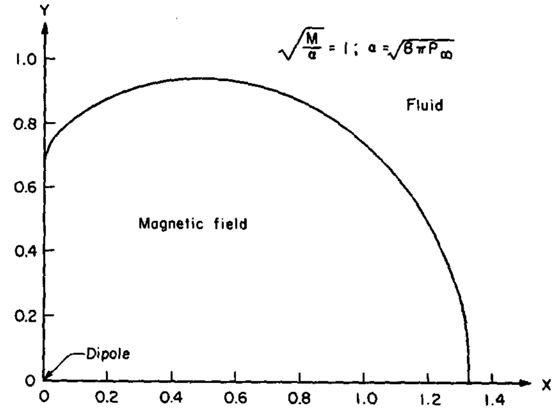

FIG. 1. Cavity surrounding a two-dimensional dipole in a perfectly conducting fluid.

PERTURBATION DUE TO A MOVING STREAM

We now inquire into the distortion of the cylindrical cavity about an isolated conductor due to an incompressible flow field. It is convenient to introduce a magnetic potential according to

$$B_\theta = -\partial\Omega/\partial r; \qquad B_r = \frac{1}{r}\frac{\partial\Omega}{\partial\theta}, \tag{6}$$

where Ω is itself harmonic in the cavity. We expand Ω and r in power series, Ω_0 and R representing the unperturbed values of the potential and cavity radius, respectively.

$$\begin{aligned} \Omega &= \Omega_0(r) + U^2\Omega_1(r, \theta), \\ r &= R_0 + U^2 g(\theta), \end{aligned} \tag{7}$$

where U is the free stream flow-velocity. The boundary conditions on the perturbed cavity wall are threefold:

tangency of the magnetic field,

$$B_r/B_\theta = (1/r)\,(dr/d\theta); \tag{8}$$

tangency of the flow,

$$q_r/q_\theta = (1/r)\,(dr/d\theta); \tag{9}$$

equality of hydrodynamic and magnetic pressures,

$$(B_\theta^2 + B_r^2)/8\pi = P_\infty + (\rho/2)(U^2 - q_r^2 - q_\theta^2), \tag{10}$$

where ρ represents the fluid density, and q_r and q_θ the components of the perturbation flow field.

By combining Eqs. (6), (7), and (10), we get

$$\Omega_1(r, \theta) = B_{\theta_0} g(\theta), \tag{11}$$

where we have neglected

$$U^2\,(\partial\Omega_1/\partial\Omega)$$

in comparison with B_{θ_0}

$$(B_{\theta_0} = 2\mu I_0/r).$$

Equations (6), (7), and (10) lead to

$$\frac{\rho}{2}\left(1 - \frac{q^2}{U^2}\right) = -\frac{B_{\theta_0}}{4\pi}\left[\frac{\partial\Omega_1}{\partial\Omega} + \frac{2\mu I}{R_0^2}\,g(\theta)\right]_{r=R_0} \tag{12}$$

to order U^4. Using Eq. (11), this becomes

$$\frac{\rho}{2}\left(1 - \frac{q^2}{U^2}\right) = -\frac{B_{\theta_0}}{4\pi}\left[\frac{\partial\Omega_1}{\partial r} + \frac{2\mu I_0}{R_0^2 B_{\theta_0}}\,\Omega_1\right]_{r=R_0}. \tag{13}$$

However, Eq. (9) leads to the result

$$q_r = (q_\theta U^2/R_0)\,dg/d\theta. \tag{14}$$

Equation (14) tells us that to order U^2, $q_r = 0$. Therefore, we may take the flow to be that around the unperturbed cylinder $r = R_0$; i.e.,

$$\frac{q^2}{U^2} = \left[1 - \frac{2R_0^2}{r^2}\,2\cos(2\theta) + \frac{R_0^4}{r^4}\right]. \tag{15}$$

Therefore, Eq. (13) leads to

$$\frac{\rho}{2}[2\cos 2\theta - 1] = -\frac{B_{\theta_0}}{4\pi}\left[\frac{\partial\Omega_1}{\partial r} + \frac{2\mu I_0}{R_0^2 B_{\theta_0}}\,\Omega_1\right]_{r=R_0}. \tag{16}$$

Since it must also satisfy Laplace's equation, Ω_1 may be taken in the form

$$\Omega_1 = C_0 + C_1 r^2 \cos(2\theta). \tag{17}$$

Substitution in Eq. (16) leads to the results

$$C_0 = \pi\rho R_0^2/\mu I_0$$
$$C_1 = 2\pi\rho/3\mu I_0. \tag{18}$$

Consequently the perturbed cavity-boundary may be expressed as

$$r = R_0\left[1 + \frac{\pi U^2 \rho R_0^2}{2\mu^2 I_0^2}\left[1 - \tfrac{2}{3}\cos(2\theta)\right]\right]. \tag{19}$$

Equation (19) tells us that the original cylindrical cavity expands slightly, and is deformed into an ellipse with its major axis transverse to the direction of flow.

CONCLUDING REMARKS

We have not discussed the stability of our solutions. However, it seems intuitively clear that the cylindrical cavity about an isolated line current will be stable, as well as the static dipole boundary.

The rather weak dependence of the dipole-cavity dimensions on P_∞ indicates that it will not be much distorted by flow. When regarded as starting points for examining the interactions between magnetic fields and moving streams, our solutions need not be regarded as applying only to an ideal conductor. For a moving stream, one need only require a high magnetic Reynolds number. The solution for the dipole cavity may then be used to estimate the extent of penetration of the magnetic field into a highly conducting moving stream. Here P_∞ could be approximately equal to the stream stagnation pressure.