Ph. 234 -- Topics in Theoretical Physics

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We shall first review rapidly the topics we covered in the Spring term last year.

No. 1. Relativistic Quantum Mechanics of a Spinless Particle

Many examples of such a particle exist in nature: $\pi^0$, $\pi^+$, $\pi^-$, $K_1^0$, $K_2^0$, $K^+$, and $K^-$. We proceeded by generalizing the Schroedinger equation to take into account the relativistic kinematics of a particle. Two modifications were made immediately: the energy zero point for a free particle is changed, and there is a new normalization. Our relativistic wave function will be $\chi(x, t)$. If we set

$$\chi = \sqrt{2m} e^{imt} \psi$$

then in the non-relativistic limit $\chi$ reduces to the ordinary $\psi$ of NR quantum mechanics. (Please note that we have adopted units such that $\hbar = c = 1$.)

To obtain the relativistic equation for $\psi$, we follow the familiar prescription:

$$\begin{align*}
E &\rightarrow i \frac{\partial}{\partial t} = \frac{\partial}{\partial x} \\
p &\rightarrow -i V = \frac{p}{\lambda}
\end{align*}$$

where $E$ denotes E operator. Remembering $E = \sqrt{p^2 + m^2}$ we write:

$$i\lambda = \sqrt{m^2 - v^2} \phi$$

This type of operator is to be interpreted in momentum space (by Fourier transforms):

$$\Phi(\vec{x}, t) = \int f(\vec{k}, t) e^{i\vec{k} \cdot \vec{x}} d^3 \vec{k}$$

$$\int m^2 - z^2 \int f(\vec{k}) e^{i\vec{k} \cdot \vec{x}} d^3 \vec{k} = \int f(\vec{k}) \sqrt{m^2 + k^2} e^{i\vec{k} \cdot \vec{x}} d^3 \vec{k}$$

$$\equiv \left( m^2 - z^2 \right)^{3/2} \int f(\vec{k}) (m^2 + k^2)^{1/2} e^{i\vec{k} \cdot \vec{x}} d^3 \vec{k}$$

where the latter form is employed in case the first does not converge. This wave equation is the square root of the Lorentz invariant Klein-Gordon equation $\Box^2 \phi = 0$. As is expected, we choose $\phi$ to be a
world scalar. We take the positive square root in the wave equation, which corresponds to choosing only positive frequencies. This procedure is relativistically invariant since if \( i p_o \) is the fourth component of a four-vector \((p, i p_o)\) for which \( p^2 - p_o^2 < 0 \), the sign of \( p_o \) is invariant under the restricted group of Lorentz transforms.

**Normalization:** Non-relativistically we can require

\[
\frac{\partial}{\partial t} \int \bar{\psi} \psi \, d^3x = 0
\]

since

\[
\frac{\partial}{\partial t} (\bar{\psi} \psi) - \frac{i}{2m} \nabla [\bar{\psi} \psi - \bar{\psi} \psi] = 0
\]

Correspondingly we note that the four-divergence of

\[
\mathcal{Q}_\nu = -i(\bar{\varphi} \sqrt{m^2 - \nabla^2} \varphi + \varphi \sqrt{m^2 - \nabla^2} \bar{\varphi})
\]

also vanishes. (Note that \( f_{,\nu} = \frac{\partial f}{\partial x} \) where \( x_1, x_2, x_3, x_4 \) are \( x, y, z, \) it). That is: \( \mathcal{Q}_{\nu,\nu} = 0 \). (Summation convention is used where it is appropriate).

This equation tells us that the integral \( \int \mathcal{Q} \, d^3x \) is a world scalar and that it is a constant in time. Thus we can define the normalization to be:

\[
| = \left( \varphi \sqrt{m^2 - \nabla^2} \varphi + \varphi \sqrt{m^2 - \nabla^2} \bar{\varphi} \bar{\varphi} \right) d^3x = 2 \int \sqrt{m^2 - \nabla^2} \varphi d^3x
\]

since the root operator is hermitian

\[
| = \frac{1}{m} \int \chi \sqrt{m^2 - \nabla^2} \chi d^3x \rightarrow \int \chi^* \chi d^3x \quad (NR)
\]

This integral is positive definite as can be seen easily by examining it in momentum space. But its integrand is not! Hence we cannot call the integrand a probability density. If the particle is charged, \( j_{\nu} = e \mathcal{Q}_{\nu} \) can be taken as a current density four-vector. Last term we constructed a positive definite quantity which we could interpret as a spatial probability density, namely

\[
2 \left( \sqrt{m^2 - \nabla^2} \varphi \right) \left( \sqrt{m^2 - \nabla^2} \bar{\varphi} \right)
\]

However the significant thing about particles is "how they interact". No particle appears to interact through a coupling involving this "position density", so it is not a fruitful concept.

Particles interact through their current densities and their stress-momentum-energy (SME) tensor among others. The non-symmetrized tensor \( T_{\mu \nu} = \delta_{\mu \nu} (m^2 \varphi^* \varphi + \varphi^* \varphi) - 2 \varphi^* \varphi_{\nu} \) can be symmetrized to
form $\theta_{uv} = \delta_{uv}(m^2 + \varphi^* + \varphi^* + \varphi^* + \varphi^* + \varphi^*).$ The symmetrization is done so that angular momentum will be conserved. The 4-4 component of this tensor is the Hamiltonian density:

$$\mathcal{H} = \Theta_{44} = m^2 \Phi^* \Phi + \nabla \Phi^* \nabla \Phi + \Phi^* \Phi$$

Thus, using the equation for $\Phi$ and integrating by parts, the total energy is

$$E = \int \Theta_{44} \, d^3x = 2 \int \Phi^* (m^2 - \nabla^2) \, \Phi \, d^3x$$

An alternative way of obtaining this is just to insert the energy operator into the normalization integral. Hence, we see also that the momentum must be:

$$P_k = \int \Theta_{k} \, d^3x = 2 \int \Phi^* \sqrt{m^2 - \nabla^2} \frac{1}{2} \frac{\partial}{\partial x_k} \Phi \, d^3x$$

It is this tensor which interacts with the gravitational field; the coupling term in the Lagrangian being $\theta_{uv} h_{uv}$ where $h_{uv}$ is the gravitational field tensor.

**Plane Waves:** Let $\sqrt{V}$ be the volume of the box representing all space. Then a normalized wave function will look like

$$\psi_k = (2E,V)^{-1/2} e^{i(k \cdot x - Et)}.$$  

**No. 2: Field Theory**

We now have to consider the possibility of creating and destroying particles. Thus we have to be able to superimpose states with different numbers of particles.

We will consider the case of a particle which is its own antiparticle, (e.g. $\pi^0$). A complete set of states for any number of such particles is:

- the vacuum $\varphi^0$
- one particle states $\varphi_k^{(1)} = (2E,V)^{-1/2} e^{i(k \cdot x - Et)}$
- two particle states $\varphi_{k,k'}^{(2)} = \frac{1}{\sqrt{2}} \left\{ \varphi_k^{(1)}(x) \varphi_{k'}^{(1)}(y) + \varphi_k^{(1)}(y) \varphi_{k'}^{(1)}(x) \right\}$

Each wave function is totally symmetric in accordance with the postulate that all particles with integral spins obey Bose-Einstein statistics. (The "proof" of this is that nature has never violated this assignment and no one has yet been able to construct a consistent theory violating this assignment).

A general state may be represented by an array (following Fock) such as

$$\begin{pmatrix} \varphi^0(t) \\ \varphi'(x,t) \\ \varphi^2(x,y,t) \\ \vdots \end{pmatrix}$$

where all entries are totally symmetric in the spatial variables. Alternatively we may simply list the expansion coefficients of the given state in terms of normalized n-particle states. The normalization condition is then $\Sigma |coefficients|^2 = 1.$
That is, we first list all the one-particle states (called levels) in our box:  
k_1, k_2, k_3, \ldots. Then \( |\psi(N_1, N_2, N_3, \ldots)\rangle \) is a normalized state with \( N_1 \) particles in level \( k_1 \), \( N_2 \) particles in level \( k_2 \), etc. The wave function associated with a state \( |\psi(1, 0, 1, 0, \ldots)\rangle \) is for example:

\[
\frac{1}{\sqrt{\lambda_k}} \left( \Phi_1(x) \Phi_3(y) + \Phi_1(x) \Phi_3(z) \right)
\]

A general state is a linear combination of our basis states:

\[
|\psi\rangle = g_0|\psi(0, 0, 0, \ldots)\rangle + g_1|\psi(1, 0, 0, \ldots)\rangle + g_2|\psi(0, 1, 0, \ldots)\rangle + \ldots + g_{11}|\psi(2, 0, 0, \ldots)\rangle + g_{12}|\psi(1, 1, 0, \ldots)\rangle + \ldots
\]

This may be represented by an array:

\[
\begin{pmatrix}
\begin{pmatrix}
g_0 \\
g_1 \\
g_2 \\
\vdots
\end{pmatrix}
\end{pmatrix}
\]

The normalization is then just:

\[
|\psi\rangle^2 + \sum_k |\psi_k(x)\rangle^2 + \sum_{k'k''} |\psi_{k'}(x', k'')\rangle^2 + \ldots = 1
\]

We can \( N_i \) the occupation number of the \( i \)-th level.

Let us now introduce creation and destruction operators for each level: \( a_n^+ \) and \( a_n \). \( a_n^+ \) is the hermitian conjugate of \( a_n \). Example: \( a_3^+ \) operates only on the third level:

\[
a_3^+ |\psi(N_1, N_2, N_3, \ldots)\rangle = \sqrt{N_3 + 1} |\psi(N_1, N_2, N_3 + 1, \ldots)\rangle
\]

also

\[
a_3 |\psi(N_1, N_2, N_3, \ldots)\rangle = \sqrt{N_3} |\psi(N_1, N_2, N_3 - 1, \ldots)\rangle
\]

\( = 0 \) if \( N_3 = 0 \) (This zero is not the vacuum state).

The operator \( a_n^+ a_n \) has as its eigenvalues the occupation numbers \( N_n \) of the \( n \)-th level. By extension of the simple harmonic oscillator treatment, we obtain the following operators:

\[
H_{\text{free}} = \sum_n E_n a_n^+ a_n \quad \text{(total energy)}
\]

and

\[
P = \sum_n k_n a_n^+ a_n \quad \text{(momentum)}
\]

\[
N = \sum_n a_n^+ a_n \quad \text{(total number of particles)}.
\]

We find that within this framework we may construct quantities which behave like classical fields in the classical limit. For example, if we define:

\[
\phi(x, t) = \sum_n a_n \varphi_n(x, t) + a_n^+ \varphi_n^*(x, t), \quad \text{and}
\]

\[
\pi(x, t) = \phi^*(x, t) = \sum_n -iE_n a_n \varphi_n(x, t) \quad \text{hermitian conjugate}
\]

We will see that they behave like fields. For these operators are sufficient to solve for each \( a_n \) and \( a_n^+ \) in terms of them. If one does this, we may then write the Hamiltonian in terms of these field operators:
\[ H_{\text{free}} = \sum_n \frac{a_n^+ a_n + a_n a_n^+}{2} - \sum_n \frac{E_n}{2} \]
\[ = \frac{1}{2} \int \left\{ (\nabla \phi)^2 + m^2 \phi^2 + \pi^2 \right\} d^3 x - \text{(infinite constant)}. \]

This is exactly the Hamiltonian for a classical field \( \phi \), with a Lagrangian density:
\[ \mathcal{L} = \frac{i}{2} \left( \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} - m^2 \phi^2 \right), \]
whence
\[ \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \]
is the field momentum conjugate to \( \phi \).

So let us compute the commutation \([n, \phi]\) by making use of the known commutation rules of \( a_n \) and \( a_n^+ \). We get as expected,
\[ \left[ \pi(x, t), \phi(x', t') \right] = -i \delta(x - x') \]

Furthermore, we see that the field operators \( \phi \) and \( \pi \) have matrix elements only between states whose number of particles differ by 1. Thus it is the existence of states with different numbers of particles, but the same energy, which give rise to a non-zero expectation value for the fields, and hence, more broadly to the existence of classical fields in nature. States which behave like classical fields cannot be eigenstates of the operator \( N \). This is analogous to the situation in the case of the N. R. harmonic oscillator potential, where states may be formed which behave like packets oscillating as a classical particle, but these states are not eigenvalues of the total energy.

The equations of motion of the field operators may be obtained by taking the commutator of the operator with the Hamiltonian. (We already know the result from our definition of \( \pi = \frac{\partial \phi}{\partial t} \), and the fact that \( \square^2 \phi = m^2 \phi \); namely, \( \pi = \frac{\partial \phi}{\partial t} \), and \( \pi' = (\nabla^2 - \gamma^2) \phi \).

Now if \( A \) is any operator, its equation of motion is \( \dot{A} = -i [A, H] \).

Therefore:
\[ \dot{\phi} = -i \left[ \phi(x, t), \frac{1}{2} \int \left\{ (\nabla \phi)(\xi, t))^2 + \pi^2(\xi, t) + m \phi^2(\xi, t) \right\} d^3 \xi \right] \]
\[ = \pi(x, t) \]
and similarly \( \dot{\pi}(x, t) = (\nabla^2 - \gamma^2) \phi(x, t) \).

Historically the field operators were not obtained in this way; rather, a classical field Lagrangian was written down, and then quantized.

**Problems.** 1. Calculate \([\phi(x, t), \phi(x', t')]\) (e.g. as a Fourier transform).

Find explicitly the case of \( m = 0 \). Also compute the vacuum expectation values of the anti-commutator, and the time-ordered product (earliest time on right, latest on the left):
\[ \langle \text{vac} | [\phi(x, t), \phi(x', t')]^+ | \text{vac} \rangle \]
\[ \langle \text{vac} | P(\phi(x, t), \phi(x', t')) | \text{vac} \rangle \]
Problem 2. What is the un-normalized amplitude in the vacuum state that \( \phi(x,t) = f(x) \), this being an arbitrary function?

Oct. 2 We want to investigate the interaction of nucleons and pions, where we will assume the elementary interaction is the Yukawa process \( N \rightarrow N + \pi \). We have already a theory for \( \pi^0 \), a particle which is its own antiparticle, providing we simply set \( m = \mu = \text{mass of } \pi^0 \) in the foregoing. Now we must generalize our field formalism to consider charged particles \( \pi^+ \) and \( \pi^- \). We will first do this in the manner we constructed the \( \pi^0 \) field, and then I want to go back and re-derive everything from the Lagrangian field formalism.

Recalling our list \( k_1, k_2, \ldots \) of the possible momenta, in this case we want to include states which have occupation numbers for either negative or positive particles. We may write the basis states of the charged pion field as:

\[
|\psi(N_1^-, N_2^-, \ldots; N_1^+, N_2^+, \ldots)\rangle;
\]

+ and - particles are distinct, so their individual wave functions are independent. We need not symmetrize between the positives and negatives.

We will introduce our creation and destruction operators for \( \pi^- \): \( b_{\lambda k}^+ \) and \( b_{\lambda k}^- \), and for \( \pi^+ \): \( c_{\lambda k}^+ \) and \( c_{\lambda k}^- \). In terms of these the free Hamiltonian for charged particles becomes:

\[
H_{\text{free}} = \sum_{\lambda k} E_{\lambda k} (b_{\lambda k}^+ b_{\lambda k}^- + c_{\lambda k}^+ c_{\lambda k}^-).
\]

Next we define our charged operators:

\[
\phi^c(x,t) = \sum_{\lambda k} \left( b_{\lambda k}^+(x,t) \phi_{\lambda k}^- + c_{\lambda k}^+(x,t) \phi_{\lambda k}^+ \right)
\]

creates positive charge, while its Hermitian conjugate:

\[
\phi^{c*}(x,t) = \sum_{\lambda k} \left( b_{\lambda k}^-(x,t) \phi_{\lambda k}^+ + c_{\lambda k}^-(x,t) \phi_{\lambda k}^- \right)
\]

creates negative charge. If one now defines \( \pi^c = \phi^c \), and \( \pi^{c*} = \phi^{c*} \), then an easy calculation gives the fundamental commutation rules:

\[
\left[ \pi^c(x,t), \phi^{c*}(y,t) \right] = -i \delta(x-y), \quad \text{and the hermitian conjugate:} \quad \left[ \pi^{c*}(x,t), \phi^c(y,t) \right] = -i \delta(x-y).
\]

Thus we get the commutation rules for conjugate field variables. Also, in terms of our field operators, we can get an expression for the Hamiltonian:

\[
H_{\text{free}} = \text{const.} + \int (\pi^c \pi^{c*} + \nabla \phi^c \cdot \nabla \phi^{c*} + \mu^2 \phi^c \phi^{c*}) d^2x.
\]

Using this Hamiltonian and the commutation rules, one can go back and construct the equations of motion for the operators. These turn out just as expected, yielding the Klein-Gordon equation.
We could use an identical formalism for neutral spinless particles which are distinct from their anti-particles. An example is afforded by $K_0$ and $\bar{K}_0$.

An alternative approach to this theory, and the historical approach is to quantize a classical field generated by a Lagrangian density. In the quantization, if the particle to be represented is its own antiparticle, the field operator will be Hermitian, and we speak of a real field. If, on the other hand, the particle has a distinct antiparticle, the hermitian conjugate of the field operator will be a different operator, and we talk of a complex field.

**Real Field:** Such a field is generated by the Lagrangian density:

$$\mathcal{L} = -\frac{1}{2} \left( \partial_t \phi + m^2 \phi \right)^2$$

From this, we immediately get the conjugate momentum $\pi = \frac{\partial L}{\partial \phi^*} = \phi^*$, with which we construct the Hamiltonian density:

$$\mathcal{H} = \phi^* \pi - \mathcal{L} = \frac{1}{2} \left( \mu^2 \phi^2 + (\nabla \phi)^2 + \phi^* \phi \right)$$

We now quantize by replacing the classical Poisson bracket relations by commutator relations; that is

$$\left[ \pi(x), \phi(y) \right] = -i \delta(x-y)$$

The resulting equations of motion clearly follow:

$$\phi(x) = -i \left[ \phi(x), \mathcal{H} \right] = \pi(x)$$

and

$$\pi(x) = -i \left[ \pi(x), \mathcal{H} \right] = (\nabla^2 - \mu^2) \phi(x) = \phi^{**}(x,t)$$

Thus we see that our field operators obey the Klein-Gordon equation.

The next thing we do is to find a representation of $\phi(x,t)$. We have two conditions which determine it: namely the fact that it is Hermitian, and its commutation rule with $\pi(x) = \phi(x)$. If we Fourier analyze $\phi(x,t)$, we write:

$$\hat{\phi}(k) = \sum_k \frac{i}{\sqrt{2}\pi} \left( a_k e^{i(k \cdot x - E_k t)} + a_k^* e^{-i(k \cdot x - E_k t)} \right)$$

When we substitute the Fourier sum into the momentum commutation, we find, to no one's surprise, that these conditions are fulfilled if:

$$\left[ a_k, a_k^* \right] = \delta_{kk'}$$

and

$$\left[ a_k, a_k^* \right] = 0 \quad \text{and} \quad \left[ a_k, a_k^* \right] = \delta_{kk'}$$

If we now substitute our expansion for $\hat{\phi}(x,t)$ into the Hamiltonian, and adjust the energy zero point, we find that:

$$\mathcal{H} = \sum_k E_k a_k^* a_{k}.$$
Complex Field. We do this one almost the same way. We treat \( \phi_c \) and \( \phi_c^* \) as independent variables and write down a Lagrangian:
\[
L = -\frac{1}{2}\dot{\phi}_c^* \dot{\phi}_c - \frac{1}{2}\phi_c \phi_c^* - \mu^2 \phi_c \phi_c^*
\]
Then:
\[
\pi_c = \frac{\delta L}{\delta \dot{\phi}_c^*} = \phi_c^* \quad \text{and} \quad \pi_c^* = \frac{\delta L}{\delta \dot{\phi}_c} = \phi_c
\]
Thus:
\[
\mathcal{H} = -L + \pi_c \phi_c^* + \pi_c^* \phi_c = \mu^2 \phi_c \phi_c^* + \nabla \phi_c^* \cdot \nabla \phi_c + \pi_c^* \pi_c
\]

To quantize we introduce the canonical variable commutators:
\[
\left[ \pi_c(x), \phi_c(y) \right] = \left[ \pi_c^+(x), \phi_c^+(y) \right] = -i \delta(x-y); \text{ etc.}
\]
Then we seek a representation of these operators; one such representation being that found previously. We could then compute the equations of motion for the operators by taking the commutator with the Hamiltonian. In this manner we are led again to operators \( b_k^+, b_k, c_k^+, c_k \), which have the properties of creation and destruction operators of independent harmonic oscillators. As an exercise it may be verified that:
\[
\mathcal{H} = \text{const.} + \int_A \mathcal{H} \, d^3 \mathbf{x} = \sum_k \left( b_k^+ b_k + c_k^+ c_k \right) E_k
\]

There is another way of looking at the complex field which is of some utility. Suppose we resolve \( \phi_c \) into its Hermitian and anti-Hermitian parts:
\[
\phi_c = \frac{\phi_c^+ + \phi_c^-}{\sqrt{2}}
\]
\[
\pi_c = \frac{\pi_c^+ - i \pi_c^-}{\sqrt{2}}
\]
Then our commutation rules reduce to:
\[
\left[ \pi_c^+ \phi_c^+, \phi_c^- \right] = -i \delta(x-y), \text{ with all other basic commutators being zero.} \quad \phi_c^+ \text{ and } \phi_c^- \text{ are now Hermitian operators, and they represent real fields.}
\]
In terms of these we can write the Hamiltonian, which is:
\[
\mathcal{H} = \text{const} + \frac{1}{2} \int \left( \pi_c^+ \pi_c^- + \mu^2 \phi_c^2 \right) d^3 \mathbf{x} + \frac{i}{2} \int \left( \pi_c^+ \phi_c^- + \phi_c^+ \pi_c^- \right) d^3 \mathbf{x} + \frac{1}{2} \int \left( \nabla \phi_c^2 + \nabla \phi_c^2 \right) d^3 \mathbf{x}
\]
Thus the complex field looks simply like a convenient way of writing the combination of two independent real fields. In fact this is true, until we put in interactions.

With the apparatus available at this point, I can only describe crudely how the nature of the interactions determine which of these two methods of description is really appropriate. Suppose we are dealing with a charged
particle pair like \( \pi^- \) and \( \pi^+ \). Then in the important interactions, e.g. \( N \rightarrow P + \pi^- \), we must create a \( \pi^- \) or destroy a \( \pi^+ \). So in the interactions, the operators which enter are \( \phi^+_\Lambda \) or \( \phi^-_\Lambda \). So it is clearly the complex field which is relevant to a description of phenomena involving such interactions.

In contrast, we might have the case of two real fields, the particles of each being their antiparticles, which happened to have the same masses. In this case the pattern of interactions is quite different. We have no existing example of this, but we do have one that exhibits both sets of properties. This is the case of \( K_0 \) and \( K^0 \). In their production, for example when \( n \rightarrow \Lambda \), on account of the conservation of strangeness, a \( K^0 \) must be created or a \( K^0 \) destroyed. Likewise when \( a \rightarrow n \), a \( K_0 \) must be created or a \( K_0 \) destroyed. Hence in these strong interactions where strangeness is conserved, the interactions involve the complex field.

However, when the \( K^0 \) decays, it does so by means of the weak interactions which do not conserve strangeness. (Such a situation cannot occur in the case of charged particles, where charge is conserved exactly.) For these weak interactions it is most convenient to work in terms of two real fields. For in the decay, the particle described by the \( \phi_x \) field, \( K_0 \), goes into two pions, while that described by the \( \phi_y \) field, \( K^0 \), goes into three pions with a different lifetime.*

We see that in the case of the \( K^0 \) particles, there are two types of interactions: one which respects the rule separating \( K^0 \) and \( K^0 \); and another which does not. In the case of charged pairs however, all the interactions respect the conservation of charge. Now, having just observed that for the \( \pi^- \) and \( \pi^+ \), the real fields are not meaningful, we shall proceed to discuss them anyway.

For all three pions, \( \pi^0, \pi^-, \pi^+ \), we have the Hamiltonian, where we call the field of \( \pi^0 \), \( \phi_2^0 \): 

\[
\hat{H} = \frac{1}{2} \int \left( \left( \frac{\partial \phi_2^0}{\partial \tau} \right)^2 + m_2^2 \phi_2^0 \right) d^3 \chi + \int \left( \left( \frac{\partial \phi_1^+}{\partial \tau} \right)^2 + \nabla \phi_1^+ \cdot \nabla \phi_1^+ + m_1^2 \phi_1^+ \phi_1^- \right) d^3 \chi
\]

Now at this point one should object that we have written the same mass for all the pions, whereas in fact \( M(\pi^0) \neq M(\pi^+) = M(\pi^-) \). It is believed by many today that this mass difference is purely electromagnetic in origin. So in the discussion of the nuclear interactions, we shall take all three masses to be equal.

If we now describe the complex field in terms of \( \phi_x \) and \( \phi_y \), we can

shorten the notation and write the three fields as a vector field. We notice
that the Hamiltonian:

$$H = \frac{1}{2} \int \left( \vec{A} \cdot \vec{B} + \nabla \phi \cdot \nabla \phi + \mu^2 \phi \cdot \phi \right) d^3x$$

is rotationally invariant in the abstract 3-space. This space is called the
isotopic spin space. The introduction of this space is useful if the rotational
symmetry of the Hamiltonian persists when the interactions are considered. This
symmetry is the symmetry associated with charge independence, which apparently
is valid for nucleons.

Let us now discuss the rotation operators in this isotopic spin space.
Let $I^z$ be the generator of an infinitesimal rotation. Then we have

$$e^{i I^z \xi} \phi^+_x = \phi^+_x \cos \xi - \frac{i}{\hbar} \phi^+_y \sin \xi$$

or,

$$i \left[ I^z, \phi^+_x \right] = -\phi^+_y$$

We see that

$$I^z = \int (\vec{\phi}_x \cdot \vec{P}) d^3x$$

generates the rotation.

$$I^z = \int (\phi_x P^y - \phi_y P^x) d^3x$$

We can rewrite $I^z$ now in terms of our creation and destruction operators. If
this is done the result becomes simply

$$I^z = \sum_k \left( c^+_k c^+_k - b^+_k b^+_k \right)$$

+1 for each $\pi^+$ particle

0 for each $\pi^0$ particle

-1 for each $\pi^-$ particle.

For example, the average value of $I^z$ in a state gives the average number of $\pi^+$
minus the average number of $\pi^-$ in the field state. In the case of the pion
triplet, the isotopic spin component $I^z$ is identical with the operator for total
charge. This identity is not always true however. The reason is that the pion
triplet is symmetrically distributed above zero charge. For nucleons this is
not true.

No. 3: Nucleon-Pion Interactions

We now shall begin an approximate discussion of the interactions of pions
with nucleons. To do this we shall suppose the nucleon to be moving slowly so
the Dirac treatment can be neglected, and we shall ignore complications induced
by a nucleon structure due to mesonic interactions other than those of the meson
we are looking at. By this approximate theory we may treat $\pi$-nucleon scattering
and $\pi$ photoproduction at low energies fairly well. We can also get a plausible
nuclear force between two nucleons at rest. But in doing so, we must "cheat" a little bit, because it is wrong to suppose the nucleons to be at rest. We must introduce "non-local" interactions — the interaction term must be spread out around the nucleon by at least its Compton wave length. This spreading is a smearing of the nucleonic core; the cloud of virtual mesons surrounding the core yields an additional smearing of the nucleon. This is "cheating" because all relativistic theories use only local interactions. The non-local nature results from a high-momentum cutoff in the interaction.

This core itself we will assume distributed about the "position" $x_n$ by a source function

$$\rho(x - x_n) = (2\pi)^{-3} \int v(k) e^{ik(x-x_n)} \, d^3k,$$

where $v \to 1$ for small $|k|$ and $v \to 0$ for large $|k|$. We have a great variety of things to try as couplings between the $n$-nucleon fields. However, fortunately, the choice is narrowed by experiment. The fact that pions are emitted in P-states requires that the pion be a pseudo-scalar particle. We will suppose, following Yukawa, that pion's are emitted in elementary nuclear acts singly. This has not been completely confirmed by experiment however. This implies that the coupling term in the Hamiltonian will be linear in $\tilde{\phi}$. The coupling will be proportional to $\tilde{\phi}$ evaluated in the region of the nucleon positions $x_n$. We have to arrange it so that charge is conserved, also it has to be right with respect to space symmetry and symmetry in isotopic spin space. If we let the spin operators and isotopic spin operators of the $n^{th}$ nucleon core be $\sigma_i^{(\mu)}$ and $\xi_i^{(\mu)}$, then the expression looks like:

$$\sum_n \frac{f_0}{M} \sum_i \sum_{j=1}^3 \int \frac{\partial \phi_i(x)}{\partial x_j} \rho(x-x_n) \xi_i^{(\mu)} \sigma_i^{(\mu)} \, d^3x.$$

**Oct. 7**

**Nature of the Interaction Hamiltonian**

In the last lecture we set up the Hamiltonian for mesons without interactions:

$$H_{\text{free}} = \frac{3}{2} \sum_i \left[ \left( \frac{\hbar^2}{2m} + (\nabla \phi_i)^2 + \mu^2 \phi_i^2 \right) \right] d^3x + \text{const}. $$

We then said that the interaction Hamiltonian would be linear in $\tilde{\phi}(x)$ and $\rho(|x|)$, the form factor for the nucleon. We decided that $H_{\text{coupling}}$ would be of the form

$$H_{\text{coupling}} = \frac{f_0}{M} \sum_i \sum_{j=1}^3 \int \frac{\partial \phi_i(x)}{\partial x_j} \rho(|x|) \xi_i \sigma_i \, d^3x.$$
for a single nucleon at the origin of coordinates. The Fourier transform of the nuclear coupling density $\rho(|x|)$ has a cut-off for high momentum values; we shall take the cut-off to be of the order of the mass of the nucleon, that is roughly $6.75\,\mu$. 

To determine the exact form of the interaction Hamiltonian we have to study its dependence upon the charge. We should like it to be invariant under rotations in isotopic spin space. We discussed the rotation operators in I-spin space for pions; they turned out to be represented by

$$\vec{I}^I = \int \vec{\phi} \times \vec{n} \, d^3 \chi$$

We can also put these into a form involving only the destruction and creation operators. When we do this $I^I_{\zeta \zeta}$ turns out to be represented simply by

$$\Sigma_k (c_k^+ a_k - b_k^+ b_k).$$

We see now that $I^I_{\zeta}$ simply measures the total charge of each state. For a single pion there are three charge states; thus we want to represent the pion by a triplet in I-spin space. In order to have rotational invariance in I-space we have to apply this concept also to all other fields and particles to which we couple the pion field. It is clear that for the nucleon we want a doublet to represent its charge states p or n, so we take $I^N_{\zeta} = 1/2$. For the proton we define $I^N_{\zeta} = 1/2$; for the neutron, $I^N_{\zeta} = -1/2$, and we see that the charge in units of $\epsilon$ is given by the operator $I^N_{\zeta} + 1/2$. An irreducible representation of degree 2 of the isotopic spin matrices is given, as we know, by the Pauli matrices, which, in this case, we designate by $(\hat{\gamma}_x, \hat{\gamma}_y, \hat{\gamma}_z)$.

$$\hat{\gamma}_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\gamma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\gamma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{\hat{I}^N_{\zeta}}{\zeta} = \frac{1}{\iota} \hat{\gamma}_z$$

It is evident that $\hat{\gamma}_z$ operating on a nucleon state does not change its charge, whereas both $\hat{\gamma}_x$ and $\hat{\gamma}_y$ interchange the proton and neutron states of the nucleon. Frequently it is convenient to employ

$$\hat{\gamma}_+ = \frac{\hat{\gamma}_x + i \hat{\gamma}_y}{2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\gamma}_- = \frac{\hat{\gamma}_x - i \hat{\gamma}_y}{2} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

as raising and lowering operators respectively for the nucleon charge. Besides the two charge states of the nucleon, it should be kept in mind that we have two spin states of the nucleon. For this degree of freedom we have another set of operators, the spin operators $\hat{\gamma}_z/2$. Our complete set of states for a system consisting of a stationary nucleon and a meson field can be represented as the direct product of a complete set of states for the stationary nucleon:

$$|\Psi^{+}_p\rangle, \quad |\Psi^{+}_p\rangle, \quad |\Psi^{+}_n\rangle, \quad \text{and} \quad |\Psi^{+}_n\rangle,$$
and our complete set of states for the meson field; as follows:

\[ | \psi^{(N)}(s) \rangle \equiv \left( \begin{array}{c} N_1^+, N_2^+, \cdots ; N_1^0, N_2^0, \cdots ; N_1^-, N_2^-, \cdots \end{array} \right) \]

With this choice of basis states, the \( I^N \) operators act only upon the \( (N) \) index; the spin operators \( \phi^\sigma \) change only the \( (s) \) index; while the field operators affect only the occupation numbers \( N_i \).

Our entire theory is to be invariant under rotations in isotopic spin space. The situation here is analogous to that in non-relativistic QM where we consider rotations in physical space. In that case our complete set of states could be taken as the direct product of a complete set of states for a spinless particle with a complete set of spin states. The unitary transformation yielding a rotation was then represented by the product of two unitary transformations, one of them affecting only the spin states, and the other acting upon only the states of a spinless particle. That is, if we consider a rotation about the \( z \)-axis by \( \theta \), the unitary transformation is:

\[ e^{i\theta J^z} = e^{i\theta L^z} e^{i\theta S^z} \]

and hence \( \tilde{J} = \tilde{L} + \tilde{S} \).

Now in this NR case the theory is to be invariant under rotations, and hence the Hamiltonian must commute with the generator of the rotation \( \tilde{J} \), the total angular momentum. Now in our case, the Hamiltonian must be invariant under rotations in isospin space, where the generator for a rotation is the total isotopic spin:

\[ \tilde{I} = \tilde{I}^N + \tilde{I}^I \]

and the unitary transformation for a rotation about, for example, the \( z \)-axis is just:

\[ e^{i\theta I^z} = e^{i\theta I^N} e^{i\theta I^I} \]

We must arrange the coupling Hamiltonian so that it is an invariant in I-spin space. We have only two sets of operators which act in this space: \( \phi \) for the meson field, and \( \chi \) for the stationary nucleon. Under rotations these transform like vectors. The only invariant we can form from them which is linear in \( \phi \) is obviously

\[ \tilde{\chi} \cdot \phi = \gamma_z \phi_z + \gamma_x \phi_x + \gamma_y \phi_y = \gamma_z \phi_z + \sqrt{2} \gamma_+ \phi_+ + \sqrt{2} \gamma_- \phi_- \]

Let us look at what each of the terms in the latter expression allow. The first term allows the neutron both to emit or absorb a \( \pi^0 \) with a relative amplitude \(-1\), and permits a proton core to emit or absorb a \( \pi^0 \) with a relative amplitude \(+1\). The second term gives the following two processes both with amplitude \( \sqrt{2} \):

\[ n \rightarrow p + \pi^-, \text{ and } n + \pi^+ \rightarrow p. \]

The final term induces the transitions
p → π⁺ + n, and p + π⁻ → n: these also have a relative amplitude of \(\sqrt{2}\).
Thus we have eight elementary processes in which a meson is created or destroyed. The relative amplitudes of these processes are characteristics of a theory of meson-nucleon interactions.

The ratios -1: 1: \(\sqrt{2}\): \(\sqrt{2}\) are characteristic of the charge independence of meson-nucleon interactions. (It may be noted that only three of these coefficients are independent since the hermiticity of the Hamiltonian requires that the third and fourth numbers be equal.) Since we are going to spend some time working with this theory of nucleon-meson interactions, it is perhaps appropriate to look briefly at some of the evidence for this concept. The original idea was generated in the 30's by an inspection of some data in nuclear physics, principally by the similarity in p-p and p-n scattering and the observation that the low energy spectrums of mirror nuclei are quite similar, and the difference in masses can be accounted for by the difference in electrostatic energies. However, physicists were not entirely convinced of the validity of the concept until after the war when meson-nucleon scattering began to be studied.

In 1952, several groups were investigating π-nucleon scattering. It was found that the results of the experiments could be analyzed quite well under the hypothesis of charge independence, or isotropy in isotopic spin space. It was soon found that the scattering could be analyzed in terms of only two amplitudes which are functions of space and spin. Let us look at the processes which might occur in the scattering of pions on nucleons. There are eight of these at low energies:

\[
\begin{align*}
\pi^+ + p &\leftrightarrow \pi^+ + p \\
\pi^0 + p &\leftrightarrow \pi^+ + n \\
\pi^- + p &\leftrightarrow \pi^- + p \\
\pi^0 + n &\leftrightarrow \pi^0 + n \\
\pi^0 + p &\leftrightarrow \pi^0 + p \\
\pi^- + p &\leftrightarrow \pi^0 + n \\
\pi^+ + n &\leftrightarrow \pi^+ + n \\
\pi^- + n &\leftrightarrow \pi^- + n.
\end{align*}
\]

However, of these eight only three are experimentally observable:

\[
\begin{align*}
\pi^+ + p &\rightarrow \pi^+ + p \\
\pi^- + p &\rightarrow \pi^- + p \\
\pi^- + p &\rightarrow \pi^0 + n.
\end{align*}
\]

Assuming isospin isotropy, it is almost obvious that there are only two independent amplitudes for all the processes, since there are only two different cases to consider: the case when the system has I = 1/2, and the case of I = 3/2. Charge independence was not completely believed until the results on scattering experiments were in, which confirmed this statement. We shall now look into the technical evaluation of the amplitudes, and in this process we shall show how the amplitude for each is just a linear combination of two basic amplitudes.
For reactions involving only a single meson and a nucleon core we are interested in matrix elements of the following form:

\[ \langle s_f \tau_i \tau_f^N | I, I^N \rangle \langle H_{\text{interaction}} | s_i \tau_i \tau_i^N \rangle \]

Such a matrix element can be expanded now in terms of a complete set which are eigenvectors of \( \tau^2 \) and \( I_z \), and becomes just:

\[
\sum \langle I_i, z_i; \tau_i^N \rangle \langle s_f, I_f \rangle \langle H_{\text{interaction}} | s_i, I_i \rangle \langle I_i, z_i; \tau_i^N \rangle \langle I_f, z_f; \tau_f^N \rangle \]

Now it should be realized that since the \( H' \) is invariant in isotropic spin spaces, the center matrix element vanishes unless \( I' = I'' \) and \( I'_z = I''_z \), and furthermore, for a given value of \( I' \) the matrix element is independent of the value of \( I'_z \). The center matrix element is proportional to \( f(I) \), the scattering lengths for the isotopic spin states. The \( f(I) \) are of course functions of the initial and final values of both the nucleon spin and the meson momentum.

All that remains is to calculate the matrix elements of the form:

\[
\langle 1, z, I \rangle | I_i, z_i, \tau_i^N \rangle \]

These are identical to the Clebsch-Gordan coefficients for the addition of ordinary angular momenta. There is only one complication: to employ these we must adhere to a sign convention (Condon, Shortley). Under this convention we have:

\[
| \tau K^+ \rangle = \frac{1}{\sqrt{2}} | \tau^+ \rangle \quad \text{and} \quad | \tau K^- \rangle = \frac{1}{\sqrt{2}} | \tau^- \rangle
\]

and

\[
| \tau K^0 \rangle = | \tau^0 \rangle = | \tau K^0 \rangle = \frac{1}{\sqrt{2}} \left( \tau^+ \right)
\]

We can now write down all the eigenstates of \( I \) and \( I_z \) of a meson-nucleon system. We shall write

\[
| \frac{1}{2}, \frac{1}{2}; 1, 1 \rangle = | \tau^+ \rangle \sim - C^+ \left| \Gamma^+_{\nu A c} \right|
\]

We write \( \sim \) instead of \( = \) since the latter represents a bare proton and a meson, while the former is to signify a real proton and an extra \( \pi^+ \). Our table of coefficients then gives:

(See next page for table)
\[ \begin{align*}
|I, I_2\rangle &= |\frac{3}{2}, \frac{3}{2}\rangle = |1^+, 1^+; \frac{1}{2}, \frac{1}{2}\rangle = |\frac{1}{2}, \frac{1}{2}; 1, 1\rangle = |p \pi^+\rangle \\
|\frac{3}{2}, \frac{1}{2}\rangle &= \sqrt{\frac{1}{3}} |\frac{1}{2}, \frac{1}{2}; 0, 0\rangle + \sqrt{\frac{2}{3}} |\frac{1}{2}, -\frac{1}{2}; 0, 1\rangle = \sqrt{\frac{1}{3}} |p \pi^0\rangle + \sqrt{\frac{2}{3}} |n \pi^+\rangle \\
|\frac{3}{2}, -\frac{1}{2}\rangle &= -\sqrt{\frac{1}{3}} |\frac{1}{2}, \frac{1}{2}; 0, 0\rangle + \sqrt{\frac{2}{3}} |\frac{1}{2}, -\frac{1}{2}; 0, 1\rangle = -\sqrt{\frac{1}{3}} |p \pi^0\rangle + \sqrt{\frac{2}{3}} |n \pi^+\rangle \\
|\frac{3}{2}, -\frac{3}{2}\rangle &= |\frac{1}{2}, -\frac{1}{2}; 1, -1\rangle = |n \pi^-\rangle \\
|\frac{1}{2}, \frac{1}{2}\rangle &= -\sqrt{\frac{1}{3}} |\frac{1}{2}, \frac{1}{2}; 1, 0\rangle + \sqrt{\frac{2}{3}} |\frac{1}{2}, -\frac{1}{2}; 1, 1\rangle = -\sqrt{\frac{1}{3}} |p \pi^0\rangle + \sqrt{\frac{2}{3}} |n \pi^+\rangle \\
|\frac{1}{2}, -\frac{1}{2}\rangle &= -\sqrt{\frac{1}{3}} |\frac{1}{2}, \frac{1}{2}; 1, 0\rangle + \sqrt{\frac{2}{3}} |\frac{1}{2}, -\frac{1}{2}; 1, 1\rangle = -\sqrt{\frac{1}{3}} |p \pi^0\rangle + \sqrt{\frac{2}{3}} |n \pi^+\rangle
\end{align*} \]

The amplitude for any of the eight scattering reactions indicated evidently depends linearly on only two scattering lengths \( f(\frac{3}{2}) \) and \( f^{1/2} \). We now have set up the machinery to get those coefficients. As an example we can write down the amplitude for the charge-exchange process: \( n^- p \rightarrow \pi^0 n \). It is:

\[
\begin{align*}
\mathcal{f}(n^- p \rightarrow \pi^0 n) &= -\sqrt{\frac{5}{3}} f^{1/2} \sqrt{\frac{3}{2}} + \sqrt{\frac{2}{3}} f^{3/2} \sqrt{\frac{2}{2}} = \frac{\sqrt{2}}{3} \left( f^{3/2} - f^{1/2} \right)
\end{align*}
\]

Note that if \( f^{3/2} = f^{1/2} \), the scattering would be independent of the isotopic spin of the system, and hence there would be no charge-exchange scattering. Analogously, there is no NR spin-flip scattering if there is no difference between the scattering lengths for \( \frac{2}{3} \) and \( \frac{1}{2} \). As another example, we may write down by inspection the scattering amplitude for the elastic scattering process: \( \pi^- p \rightarrow \pi^- p \). We may finally write down the amplitude for the third of the physically observable processes: \( \pi^- p \rightarrow \pi^- p \). They are:

\[
\begin{align*}
\mathcal{f}(n^- p \rightarrow n^- p) &= \frac{2}{3} f^{1/2} + \frac{1}{3} f^{3/2} \\
\mathcal{f}(n^+ p \rightarrow n^+ p) &= f^{3/2}
\end{align*}
\]

**Problem 3:** Consider \( \frac{1}{2} \) \( \bar{\chi}^2 = (\bar{\chi}^\pi + \bar{\chi}^N)^2 \). Show explicitly

\[
\bar{\chi}^\pi \left( -\sqrt{\frac{2}{3}} |\bar{\pi} p\rangle + \sqrt{\frac{1}{3}} |\pi^0 n\rangle \right) = \frac{3}{4} \left( -\sqrt{\frac{2}{3}} |\bar{\pi} p\rangle + \sqrt{\frac{1}{3}} |\pi^0 n\rangle \right)
\]

Before the correct theory of mesons was known, i.e. before the \( \pi^0 \) was found, a "meson-pair" theory was popular, which explained nuclear forces as being due to the exchange of a pair of fermion mesons, so that \( \mu \) mesons could be used. However the theory predicted strong nuclear scattering of \( \mu \)'s, and had to be abandoned because, as is well known, the coupling between \( \mu \)'s and nucleons is very small.
In response to a question: If we had not found the \( \pi^0 \) experimentally, assuming that the coupling would remain linear, we could not have represented the \( \pi^\pm \) states of a pion by a spinor in isospin space, and formed an interaction which would preserve the number of nucleons. For there is no operator in isospin space acting on the nucleon which transforms like a spinor.

There are two other theories which do not correspond to nature but are nevertheless interesting. Suppose that only the neutral pion existed; then we would have the coupling Hamiltonian of the form \( \gamma \phi \). On the other hand, if we had only charged pions, the coupling would be of the form:

\[
\sqrt{2} \left( \gamma + \phi^+ \right) \]

The theories are wrong, but it is convenient to try out mathematical procedures on them as exercises because they are simpler than a correct theory.

So much for the charge dependence of the theory; now we want to discuss the space dependence of the interaction, i.e. the scalarity or the pseudo-scalarity of the coupling. Originally the chief piece of evidence for the fact that the spin of the pion is zero came from the ratio of the statistical weights for a pair of inverse reactions:

\[
\begin{align*}
p + p & \rightarrow \pi^+ + d \\
\pi^+ + d & \rightarrow p + p.
\end{align*}
\]

The original experiment gave \( S_\pi = -0.2 \pm 0.4 \). But today the evidence is overwhelming that the pion is spinless.

A rather abstruse question remains: What is the intrinsic parity of the pion-nucleon coupling? Is it scalar or pseudo-scalar? The best way to understand the physical significance of the fact is to look at the experiment which determined the answer. It was observed that the reaction \( \pi^- + d \rightarrow n + n \) occurs quite often when the \( \pi^- \) is bound by the deuteron. What must be the nature of the coupling for this process to occur?

We assume that the \( \pi^- \) has fallen to an atomic \( S \) state before it is captured by the proton in deuterium. The deuteron is known to be a triplet state with \( J = 1 \). Its ground state is mostly \( ^3 S_1 \) with a small admixture of \( ^3 D_1 \); so the deuteron has parity (+), assuming that a nucleon has an intrinsic parity (+). The pion-deuteron \( S \)-state has (+) parity, so that if the pion-nucleon coupling Hamiltonian does not change the parity of the system, then the two neutron system must have (+) parity.

But, assuming the conservation of angular momentum, the two neutron can only be in the following two states if the parity is (+): \( ^3 S_1 \) or \( ^3 D_1 \). But such
But such states are totally symmetric, and two neutrons are forbidden to occupy them. With the Pauli principle in mind, it is seen at once that the two neutrons must be in a \(^3\)P\(_1\)-state which has parity (\(-\)). Hence the coupling Hamiltonian changes the parity of the initial and final states. This result can be expressed slightly differently by saying that the pion has intrinsic parity (\(-\)) and that parity is conserved in any nucleon-meson interaction, i.e. the pion is pseudoscalar.

However it is necessary to go back and check our assumption that the absorption of the \(\pi^-\) occurs from an S-state and not an atomic P-state. This was checked by getting the strength of the pion-nucleon interaction in a P-state from low energy scattering experiments. With this strength it was then shown that X-ray emission was strongly favored over absorption in the P-state. Hence almost all of the absorptions occur from S-states. The reaction can also go to \(n + n + \gamma\), and \(n + n + \pi^0\) with almost equal probabilities, but with very small probability compared to \(n + n\).

It is a direct consequence of the pseudoscalarity of the pion-nucleon coupling that a single pion is always emitted into a p-state. That is, in the process: \(N \rightarrow N + \pi\), the \(\pi\) is in a \(p_{1/2}\) state. This holds as long as we do not consider nucleon pair production.

This tells us what we need to know about the form of the meson-nucleon coupling Hamiltonian in isospin-space and physical space. We know how the spatial coupling must be pseudo-scalar. Hence we must introduce an axial vector into the coupling. This vector operator would be the spin operator for the nucleon. We could take \(\vec{\sigma} \cdot \nabla\) or \(\vec{\sigma} \cdot \vec{r}\), but the latter would vanish in the limit of local interactions, so we take the form \(\vec{\sigma} \cdot \nabla\). Hence the coupling Hamiltonian is:

\[
\frac{f_0}{\lambda} \sum_j \phi_j < \vec{\tau} \cdot \nabla \phi_j > \int \frac{\partial \phi(\xi)}{\partial \xi_j} \phi(\xi) d^3 \xi
\]

The \(f_0\) is the bare coupling constant in this theory; later on we shall get another \(f\).

Let us recall that we have a mathematically interesting charged scalar theory of \(\pi^+\) and \(\pi^-\).

\[
H_{\text{free}} = \int \left( (\pi^+ c \bar{\pi} c + \nabla \phi^+_c \cdot \nabla \phi^+_c + \mu^+ \phi^+_c \phi^+ c) d^3 \xi \right)
\]

\[
H_{\text{coupling}} = g_0 \left\{ \zeta_+ \int \phi^+_c(\xi) \rho(\xi) d^3 \xi + \zeta_- \int \phi^- c(\xi) \rho(\xi) d^3 \xi \right\}
\]

Now as a continuing problem, I would like you to carry out all the developments we make on the pseudoscalar theory for this charged scalar theory also.
Ph 234

October 9, 1958

No. 4: Low Energy π-N Scattering

We have set up a charge-independent, pseudoscalar theory of the interaction of mesons with stationary nucleons. Let us now start to calculate with this theory. In particular, let us now consider problems involving only one stationary nucleon. We shall calculate, for example, the scattering of a single meson by a nucleon. Later on, we will consider problems involving several nucleons, including the problem of nuclear forces.

Last time we saw that in the elementary act, \( N \rightarrow N + \pi \), the pion involved must be in a p-state. This result comes simply from the conservation of angular momentum and of parity in the Yukawa process. It is of course a direct consequence mathematically of the form of our interaction Hamiltonian. Let us demonstrate this fact.

The total Hamiltonian is: \( H = H^{\text{coup.}} + H^{\text{free}} \), where \( M \) is the "mass of the bare nucleon". Last time we set up \( H^{\text{coup.}} \):

\[
H^{\text{coup.}} = \frac{f}{\mu} \int \sigma^i \tau^i \frac{\partial}{\partial \chi_j} \rho(x) \, d^3 \chi
\]

Now we had been expanding the field operators in a complete set of functions representing levels with definite energies and linear momenta \( k \):

\[
\phi_i = \sum_k \lambda_i^k \phi_k \frac{\lambda_i^k}{\sqrt{2 \xi_k V}} + H. A.
\]

However, we can equally well expand our field operators in terms of a complete set of functions which are eigenvalues of the energy, angular momentum \( \ell \), and its z-component \( m \):

\[
\phi_i(x) = \sum_{k \ell m} \xi_{k \ell m} \phi_{k \ell m} \frac{\phi_{k \ell m}}{\sqrt{2 \xi_k V}} + H. A.
\]

If we do this and substitute our expression for the field operators into the coupling Hamiltonian, it is mathematically immediately evident that, in this theory, only the terms with \( \ell = 1 \) do not yield zero upon doing the integration. Therefore only p-wave mesons are coupled. In this theory all the mesons with values of the angular momentum other than 1 behave as if they are free. This is false physically. Experimentally the p-wave scattering is dominant at low energies, but there is also a small amount of s-wave interaction. We are not accounting for the s-wave scattering in this model. The s-wave scattering has
has not yet been completely accounted for by theory. When anti-nucleons or pair-formation is taken into account, one gets s-wave scattering. However, no one has been able to account for the observed amount up to now.

We can describe the elastic scattering, in this theory, by just four numbers, the four phase shifts corresponding to the four sets of allowed values of the total angular momentum $J$ and the total isotopic spin $I$. The standard notation for these numbers is due to Fermi; they are:

$$\delta_{33}, \delta_{31}, \delta_{13}, \delta_{11}, \ldots$$

where the first subscript is $2I$ and the last $2J$. In our particular model, though not in general, two of these are equal: $\delta_{31} = \delta_{13}$. This can be made plausible by observing that essentially the interaction Hamiltonian is invariant in a certain sense under the interchange of $\pi_{\uparrow}$ and $\pi_{\downarrow}$. It turns out that all the answers in this formalism are unchanged if you interchange the values of $J$ and $I$. This is not a proof, but the result may be verified in the perturbation calculations we shall make. The phase shifts are real as long as the scattering is elastic, i.e., at low energies where nothing else can occur. For the time being, we shall restrict ourselves to this case.

We will calculate the scattering in terms of the $R$-matrix:

$$R_{fi} = \sum_{m} \frac{H_{fm}^* H_{mi}^*}{E_f + i\epsilon - E_m} + \cdots$$

However, all the experimental results are expressed in terms of phase shifts. We therefore want to be able to convert our answers to phase shifts. In terms of the phase shifts:

$$f(k,\theta) = \frac{1}{k} \sum_{l} (2l+1) P_{2l+1}(\cos \theta) e^{i\delta_{2l}} \sin \delta_{2l}$$

where $f(k,\theta)$ is the scattering amplitude. In the NR approximation, we found last year that

$$f(k,\theta) = -\frac{mV}{2m} R_{fi}$$

We can generalize this now to a relativistic expression by using the Golden Rule:

$$\frac{d\sigma}{d\omega} = |f^2| = \frac{|R|^2}{K/\omega} \frac{V^2}{(2\pi)^3} \frac{K^2 dK}{d\omega} = \frac{|R|^2 V^2 \omega^2}{(2\pi)^2}$$

since the Einstein relation gives $\omega d\omega = k dk$. Thus the relativistic generalization
is obtained simply by replacing $m$ by $\omega$:

$$f(k, \theta) = - R_{k'k} \frac{\omega \sqrt{2\pi}}{2\pi}$$

The R-matrix is determined in form by the parts available for constructing it. If we are considering the scattering of a meson with initial momentum $k$ and charge index $(i)$ to a final momentum $k'$ and charge index $(f)$, on the energy shell, the R matrix will take the form:

$$R_{k'k;fi} = F_1(k) \left( \bar{\sigma} \cdot \bar{k} \cdot \sigma \cdot k \right) \tau_i \tau_f$$

$$+ F_2(k) \left( \bar{\sigma} \cdot \bar{k} \cdot \sigma \cdot k \right) \tau_i \tau_f$$

$$+ F_3(k) \left( \bar{\sigma} \cdot \bar{k} \cdot \sigma \cdot k \right) \tau_i \tau_f$$

Due to that symmetry between the interchange of spatial and isospin subscripts in our model, it turns out that there are only three arbitrary functions $F(k)$ in the R-matrix. In a more general model, one surmises that there will be four functions $F(k)$: One for each of the four possible orderings of the relevant spin and isospin operators.

In the absence of nucleon spin and isospin, the form that the R-matrix can take involves only one function $F(k)$. We will have

$$R_{k'k} = F(k) \cdot k' \cdot k$$

In this case it is easy to write down the phase shift by writing out the scattering length $f(k, \theta)$:

$$f(k, \theta) = \frac{3}{k} \cos \Theta \ e^{i \delta_3} \sin \delta_3 = - \frac{\omega \sqrt{2\pi}}{2\pi} F(k) \cos \Theta K^2$$

from which we have:

$$e^{i \delta_3} \sin \delta_3 = - \frac{\omega K^3 \sqrt{2\pi}}{6 \pi} F(k)$$

Problem 4: Show that, taking into account spin and isospin, the phase shift relations are:

$$e^{i\delta_3} \sin \delta_3 = - (\omega K^3 \sqrt{2\pi} / 6 \pi) (4 F_3)$$

$$e^{i\delta_3} \sin \delta_3 = e^{i\delta_3} \sin \delta_3 = - (\omega K^3 \sqrt{2\pi} / 6 \pi) (6 F_2 - 2 F_3)$$

$$e^{i\delta_3} \sin \delta_3 = - (\omega K^3 \sqrt{2\pi} / 6 \pi) (4 F_1 - 6 F_2 + F_3)$$

One notices readily that the sums of the coefficients of the $F_i$'s is always four. This is to be expected, since if the $F_i$'s are all equal, the
R-matrix contains no spin nor isospin operators (except unit operators) and simply reduces to $4 F_{1}(k) k' \cdot k$.

**Perturbation Calculation of π-N Scattering**

Last year we derived the formalism of calculating scattering by means of the R-matrix. In field theory, however, things are more complicated, and we will have to modify the theory of using the R-matrix. We can use what we know to get the lowest order term in the perturbation calculation, and let us proceed now to do this. Later, when we come to calculate higher order corrections, we shall put in the necessary modifications.

For the time being we will take the $H'$ occurring in the perturbation expansion of the R-matrix to be the coupling Hamiltonian: $H_{\text{coup}}$. (This choice will be modified later.) Then it is easy to see that the first order term in the R-matrix is zero, since $H_{\text{coup}}$ is linear in $\phi$, and therefore creates or destroys one meson. But the initial and final state both have one meson, hence $(H'_{\text{coup}} f) i = 0$. The second order term though does contribute, by way of 2 or 0 meson intermediate states. Indeed, it turns out that, when the correct $H'$ is employed, only the even orders give non-vanishing contributions.

Let us expand $\phi$ in plane waves. Then by Fourier transform, the coupling Hamiltonian is:

$$H_{\text{coup}} = \int \frac{d^3 \kappa}{(2\pi)^3} \frac{\varphi(\kappa)}{\sqrt{2\omega_\kappa}} \left( \tilde{\sigma}_\lambda^\dagger \tilde{\kappa}^\dagger \tilde{\chi}_m^\lambda \tilde{c}_m^\lambda - \tilde{\sigma}_\lambda \tilde{\kappa} \tilde{\chi}_m^\lambda \tilde{c}_m^\lambda \right)$$

In calculating it is convenient to represent our matrix elements by diagrams. Since in the matrix elements we write the initial state on the right and the final state on the left, we shall draw the diagrams with time proceeding from right to left also. Thus the R-matrix to second order is diagrammed:

$$R_{k'f; k'i} = \tilde{\kappa}^f \tilde{\kappa}^i + \tilde{\kappa}^f \tilde{\kappa}^i$$

Or analytically,

$$R_{k'f; k'i} = \frac{\int_{0}^{\infty} \varphi(\kappa) (-\ii \kappa)^{-1}}{\sqrt{2\omega_\kappa}} \left( \tilde{\sigma}_\lambda^\dagger \tilde{\kappa}^\dagger \tilde{\chi}_m^\lambda \tilde{c}_m^\lambda - \tilde{\sigma}_\lambda \tilde{\kappa} \tilde{\chi}_m^\lambda \tilde{c}_m^\lambda \right)$$

Now by comparing this expression for $R$ with the standard form, we can read off...
the three $F'$s:

$$F_1(k) = -F_3(k) = \frac{f_0^2}{\mu^2} \frac{v^2(k)}{2\omega_k^2}; \quad F_2 = 0.$$  

And using the usual approximations which are associated with lowest order perturbation calculations, we can write the phase shifts as:

$$e^{i\delta_{33}} \sin \delta_{33} = \delta_{33} \simeq \tan \delta_{33} \simeq \frac{4}{3} \left( \frac{f_0^2}{4\pi} \frac{k^2 v^2(k)}{\omega_k^2 \mu^2} \right) = 2 \gamma,$$

$$e^{i\delta_{31}} \sin \delta_{31} = -\frac{2}{3} \left( \frac{s_2 w_c}{s_2 w_c} \right) = -\gamma,$$

$$e^{i\delta_{11}} \sin \delta_{11} = -\frac{8}{3} \left( \frac{s_2 w_c}{s_2 w_c} \right) = -4 \gamma,$$

where $y = \frac{2}{3} \left( \frac{f_0^2}{4\pi} \right) v^2(k) k^2 / \mu^2 \omega_k^2$.

In the rationalized units we are using, $f_0^2/4\pi$ is analogous to the fine structure constant of electrodynamics: $e^2/4\pi = 1/137$.

These expressions are not to be compared with experiment, since the higher order corrections are very important. Indeed experimentally, both $\delta_{31}$ and $\delta_{11}$ are small compared to $\delta_{33}$, which contrasts sharply with our second order estimates. However, we can make it plausible that the corrections will improve the agreement with experiment.

In NR quantum mechanics, an attractive potential, in the absence of a bound state, gives a positive phase shift which is enhanced in higher approximations. This is so because the wave function is sucked into the potential region. (Such a statement is not valid when there are bound states because of interference effects.) Thus we may well hope that $\delta_{33}$ will become larger when higher approximations are calculated.

In contrast, in NR QM, a repulsive potential gives rise to negative phase shifts (if the repulsion is not too strong) which are decreased in magnitude, because the wave function is pushed out of the potential region. We have some reason to expect better agreement for $\delta_{13}$ when higher orders are computed. In the case of $\delta_{11}$, we have a bound state, namely the nucleon. Non-relativistically this is analogous to a bound state just below zero energy.

**Modification of the R-Matrix Theory**

We shall proceed now to calculate the fourth-order terms in the R-matrix. Incidentally, in a perturbation calculation, all terms of order higher than the first non-vanishing term are called radiative corrections. In electrodynamics
these corrections are small, but in mesodynamics they are quite large. In this particular case, since the perturbation series does not converge rapidly, the fourth order corrections will still not indicate the true behavior of the scattering.

But before we can calculate radiative corrections, we must modify the R-matrix theory. There are three subtleties: (1) wave function renormalization, (2) mass renormalization, and (3) mesonic charge renormalization.

Wave-Function Renormalization. In NR perturbation theory, if we were dealing with a scattering problem with a Hamiltonian $H = H^0 + H'$, we would choose $\tilde{\psi}_i$ to represent an eigenstate of the total Hamiltonian with energy $E_i$. If we expand $\tilde{\psi}_i$ in terms of eigenfunctions $\phi_j$ of $H^0$, we have:

$$\tilde{\psi}_i = \phi_i + (E_i + i\epsilon - H^0)\psi_i$$

$$\psi_i = \phi_i + \sum_j (E_i + i\epsilon - E_j)^{-1} \phi_j R_{ji},$$

where $R_{ji} = \langle \phi_j | H' | \phi_i \rangle$.

Non-relativistically, $R_{ji}$ is of order 1/V so when we pass to the limits $V \to \infty$, we need not renormalize, providing the passage to the limit $\epsilon \to 0$ is done in such a way that $1/\epsilon V \to 0$.

Relativistically, however, $R_{ji}$ will be of order $1/\sqrt{V}$, and we will have to renormalize our expansions. As an example, let us compute the first order expansion of a state representing a physical nucleon.

$$\Psi_{\text{comp. nuc.}} = C \left\{ \phi_{\text{bare}} + \sum_{k,m} \frac{\Psi_{k,m}^H}{E_{\text{bare}} - E_{k,m}} \right\}$$

$$\frac{1}{C} \Psi_{\text{comp. nuc.}}^{NS} = \phi_{\text{bare}}^{NS} + \sum_{N'S,K'i} \frac{f_o \Gamma(K)(-1)}{\sqrt{2} \omega_k \sqrt{-\omega_k}} \langle \Psi_{N'S'}| \hat{\phi}_{K'i} | \Psi_{N'S} \rangle \frac{\Psi_{K'i}^{NS'}}{\Psi_{K'i}^{NS}}$$

From this calculation we see that to lowest order, the relative probability that there is 1 meson in the nucleon cloud is:

$$R^2 = 3 \sum_{k} \frac{f_o^2}{\omega_k^2} \frac{\Gamma(K)}{2 \omega_k \sqrt{-\omega_k}} = \frac{3}{4\pi^3} \frac{f_o^2}{\omega_k^2} \left( \frac{\Gamma(K)}{\omega_k} \right) d^3k$$

$$C^2 = \frac{1}{1 + R}$$

In field theory, corrections of this nature are of order (1) and not of order $(1/V)$. In our cut-off theory, such corrections are finite numbers, but they depend upon the cut-off. If there was no cut-off, this correction would diverge.

---

More generally, we will define a number \( Z_2 \) by the relation:

\[
\Psi_{\text{comp.}} = \sqrt{Z_2} \Psi_{\text{bare}} + \frac{1}{M - \hat{H}_0} \left( 1 - \frac{\hat{P}_{\text{bare}}}{\hat{H}_0} \right) \hat{H}' \Psi_{\text{comp.}}.
\]

where \( \hat{P}_{\text{bare}} \) is the projection operator onto \( \Phi_{\text{bare}} \), i.e.

\[
\hat{P}_{\text{bare}} = \left| \Phi_{\text{bare}} \right> \left< \Phi_{\text{bare}} \right|.
\]

We have just computed \( Z_2 \) to lowest order:

\[
Z_2 = 1 - A = 1 - \frac{3}{(2\pi)^3} \frac{f_0^2}{\mu^2} \int \frac{k^2 v'(k)}{2 \omega_k^3} \, \, d^3 k.
\]

**Mass Renormalization.** This second subtlety in the theory of the R-matrix again appears in NR QM as a correction of order \( 1/V \), and is thus dismissed. However, in field theory, because of the different order of the R-matrix, we have mass (or energy) corrections of order \( (1) \). We have a uniform energy shift of all the states. If we define \( \Delta M \) as the mass of the complete nucleon: \( M \), minus the mass of the "bare" nucleon \( M_0 \), i.e., \( \Delta M = M - M_0 \), and \( \Delta E \) as the energy shift of any state, we have:

\[
\Delta M^{(2)} = \Delta E^{(2)} = - \frac{3}{(2\pi)^3} \frac{f_0^2}{\mu^2} \int \frac{v'(k)}{2 \omega_k^3} \, \, d^3 k + O(1/V).
\]

Now in the theory of the R-matrix, our expressions are derived under the assumption that the unperturbed Hamiltonian has the same energy spectrum as the complete Hamiltonian. We can satisfy this requirement easily by splitting up the Hamiltonian as follows:

\[
H = M_o + H_{\text{free}} + H_{\text{coup}} = (M_o + \Delta M + H_{\text{free}}) + (H_{\text{coup}} - \Delta M)
\]

\[
= H_0 + H'.
\]

With this simple choice \( H_0 \) has the same spectrum as \( H \), and we can calculate finite radiative corrections.

It is interesting now to note that in drawing diagrams of second order scattering processes, we left out a diagram in the case where the initial meson state is the same as the final meson state:
But now we see that when using the correct $H'$ in the R-matrix, this term is cancelled exactly (to order $(1/V)$) by the following matrix element:

\[ -\Delta M^{(1)} \]

**Charge Renormalization.** The renormalization of the coupling constant $f_c$ is a convenient but not necessary change when calculating. We shall discuss this more fully later on.

October 14.

In the last lecture, we saw that two modifications were necessary in our calculations. One of these is due to a change in normalization, represented by $Z_2$, of the states representing a real nucleon plus free mesons. We saw, by perturbation expansion, that this change in normalization is the same for all states, to terms of order $1/V$. The other is needed because all the real states are shifted by an amount $\Delta M$ compared to the states with a "bare" nucleon. We are familiar with energy shifts in discrete states non-relativistically when an interaction is turned on, for example, the Zeeman shift. However the shift of the energies of the continuum levels also had not been investigated last year.

Last time we found the lowest order terms in the expansions of $Z_2$ and $\Delta M$. $Z_2$ can be described as the probability that a complete nucleon is a bare nucleon.

\[
Z_2 = 1 - \frac{3f_c^2}{\mu^2(2\pi)^3} \int \frac{k^2 v^2(k)}{2\omega_k^3} \, d^3k \quad \ldots
\]

The energy shift $\Delta M$ appears as a shift of the spectrum of the complete Hamiltonian as compared to the spectrum of the uncoupled Hamiltonian.

\[
\Delta M = -\frac{3f_c^2}{\mu^2(2\pi)^3} \int \frac{k^2 v^2(k)}{2\omega_k^3} \, d^3k \quad + \ldots
\]

To employ the theory of the R-matrix, we saw that we must take $H' = H_{\text{coup}} - \Delta M$. $\Delta M$ can be calculated by means of the perturbation theory of the energy shift of a bound state: the Rayleigh-Schrödinger perturbation formula. It of course comes out in pieces of various order, and we use only those pieces which are of order not greater than the order to which we calculate $R_{\text{f4}}$. The second place where care must be exercised is taking $Z_2$ into account. We obtained
the perturbation expansion
\[ R_{fi} = H_{fi} + \sum_{m} \frac{H_{fm} H_{mi}}{E_i + i\epsilon - E_m} + \cdots \]
from the iteration of the equation
\[ \Phi_{\epsilon} = \frac{1}{H_{fi}} + \frac{1}{E_i + i\epsilon - H_{fi}} \cdot \Phi_{\epsilon} \]

We see that we should multiply that expansion of \( R_{fi} \) by a constant in order to correct for wave function renormalization. If we are very careful, later on we will see this constant cancel out if we maintain some care about the energies in the denominator. In any physical problem we are interested in the R-matrix only on the "energy shell", that is, where the final energy equals the initial energy. It turns out that if we are careful and do not identify the final energy with the initial energy until the end of the problem, we will not run into any problem with wave function renormalization; the constant will cancel out.

The K-Matrix Formalism

Another convenience in calculating is to employ the K-matrix instead of the R-matrix. There is a very simple algebraic relation between these two matrices, which enables a simple conversion of one into the other. The R-matrix is:
\[ R_{fi} = H_{fi} + \sum_{m} \frac{H_{fm} H_{mi}}{E_i + i\epsilon - E_m} + \sum_{m,n} \frac{H_{fm} H_{mn} H_{ni}}{(E_i + i\epsilon - E_m)(E_i + i\epsilon - E_n)} + \cdots \]
while the K-matrix is defined similarly except with the prescription that any term in the sum is dropped if the energy of any intermediate state equals the energy of the initial state. That is, if we turn these sums into integrals, we drop the \( \epsilon \) and take the principal values of the integrals.
\[ K_{fi} = H_{fi} + \sum_{m} \frac{H_{fm} H_{mi}}{(E_i - E_m)} + \sum_{m,n} \frac{H_{fm} H_{mn} H_{ni}}{(E_i - E_m)(E_i - E_n)} + \cdots \]

We can see the relation between the K- and the R-matrices from their series expansion. The relation is very simple on the energy shell, which is where we are interested in \( R \) and \( K \).
\[ R_{fi} = K_{fi} - i \pi \sum_{m} H_{fm} \delta(E_i - E_m) H_{mi} - i \pi \sum_{m,n} H_{fm} \delta(E_i - E_m) \frac{H_{mn} H_{ni}}{E_i - E_n} \]
\[ - i \pi \sum_{m,n} H_{fm} \delta(E_i - E_m) H_{mn} \delta(E_i - E_n) H_{ni} + (-i \pi)^2 \sum_{m,n} H_{fm} \delta(E_i - E_m) H_{mn} \delta(E_i - E_n) H_{ni} + \text{higher order terms} \]
Now one will notice that the third term is just the first order correction to the second which arises when $H'_{ij}$ is replaced by $K_{ij}$; similarly the delta function in the fourth sum tells us the $E_i = E_n$ in all the terms of that sum, and hence that the fourth term is the first order correction to the second term which arises when $H'_{ij}$ is replaced by $K_{ij}$. The fifth term can be viewed as the zeroth order term obtained when all the $H'$ matrix elements are replaced by $K$ matrix elements. Indeed, further terms in the expansion all can be compressed to yield the following expansion for $R$ in terms of $K$:

$$R_{fi} = K_{fi} - i\pi \sum_m K_{fm} \delta(E_m - E_i) K_{mn} + (i\pi)^2 \sum_{m,n} K_{fm} \delta(E_m - E_n) K_{nm} \delta(E_n - E_i) K_{ni} + \ldots$$

If we now define $\tilde{R}_{fi} = K_{fi} \delta(E_f - E_i)$, and $\tilde{R}_{fi} = \delta(E_f - E_i) R_{fi}$, we have

$$\tilde{R} = \tilde{K} (1 + i\tilde{K})^{-1}.$$  

Now the scattering matrix $S$ becomes:

$$S = 1 - 2\pi i \tilde{R} = (1 - i\tilde{K})/(1 + i\tilde{K}).$$

From the series expansion we see that $K$ is Hermitian, hence the $S$-matrix is unitary. Because $S$ is unitary, its eigenvalues must be $e^{i\delta}$. Thus we get for the eigenvalues:

$$R \rightarrow - (e^{i\delta} \sin \delta)/\pi$$  

and

$$\tilde{K} \rightarrow - \frac{1}{\pi} \tan \delta.$$

When we consider scattering states of definite angular momentum and isotopic spin, the $S$-matrix is diagonal and the $\delta$'s are just the phase shifts. We will find it most convenient to work in terms of the $K$-matrix for we can throw out any term for which the energy of an intermediate state equals the energy of the initial state, being careful, however, not to identify the final state energy with the initial energy until the end of the problem. When we calculate with the $K$-matrix instead of the $R$-matrix, everything is the same in our answers except we get $\tan \delta$ instead of $e^{i\delta} \sin \delta$.

Thus we can rewrite the results of problem 4:

If $K_{k'f;ki}$ then

$$t_{2n} = -\left(\omega K^3\sqrt{6}/6\right) (4G_3)$$  

$$t_{51} = -\left(\frac{3}{2}\right) (6G_2 - 2G_3)$$  

$$t_{51} = -\left(\frac{3}{2}\right) (9G_1 - 6G_2 + G_3).$$
In the lowest order the K-matrix and the R-matrix are the same, so we can write down the second order results for the scattering of a meson in the K-matrix formalism. We had four diagrams, of which the last two were cancelled by each other. In none of these diagrams is an intermediate energy equal to the initial energy.

\[ K'_{\mathbf{k}_0^{f}, \mathbf{k}_0^{i}} = \frac{\hat{f}^2}{\mu^2} \frac{\omega_{\mathbf{k}}}{k_{\mathbf{k}}} \left\{ \frac{\hat{t}_f \hat{t}_i \hat{r}^*_{\mathbf{k}'_0} \hat{r}_{\mathbf{k}_0^{i}}}{\omega_{\mathbf{k}}} + \frac{\hat{t}_i \hat{t}_f \hat{r}^*_{\mathbf{k}} \hat{r}_{\mathbf{k}'_0}}{-\omega_{\mathbf{k}}} \right\} \]

Thus if we set \( y = \frac{2}{3} (\hat{f}^2/4\pi) \left( k_{\mathbf{k}} \omega_{\mathbf{k}} / \mu^2 \right) \), \( G_1 = -G_3 = \frac{y}{2} \left( \frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right) \), \( G_2 = 0 \), hence:

\[
\begin{align*}
\tan \delta_{33} & = 2y + \ldots \\
\tan \delta_{31} & = -y + \ldots \\
\tan \delta_{11} & = -4y + \ldots
\end{align*}
\]

**Fourth Order Corrections to π-N Scattering**

To calculate these corrections, we will first diagram all the possible uncrossed matrix elements. "Uncrossed" matrix elements are those in which the initial meson is destroyed before the final one is created. The matrix elements for the crossed diagrams are easily obtained from the corresponding uncrossed matrix element by simply reversing the sign of \( \omega_{\mathbf{k}} \) in the energy denominators, interchanging \( \mathbf{k} \) and \( \mathbf{k}' \) and interchanging \( \hat{r}_f \) and \( \hat{r}_i \).
Of these matrix elements we can forget about A, G, L, M, P, Q, since in each of them there is an intermediate state with the same energy as the initial state.

Next we will show that the sum of all the diagrams, in which the initial meson just passes by without interacting, vanishes (at least to order 1/V). To do this we will have to get an expression for ΔM good up to at least the fourth order. This can be generated easily by the following argument. Remembering that $H = H^0 + H'$, and defining $Z_2$ as the probability that a complete nucleon is a bare nucleon, I can formally write the state of a real nucleon $i$ as:

$$\bar{\Phi} = \sqrt{Z_2} \phi_{\text{bare}} + (M - H^0)^{-1} (1 - P_{\Phi}) H' \bar{\Phi},$$

provided only that it be required that $P_{\Phi} H' \bar{\Phi} = 0$. This last condition, namely that

$$\langle \phi_{\text{bare}} | H_{\text{coup}} - \Delta M | \bar{\Phi} \rangle = 0$$

determines for me the mass shift ΔM.

This is the same procedure which determines the level shift of a bound state non-relativistically when a perturbing potential is applied to a system. "This is the most convenient way of writing the Rayleigh-Schrödinger perturbation formula." If we iterate the equation for $\bar{\Phi}$ and plug this into our condition, we have:

$$0 = \sqrt{Z_2} \left\{ \langle \Phi | H' | \Phi \rangle + \langle \Phi | H' \frac{1 - P_{\Phi}}{M - H^0} H' | \Phi \rangle + \cdots \right\}.$$ 

We can now use this series to determine ΔM to any order in the coupling constant we are interested in. One at this point should keep in mind that $H_{\text{coup}}$ changes the number of mesons by one when it acts. Remembering this, the expansion yields immediately:

$$0 = -\Delta M(2) + \langle \Phi | H_{\text{coup}} \frac{1 - P_{\Phi}}{M - H^0} H_{\text{coup}} | \Phi \rangle$$

to second order.

This is exactly what we found for $\Delta M(2)$ in the last lecture. Now a little thought will convince one of the significance of this formula. Since the formula holds for any eigenstate of energy (at least to order 1/V), the formula says that the sum of all the diagrams of a given order (and a fortiori, the sum of all diagrams) in which no intermediate state energy equals the initial energy, and in which no initial or final meson interacts with the nucleon, is zero!

On account of this fact, we can now cross out diagrams H, J, K, and R.

We are thus left with only the seven following diagrams and the
corresponding crossed diagrams.

The last four diagrams in which we either have two intermediate states of the same energy or an intermediate state with the same energy as the final state have to be treated carefully because of the form of the energy denominators. Otherwise it is very easy to calculate the matrix elements.

October 16, 1958

In the last lecture we eliminated all the diagrams of the fourth order which either are to be omitted in the K-matrix or are cancelled by one another. Our task now is to evaluate the remaining diagrams.

We start with the sum:

\[ S_{67} = \left( \frac{\partial}{\partial t} \right) + \frac{1}{\omega_k - \omega_{k'}} \left( \begin{array}{c} 1 \\ \omega_k - \omega_{k'} \end{array} \right) \]

where \( I = \frac{3 f_0^2}{(2 \pi)^3 \mu^2} \int n^2 \frac{d^3 n}{2 \omega_n (\omega_k - \omega_{k'} - \omega_n)} \); \((\omega_k - \omega_{k'}) \approx 0\)

Now using our integral for \( \Delta M^{(2)} \):  

\[ \Delta M^{(2)} = - \frac{3 f_0^2}{(2 \pi)^3 \mu^2} \int n^2 \frac{d^3 n}{2 \omega_n^2} \]

and dropping terms of order \((\omega_k - \omega_{k'})\) since we are concerned at the end only with the K-matrix on the energy shell, we have:

\[ S_{67} = -A \left( \begin{array}{c} 1 \\ \omega_k - \omega_{k'} \end{array} \right) \]

One should notice that \(-A\) is the lowest order term in the perturbation expansion of \( Z_2 \): i.e., \( Z_2 = 1 - A + \ldots \). It should also be noted that the singularities in the two diagrams when \( \omega_k = \omega_{k'} \) cancel one another. This is just an example of a general theorem which states that:

\[ \Sigma(\text{all diagrams of the form}) \left( \begin{array}{c} \text{junk} \\ \delta(E_i - E_f) \end{array} \right) = Z_2 \left( \begin{array}{c} 1 \\ \delta(E_i - E_f) \end{array} \right) \]

This theorem is obviously the same regardless of the nature of the primary process to the right of the junk.
Now let us do $S_{45} = \frac{\omega_k}{\omega_h} \left( \frac{1}{\omega_h - \omega_n} \right) + \frac{1}{\omega_h} \left( \frac{1}{\omega_h - \omega_n} \right)$.

$$S_{45} = \left( \frac{3 f_0^2}{(2\pi)^3 \mu^2} \right) \left( \frac{\omega_k}{2 \omega_h} \right) \left( \frac{\omega_k - \omega_n}{2 \omega_h} \right) \int \frac{d^3 n}{\omega_n} \left( \frac{1}{\omega_h - \omega_n} \right)$$

$$S_{45} = \left( \frac{3 f_0^2}{(2\pi)^3 \mu^2} \right) \left( \frac{\omega_k}{2 \omega_h} \right) \left( \frac{\omega_k - \omega_n}{2 \omega_h} \right) \int \frac{d^3 n}{\omega_n} \left( \frac{1}{\omega_h - \omega_n} \right)$$

If we now define

$$\beta(\omega_k) = \frac{3 f_0^2}{(2\pi)^3 \mu^2} \left( \frac{\omega_k}{2 \omega_h} \right) \left( \frac{\omega_k - \omega_n}{2 \omega_h} \right) \int \frac{d^3 n}{\omega_n} \left( \frac{1}{\omega_h - \omega_n} \right)$$

then $S_{45} = \left( -A + \beta(\omega_k) \right)$

Now we evaluate $S_2 = \frac{\omega_k}{\omega_h} \left( \frac{1}{\omega_h - \omega_n} \right)$.

$$S_2 = \frac{f_0^2}{\mu^2 (2\pi)^3} \int \frac{d^3 n}{\omega_n} \left( \frac{1}{\omega_h - \omega_n} \right)$$

But the application of the commutation rules gives:

$$\sum_j \zeta_j \zeta_j = -\zeta_\sigma \zeta_\sigma$$

and

$$\int \sigma \cdot n \sigma \cdot k' \sigma \cdot n \sigma \cdot k' \frac{d^3 n}{\omega_n} \left( \frac{1}{\omega_h - \omega_n} \right)$$

Thus $S_2 = \frac{f_0^2}{\mu^2 (2\pi)^3} \int \frac{d^3 n}{\omega_n} \left( \frac{1}{\omega_h - \omega_n} \right)$

$$S_2 = \frac{f_0^2}{\mu^2 (2\pi)^3} \int \frac{d^3 n}{\omega_n} \left( \frac{1}{\omega_h - \omega_n} \right)$$

$$S_2 = \left( \beta - \frac{1}{4} \beta(\omega_{k'}) \right)$$

It proves convenient to define the constant $B$, although $B = A/9$.

The mirror image diagram $S_3$ gives the same result, as a little thought will verify. Hence the sum of these two diagrams is just:

$$S_{32} = \left( \frac{2 B - \frac{2}{3} \beta(\omega_k)}{\omega_h} \right) \left( \frac{1}{\omega_h - \omega_n} \right)$$

on the energy shell.
The net result of the six diagrams in which no virtual meson crosses both 
real mesons is just a multiple of the second order diagram. Clearly the same 
result will hold for the sum of the corresponding crossed diagrams. I should like 
to point out a few things about the situation as we have it so far. Including the 
second order uncrossed diagram, at this juncture we have:

\[
(1 - 2A + 2B + \frac{7}{9} \beta(\omega)) \left(\begin{array}{c}
\vdots \\
\vdots \\
\vdots 
\end{array}\right)
\]

To this order, \(Z_2\) corresponds to \(1 - A\). If we consider higher order processes, 

it will be found that \(1 - 2A\) corresponds to the first term in the expansion of \(Z_2^2\). 

We get a factor of \(Z_2\) from junk added onto the left end, and in addition a factor 
of \(Z_2\) from junk in the middle. But what about the \(2B\)? Each factor of \(B\) arose 

from complicating a single vertex of a real meson. We can imagine adding up all 
the diagrams corresponding to junk encumbering a single vertex of a real meson. 

The result will just be a function of the real meson energy \(F(\omega)\) times the un-

encumbered vertex. Then if we define \(Z_1 = 1/F(0)\), the generalization of the term 
\(1 + 2B\) to all higher orders is \(1/Z_1^2\). Accepting the fact that these generalizations 
can be made, the net result of our computations so far is:

\[
\left(\frac{Z_2}{Z_1}\right)^2 \left(1 + \frac{7}{9} \beta(\omega)\right) \left(\begin{array}{c}
\vdots \\
\vdots \\
\vdots 
\end{array}\right).
\]

Why is it that we chop up the matrix elements by factoring out \(\left(\frac{Z_2}{Z_1}\right)^2\)? 
The reason is that this factor will contain all the quantities which are quadrati-
cally dependent on the cut-off point. (Note that \(\beta(\omega)\) is only linearly depen-
don the cut-off point.) In the relativistic theory, things are improved, for 
once one factors out the corresponding \(\left(\frac{Z_2}{Z_1}\right)^2\), the remaining quantities are 
finite and independent of any cut-off.

In a general process, the number of times the factor \(\left(\frac{Z_2}{Z_1}\right)\) appears is 
equal to the number of times an unencumbered vertex is present. It is evident 
that the number of times the factor \(Z_2/Z_1\) appears is equal to the number of times 
the coupling constant \(f_0\) appears. For this reason, it is convenient to define a 
new coupling constant \(f_1 = f_0 \frac{Z_2}{Z_1}\), and calculate our corrections to a given 
order in \(f_1\). We may consider therefore any experiment as one determining the 
value of \(f_1\). This procedure is called charge renormalization. I should like to 
emphasize that this renormalization is not a necessary procedure in this type of 
theory. In this respect it is unlike the other two renormalization processes, 
wave function and mass, which we carried out previously. We have not proved here 
that this process can be carried out to all orders, but we have shown that it can
be done in fourth order. Later on in the year we will come back to this and construct a proof. An elegant, but concise, proof that charge renormalization can be carried out was given by Ward (Phys. Rev. 84, 897 (1951)).

This renormalized coupling constant has the significance that if one extrapolates the results of experiments down to estimate the effect mathematically of "zero energy mesons", one will get a value for \( f_\perp^2/4\pi \). This can be done for several types of experiments, and one can compare the results. Experiments on scattering, photoproduction, and nuclear forces agree well in this matter, and yield a value for \( f_\perp^2/4\pi \) of about .08.

Resuming our calculations, we will consider now the crossed diagrams we have already considered. We found in the last lecture that the crossed matrix elements are obtained from the uncrossed ones by reversing the sign of \( \omega \) in the energy denominators, and interchanging \( k_f \) and \( r_i \). Then if we define

\[
\gamma(\omega) = \beta(-\omega),
\]

we have the total matrix element of \( K \) up to the fourth order in the form:

\[
\begin{align*}
\text{Diagram 1} & \quad + \quad \text{Diagram 2} = \left( 1 - 2A + 2B + \frac{2}{q} \beta(\omega) \right) \\
\text{Diagram 3} & \quad + \quad \text{Diagram 4} = \left( 1 - 2A + 2B + \frac{2}{q} \gamma(\omega) \right)
\end{align*}
\]

Now in the evaluation of the remaining two diagrams, we shall replace \( f_0 \) by \( f_\perp \) to represent the effect of charge renormalization which is to be performed in each successive term. Again this is not necessary here, for these remaining matrix elements will be only linearly dependent on the cut-off point, but we do so for convenience.

We have:

\[
S_1 = \frac{f_\perp^4}{(2\pi)^3} \sum_{\infty} \left( \frac{U(n) \sigma \cdot n \tau_\alpha \sigma \cdot K \tau_\alpha \sigma \cdot K \tau_f \sigma \cdot \Delta \sigma \cdot n \Delta \right) \frac{1}{2 \omega_f \omega_i \omega_n \omega_{\perp}}
\]

The only problem is to evaluate the sum \( \sum_{\infty} \tau_\alpha \tau_f \tau_i \tau_\alpha \) and the integral \( \int \sigma \cdot n \sigma \cdot K \sigma \cdot \Delta \sigma \cdot n \Delta \). To accomplish this we write:

\[
\tau_f \tau_i = \frac{1}{2} \left( \tau_f \tau_i + \tau_i \tau_f \right) + \frac{1}{2} \left( \tau_f \tau_i - \tau_i \tau_f \right) = \frac{1}{2} \left( \tau_f \tau_i + \tau_i \tau_f \right) \quad \text{where } \tau_\alpha, \tau_i, \tau_k \text{ are in cyclic order.}
\]

\[
\sum_{\infty} \tau_f \tau_i \tau_\alpha \tau_\alpha = \frac{3}{2} \left( \tau_f \tau_i + \tau_i \tau_f \right) - \frac{1}{2} \left( \tau_f \tau_i - \tau_i \tau_f \right)
\]

\[
= \tau_f \tau_i + 2 \tau_i \tau_f
\]
A similar result holds also for the spin operators:
\[ \int \sigma' \cdot n \sigma' \cdot k' \sigma' \cdot k \cdot dS \cdot n = \frac{4\pi}{3} n^2 \left\{ \sigma' \cdot k' \sigma' \cdot k + 2 \sigma' \cdot k \sigma' \cdot k' \right\} \]

Noting that
\[ \frac{1}{3} \frac{f_0^2}{\mu^2} \int \frac{v^2(k) n^2 d^3 n}{2 \omega_n^3 (\omega_n - \omega_n)} = \frac{1}{2} \beta(\omega) \]

we obtain $S_1$ in the form:
\[ S_1 = \frac{1}{2} \frac{\beta(\omega)}{\omega} \int \frac{v^2(k)}{\mu^2} \frac{d^3 n}{2 \omega_k} \left\{ \frac{T_k' T_k + 2 T_k' T_k}{\omega_k} \right\} \left\{ \sigma' \cdot k' \sigma' \cdot k + 2 \sigma' \cdot k \sigma' \cdot k' \right\} \]

Now we can compare the results of a fourth order charge-renormalized expression for the $K$-matrix with the second order expression. To the second order we had:
\[ K_{k', f, k} = \frac{f_1^2}{\mu^2} \int \frac{v^2(k)}{2 \omega_k} \left\{ \frac{T_k T_k' \sigma' \cdot k' \sigma' \cdot k}{\omega_k} + \frac{T_k T_k' \sigma' \cdot k \sigma' \cdot k'}{\omega_k} \right\} \]

The total effect of our corrections was to change $f_0$ to $f_1$, introduce the factors $(1 + \frac{7}{9} \beta_1(\omega))$, $(1 + \frac{7}{9} \gamma_1(\omega))$, where to the lowest order
\[ \beta_1(\omega) = \left( \frac{f_1}{f_0} \right)^2 \beta(\omega) \]

and similarly for $\gamma_1(\omega)$, and add the two irreducible diagrams:

\[ \text{Diagram 1} \quad \text{and} \quad \text{Diagram 2} \]

We obtain:
\[ K_{k', f, k} = \frac{f_1^2}{\mu^2} \int \frac{v^2(k)}{2 \omega_k} \left\{ \frac{T_k T_k' \sigma' \cdot k' \sigma' \cdot k}{\omega} \left( 1 + \frac{7}{9} \beta_1(\omega) \right) + \frac{T_k T_k' \sigma' \cdot k \sigma' \cdot k'}{\omega} \left( 1 + \frac{7}{9} \gamma_1(\omega) \right) \right\} \]

Next time we shall convert this to phase shifts and show that our expectations are fulfilled. The 33 phase shift is indeed enhanced, while the magnitudes of the 31, 13, and 11 phase shifts are reduced. Thus we get more towards the experimental situation, where the 33 phase shift is the dominant feature of the $\pi$-$N$ scattering at low energies. Of course, we will have to do something better than perturbation theory since it does not give rapidly converging results.
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To complete our fourth-order calculation of \( \pi-N \) scattering, we have now to express our result in terms of the three independent phase shifts. If we imagine the \( K \)-matrix in our standard form, then:

\[
G_1(k) = \left( \frac{f_1^2 v^2(k)}{2} \mu^2 \omega^2 \right) \left[ 1 + \frac{8}{9} \beta_1(\omega) - \frac{4}{9} \gamma_1(\omega) \right].
\]

\[
G_2(k) = \left( \frac{f_1^2 v^2(k)}{2} \mu^2 \omega^2 \right) \left[ \frac{2}{9} \beta_1(\omega) - \frac{2}{9} \gamma_1(\omega) \right].
\]

\[
G_3(k) = -\left( \frac{f_1^2 v^2(k)}{2} \mu^2 \omega^2 \right) \left[ 1 + \frac{8}{9} \gamma_1(\omega) - \frac{4}{9} \beta_1(\omega) \right].
\]

By application of the results of problem 4, we then obtain the phase shifts:

\[
\tan \delta_{33} = 2y_1 \left( 1 + \frac{2}{9} \gamma_1 - \frac{4}{9} \beta_1 \right),
\]

\[
\tan \delta_{31} = -y_1 \left( 1 + \frac{2}{9} \gamma_1 + \frac{2}{9} \beta_1 \right),
\]

\[
\tan \delta_{11} = -4y_1 \left( 1 - \frac{4}{9} \gamma_1 + \frac{8}{9} \beta_1 \right),
\]
where

\[
y_1 = \left( \frac{2}{3} \right) \left( \frac{f_1^2}{4\pi} \right) \left( k^3 v^2(k) / \omega \mu^2 \right).
\]

**Interpretation of the Results**

Experimentally, the dominant feature of the scattering is that the \( 33 \) scattering is much greater than those of the \( 31, 13, \) and \( 11 \) states. This is the feature that we are hoping can be made plausible by our fourth-order correction to the scattering. We can not expect that our perturbation calculation will tell us very much quantitatively for the following reason. At the lowest possible real meson energy, namely \( \omega = \mu \), we see that the ratio of the fourth order charge-renormalized phase shift to the second order charge-renormalized phase shift is of order

\[
\beta_1(\mu) \approx \left( \frac{f_1^2}{4\pi} \right) \left( \frac{M}{\mu} \right) \approx 1/2.
\]

With the importance of higher order terms being of this degree, we cannot expect that our fourth order calculation will tell us much unaided. But we can force some answers out of our analysis if we can guess the nature of the contributions of the higher order terms. Let us make our guess after we take a glimpse of the situation experimentally.

The \( 31 \) and \( 11 \) phase shifts are in reality too small to give us the type of information we now want. However, data on the \( 33 \) scattering indicates the following relationship, which is almost linear at low meson
According to our theory, since $\beta_1(0)$ and $\gamma_1(0)$ are zero, the extrapolated intercept of the curve with the axis $\omega = 0$ gives the value of the coupling constant $(r_1^2/4\pi)$. (Actually the extrapolation is made to $\omega = -\mu^2/2M$ to correct for recoil.) But the important piece of information we are going to pull out of this diagram is that taking out the factor $(k^3/\omega)$, $\tan \delta_{33}$ goes like the inverse of a linear function of $\omega$, for small meson energies. Now for small kinetic energies, $\beta_1(\omega)$ and $\gamma_1(\omega)$ are both almost linear functions of $\omega$. Indeed we may put:

$$
\beta_1(\omega) \approx -\gamma_1(\omega) \approx -\omega \frac{f_1^2}{4\pi} \frac{M}{\mu^2} = -C \omega \quad \text{for} \quad \omega \ll M
$$

Now then we set $\tan \delta_{33} = 2\gamma_1(1 + 4C\omega/3)$. But the clue from the experimental data indicates that $4C\omega/3$ may well be the first perturbation term in the expansion of something like:

$$
\frac{1}{1 - 4C\omega/3}.
$$

Now suppose we follow through on this guess. Then we would estimate that near $\omega = \mu$, $C\omega \approx 1/2$, if we are bold, we might scribble that:

$$
\tan \delta_{33} \approx 2\gamma_1 / (1 - \frac{4}{3} \frac{1}{2}) \approx 6\gamma_1.
$$

$$
\tan \delta_{31} \approx -\gamma_1 / (1 + 0) \approx -\gamma_1.
$$

$$
\tan \delta_{11} \approx -4\gamma_1 / (1 + \frac{4}{3} \frac{1}{2}) \approx -2\gamma_1.
$$

Now nobody in his right mind is going to claim that the numbers have much, if any, validity. They are set down only to demonstrate the importance of how higher order corrections may be expected to contain the feature that they make the $33$ scattering dominant at low meson kinetic energies.

**Charge Renormalization**

Let us consider in a little more detail the process of charge renormalization which we carried out for our convenience. Consider the
sum of all diagrams of the following form in which there is a bare nucleon line between the two real vertices separating the junk:

\[
\sum \text{ all diagrams of this form}
\]

Any diagram of this nature is reducible to the basic second-order diagram and furthermore we may consider this sum as the product:

\[
\left( \sum \right) \left( \sum \right) \left( \sum \right) \left( \sum \right)
\]

where the light dotted lines are added to show the relative position of the junk.

(on a vertex means the junk must be entwined with the vertex. The junk includes $\Delta M$ crosses in addition to virtual mesons. We notice that we exclude all junk which is separated from the first vertex and sitting on the far right segment of the nucleon line, since such terms are excluded from the $K$-matrix).

Each of the four sums is just a function of the relevant energies times the basic second order diagram:

\[
\sum = f_2(\omega_k - \omega_{k'})
\]
\[
\sum = f_1(\omega_k - \omega_{k'})
\]
\[
\sum = f_2(\omega_k)
\]
\[
\sum = f_1(\omega_k, 0)
\]

where \( f_1(x, y) = f_1(y, x) \)

Hence \[
\sum = f_2(0)f_1(0, \omega)f_2(\omega)f_1(0, \omega)
\]

since at the end we put \( \omega_k = \omega_{k'} = 0 \).
A similar argument holds for the crossed diagrams which are reducible, and it yields:

\[ \sum \text{\(\text{crossed diagrams}\)} = f_2(0)f_1(0,-\omega)f_2(-\omega)f_1(0,-\omega) \]

Now in our theory it proves convenient to examine functions of \(\omega\) like \((3/4) (\omega^2/k^3) \tan \delta_{22}\), because the range of these functions can be extended down to \(\omega = 0\). If we construct perturbation expansions of functions like this, near \(\omega = 0\) our expansions will be exact if we include all terms which come from diagrams reducible to the two basic second-order diagrams. For to get the exact limit near \(\omega = 0\), we need include precisely those diagrams which have an energy denominator of \(\omega\). But these are just the diagrams which are reducible to second order diagrams. (In all other diagrams, there is always a virtual meson present, and since this meson "likes to have a rather high energy", the energy denominator behaves like \(\omega \approx \omega_n \).) But the result of including all diagrams of this nature is just a multiple of the second order diagram. We therefore will get the second order approximation to be exact in the limit \(\omega = 0\) if only we multiply the coupling constant \(f_0^2/4\pi\) by \(f_2^2(0) f_1^2(0,0)\). Now we had previously defined \(f_1(0,0) = 1/Z_1\), and I stated that \(f_2(0) = Z_2\). This procedure is equivalent to renormalizing the mesonic charge:

\[ f_1 = f_0 (Z_2/Z_1) \]

When we extrapolate our scattering data down to \(\omega = 0\), it is a value of \(f_1^2\) that we measure. The same thing is true in electrodynamics; the charge that is measured in experiments is not the "bare" charge, but rather the effective value at very low frequencies. For the pion, this extrapolation procedure used in measuring the coupling constant is not too bad, since the pion mass is not too large. Results of several different types of measurements, e.g., \(\pi-N\) scattering, \(\pi\) photoproduction, and theories of nuclear forces give good agreement on a value of \((f_1^2/4\pi) \approx 0.08\).

Let us now take a look at an irreducible diagram, e.g.
We may consider the sum of all diagrams which reduce to this fourth-order diagram:

\[
\sum \Gamma_{\omega_{k'} \omega_{k}} \Gamma_{\omega_{n}} \Gamma_{i \omega_{k}, i} \\
\binom{\omega_{k'}}{\omega_{k'}} - \binom{\omega_{n}}{\omega_{n}} - \binom{i \omega_{k}, i}{i \omega_{k}, i}
\]

This sum will just equal a multiple of the basic fourth-order diagram:

\[
\sum = \frac{f_2(\omega_k - \omega_k') f_1(\omega_k - \omega_{k'}, \omega_{k'} - \omega_{k'}') f_2(\omega_{k'} - \omega_{k'}) f_1(\omega_{k'} - \omega_{n}, \omega_{n})}{f_1(\omega_k - \omega_{n}, \omega_{k'} - \omega_{n}) f_2(-\omega_{n}) f_1(-\omega_{n}, 0)}
\]

It proves convenient again to factor out \(f_2^4(\omega)/f_2^4(0) = (Z_2/Z_1)^4\) from this expression and absorb it into the coupling constant, since the coupling constant appears in the irreducible fourth-order diagram exactly four times. It turns out that when this is carried out, the remaining functions of energy, e.g.,

\[
f_2(\omega)/f_2(0)
\]

depend only linearly on the cut-off point.
I suggested some time back that it would be a good exercise to carry through our calculations using the charged scalar theory of mesons. The physics is very different, and the mathematics is much simpler. One may notice that after the charge renormalization is performed in this false theory, everything else is independent of the cut-off. This is due to the absence of the gradient coupling in the theory, which gave us always a factor of $k^2$ in our integrals.

**Improved Methods of Calculation**

We must try to obtain results on $n$-$N$ scattering which are in closer agreement with experiment by using more sophisticated schemes of calculation than perturbation theory. The total number of methods available for dealing with a Hamiltonian such as we have consist of strong coupling methods, by which it is possible to evaluate exactly what happens in the limit of a very large coupling constant, and weak coupling methods, which start with perturbation theory and get more sophisticated. Strong coupling methods are not too bad, but weak coupling methods are closer to applicability. Actually the experimental situation is somewhat mid-between, so that both of them give fairly good ideas on what is going on. However, in scattering, weak coupling is better, so we will explore some of the improvements on perturbation calculations.

I want to state the latest improvement first, because it is very simple and improves the value of any perturbation calculation enormously. It is applicable to any problem involving a perturbation calculation, not just field theory, and I want to strongly encourage its use. I won't justify it particularly; if we have time later on we may return to it and give an exhaustive analysis.

In the Born approximation, with the K-matrix, we generate a series for $\tan \delta$:

$$\tan \delta = B^{(2)} + B^{(4)} + \ldots.$$  

But in the lowest approximation it does not matter whether we equate the $B^{(2)}$ to $e^{i\theta} \sin \delta$, $\sin \delta$, or just $\delta$, if $B^{(2)}$ is very small. But as $B^{(2)}$ gets larger, it matters a great deal how one uses the series to get the phase shift. In general, we stated that the best thing to do was to employ the series as an expansion of $\tan \delta$. But the point is, that a better way of proceeding from the perturbation series to the phase shifts has been found. What was lacking in the Born approximation was an estimate of the enhancement or of the
reduction of the wave function in the vicinity of the potential region. For instance in the 31 case there is a repulsive potential. On account of this there is a great diminution of the wave function in the potential region, which results in a substantial reduction of the magnitude of $\delta_{31}$. Nevertheless, the fourth Born approximation does not particularly take this diminution into account. The fourth Born approximation has a piece of this in it, but it is a rather feeble attempt at correcting for the diminution.

A formula has been found which takes into account a great deal of the effect of the enhancement or diminution of the wave function. It is:

$$\tan \delta(\omega) = \frac{f(\omega)}{1 + \frac{1}{\pi} \int_\mu^{\infty} \frac{f(\omega')}{\omega - \omega'} d\omega'},$$

where $\mathcal{P}$ signifies the principal value.

The procedure to be used is to employ the Born approximations for $\tan \delta$ to calculate a perturbation expansion for $f(\omega)$, and then plug that perturbation expansion for $f(\omega)$ back into the formula to calculate $\tan \delta$. Because of the form of the denominator, this procedure brings in partial contributions from terms of higher order than the perturbation calculation.

Let's try this procedure out now on our $\pi$-N scattering problem. First we shall look at what it does to our second order calculation. We had:

$$\begin{pmatrix} \delta_{33} \\ \delta_{31} \\ \delta_{11} \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} y_1(\omega) \text{ where } y_1(\omega) = \frac{2}{3} \frac{k^2 v(k)}{\omega \mu^2}.$$ 

Since we know the behavior of $\tan \delta$ near $\omega = 0$, it is convenient to modify our formula by defining

$$f_1(\omega) = \frac{f(\omega)}{1 - \frac{1}{\pi} \int_\mu^{\infty} \frac{f(\omega')}{\omega'} d\omega'}.$$ 

If we do this and write $1/(\omega - \omega') = -1/\omega' + \omega/\omega'(\omega - \omega')$, then our "magic" formula can be expressed as:

$$\tan \delta = \frac{f_1(\omega)}{1 + \frac{\omega}{\pi} \mathcal{P} \int_\mu^{\infty} \frac{f_1(\omega')}{\omega'(\omega - \omega')} d\omega'}.$$ 

We do this because we know the exact behavior of $\tan \delta$ near $\omega = 0$ from our theory, and it is much easier to expand $f_1(\omega)$ near where we know $\tan \delta$ best because the integral term vanishes at this point. Proceeding now with the computation, we find:
\[
\frac{\omega}{\pi} \int_0^\infty \frac{f(w') \, dw'}{w' (\omega - w')} = \frac{\omega}{\pi} \frac{2}{3} \frac{f_1^2}{4\pi \mu^2} \int_0^\infty \frac{K'^3 \gamma^2(k') \, dk'}{\omega'^2 (\omega - \omega')} = \frac{\omega}{\pi} \frac{2}{3} \frac{f_1^2}{(4\pi)^2 \mu^2} \int_0^\infty \frac{K'^2 \gamma^2(k') \, dk'}{\omega'^2 (\omega - \omega')} = \frac{2}{\alpha} \beta_1(\omega)
\]

Therefore, our improved estimates for \( \tan \delta \) are:

\[
\tan \delta_{\text{improved}} = \begin{pmatrix}
\frac{2}{1 + \frac{4}{9} \beta_1} \\
\frac{-1}{1 - \frac{8}{9} \beta_1} \\
\frac{-4}{1 - \frac{8}{9} \beta_1}
\end{pmatrix} \gamma_1(\omega)
\]

If we compare this improved 2nd order calculation with our results from 1st order perturbation theory, we notice that the fourth order corrections involving \( \beta_1(\omega) \) are included in our present results. The only fourth order correction not included involves \( \gamma_1(\omega) \) which does not have a singularity in its integrand. But in addition, since \( \beta_1(\omega) \) is negative at low energies, we notice also that our improved calculation indicates the possibility of a resonance in the 33 scattering. In the last lecture we calculated that for small \( \omega \),

\[\beta_1(\omega) = -\omega/2\mu \approx -\gamma_1(\omega)\]

The improved calculation indicates that for not too large \( \omega/\mu \):

\[
\frac{4}{3} \frac{k^2}{\omega^2} \cot \delta_{33} = \frac{4\pi}{f_1^2} \left(1 - 2\omega/9\mu\right)
\]

We obtain the linear plot observed experimentally, and we get a resonance at \( \omega = 4.5\mu \). This value is too large; the observed resonance in the 33 scattering occurs at \( \omega \approx 2.4\mu \). Let us now take a look at the ratios of the phase shifts near \( \omega = \mu \). A little arithmetic gives:
\[
\tan \delta_{33} : \tan \delta_{31} : \tan \delta_{11} = 18/7 : -9/10 : -36/13 = 2.6 : -0.9 : -2.8.
\]

This is an improvement over our result from second order perturbation theory, but is still in poor agreement with experiment.

We have carried out our perturbation calculations to fourth order, so let's use the "magic" formula to improve them also. Our fourth order results indicate that:

\[
f_1^{(4)}(\omega) = \begin{pmatrix}
16/9 \\
-2/9 \\
16/9
\end{pmatrix} \gamma_1(\omega) \gamma_1(\omega)
\]

The fourth order perturbation terms in \(f_1^{(4)}(\omega)\) do not appear here because they are part of the improved second order.

If we define \(\varepsilon_1(\omega) = \frac{\omega}{\pi} \int_{\mu}^{\infty} \frac{\gamma_1^2}{\omega^2}(\omega - \omega') d\omega'\), then our improved fourth order estimates are:

\[
\tan \delta_{\text{improved}} = \gamma_1(\omega) \begin{pmatrix}
\frac{2(1 + 8\gamma_1/9)}{(1 + 4\beta_1/9 + 16\varepsilon_1/9)} \\
-1(1 + 2\gamma_1/9)/(1 - 2\beta_1/9 - 2\varepsilon_1/9) \\
-4(1 - 4\gamma_1/9)/(1 - 8\beta_1/9 + 16\varepsilon_1/9)
\end{pmatrix}
\]

Note that for small \(\omega\), \(\varepsilon_1(\omega)\) looks like a negative constant times \(\omega\). This will tend to bring the resonance energy down closer to where it really is.

**PROBLEM 5:** Examine the phase shifts for small values of \(\omega\), and the location of the 33 resonance.

A similar approximation was arrived at many years ago by Tamm and Dancoff independently of each other. This method, however, is much more complicated. Essentially, the method consists of considering a limited subset of the possible states, and diagonalizing the Hamiltonian with respect to the subset. The higher approximations are calculated by taking larger and larger subsets in a physically significant fashion. This is the same procedure which is employed in many problems in atomic physics dealing with the changes in bound states due to the application of a perturbing potential, for example, the Zeeman and Stark effects. In essence, we neglect states which are far removed in energy from the states we are interested in.

In the application of the Tamm-Dancoff method to meson scattering, the initial approximation consists in considering only that subset of states which
have no more than one meson in them. We calculate the sum of all diagrams of the following nature:

\[
\begin{align*}
\text{\includegraphics[width=0.5\textwidth]{diagram1.png}}
\end{align*}
\]

We include all diagrams corresponding to the above which have mass renormalization X's in place of a bubble. Essentially, the first order Tamm-Dancoff approximation adds to the lowest order uncrossed diagram all the diagrams which are iterations of the bubble on the bare nucleon.

This first approximation excludes the fundamental crossed diagram which

\[
\begin{align*}
\text{\includegraphics[width=0.2\textwidth]{diagram2.png}}
\end{align*}
\]

is of the lowest order in the Born approximation. The second Tamm-Dancoff approximation sums all diagrams which have states with no more than two mesons. In so doing, we include the fundamental crossed diagram of second order and the fourth order diagrams except this one:

\[
\begin{align*}
\text{\includegraphics[width=0.2\textwidth]{diagram3.png}}
\end{align*}
\]

In the Tamm-Dancoff approximation, the iteration of bubbles gives rise to the same type of term in the denominator as we get in the "improved" perturbation calculation. However, because the Tamm-Dancoff approximation of a given order does not include a piece of the Born approximation of the same order, the results are not quite as good as the "improved" Born approximation. There have been numerous attempts to fix up the Tamm-Dancoff approximation by putting in some extra pieces, like the omitted crossed diagram. These have been moderately successful in obtaining results which agree with experiment.

Whether these approximations actually indicate what the theory predicts is a matter which is not to be dismissed lightly. It is a very dangerous thing in theoretical physics to take a Hamiltonian, which you dream up and are not absolutely sure if it's right or wrong, and then find an approximation method and grind out approximations until you get agreement with experiment, and then conclude that the theory is in agreement with experiment. For you do not know for sure that, maybe because the Hamiltonian is wrong and the approximation is
bad, you may reach agreement with experiment on account of cancelling errors! In our perturbation calculations we have not shown that if one calculates the sixth, eighth, and higher orders the agreement with experiment will not be destroyed. We hope it will not, but we are not sure about it since the convergence of the Born approximation is so slow.

An approach of a different nature has come into wide use in the past three years, which is not based on solving the problem in approximations. Rather, it derives from the theory a set of exact relations among experimental quantities, which must be satisfied precisely if the theory is correct. This is the method of Low, and is related to the theory of dispersion relations employed in relativistic theories. It was a great advance to derive from a very complicated theory a set of exact relations which could be used to check the theory. The first inkling we had of such a possibility was the theorem that the lowest order Born approximation with a renormalized mesonic charge gave the exact phase shifts in the limit of zero energy mesons. This is the statement that if the observed phase shifts are extrapolated down to zero energy, they should have the characteristic ratios $2:1:4$. We would be in a position to get a decisive check on our theory were it not for the fact that in the physical region, $(\omega > \mu)$, the $31$, $13$, and $11$ are too small to allow such an extrapolation procedure. However, we still have something of value for there are other problems also. From the extrapolation of the $33$ phase shift we get a rigorously correct value for $f_{1}^{2}/4\pi$ on our theory. We then can consider photoproduction, where we will see a similar exact prediction of the theory. A similar extrapolation of photoproduction data gives a rigorous value for $f_{1}^{2}/4\pi$ which can be compared with the scattering value. In addition, a study of nuclear forces can be made to yield the same thing. The fact is that these results for $f_{1}^{2}/4\pi$ agree. Thus we see the first example of a set of results which do not depend upon an approximate calculation. The dispersion relations or Low equations are another example of a set of rigorously correct relations. We shall take up this topic in the next lectures.
An excellent reference to this method of calculating meson scattering is the paper by Wick (Rev. Mod. Ph. 27, 339 (1955)). In this paper Low's derivation of the method is presented, and we shall study it in a little while. However, there is another approach to the Low equation, which relies on the concept of causality. This latter approach is better heuristically, and can be exploited in the relativistic theories. Therefore, we shall begin our discussion of the Chew-Low method by investigating certain dispersion relations which are a result of the principle of causality.

The relations we shall derive are consequences of the assumption that effects follow causes in time. This statement at first seems obviously true, but actually it need not be true. In the future we may have to admit that the principle of causality is violated over very short time intervals, or that the times associated with causes and effects are smeared out a little so that one cannot precisely define a temporal sequence of cause and effect if these are too close to each other. However up to now, all our successful theories have been causal and we shall proceed under this assumption.

In the theory of special relativity the principle of causality has the consequence that the upper limit on signal velocities must be $c$. For if there existed a signal which propagated faster than $c$, then one could find a Lorentz frame where the receipt of the signal occurred before the signal was sent.

There are simpler applications of the principle of causality than those of special relativity. For example, in electrical networks, the principle requires that the response of the network, $F(t)$, must vanish for times earlier than the application of the disturbance, say $t = 0$. Let us look at some mathematical consequences of the simple relation:

$$F(t) = 0 \quad \text{for} \quad t < 0.$$

This condition can be exploited by looking at the Fourier transform of the response function:

$$f(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} F(t) e^{i\omega t} dt.$$

Assuming that $F(t)$ is a bounded function, since $F(t) = 0$ for $t < 0$, $f(\omega)$ is an analytic function which has no poles in the upper half plane of $\omega$.\footnote{Titchmarsh, Theory of Fourier Integrals.}
Indeed for real \( u, v, f(u + iv) \xrightarrow{v \to \infty} 0 \), uniformly. Therefore, we may write:

\[
f(\omega) = \frac{i}{\pi} \int_{-\infty}^{\infty} f(\omega') \frac{d\omega'}{(\omega - \omega')}.
\]

This relation is a typical dispersion relation. We may rewrite it in several forms:

\[
\begin{align*}
\text{Re } f(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \text{Im } f(\omega') \frac{d\omega'}{(\omega' - \omega)} \\
\text{Im } f(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \text{Re } f(\omega') \frac{d\omega'}{(\omega - \omega')}
\end{align*}
\]

\[
f(\omega) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \text{Im } f(\omega') \frac{d\omega'}{(\omega' - \omega - i\epsilon)}.
\]

In our static model, because the nucleon is smeared out (i.e., \( v(k) \neq 1 \)), we cannot say that the theory is causal. If it were not for the \( v(k) \), we would expect that the scattering amplitudes, \( f^a_\omega(k) \) where \( \alpha = 33, 31, 11 \), would obey dispersion relations, since they indicate the response when an incident meson is scattered by a nucleon. It turns out that we may get around this by writing a dispersion relation for \( f^a_\omega(k) / k^2 v^2(k) \):

\[
\frac{f^a_\omega(k)}{k^2 v^2(k)} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega' (\omega' - \omega - i\epsilon)^{-1} \text{Im } \frac{f^a_\omega(k')}{k'^2 v^2(k')}.
\]

This turns out to be the content of the Low equation for meson scattering. Of course, we will have to rewrite it in a different form, but this mathematical statement concerning the principle of causality is the essential reason why the Low equation is true.

Now this equation is deceptively simple, and we will have to scrutinize it carefully to get a useful form for it. The first thing one notices is that

\[
f^a_\omega(k) / k^2 v^2(k) = e^{ia} \sin \frac{\omega}{k^2 v^2(k)}
\]

is supposed to be defined (analytically continued) for all values of \( \omega \), at least in the upper half plane of complex \( \omega \). The important question is what do we put into the equation for non-physical values of \( \omega' \); those are \( \omega' < \mu \). It would be very nice if we could rewrite the relation so that it involved only physically real values of \( \omega \). It turns out that this can almost be done, but not quite. The integral over negative values of \( \omega' \) can be converted into one over positive \( \omega' \) by means of the "crossing theorem", which is a relation between \( f(-\omega) \) and \( f(\omega) \). This theorem enables us to rewrite the integral so that it runs only over positive \( \omega \). This still leaves a contribution from
the non-physical region $0 < \omega < \mu$, which turns out to be representable simply by an additive term, which can be integrated out. We will be left with an integral over only physical values of $\omega$, i.e. $\omega > \mu$.

We now can proceed further and find a relation which gives us the imaginary part of the scattering amplitude as a bilinear form in the scattering amplitude. We are familiar with such a relation, since last year we derived the optical theorem in non-relativistic QM:

$$\sigma_{tot}^* = \frac{4\pi}{k} \text{Imag.} f(k, \theta = 0).$$

This relation can be generalized very easily by setting down the requirement that the $S$-matrix must be unitary in order that probabilities be conserved:

$$S_{\alpha\beta} = \delta_{\alpha\beta} - 2\pi i R_{\alpha\beta} \delta(E_{\alpha} - E_{\beta})$$

$$\delta_{\alpha\gamma} = \sum_{\beta} S_{\alpha\beta} S_{\beta\gamma} = \delta_{\alpha\gamma} + 2\pi i \delta(E_{\alpha} - E_{\gamma})(R_{\alpha\gamma}^+ - R_{\alpha\gamma})$$

$$+ 4\pi^2 \sum_{\beta} R_{\alpha\beta}^+ R_{\beta\gamma} \delta(E_{\alpha} - E_{\beta}) \delta(E_{\alpha} - E_{\gamma})$$

Now if we agree in the following to consider matrix elements of $R$ only between states of the same energy, i.e. on the energy shell, we have:

$$(R_{\alpha\gamma} - R_{\alpha\gamma}^+) = -2\pi i \sum_{\beta} R_{\alpha\beta}^+ R_{\beta\gamma} \delta(E_{\alpha} - E_{\beta}).$$

In particular this equation yields the generalization of the optical theorem:

$$\text{Imag} R_{\alpha\alpha} = -\pi \sum_{\beta} R_{\alpha\beta}^+ R_{\beta\alpha} \delta(E_{\alpha} - E_{\beta})$$

which indeed is a relation between the imaginary part of the forward scattering amplitude and the total cross-section.

Now we can substitute this bilinear form for $f_a(\omega')$ into our dispersion relation. The bilinear form amounts to adding up the cross-sections for all possible reactions, which are energy conserving. If we start from a state where there is one free meson and a complete nucleon, what are the possible processes? Now since we are considering only real processes, we may expect that in the range $0 < \omega < \mu$, the only term in $\text{Imag} f_a(\omega)$ will be a term proportional to $\delta(\omega)$ coming from the $R$-matrix element representing the process in which a zero energy pion is absorbed by the nucleus. In the range of physical energies, the only process that can occur for $\mu < \omega < 2\mu$ is the elastic scattering of the pion. However, above $\omega = 2\mu$ we can have processes in which additional pions are created. We can write:
for $\mu < \omega' < 2\mu$ \hspace{1cm} \text{Imag } f_a = \text{const.} \hspace{0.5cm} |f_a|^2$

for $2\mu < \omega'$ \hspace{1cm} \text{Imag } f_a = \text{const.} \hspace{0.5cm} |f_a|^2$

+ higher terms from $\pi \rightarrow 2\pi$, etc.

Low employs the approximation of neglecting all the cross sections for processes representing the production of additional pions. Hence, his method is called the one-meson approximation.

To recast our formula, we now note the non-physical process for a zero energy $\pi$ contributes a term like $1/\omega$ to the scattering length, and it is the only place from which such a singular term can arise. We can therefore omit the integral from 0 to $\mu$ and add in:

\[
\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \left( \frac{2}{3} \right) \left( \frac{e^2}{4\mu^2} \right) \left( \frac{1}{\omega} \right). 
\]

The proportionality factor between $\text{Imag } f_a(\omega)$ and $|f_a(\omega)|^2$ can be obtained quickly by writing

\[ k f_a(\omega) = e^{i\delta_a} \sin \delta_a \]

and noting that $\delta_a$ is real for elastic scattering. Hence

\[ \text{Imag } e^{i\delta_a} \sin \delta_a = \sin^2 \delta_a = \left| e^{i\delta_a} \sin \delta_a \right|^2. \]

Therefore, we get as a result of the dispersion relation:

\[
\frac{f_a(\omega)}{v^2(k)k^2} = \left( \frac{2}{1} \right) \left( \frac{e^2}{3} \right) \left( \frac{1}{4\mu^2} \right) \left( \frac{1}{\pi} \right) \int_{\omega' - \omega - i\epsilon}^{\infty} \frac{k^4 v^2(k')}{v^2(k')k'^2} \left| \frac{f_a(\omega')}{v^2(k')k'^2} \right|^2 d\omega',
\]

+ crossed term for integral $-\infty$ to $-\mu$

+ higher terms from the processes $\pi \rightarrow 2\pi$, etc.

To be consistent with our heuristic derivation we should put $1/(\omega + i\epsilon)$ or $(1/\omega - i\epsilon)$ instead of $1/\omega$ in the first term; but we do not do so since this minor change cuts no ice anywhere the formula is employed.

Let us now look into the method of relating the integral over negative values of $\omega'$ to an integral over positive values of $\omega'$. Remember that in computing the K-matrix for the meson elastic scattering, for every uncrossed diagram there existed a crossed diagram whose matrix element could be obtained simply from the uncrossed one by (1) interchanging $\tau_f$ and $\tau_i$, and (2) by interchanging $K$ and $K'$, and (3) changing the sign of $\omega$ in the energy denominators. Hence if we perform these three operations on the K-matrix, the matrix must be unchanged. Since the only place that $\omega$ appears outside an energy denominator
is in the normalization factor for the free meson wave functions which produces just the factor \( 1/\omega \) in the K-matrix, the following relations must hold:

\[
G_1(-\omega) = -G_3(\omega) \\
G_2(-\omega) = -G_2(\omega)
\]

This is called a crossing theorem*. We have just shown that on the energy shell the operations of replacing \( \omega \) by \(-\omega\), \( \tau_1 \) by \( \tau_f \) and \( K' \) by \( K \) just changes the sign of the K-matrix. Now since the K-matrix is Hermitian, and

\[
\bar{R} = \bar{K} (1 + i\pi K)^{-1};
\]

we see that if we apply the same operations to the \( \bar{R} \) matrix on the energy shell, we get just

\[
-\bar{K}(1 - i\pi K)^{-1} = -\bar{R}^+.
\]

Therefore, the following relations hold:

\[
F_1(-\omega) = -F_3(\omega) \\
F_2(-\omega) = -F_2(\omega).
\]

From these crossing relations we may deduce the corresponding crossing relations for the scattering amplitudes in the 33, 31 and 11 states. We will obtain:

\[
\text{Imag } f_\alpha(-\omega) = \sum \frac{A_{\alpha\beta}}{\beta} \text{ Imag } f_\beta(\omega).
\]

Problem 6: Find the 3 x 3 matrix \( A_{\alpha\beta} \).

With the aid of this matrix we may rewrite the integral from \(-\infty\) to \(-\mu\) as an integral over positive frequencies:

\[
f_\alpha(\omega)/k^2 v^2(k) = \left( \begin{array}{c} 2 \\ -1 \end{array} \right) 2 \frac{f_\alpha^2}{4\pi^2} \frac{1}{\omega + i\pi} \int_{-\mu}^{\infty} \frac{\text{Imag } f_\alpha(\omega')}{\omega' - \omega - i\pi k_1^2 v^2(k')} d\omega'
\]

\[
- \frac{1}{\pi} \int_{-\mu}^{\infty} \frac{1}{\omega' + \omega} A_{\alpha\beta} \text{ Imag } \left( \frac{f_\beta(\omega')}{v^2(k')k_1^2} \right) d\omega'
\]

As this equation stands, it is an exact prediction of the theory, based upon the application of a causality principle and the crossing theorem. Chew and Low then write the imaginary part of the scattering amplitude as a bilinear form involving the cross-sections for all possible processes at the given energy. Then to get a set of equations which they can attempt to solve, they employ the one-meson approximation which neglects all processes other than

elastic scattering. As a result they obtain the following three non-linear integral equations for the scattering amplitudes:

\[
\frac{f_a(\omega)}{v^2(k)k^2} = \left( \begin{array}{c} 2 \\ -1 \\ -4 \end{array} \right) \frac{f^2_{a1}}{4\pi^2} \frac{\rho^2}{3} (\omega) + \frac{1}{\pi} \int_{\omega' = \omega - i\epsilon}^{\infty} \frac{k^3 v^2(k')}{v^2(k')k'^2} \left| \frac{f_a(\omega')}{v^2(k')k'^2} \right|^2 d\omega' + \frac{1}{\pi} \int_{\omega' = \omega + i\epsilon}^{\infty} \frac{k^3 v^2(k')}{v^2(k')k'^2} \left| \frac{f_{a1}(\omega')}{v^2(k')k'^2} \right|^2 d\omega'
\]

This system of equations in which the higher processes are neglected gives a pretty good account of the features of \(\pi-N\) scattering at low energies, including the \(3\bar{3}\) resonance. However, we are still subject to the same bugaboo of not being exactly sure of what the theory predicts, since we have made an approximation without knowing that the neglected part can truly be neglected. We cannot claim therefore that the agreement with experiment validates the theory, but certainly we hope that the agreement indicates that the theory may be OK.

There may exist a connection between the Low one-meson approximation and the method of the "magic formula" for improving the Born approximation. It must be kept in mind that the following is a conjecture and may very possibly be false.

Recall our magic formula:

\[
\tan \delta_a(\omega) = \frac{f_{1a}(\omega)}{1 + \omega} \int_{\omega}^{\infty} \frac{f_{1a}(\omega')}{(\omega - \omega')d\omega'}
\]

As we have seen in the application of this formula to the second order perturbation calculation, the effect of the integral is to add up certain higher order terms. The improved second-order perturbation calculation includes all those fourth order terms which yield a term containing \(\beta_1(\omega)\). Forgetting certain complications due to the mass renormalization, these terms are precisely just the terms which are iterations of the second order terms which we feed into the "magic formula". For example, this fourth order diagram

\[
\begin{array}{c}
\end{array}
\]

can be regarded as the iteration of these two second order diagrams:
In addition, all of the terms involving $\beta_1(\omega)$ come from "uncrossed" diagrams. Now it appears promising that one can say that the magic formula sums all those diagrams which are iterations of any set of the diagrams fed into it. Furthermore, it may be true that all the higher order diagrams which are iterations of lower order diagrams are "uncrossed" diagrams.

If these conjectures be true, then a very easy method of calculation suggests itself. (1) Take the results of a perturbation calculation to some definite order, (2) use the magic formula to generate a piece of the next higher order, (3) use the crossing theorem to generate that part of the higher order which arises from the crossed diagrams corresponding to those which are generated in the "magic formula", (4) repeat the cycle. Does this procedure succeed in accounting for all the terms? If one starts with the second order, it will be found that this procedure includes all the fourth order perturbation terms in the sum. (Incidentally, it would be a good exercise to carry out this prescription for an order or so.) But when the prescription is carried out in the sixth-order, apparently some of the diagrams are not included, e. g.

If this diagram is sliced in half, the right half corresponds to the process $\pi \rightarrow 2\pi$. This is just the type of term which is not included in the Low one-meson approximation. Our final conjecture is that the approximation generated by the prescription presented may be the Low one-meson approximation. A final warning; the above is only a guess and there may very well be something wrong with it.

The derivation of the Low equation based upon the causality condition has not been a proof, because we have not shown explicitly that $f_\omega(k^2v^2(k)$ must satisfy a dispersion relation, nor have we treated carefully what happens in the region $-\mu < \omega < \mu$. The derivation has nevertheless been presented in considerable detail because it is more fundamental than Low's derivation.
We shall now take up a rigorous derivation of the Low equation; essentially Wick's reformulation shall be presented.

To facilitate reference to Wick's paper we shall introduce some of his notation. He always writes the coupling Hamiltonian in terms of

\[ V_{\alpha p \lambda} = \frac{f_0}{\mu} \frac{\hbar^2}{\tau_{\lambda}} \int p(x) \psi_\lambda(p(x) \hat{d}^3x = \frac{f_0}{\mu} \frac{\hbar^2}{\tau_{\lambda}} \frac{i}{\sqrt{2\omega_k}} \sigma^i k V(k) \]

for momentum states, as

\[ H_{\text{coup}} = \sum_{p\lambda} \left( c_{\alpha p \lambda} V_{\alpha p \lambda} + c^+_{\alpha p \lambda} V^+_{\alpha p \lambda} \right). \]

Throughout the derivation of the Low equation, constant use is made of the following commutation rules:

\[ [H^0_{\alpha \lambda}, c_{\alpha p \lambda}] = -\omega_p c_{\alpha p \lambda} \]
\[ [H^0_{\alpha \lambda}, c^+_{\alpha p \lambda}] = +\omega_p c^+_{\alpha p \lambda} \]
\[ [H^1_{\alpha \lambda}, c_{\alpha p \lambda}] = -V^+_{\alpha p \lambda} \]
\[ [H^1_{\alpha \lambda}, c^+_{\alpha p \lambda}] = V^+_{\alpha p \lambda} \]

He constructs a scattering theory by analogy with the non-relativistic potential scattering theory, in which one employs the equations:

\[ (H^0_{\lambda} + V) \psi_i = E\psi_i \]
\[ H^0_{\lambda} \phi_i = E \phi_i, \]

to write a solution \( \psi_i \) which represents an incoming plane wave plus outgoing spherical waves:

\[ \psi_{i+} = \phi_i + \frac{1}{E - H^0 + i\epsilon} V \phi_i = \phi_i + \frac{1}{E - H^0 + i\epsilon} V \psi_{i+}. \]

In the application of this procedure to scattering problems in field theory, some care must be exercised because of difficulties due to wave-function renormalization. \( \phi_i \) is not to be taken as an eigenfunction of the uncoupled Hamiltonian, for we do not want the initial state as a free meson incident onto a "bare nucleon". Rather we wish to take as the initial state one which corresponds to a free meson incident onto a real, complete nucleon. Therefore, the states \( |a\rangle \) are introduced, which symbolize the four states of a complete nucleon. These have \( M \) as their energy eigenvalue.

By the construction of wave packet states for the incoming meson, it can very easily be shown that the quantity analogous to \( \phi_i \) is:

\[ |p\lambda a\rangle \equiv c^+_{\alpha p \lambda} |a\rangle. \]

The construction of the proper scattering state, which equals the initial state plus a sum of states representing outgoing free mesons, proceeds as follows:

\[ (H - \omega_p - M) c^+_{\alpha p \lambda} |a\rangle = \{ c^+_{\alpha p \lambda} (H - \omega_p - M) + [H^0_{\alpha \lambda} + H^1_{\alpha \lambda}, c^+_{\alpha p \lambda}] \} |a\rangle \]
\[ = V_{\alpha p \lambda} |a\rangle \]
From this relation, we see that
\[ V_{p}^{\lambda} a = c_{p}^{+} |a\rangle + \frac{1}{\omega_{p} + M - H + i\epsilon} V_{p}^{\lambda} |a\rangle \]

\[ |p\rangle |p\alpha+\rangle = V_{p}^{\lambda} \]

must be the correct scattering state.

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In order to get the Low equation, all that remains is to get an expression for the R-matrix, and to transform that expression so that we obtain an equation for the scattering.

The R-matrix can be written in the following form

\[ \langle q | \mu | R | p \lambda \alpha \rangle = \langle q | \mu | p \lambda \alpha \rangle \]

But what is to be inserted for \( \gamma \lambda \)? In this formulation of the scattering problem, \( \gamma \lambda \) is not \( H = H_{\text{coup}} - \Delta M \), because our unperturbed Hamiltonian is not \( H_{\text{coup}} \). Rather we have taken as the unperturbed Hamiltonian the Hamiltonian operator \( H_{0} \) which is such that: \( H_{0} |p\alpha\rangle = (M + \omega_{p}) |p\alpha\rangle \).

Therefore, since \( H_{0} = H_{\text{coup}} - \Delta M \),

\[ \langle q | \mu | R | p \lambda \alpha \rangle = \langle q | \mu | p \lambda \alpha \rangle \]

Thus, one can write the R-matrix element as

\[ \langle q | \mu | R | p \lambda \alpha \rangle = \langle q | \mu | p \lambda \alpha \rangle \]

where, of course, \( c_{p}^{+} \lambda \mu \) and \( V_{p}^{+} \lambda \mu \) commute. Hence, since

\[ \langle q | \mu | p \lambda \alpha \rangle = \langle q | \mu | p \lambda \alpha \rangle \]

and since the operator \( H + \omega_{p} - M \) has a well defined inverse,

\[ \langle q | \mu | R | p \lambda \alpha \rangle = \langle q | \mu | p \lambda \alpha \rangle \]

If the form of the R-matrix is examined, the crossing theorem becomes obvious. For the R-matrix is invariant under the compound transformation of complex conjugation, interchanging \( p \lambda \) and \( q \mu \), and reversing the sign of \( \omega \). Note that the \( \zeta \)'s which are a nuisance, can be dispensed with in the first term, since the lowest value that \( H \) can take is defined as \( M \), and thus there is no pole in the first term. The second term just is an implicit formula for the sum of all the uncrossed diagrams as can be seen by expanding the second term in a perturbation series. Likewise, the first term represents all the crossed diagrams.
This is the formula Low employs to derive his integral equation for meson scattering. To put it into the same form as was deduced heuristically from causality relations, a set of eigenstates of the total Hamiltonian $H$ are introduced, which represent waves incident onto the nucleon plus outgoing scattered waves. For our purposes we will only specify these with the symbols $|n^+\rangle$.

In this complete set of states, for example $|0^+\rangle$ are the four complete nucleon states $|a\rangle$, and the set $|1^+\rangle$ is the complete set of states of the form $|p^+a\rangle$, and so forth. Then

$$\langle q\mu\beta | R | p^+a \rangle = \sum_n \frac{\langle \beta | V_n^+ | n^+ \rangle \langle n^+ | V_q \mu^+ | a \rangle}{M - \omega - E_n} + \sum_n \frac{\langle \beta | V_q^+ | n^+ \rangle \langle n^+ | V_p \mu^+ | a \rangle}{M + \omega_p - E_n + i\epsilon}.$$

We shall abandon the algebra at this point and concentrate on showing the features of this equation. If one recalls the Low equation as we had set it up in the last lecture:

$$\frac{f_a(\omega)}{v^2(k)k^2} = \sum_{\mu} \frac{2}{3} \frac{f_n^2}{4\pi^2} \omega \left(\frac{1}{\omega + \omega_{10}}\right) \left\{ \frac{1}{\omega' - \omega - i\epsilon} \frac{f_n(\omega')}{v^2(k')k'^2} \right\}^2 = \frac{1}{\omega + \omega_{10}} \sum_{\mu} \frac{A_{\alpha\beta} f_n(\omega')}{v^2(k')k'^2} \left(\frac{1}{\omega + \omega_{10}}\right)^2 d\omega' + \text{higher terms.}$$

It becomes evident that the term in $1/\omega$ comes just out of the sum over the intermediate states $|0^+\rangle$ which are the complete nucleon states with no free mesons. To see this, it is necessary to remember that the scattering amplitudes $f_a(\omega)$ are equal to some constant times $\omega R$.

There exists a vital difference in general between the Low equation and the equation resulting from dispersion theory. However, in the case of the static model which is being used here, this difference is non-existent. Only the matrix elements involving $|1^+\rangle$ as intermediate states are employed in the one-meson approximation; in general these matrix elements

$$\langle \beta | V_q^+ | n^+ \rangle$$

cannot be identified as matrix elements for the scattering amplitude, for the matrix elements are not on the energy shell; that is $\omega_n \neq \omega_q$. However, in this simple static theory, we can fix this up very easily so that we can equate these matrix elements to constants (dependent on $\omega$) times $R$-matrix elements which are on the energy shell. Namely, we can write

$$\langle \beta | V_q^+ | n^+ \rangle = \text{const.} \left(\frac{-f_n(\omega_n)}{\omega_n} \frac{q v(q)}{\omega_n v(q_n)} \sqrt{\frac{\omega_n}{\omega_n}}\right).$$
With this trick in mind it can now be understood why the Low equation is written in terms of the (somewhat) unexpected combination $f_a(\omega)/k^2 v^2(k)$. This trick can only be pulled in a simple theory such as this where the energy dependence of the matrix elements is so trivial. In the relativistic theories such a maneuver cannot be employed. Where this procedure is impossible, the equation generated from dispersion theory differs from the Low equation, for clearly the Low equation is a relation between matrix elements which are off the energy shell, while the matrix elements occurring in the dispersion equation are on the energy shell. In the static theory, Low's method leads to the interesting set of equations, but in the relativistic theory the interesting equations must be obtained in another way, namely by the method of dispersion relations.

The reason we have gone through all the trouble of deriving the Low equation is that it has been used to describe $\pi$ meson-nucleon scattering. The equation has been solved numerically by iteration, and with a cut-off at about $M$ and a coupling constant of around .08, results have been obtained in the one-meson approximation which are in quite good agreement with experiment. The one-meson approximation is slightly better than the fourth-order improved approximation which we have derived in this course. We could go on and work out numerically this one-meson approximation, but since we would not obtain substantially different results, we will instead just refer the interested reader to the literature.* What the effect of the higher terms which are neglected in the approximation may be has been estimated only crudely; it looked as though they would not contribute very much, but this hasn't been checked too well. However, it is not clear that this static theory is good enough to warrant beating it to death! So we will leave the problem of meson scattering at this point and go on to other problems where the static theory gives a reasonably good approximate description of the phenomena.

* Chew and Low, Phys. Rev. 101, 1570 (1956)
Two other interesting problems which can be treated in the static theory are the phenomenon of photoproduction, that is \( \gamma + N \rightarrow N + \pi \), and nuclear forces. Of these, we shall now discuss the interaction of a photon with a fixed nucleon in which a pion is produced.

We have to introduce the quantum field theory for electromagnetic fields, photons. Such a theory was constructed last year for free photons by choosing the gauge in which the fourth component \( \phi \) of the electromagnetic fields vanish. Also the \( A \) field was chosen so that \( V \cdot A = 0 \), that is, the photons are polarized transverse to their momentum. Under these conditions, the free photon field can be treated quantum mechanically by just carrying over our formalism for a neutral spinless particle but making \( \phi \) into a vector field \( A \) with two polarization degrees of freedom. It is clear that we are not setting up a manifestly covariant photon field theory, but the theory will be sufficient for our present purposes. In the presence of charges, we should have to introduce the fourth component of the electromagnetic field, or else longitudinal photon states to do the theory correctly; however, we shall neglect the coulomb interaction between charged particles here.

In the absence of an interaction between the nucleon-meson system and the Photon field the Hamiltonian is: \( H = M + H_{\text{free } \pi} + H_{\text{free } \gamma} + H_{\text{free } \gamma} \) where we are familiar with the first three operators, and the fourth is:

\[
H_{\text{free } \gamma} = \sum_{q, \epsilon} \frac{1}{2q^2} (\mathbf{p} \cdot x - \mathbf{p} \cdot q) + H.A.
\]

To fix the notation, the levels of a photon will be designated by its momentum \( q \) and polarization \( \epsilon \); the destruction and creation operators by \( a_{\epsilon q} \) and \( a_{\epsilon q}^\dagger \), respectively; and the field operator by

\[
A(x) = \sum_{q, \epsilon} a_{\epsilon q} \frac{\mathbf{p} \cdot \mathbf{x} - \mathbf{p} \cdot q}{\sqrt{2q^2}} a_{\epsilon q}^\dagger + H.A.
\]

The problem now is to write down the coupling term between the photon field and our nucleon-meson field system. To repeat, we are neglecting the terms representing coulomb interactions between mesons and bare nucleons. We are going to neglect any other coupling of order \( e^2 \) also. The reason is that in our cut-off theory of static nucleons, it is just not worthwhile to worry about higher order electromagnetic effects when we have to use only an approximate
theory of the much stronger mesonic interactions.

There will be a coupling Hamiltonian between the stationary nucleon and the free photons. For a "stationary" fermion this will consist of the two terms \( e^2 \frac{A^2}{2M} \) and \( +e/2M \frac{\alpha}{2} \cdot \vec{B} \) (where \( \vec{B} = \nabla \times \vec{A} \)). The first term will be disregarded because it is of order \( e^2 \); it is, of course, just the term which gives the Thompson cross-section for the scattering of photons by a free nucleon in analogy with electrodynamics. The second term just represents the interaction of the Dirac magnetic moment with the magnetic field. It is often an ambiguous task to decide when to keep or drop a small term in making an approximation. Since the Dirac term is linear in \( e \), we will carry it along. (Since it goes like \( 1/M \), however, we may also feel free to drop it for convenience.) One should notice that only the Dirac moment is put into the coupling term between the bare nucleon and the photon field. The anomalous part of the real nucleon magnetic moment presumably arises from the interaction of the meson cloud current of a complete nucleon with photons, so that it should be a quantity which may be calculated from the theory, and not an arbitrary number which is to be fed into the theory.

When an attempt is made to set up any electromagnetic interaction term, it must be done in such a way that the coupling is gauge invariant. Gauge invariance does not determine the interaction, but does specify the presence of certain terms in the coupling Hamiltonian when other terms are known to be included. Any interaction term involving the field strengths is automatically gauge invariant. Up to the present time, however, it has been possible to account for phenomena involving electromagnetism by just taking the coupling to be as simple as possible consistent with the requirement of gauge invariance. It appears that although gauge invariance allows the presence of terms involving any function of the electromagnetic field strengths, only the most simple term in the potentials actually exists in nature. This principle is fondly called "the principle of minimal electromagnetic interaction".

The minimal interaction which is gauge invariant and which couples the electromagnetic vector potential is simply to replace every \( p \) in a Hamiltonian by \( (\vec{p} - e\hat{\alpha}) \) or \( \nabla \) by \( \nabla - i e \hat{\alpha} \). It can be seen that this combination is gauge invariant in the following way. Suppose a gauge transformation is given:

\[
\vec{A}' = \vec{A} + \nabla \chi,
\]

then if one performs a unitary transformation on the wave functions:

\[
U = e^{i e \chi},
\]
we have:
\[ e^{i\chi_0} (V - ie\hat{A}) e^{-ie\chi} = V - ie\hat{A}. \]

The introduction of this substitution of course modifies the Hamiltonian for free pions. This produces just the term representing the meson current. When we do the same to the Dirac equation, we get just a term for the nucleon current, plus a term representing the Dirac moment. But it does much more, for there is a gradient in the interaction term between mesons and nucleons. In this way we get a "direct photoproduction term"

\[ \frac{-ief}{\mu} \mathbf{\hat{\sigma}} \cdot \mathbf{\hat{A}} \cdot \mathbf{\hat{\tau}} \cdot \phi(0) \]

This direct term is characteristic of a theory in which there is gradient coupling between mesons and nucleons. There are two relativistic meson theories; one of these involves gradient coupling, while the other, the \( \gamma_5 \) theory, does not. However, in the static or nonrelativistic limit of both these theories, we get the gradient coupling and thus also the direct photoproduction term.

**November 11, 1958**

In the next two lectures we are going to give a brief introduction to the theory of photo-production using the static model. However, we shall not be able to treat this phenomenon in as much detail as was put into the discussion of meson scattering. One of the principle goals of this discussion is to demonstrate how the renormalized meson-nucleon coupling constant can be obtained from data on photoproduction.

It was asserted last time that the proper way to put in the electromagnetic interaction term was to replace \( p \) by \( p - e\hat{A} \). This, however, leads to many complications because we are employing a theory with a cutoff, that is, a theory in which the interactions are not local. A non-local interaction can be expressed as one which depends on all powers of the gradient operator, therefore ambiguities arise when one starts to make the substitution of \( p - e\hat{A} \) for \( p \). Since we are not interested in results which depend in a complicated way on the source function, these complications will be ignored. People have attempted to use elaborate hypotheses concerning currents in the source, but none of these calculations are reliable. It just is not worthwhile to attempt to put that much detail into the static theory; one should wait until a correct local relativistic theory is developed.

When the electromagnetic interaction is inserted into our static theory,
certain terms can be correlated with the three parts of the π-N Hamiltonian as follows:

<table>
<thead>
<tr>
<th>Term of π-N Hamiltonian</th>
<th>Electromagnetic Interaction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{\text{free } \pi}$</td>
<td>Meson current interaction</td>
</tr>
<tr>
<td>$H_{\text{coupling}}$ (in Dirac theory)</td>
<td>Direct Photoproduction term</td>
</tr>
<tr>
<td>$H_{\text{free } N}$</td>
<td>Dirac Magnetic Moment</td>
</tr>
</tbody>
</table>

The first task is to write down the expressions for these interaction terms. As we said last time, we shall include only terms which are linear in $e$, and consequently omit the coupling with the time component of the electromagnetic field which carries the coulomb interactions between charged particles. I should like to point out explicitly that the introduction of the electromagnetic coupling destroys the invariance of the Hamiltonian in isotopic spin space. Thus the total isotopic spin can change under the influence of electromagnetic interactions, although $I_z$ remains a good quantum number.

Now to first order in $e$, the meson current operator is:

$$j = \frac{e}{2i} \left( \psi^+_C \gamma \gamma^+ \psi - \phi^+_C \gamma \gamma^+ \phi \right).$$

Since the interaction term is $-\int j \cdot A \, d^3x$, the following expression results:

$$H''_{\text{W}} = \frac{ie}{\gamma} \int (A^+_\gamma \cdot \gamma \cdot \phi - A \cdot \gamma \cdot \phi^+_\gamma) \, d^3x.$$

Next, disregarding the complications due to the cutoff, the following part of $H_{\text{coup}}$:

$$\sqrt{2} \frac{f_0}{\mu} \int \gamma \cdot V (\tau \phi^+_c(0) + \tau \phi^+_c(0)) \, d^3x \approx \sqrt{2} \frac{f_0}{\mu} \int \gamma \cdot V (\tau \phi^+_c(0) + \tau \phi^+_c(0)) \, d^3x$$

yields the direct interaction term:

$$H''_{\text{direct}} = -ie \sqrt{2} \frac{f_0}{\mu} \left\{ \tau \phi^+_c(0) \gamma \cdot A(0) - \tau \phi^+_c(0) \gamma \cdot A(0) \right\}.$$

And, finally, the Dirac moment of the nucleon gives rise to this interaction term:

$$H''_{\text{Dirac}} = \frac{e}{2M} \frac{1 + \tau^z}{2} \gamma \cdot B(0).$$

The experiments on photoproduction are usually analyzed according to the multipoles of the radiation. Since angular momentum and parity are both conserved (treating the π as having a negative intrinsic parity) by virtue of the symmetries of the interaction, from a given multipole, pions can be produced in only a few states, and vice-versa. Keeping track of the parity and angular momentum, the following table can be constructed.*

---

*For a complete discussion of vector multipole expansions, see Blatt and Weisskopf, *Theoretical Nuclear Physics*, page 796 ff.
The meson partial wave \( S_{1/2} \) with \( J^\pi = 1/2^- \) comes from El photons.

- \( P_{1/2} \) with \( 1/2^+ \) is \( M_1 \).
- \( P_{3/2} \) with \( 3/2^+ \) is \( M_1, E_2 \).
- \( D_{3/2} \) with \( 3/2^- \) is \( E_1, M_2 \).

For the present, we are going to investigate the production of mesons only near threshold. Thus we are concerned today only with S-wave mesons, and accordingly only the El part of the radiation. Consider for the moment the production of an S-D-F-G-... wave meson by a photon. As we have repeated many times, in our processes we shall restrict our calculations to be of first order in \( e \). With this restriction, once such a meson is made, it is free. Only P wave mesons are coupled in the static theory, and thus we need not worry about any higher order corrections in \( f \) due to S-D-F... wave mesons. (However, this does not hold in a correct relativistic theory, because recoil effects produce an effective S-wave interaction.)

Let us now treat the production of S-D-F-G... wave mesons; in the following lecture we shall take up Chew and Low's treatment of the production of P-wave mesons. Clearly the Dirac moment interaction involves only the M1 component of the photon and thus gives rise only to P-wave mesons. Therefore, it need not be considered in the current calculation. Consider now the term in the R-matrix due to the "direct photoproduction term". Since we take the value of the A operator at a point, we will get only S-wave mesons. For the process \( \gamma + p \rightarrow \pi^+ + n \) the first order diagram (in \( e \)) for the direct term is

![Diagram](image)

Analytically the corresponding term in the R-matrix is:

\[
\langle ns'| H^\text{Direct} | \gamma p\rangle = \langle ns' | -i \frac{e^2}{2 \mu} \tau_\mu \frac{-\tau}{\sqrt{2m_kV}} \frac{\hat{\sigma} \cdot \hat{e}}{\sqrt{2qV}} | ps \rangle
\]

\[
= \frac{i e}{\sqrt{2m_V}} \frac{1}{\sqrt{\omega_k q}} \ f_\circ \langle ns' | \tau_\mu \hat{\sigma} \cdot \hat{e} | ps \rangle .
\]

Recalling that the renormalization constants were defined in such a way that

\[
f_\circ \langle ns' | \tau_\mu \hat{\sigma} \cdot \hat{e} | ps \rangle = f_1 \langle \Phi_{\text{bare}} | \tau_\mu \hat{\sigma} \cdot \hat{e} | \Phi_{\text{bare}} | ps \rangle,
\]
the following simple result is obtained:

$$\langle ns'\pi^+|H^\pi_{\text{Direct}}^\pi|\gamma p\rangle = \langle s'|(\sigma \cdot \hat{e})s\rangle \frac{ie^2}{\sqrt{2}\mu} \frac{1}{\sqrt{\omega_k q}} = \langle ps'\pi^-|H^\pi_{\text{Direct}}^\pi|\gamma ns\rangle.$$  

It is easy to trace through the calculation and verify that the same matrix element is obtained for the process $\gamma + n \rightarrow p^+ p$. You will notice that in the production of non-interacting mesons, (those are S-D-F-G...), this theory indicates an absence of $\pi^0$'s. The theory indicates that only P-wave $\pi^0$'s are made, and then only by a charge-exchange scattering process. Experimentally the S-wave $\pi^0$'s are quite scarce; this is in agreement with the experimental evidence for small S-wave scattering phase shifts.

The next step is to compute the R-matrix element (in first order) employing the meson current electromagnetic interaction. The diagram is the same:

but there is a difference analytically. Since the matrix element will be computed for convenience with a plane wave meson wave function, we will get a piece of the P-wave amplitude in the following calculation. In a more detailed calculation, this piece may be extracted, since the P-wave part must be treated differently due to the interaction of P-wave mesons with nucleons. However, we shall not bother with this now, since our aim is to look at the photoproduction near threshold where the P-wave part of the amplitude vanishes. (In the following algebra, there are two subtleties. First, with the operator $\hat{c}_c$, we must create only a $\pi^+$ with momentum $k$, and thus we may do the integral quite easily. Second, remember that in our original notation $|\pi^+\rangle = \frac{-c^+}{\sqrt{2}}|\text{vac}\rangle$.)

$$\langle ns'\pi^+|H^\pi_{\text{Direct}}^\pi|\gamma p\rangle = \langle ns'\pi^+\rangle \frac{e^2}{\sqrt{2}\omega_k} \left\{ e^{i\frac{\hat{q} \cdot \hat{x}}{\omega_k}} \right\} \frac{d^3x + HA}{\omega_k} |\gamma p\rangle$$

$$= \langle ns'\pi^+\rangle \frac{e^2}{\sqrt{2}\omega_k} \left\{ e^{i\frac{\hat{q} \cdot \hat{x}}{\omega_k}} \right\} \frac{d^3x + HA}{\omega_k}$$

These two matrix elements have a simple approximate physical interpretation. For example, $\langle ns'|c^+_p|ps\rangle$ is roughly the amplitude that a neutron is dissociated virtually into a proton and a $\pi^-$. Similarly, $\langle ns'|c^-_n|ps\rangle$ can be thought of as roughly the amplitude that a neutron is virtually a proton and a $\pi^+$. The interaction can then be thought of as the interaction of the photon with the
virtual meson current in a physical nucleon. However, it is simpler to calculate these matrix elements exactly. One only has to recall the relations
\[
\langle \pi^+(k-q) | p_s \rangle = \frac{1}{M - \omega_{k-q} - H} V^+_{\pi^+(k-q)} | p_s \rangle
\]
\[
\langle n_s^1 | c^+ \rangle = \langle ns^1 | V \rangle_{\pi^- (q-k)} \frac{1}{M - \omega_{q-k} - H}
\]
which were used to derive the Low equation. Employing these, the answer is obtained quite easily
\[
\langle ns^1 | c^+ \rangle_{\pi^- (q-k)} = \frac{f_0}{\mu} \langle ns^1 | \frac{1}{2\omega} \frac{1}{q-k} V (q-k) \rangle
\]
\[
\langle \pi^+ | p_s \rangle = \frac{i \frac{1}{2} \sigma \cdot (q-k) v(q-k)}{\sqrt{2\omega} q-k} | p_s \rangle
\]
\[
\langle s^1 | \sigma \cdot (k-q) | s \rangle.
\]
It may again be verified that the matrix element for the process \( \gamma + n \rightarrow \pi^- + p \) is the same as the one just calculated. Collecting our results, we now write the R-matrix element to first order in \( e \) for the process \( \gamma + p \rightarrow \pi^+ + n \) or \( \gamma + n \rightarrow \pi^- + p \). This is the prediction of the static model for the photoproduction of S-D-F-G-... wave mesons. (Remember that there is a piece of the P-wave amplitude in here also.)
\[
\langle k \pi^+ n s^1 | R | q \rangle
\]
\[
\frac{i \omega f_1}{2} \langle s^1 | \sigma \cdot \{ e - \frac{2k \cdot e(k-q)}{\omega_{k-q}} \} | s \rangle.
\]
At threshold, in this static model, \( k = 0 \), and the dominant contribution to the photoproduction amplitude comes from the direct term, which produces only S-wave charged mesons. In computing the cross-section, the final nucleon spin states will be summed over and the initial spin states will be averaged, since the experiments have not yet been done with polarized nucleons. Therefore, at threshold, \( \omega = \mu \), by the Golden Rule:
\[
\frac{d\sigma}{d\Omega} = V \frac{2\pi V}{(2\pi)^3} \frac{k^2}{dk} \frac{1}{2} \Sigma_{\text{spins}} \Sigma_{\text{spins}} |R|^2
\]
But
\[
\frac{1}{2} \Sigma_{\text{initial}} \Sigma_{\text{final}} \langle s^1 | \sigma \cdot \{ e \} | s \rangle^2 = \frac{1}{2} \text{Trace} (\sigma \cdot \cdot e)^2 = \frac{1}{2} \text{Trace} (\cdot \cdot e) = 1.
\]
And thus,
\[
\frac{d\sigma}{d\Omega_{\text{photo}}} = \frac{2}{\mu^2} \left( \frac{1}{4\pi} \right) \frac{1}{q} \left( \frac{1}{4\pi} \right) \frac{2}{k}
\]
\[
\sigma_{\text{Tot.}}(\text{threshold}) = \frac{8\pi}{\mu^2} \frac{1}{137} (0.08) = \frac{1}{308} \text{ (millibarns)}
\]
where $\beta$ is the velocity of the meson. The experimental value for the cross-section is a little less than $0.30\beta$; it is about $\frac{1}{4}\beta$. The analysis can be improved by keeping track of recoil considerations; that is, if one just puts in the relativistic kinematics, the calculation can be improved by 30% or so. Excellent agreement with experiment near threshold can then be obtained, and the determination of the renormalized coupling constant gives a result which is quite compatible with the result from meson scattering.

Thus, having determined the renormalized coupling constant from scattering data, we can make several predictions on the behavior of the photoproduction cross-section at low energies. Namely, the static theory gives a prediction of the threshold cross-section. It says that the energy dependence should go as $k/q$. And there should be no neutral pions produced, while the threshold cross-sections for the processes $\gamma + p \rightarrow \pi^+ + n$ and $\gamma + n \rightarrow \pi^- + p$ should be the same. All of these predictions agree reasonably well with experiment.

If we extract from our scattering amplitude only the S-wave part, we have included in our calculation all the first-order terms which produce S-wave mesons. Thus we have a valid prediction of the theory for the production of S-wave mesons up to energies of a couple hundred mev. This prediction is not borne out as well as the threshold predictions. The S-wave cross-section follows more or less the behavior predicted by the static model, but cuts out eventually at higher energies. All along the range in energies, there are relatively few neutral pions produced.

The interesting and complicated question now is to calculate the P-wave amplitude. Part of the P-wave amplitude we have already obtained, but this is a trivial part of the problem. Likewise we will get another piece of the P-wave amplitude from the Dirac moment term, but that also is an easy part of the calculation. The important part of the P-wave problem is to put in the higher order corrections which come from the meson-nucleon interaction in the P-wave state. This will be sketched roughly according to the methods of Chew and Low and a simple physical picture of what is going on. It will be found that by this simple argument we will pick up the lion's share of the P-wave amplitude.

For the benefit of those who are not familiar with the experimental results on photoproduction, the following two graphs show the features that a theory should be able to predict.

$$\gamma + p \rightarrow \pi^+ + n$$
$$\gamma + p \rightarrow \pi^0 + p$$
November 13, 1958

Last time we calculated the exact matrix element (to first order in e) for the photoproduction of S-D-F-G-... wave mesons, using the static theory. At threshold, the only important term is the S-wave term, and thus the calculation gives an exact prediction of the cross-section near threshold. This prediction is in fairly good agreement with experiment, especially when recoil kinematics is used to improve the calculation. However, as soon as we go to higher energies, the effect of the production of higher partial waves has to be included. By far the most important of these is the P-wave. Unfortunately, the very factor which makes it so important, also makes it extremely difficult to calculate. That factor is, of course, that P-wave mesons interact strongly with nucleons.

The best calculation of the P-wave photoproduction up to date has been made by Chew and Low. In this lecture a sketch of the method and its results will be given. The mathematical arguments for the justification of the approximations employed will not be repeated here. The interested reader will find them in the original paper. After the mathematical sketch of the method has been given, a few physical arguments will be advanced on why the approximation is as good as it appears to be.

It was shown that the direct term gave only S-wave mesons, the Dirac term gives only P-wave mesons, and the meson current term produces mesons in all of the partial waves. The exact amplitude for the S-D-F-... wave production in the static theory was found to be the S-D-F-... part of:

$$\langle k\lambda\beta|H_B^0|\vec{q}\vec{a}\rangle = \langle\beta| \frac{ie\sigma}{\sqrt{2a_1 q}} \cdot \left\{ \frac{e^2 - 2(k\cdot q)k\cdot e}{a_1^2 q^2} \right\} \tau^+ \lambda |a\rangle$$

In these partial waves, there are no neutral mesons predicted. Now in calculating the P-wave production, corrections of higher order in $f_1$ have to be included, due to the strong coupling. The P-waves are produced through both the Dirac moment term and the meson current term. They are produced only by the M1 and E2 components of the incident photon. Now Chew and Low have given what they believe to be the principal term in the R-matrix element for P-waves. They allege that the most important extra effect of the meson current term can be taken into account by adding to the Dirac moment the anomalous nucleon moments. That is, the principle contribution to P-wave

*Chew and Low, Phys. Rev. 101, 1579 (1956)
photoproduction comes through the coupling of the observed magnetic moments with the M1 radiation.

In calculating the R-matrix for P-wave photoproduction they say that it is a good approximation to take the P-wave amplitude we obtained in part last time and add the term resulting from

$$H''_{\mu\gamma} = \frac{f_0}{f_1} \left\{ \left( \frac{1+\tau_z}{2} \right) - \left( \frac{1-\tau_z}{2} \right) \right\} \vec{\sigma} \cdot \vec{B}(0),$$

where $\mu_p$ and $\mu_n$ are the observed proton and neutron magnetic moments. We can rewrite $H''$ in the following manner:

$$H''_{\mu\gamma} = -\frac{f_0}{f_1} \left\{ \left( \frac{\mu_p+\mu_n}{2} \right) \left( \frac{\mu_p-\mu_n}{2} \right) \tau_z \right\} \vec{\sigma} \cdot \vec{B}(0) = H''_s + H''_v.$$  

The term involving $\tau_z$, that is $H''_v$, is very simple to calculate; in fact, it turns out to be expressible as an elastic scattering matrix element. On the other hand, the term not involving a nucleon isospin operator, $H''_s$, is quite difficult to calculate. However, since $\mu_p = 2.79(-\alpha/2M)$ and $\mu_n = -1.91(-\alpha/2M)$, the term which is difficult to compute is roughly 1/5 as large as the term with $\tau_z$. (To understand the signs in some places, it should be remembered that in these notes $e = -|e|.$) We will disregard the smaller term in the following because of the 1/5 factor and because it does not have resonant behavior as does the term kept. The fact that it does not participate in the resonance is easy to see. The initial state has $I = 1/2$. Since there is no isospin operator, there is no way we can pass through the resonant $I = 3/2$ state.

Thus, according to Chew and Low, it is a good approximation to set

$$\langle \hat{k}\lambda\beta | R_{\text{photo}} | q\hat{\sigma}\alpha \rangle = \langle \hat{k}\lambda\beta | H''_B | q\hat{\sigma}\alpha \rangle + \langle \hat{k}\lambda\beta - | H''_V | q\hat{\sigma}\alpha \rangle$$

(The minus sign is to signify that the state has only incoming waves in addition to the plane wave.)

Now let us define $\hat{q}' = \hat{q}k/q$ and make use of the fact that energy conservation demands that $q = \omega_k$. Then, employing the same trick as in the Low derivation of Low's scattering equations, we write

$$\langle \hat{k}\lambda\beta - | H''_V | q\hat{\sigma}\alpha \rangle = -\frac{\mu}{f_1} \frac{q}{k} \frac{\mu_p-\mu_n}{2} \langle \hat{k}\lambda\beta - | \frac{f_0}{\mu} \tau_z \vec{\sigma} \cdot \frac{\hat{q}' \times \hat{e}}{\sqrt{2qV}} | \alpha \rangle$$

and notice that the matrix element is just the elastic scattering R-matrix element representing an incident $\pi^0$ with momentum $\vec{m} = \hat{q}' \times \hat{e}$ being scattered
into the meson state $\vec{k}$, $\lambda$. That is:

$$\langle \vec{k}\lambda\beta | H'' | \vec{q}_{\text{qea}} \rangle = -\frac{\mu}{n_1} \frac{q}{\vec{k}} \frac{\mu_p - \mu_n}{2} \langle \vec{k}\lambda\beta | R^{\text{scat}} | \vec{m}_{\text{za}} \rangle.$$ 

At this point, still another approximation is employed in the evaluation of the scattering matrix element. By transformation theory, one can write

$$\langle \vec{k}\lambda\beta | R^{\text{scat}} | \vec{m}_{\text{za}} \rangle = \sum_{J_z, J} \langle \vec{k}\lambda\beta | k J J_z \rangle R^{\text{scat}} \langle k J J_z | \vec{m}_{\text{za}} \rangle.$$ 

By far the largest of the $R^{\text{scat}}_{kJ/J}$ is $R^{\text{scat}}_{k,3/2,3/2}$ especially near the $33$ resonance. It proves admissible to write

$$\langle \vec{k}\lambda\beta | R^{\text{scat}} | \vec{m}_{\text{za}} \rangle \approx \sum_{J_z} \langle \vec{k}\lambda\beta | k \frac{2}{2} J \frac{2}{2} J_z \rangle R^{\text{scat}} \langle k \frac{2}{2} J \frac{2}{2} J_z | \vec{m}_{\text{za}} \rangle.$$ 

With the expansion and neglect of all except the $33$ scattering matrix elements, it is easy to make an important statement. Consider the process $\gamma + p \rightarrow n + N$. Under these approximations, we say that the process goes through the $I = J/2$, $I_z = 1/2$ scattering state. Now, a glance at the Clebsch-Gordan coefficients on page 16 shows that the amplitude for the production of neutral mesons is $\sqrt{2}$ times the amplitude for producing charged mesons. The same statement holds for the process $\gamma + n \rightarrow \pi + N$. Thus, whenever it is possible to neglect the part of the amplitude from $H''_B$, e.g. at the $33$ resonance, the theory predicts that the cross-section for producing $\pi^0$ is double the cross-section for making charged $\pi$'s. Indeed this statement has been verified experimentally.

For the production of neutral pions, we have an estimate of the total amplitude in the single term due to the moments, for the term in $H''_B$ does not yield $\pi$'s. It is an easy arithmetical task to invert the formulae on page 21, to relate the $R$-matrix for scattering to the $33$ phase shift (neglecting the other phase shifts).

$$\langle \vec{k} \lambda \beta | R^{\text{scat}} | \vec{m}_{\text{za}} \rangle \approx -\frac{2m}{3} \frac{e \sin \delta_{33}}{\omega_k k^3} \langle s' | \vec{m} \vec{m} \cdot \vec{m} + 3 \vec{m} \vec{m} \cdot \vec{m} \rangle.$$ 

In computing the cross section, we will sum over the final nucleon spins, and average over initial nucleon spins. Therefore we want to compute

$$\frac{1}{2} \sum_s \langle s | \vec{m} \vec{m} \cdot \vec{m} + 3 \vec{m} \vec{m} \cdot \vec{m} | s' \rangle \langle s' | \vec{m} \vec{m} \cdot \vec{m} + 3 \vec{m} \vec{m} \cdot \vec{m} | s \rangle.$$ 

$$= \frac{1}{2} \text{Tr} \{4(\vec{m} \cdot \vec{m})^2 + (\vec{k} \cdot \vec{m})^2 \} = 2 \text{Tr} \{ 4(\vec{m} \cdot \vec{m})^2 + (\vec{k} \cdot \vec{m})^2 \} = 4k^2 m^2 + 3(k \cdot \vec{m})^2.$$ 

$$= 4k^2 \{1 + 3 \sin^2 \theta \cos^2 \phi\}.$$
Unless the photons are polarized, it is appropriate to average over the photon polarizations. This amounts to averaging over the angle \( \varphi \). Therefore the cross-section for the production of neutral pions is

\[
\frac{d\sigma}{dQ} = \frac{1}{(2\pi)^2} \left( \frac{s}{4} \right)^{1/2} \frac{1}{s} \left| R \right|^2 = \frac{1}{18} \frac{2 \times (\mu_p - \mu_n)^2}{f_1^2} \frac{\omega}{k^3} \sin^2 \delta_{33} (5 - 3 \cos^2 \theta)
\]

Our calculation also yields a prediction for the charged cross-section, which is

\[
\frac{d\sigma}{dQ} = \frac{1}{\mu^2} \left( \frac{s}{4\pi} \right) \left( \frac{f_1^2}{4\pi} \right) \frac{k}{\omega} + \frac{1}{2} \frac{d\sigma}{dQ} + \text{interference terms.}
\]

The prediction for neutral mesons is in good agreement with experiment in that it gives very beautifully the resonant behavior of the cross-section, gives the correct angular distribution, and gives the magnitude of the cross-section at resonance to about 15% or so. As was mentioned previously, the charged cross-section gives the correct S-wave production near threshold. It also gives the observed ratio of charges to neutral mesons at the resonance. However, the interference behavior is not described well. This is attributed to the critical influence of small terms, such as were neglected, on interference effects.

Now, the interesting question is why does this series of approximations give such impressive agreement. Physically, the important point is that a meson is produced virtually in a state in which there is a very strong interaction with nucleons. By far the strongest interaction is in the 33 state, in which there is a strong attractive potential. There is a tremendous enhancement of the correct meson wave function near the origin. This enhancement is taken care of by the factor \( \sin^2 \delta_{33} k^3 \). This is the physical reason for the overwhelming importance of the 33 state.

The remaining question is why were Chew and Low able to get away with considering only magnetic dipole transitions rather than including also electric quadrupole transitions. The reason here is that the quadrupole matrix element gets its principal contribution from relatively larger distances than does the magnetic dipole. But the enhancement is a big consideration only over short distances. Thus, the nature of the enhancement discriminates against the E2. Now the sharply localized effect of the meson current is to produce the anomalous magnetic moment. Hence the principal modification for the P-wave scattering is to put in the effect of the observed magnetic moments.
On the static theory, we have seen that the threshold ratio of \( \pi^- \) to \( \pi^+ \) production is 1. When the relativistic kinematics is considered, that ratio becomes \( 1 + 2\mu/M = 1.3 \). This ratio comes out of any approximation used on the relativistic meson theories, and can be obtained classically from recoil considerations.

We are now going to leave this static theory, and go on to investigate relativistic theories. By this time, its generally unsatisfactory character should be evident — the fact that neglecting recoil and putting in a cutoff we never know quite what we're doing, and that when we get disagreement with experiment, we don't know whether to attribute it to the crudity of the model or to the poorness of the approximations. The relativistic theories, as applied to meson-nucleon interactions, is even more unsatisfactory because, although they can be formulated in very beautiful ways, nothing at all can be done in using them to calculate answers except to employ perturbation theory. But as an expansion in powers of \( 1/5 \), perturbation theory is worthless. However, we do have a relativistic theory of photons, quantum electrodynamics. This theory is in very beautiful shape, thanks to the fact that the expansion parameter is \( 1/137 \). The expansion can be performed very well (actually it is an asymptotic expansion), and we get perfect agreement with experiment. This theory is therefore very satisfactory, and we shall take it up in the next lectures.

Time took the square root of the second degree equation is what turned out to be the right way. As we had to do it by a number of cumbersome arguments which we hadn't enter into. It is really a quite straightforward procedure to take the square root operator. We want to find an operator which is linear in \( \frac{p}{m} \), that is, of the form

\[
\hat{O} = \frac{p}{m} + \alpha
\]

such that \( \hat{O}^2 = \frac{p^2}{m^2} + \alpha^2 = \alpha \), and such that \( \hat{O} \) involves something which acts upon the components which describe spin degrees of freedom. It is generally to think that you could take a square root operator which was linear not only

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November 18, 1958

No. 6 Relativistic Quantum Mechanics of a Spin 1/2 Particle

We have learned how to treat spinless bosons relativistically. We saw how to write the relativistic wave equations, how to introduce the statistics into the theory, and how to couple the boson field with other fields. The problem still remains of how does one treat fermions relativistically. We shall treat this problem in the same way as we treated the relativistic wave mechanics of bosons. First, we will discuss one single uncoupled fermion, then we will take up the many-particle formalism and put in the statistics, and finally we shall couple the fermion field to other fields. All elementary fermions that we know of have spin 1/2, and so we shall not bother to demonstrate the possibility of constructing a wave mechanics for particles with spin 3/2, 5/2, etc. This can be done, but it is very complicated. All the known fermions of spin 3/2, 5/2, ... are regarded as composite particles.

Since for a free particle, we still desire that the Einstein relation $E^2 = k^2 + m^2$ should hold, and since in quantum mechanics, we want the operators for $E$ and $k$ to be represented by $i \frac{\partial}{\partial t}$ and $-i\nabla$ respectively, we must find a way to take the square root of the free particle second order equation

$$(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2)\psi = 0.$$ 

When $\psi$ was a world scalar, we could take the square root of this operator equation directly, and we obtained just the Schrödinger relativistic wave equation for spinless particles. This way of taking the square root obviously won't work for spin 1/2 particles, where we need a many component wave function to describe the spin degree of freedom, and for which the square root operator must involve something that operates on these components.

Dirac took the square root of the second degree equation in what turned out to be the right way. He was led to do it by a number of erroneous arguments which we needn't enter into. It is really a quite straightforward procedure to take the correct square root. We want to find an operator which is linear in $\frac{\partial}{\partial t}$, that is, of the form

$$i \frac{\partial}{\partial t} - H$$

such that $H^2 = p^2 + m^2 = -\nabla^2 + m^2$, and such that $H$ involves something which acts upon the components which describe spin degrees of freedom. It occurred to Dirac that one could form a square root operator which was linear not only
in \( \partial/\partial t \) but also in \( \mathbf{\hat{p}} = -i\nabla \). To Dirac, this appears to be the most likely way to set up a relativistically covariant theory. So he wrote down the following square root equation

\[
\mathbf{E}\mathbf{\psi} = (\mathbf{\hat{a}} \cdot \mathbf{\hat{p}} + \mathbf{\beta} m)\mathbf{\psi}
\]

where \( \mathbf{\hat{a}} \) and \( \mathbf{\beta} \) are operators which act only on the components of \( \mathbf{\psi} \) and which, being independent of \( \mathbf{\hat{r}} \) and \( \mathbf{\hat{p}} \), commute with the position and momentum operators.

(Note: In these lectures, we are going to use the coordinates \( x_1, x_2, x_3, x_4 = x, y, z, i \) to preserve the simplicities inherent in a Euclidean metric. We shall use the symbol \( \rightarrow \) to denote four vectors.)

Dirac was motivated in part by the classical equation

\[
E = \mathbf{\hat{v}} \cdot \mathbf{\hat{p}} + \sqrt{1 - \mathbf{\hat{v}}^2} m
\]

and was led to write down a formal operator expression of the same kind. We will see later that the expectation values of the operators \( \mathbf{\hat{a}} \) and \( \mathbf{\beta} \) are just \( \mathbf{\hat{v}} \) and \( \sqrt{1 - \mathbf{\hat{v}}^2} \), but it is best just to regard this analogy as a useful psychological piece of motivation, because, as I want to emphasize, \( \mathbf{\hat{a}} \) and \( \mathbf{\beta} \) are not functions of \( \mathbf{\hat{p}} \). The \( \mathbf{\hat{a}} \) and \( \mathbf{\beta} \) are \( n \times n \) matrices which act upon the component wave function and which commute with \( \mathbf{\hat{p}} \).

What are the properties of these operators \( \mathbf{\hat{a}}_1, \mathbf{\hat{a}}_2, \mathbf{\hat{a}}_3, \) and \( \mathbf{\beta} \)? Well, the determining factor is that this equation really be a square root of the second order equation, that is

\[
H^2 = (\mathbf{\hat{a}} \cdot \mathbf{\hat{p}} + \mathbf{\beta} m)^2 = p^2 + m^2.
\]

This single requirement determines that

\[
\mathbf{\hat{a}}_j \mathbf{\hat{a}}_i = -\mathbf{\hat{a}}_i \mathbf{\hat{a}}_j \quad \text{for } i \neq j, \quad \mathbf{\hat{a}}_i \mathbf{\beta} = -\mathbf{\beta} \mathbf{\hat{a}}_i \quad \mathbf{\hat{a}}_i^2 = \beta^2 = 1.
\]

Now, if we consider the group ring generated by these four \( n \times n \) matrices, we see that there can be at most 16 independent matrices in the group ring, because the algebraic properties determined above make it possible to simplify every product containing two identical matrices. Our problem is to find the irreducible representations of this group ring. There certainly exists no representation of degree 1 (i.e. \( n = 1 \)), for all matrices of degree 1 commute. We are very familiar with three anticommuting matrices of degree 2, viz. the Pauli matrices \( \sigma_x, \sigma_y, \) and \( \sigma_z \). But there are only four linearly independent \( 2 \times 2 \) matrices: \( 1, \sigma_x, \sigma_y, \sigma_z \). Since the unit matrix commutes with everything, there exists no representation of the group ring of second degree. It is very easy to show that there exists no representation of an odd degree. For, consider
the determinant of the following product
\[ \text{Det } a_1 \beta = (\text{Det } a_1)(\text{Det } \beta). \]

But since the determinant of any of these four matrices cannot vanish (they are their own inverses and hence their determinants are \( \pm 1 \)), the following identity
\[ \text{Det } a_1 \beta = \text{Det } (-a_1) = (\text{Det } -I)(\text{Det } \beta)(\text{Det } a_1) = (-)^n(\text{Det } a_1)(\text{Det } \beta) \]
indicates that \( n \) must be an even number.

Perhaps the simplest way to prove that there exists a representation of degree 4 is to exhibit one! We find it convenient to write 4x4 matrices using 2x2 matrices as elements in another 2x2 matrix.

\[ \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \dd = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \]

We have shown that a representation by 4x4 matrices is possible. It may be verified also that the 16 possible products of \( a_1, \beta \) are represented by 16 linearly independent matrices. It then follows that the hyper-complex number system, of which the sixteen possible products are basic elements, is isomorphic to the complete matrix ring of all 4x4 matrices. Because of this isomorphism, theorems from group theory and the theory of rings and ideals can be applied. These theorems guarantee that, up to equivalence transformations (which in quantum mechanics are always unitary transformations), there is only one irreducible representation, and that every reducible representation can be decomposed completely into irreducible ones, which are all equivalent to the one exhibited.

It may now be remarked that we might have set the requirement that matrices \( \dd \) and \( \beta \) are to be Hermitian in order that the free fermion Hamiltonian be Hermitian. Since we have exhibited a representation in which the matrices are Hermitian, and since all the representations may be obtained from the given one by compounding them and performing unitary transformations, all the representations are automatically Hermitian representations.

For the most part, it is best in working with the Dirac equation to just use the algebraic properties of the matrix operators, rather than to do a lot of matrix arithmetic. However, when we are forced to use a definite representation, it will probably be the one displayed above. This will be called the standard representation (SR) in these notes, because it is the one commonly
employed in the literature. There is one other representation which is
commonly employed in problems dealing with charge conjugation. We will
discuss it when the need arises.

**Momentum Eigenstates**

We now want to consider the physics of a spin $1/2$ particle. The best
way to do this is to solve problems, of which the simplest is the problem
of a free particle. What are the eigensolutions of the Dirac equation which
represent free particles with a definite momentum and energy?

The solutions will be the product of a four-component spinor and a plane
wave:

$$\psi = \frac{u e^{i(k \cdot x - \omega t)}}{\sqrt{V}}.$$ 

From the consideration of the squared equation, it can be seen easily that
$$\omega^2 = m^2 + k^2.$$ The first order equation which must be satisfied by the spinor
is:

$$\omega u = (\alpha \cdot \hat{k} + \beta m) u.$$ 

The best way to find out the nature of the solutions of this equation is to
look at the case of a particle at rest, i.e. find the solutions for $k = 0$.
In this case, we have two solutions for $\omega = m$.

$$u_{\uparrow} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_{\downarrow} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

We expect that these two solutions will represent the two possible spin
states of a spin $1/2$ particle. Indeed, this is true; we will show this
later when we come to discuss the angular momentum operator.

There is an easy way to construct the solutions for a particle in motion.
If we define $E = \sqrt{m^2 + k^2}$ and consider the identity

$$w = \frac{E + \alpha \cdot \hat{k} + \beta m}{2E} w + \frac{E - \alpha \cdot \hat{k} - \beta m}{2E} w = u + v$$

we see that an arbitrary spinor $w$ can be split up into two parts -- $u$ and $v$ --
in such a way that the spinor $u$ is a solution of the Dirac equation for a
particle with momentum $k$ and energy $\omega = E \frac{E}{k}$. For,

$$(\omega - \alpha \cdot \hat{k} - \beta m) u = \frac{1}{2E} (E^2 - k^2 - m^2) w = 0.$$ 

If we normalize the spinor $u$, so that $u u = \frac{E}{m}$, then we can generate the two
solutions for $\omega = E$:
Also, it may be verified that there are only two solutions for \( \omega = E_k \) of the eigenvalue equation by noticing that \((E + \hat{a} \cdot \vec{k} + \beta m)\) operating on the remaining two basis spinors

\[
\begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix}
\]
and

\[
\begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix}
\]
yields just a linear combination of the two spinors \( u_k^\uparrow \) and \( u_k^\downarrow \).

It should be noticed that the \( v \) spinors which are generated above also satisfy the Dirac equation, but for \( \omega = -E_k \). There are two additional solutions of the Dirac equation for a given momentum \( k \). In the case of a particle at rest, these solutions may be taken to be

\[
v_0^\uparrow = \begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix}, \quad \text{and} \quad v_0^\downarrow = \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}.
\]

The two spinors for \( k \) with negative frequencies may be obtained in the same manner we obtained the positive frequency spinors:

\[
v_k^\uparrow = \frac{E - a \cdot k - \beta m}{\sqrt{2m(E+m)}} \quad v_0^\uparrow = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix}
-k \\
-1 \\
0 \\
0
\end{pmatrix};
\]

\[
v_k^\downarrow = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix}
-k_x + ik_y \\
0 \\
0 \\
k_z \\
E + m
\end{pmatrix}, \quad \text{and} \quad v_0^\downarrow = \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}.
\]

It is an easy exercise to verify that the four solutions for a given value of the momentum \( k \) are orthogonal to one another, and thus form an orthonormal basis for the four dimensional spinor space if \( u^+u \) is taken to be 1. This is another way of normalizing these spinors, which is widely used. However, our method has the advantage that it is relativistically invariant. We normalize so that \( u^+u = E/m \).
Thus for each momentum $k$, there are four solutions of the Dirac equation. Two of these are for a positive frequency, that is, $\omega = \pm E$, while there are two also for $\omega = -E$. This, of course, was very traumatic in the days of Dirac. To eliminate the possibility of endless transitions to electron states of lower and lower energies, Dirac created the famous "hole theory". In this theory, it was imagined that all the negative energy states were occupied for the most part, and that this "sea" of negative energy electrons did not give rise to observable fields, etc. Thus a positive energy electron was prevented, in general, from making a transition to a negative energy state on account of the exclusion principle. However, the possibility remained that an electron in a negative energy state could undergo a transition to a positive energy state. The hole in the sea would behave in every respect like a particle with a positive charge, the positron. By this mechanism, the Dirac hole theory could explain pair production, and pair annihilation was envisioned as just the inverse process.

But today, we don't have to go through this mysticism, although it is still perfectly correct. You can, if you wish, use Dirac's concepts to interpret the negative frequency solutions. However, I think it is much more sensible these days to say we know what to do with these equations which have positive and negative frequency solutions. We've seen what to do in the case of Klein-Gordon equation, the Maxwell equation, and so on. Namely, there are two distinct classes of particles in nature. Particles in Class I have distinct antiparticles; examples are $\pi^-$, $K^-$, and $e^-$. On the other hand, the particles of Class II are identical with their antiparticles; examples are $\pi^+$ and $\gamma$. In both classes the solutions with positive frequencies are associated with the destruction of something, while the negative frequency solutions are associated with the creation of something. In Class I, the positive frequency solutions are correlated to the destruction of the particle, and those with negative frequencies are correlated to the creation of the anti-particle. The complex conjugate of a positive frequency solution then is associated with the creation of the particle, and the complex conjugate of a negative frequency solution with the destruction of the anti-particle. In Class II, the situation is a little simpler, for the positive frequency solutions are associated with the destruction of the particle, and the negative frequency solutions, which are the complex conjugates of the positive frequency solutions, are correlated to the creation of the particle.

Exactly the same situation holds for the solutions of the Dirac equation. However, no fermions have been discovered which are their own anti-particles. In all the cases of fermions we know of, the particle and anti-particle are separated not only by being two distinct degrees of freedom, but also by an absolute conservation law, which forbids particles from turning into anti-particles, and vice-versa. You remember in the case of bosons, this was not always so; for the gamma ray and the $\nu^0$, the particle and the anti-particle were the same. In the case of the $K^0$ and the anti-$K^0$ the particle and the anti-particle were distinct, but the law which prevents one from turning into the other, the law of conservation of strangeness, is not absolute, which means that $K^0$ and $\bar{K}^0$ can turn into each other -- weakly -- and that, therefore, it is sometimes worthwhile to consider linear combinations of them, which are their own antiparticles, $K^0_1$ and $K^0_2$. In the case of fermions this has never occurred.
Another way of describing what is to be done, which is applicable to one particle phenomena, is that the positive frequency solutions are to represent the particle, and the complex conjugates of the negative frequency solutions are to describe the anti-particles. Thus, if we want to look at one electron or positron, we say that, starting arbitrarily with the notion that the electron is the particle and the positron the anti-particle, the positive frequency solutions of the Dirac equation will describe the electron, free or in an external field. The negative frequency solutions, complex conjugated, are the positron solutions in the same field. (Actually, depending upon the representation, we may have to shuffle the spinor components a little also, but this is a trivial complication here.) This is the same procedure we used in the case of the charged pions, and it works here just as it worked in that case. So there is no need to construct an elaborate "mystic" for the Dirac equation.

Thus we have constructed the two possible solutions for an electron with momentum $k$, which represent the two possible orientations of the spin degree of freedom. They are:

$$u_k^{\uparrow} = \left(\begin{array}{c} e^{i(k \cdot \mathbf{x} - Et)} \\ \sqrt{V} \end{array}\right) \quad \text{and} \quad u_k^{\downarrow} = \left(\begin{array}{c} e^{i(k \cdot \mathbf{x} + Et)} \\ \sqrt{V} \end{array}\right).$$

On the other hand,

$$v_k^{\uparrow} = \left(\begin{array}{c} e^{i(k \cdot \mathbf{x} + Et)} \\ \sqrt{V} \end{array}\right) \quad \text{and} \quad v_k^{\downarrow} = \left(\begin{array}{c} e^{i(k \cdot \mathbf{x} + Et)} \\ \sqrt{V} \end{array}\right),$$

except for a shuffling of components, are the complex conjugates of the solutions for a positron with momentum $-k$. In this way the extra set of two solutions are seen to be useful in describing a positron, just as in the Klein-Gordon equation, where the extra solution was useful in describing the $\pi^+$. In the case of the free particle where both positrons and electrons behave alike, this is not very instructive. But as soon as an external electromagnetic field is applied, the positive and negative frequency solutions behave quite differently.

We will come back later to the form of the solutions, gaining practice in manipulating them, when we come to solve problems in electrodynamics.
Relativistic Invariance of the Dirac Equation

The next obvious theoretical question is what are the relativistic transformation properties of the spinor quantities. As the Dirac equation stands, it doesn't look terribly Lorentz invariant. Pauli suggested that it was possible to write the equation so that it "looked" Lorentz invariant. This is done by multiplying the Dirac equation by \( \beta \) and defining a new set of four matrices \( \gamma'_\mu \):

\[
\gamma'_\mu = -i\beta \gamma^\mu.
\]

Then the Dirac equation becomes simply

\[
(\gamma'_\mu \partial^ \mu + m)\psi = 0
\]

where

\[
\partial^\mu = \frac{\partial}{\partial x^\mu}.
\]

One should notice that all the \( \gamma' \)'s are hermitian, and that the following relation holds:

\[
\gamma'_\mu \gamma'_{\nu} + \gamma'_{\nu} \gamma'_\mu = 2\delta_{\mu\nu}
\]

Suppose we consider a given Lorentz transformation as specified by an orthogonal, not unitary, transformation of coordinates:

\[
x'^\mu = \sum_{\nu} L_{\nu\mu} x^\nu
\]

where

\[
(L^{-1})_{\nu\mu} = L_{\nu\mu}.
\]

Under transformation, we expect that the components of the spinor wave function \( \psi \) will be shuffled as, for example, the components of the electromagnetic field tensor undergo a shuffling under a Lorentz transformation. That is, in the primed coordinate system, the new wave function \( \psi' \) will be obtained from \( \psi \) by a linear transformation \( S \):

\[
\psi' = S\psi.
\]

The transformation matrix \( S \) is to depend only upon the elements of the Lorentz transformation \( L \).

The condition which serves to pick out the \( S \) is that \( \psi' \) must satisfy the Dirac equation in the new coordinate system, i.e.,

\[
(\gamma'_{\nu} \partial'_{\nu} + m)\psi' = 0.
\]

By multiplying the Dirac equation \( \psi' \) by \( S \), and writing \( \psi = S^{-1}\psi' \), the following equation for \( \psi' \) may be deduced:

\[
(S_{\nu\mu} S^{-1}_{\nu'\mu'} \partial'_{\mu'} + m)\psi' = 0.
\]

In order that these two equations may be identical,

\[
S_{\nu\mu} S^{-1}_{\nu'\mu'} = \gamma'_{\nu} L_{\nu'\mu'},
\]

since

\[
\partial'_{\nu} = \sum_{\mu} L_{\nu\mu} \partial^\mu.
\]
Finally, by making use of the orthogonality condition $L_{\mu} \sigma_{\mu} = \delta_{\lambda} \sigma_{\lambda}$, it is possible to write the condition on the $S$ as:

$$S^{-1} \gamma_{\mu} S = L_{\mu \nu} \gamma_{\nu}.$$ 

We shall now construct the matrices $S$ for the Lorentz transformations which belong to the restricted group in which the direction of time is left unchanged, that is, $L_{44} > 0$. All of these transformations are either proper Lorentz transformations, that is, $|L| = +1$, and $L_{44} > 0$, or can be obtained as the products of space reversal and the proper Lorentz transformations.

The proper Lorentz transformations can be built up from infinitesimal Lorentz transformations. We may thus consider the problem of determining $S$ for an infinitesimal transformation, realizing that the $S$ for the full transformation can be constructed easily by exponentiation. Suppose the infinitesimal Lorentz transformation $L_{\mu \nu} = \delta_{\mu \nu} + \xi_{\mu \nu}$ is given. Then $\xi_{\mu \nu}$ must be antisymmetric, since $L_{\mu \nu} L_{\lambda \nu} = \delta_{\mu \lambda}$ to first order. Now write

$$S = 1 + \theta,$$

where $\theta$ is of first order. Then, to first order,

$$S^{-1} = 1 - \theta.$$

The equation $S^{-1} \gamma_{\mu} S = L_{\mu \nu} \gamma_{\nu}$ then yields:

$$[\gamma_{\mu}, \theta] = \xi_{\mu \nu} \gamma_{\nu}.$$

Our problem is to determine the operator $\theta$. Now it clearly must be linear in $L_{\mu \rho}$, and quadratic in the $\gamma$ matrices. Also we expect that it depends on a systematic correlation between the $\gamma_{\rho} \gamma_{\lambda}$ and $\xi_{\rho \lambda}$. The only thing that fits the bill is

$$\text{const.}(\xi_{\rho \lambda} \gamma_{\rho} \gamma_{\lambda}).$$

In trying this out, since the $\gamma$'s obey anticommutation relations, it is convenient to employ the identity

$$[A, BC] = \{A, B\} C - B \{A, C\}.$$

Hence,

$$[\gamma_{\mu}, \xi_{\rho \lambda} \gamma_{\rho} \gamma_{\lambda}] = 2 \xi_{\rho \lambda} \delta_{\mu \rho} \gamma_{\lambda} - 2 \gamma_{\rho} \xi_{\rho \lambda} \delta_{\mu \lambda} = 4 \xi_{\mu \lambda} \gamma_{\lambda}$$

since $\xi_{\nu \mu} = - \xi_{\mu \nu}$.

If, for convenience, we define $\sigma_{\mu \nu} = \frac{1}{21} (\gamma_{\mu} \gamma_{\nu} - \gamma_{\nu} \gamma_{\mu})$, then

$$\theta = \frac{1}{4} \xi_{\mu \nu} \sigma_{\mu \nu}.$$
By examining the form of $S$ for an infinitesimal spatial rotation, we can find the form of the spin operators:

$$
\xi_{\mu\nu} = \begin{pmatrix}
0 & d\theta_x & -d\theta_y \\
-d\theta_x & 0 & d\theta_z \\
d\theta_y & -d\theta_z & 0
\end{pmatrix}
$$

We have:

$$
S = 1 + \frac{i}{2} \left( \sigma_{yx} \frac{d\theta_x}{2} + \sigma_{zz} \frac{d\theta_y}{2} + \sigma_{xy} \frac{d\theta_z}{2} \right).
$$

Let us now concentrate on a rotation about the z-axis. The generator of the rotation matrix is then the z-component of the angular momentum operator $J_z$. In this discussion, we have not considered the change in the wave function due to an infinitesimal change in the position under the rotation. As we know, that is generated by $L_z$. Hence,

$$
J_z = L_z + S_z = L_z + \frac{1}{2} \sigma_{xy}.
$$

But

$$
\frac{\sigma_{xy}}{\lambda} = \begin{pmatrix}
\sigma_x & 0 \\
0 & \sigma_z
\end{pmatrix} = \Sigma_z.
$$

Since $\Sigma_z^2 = 1$, we see that the Dirac equation does indeed represent a particle which has spin 1/2.

As examples of the exponentiation technique for getting the full transformation from the infinitesimal one, we will obtain the matrices for a rotation about the z-axis by an angle $\theta$, and the transformation for the switch to a system moving with velocity $v = \tanh \eta$ along the x-axis. First, however, let us identify the $\sigma_{\mu\nu}$:

$$
\sigma_{\mu\nu} = \begin{pmatrix}
0 & \Sigma_z & -\Sigma_y & a_x \\
-\Sigma_z & 0 & \Sigma_x & a_y \\
\Sigma_y & -\Sigma_x & 0 & a_z \\
a_x & -a_y & -a_z & 0
\end{pmatrix}
$$

Then, for the spatial rotation,

$$
S(\theta_z) = \exp(i\theta_z \Sigma_z/2) = \cos \theta_z/2 + i\Sigma_z \sin \theta_z/2.
$$

And for an infinitesimal velocity transformation: $\xi_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & \text{id}y \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\text{id}y & 0 & 0 & 0
\end{pmatrix}$

Thus:

$$
S(\eta) = \exp(\frac{i}{2} \text{id}_x \eta) = \cosh \frac{\eta}{2} - a_x \sinh \frac{\eta}{2}.
$$
Finally, we consider $S$ transformation matrix for a spatial inversion of the coordinates. It is required that
\[ S^{-1} Y S = \tilde{Y}, \quad \text{and} \quad S^{-1} \gamma_4 S = \gamma_4. \]
Evidently, $S = \beta$ fits the bill.

It should be observed that the transformation matrix $S$ is not unitary whenever we deal with a transformation to a moving system. This is so for the space-time components of the Lorentz transformation tensor $L_{\mu\nu}$ are imaginary. Notice, however, that $\beta$ commutes with the space-space components of $\sigma_{\mu\nu}$, and anticommutes with the space-time components. Therefore, if
\[ S = 1 + \frac{1}{4} \xi_{\mu\nu} \sigma_{\mu\nu}, \]
\[ S^{-1} = 1 - \frac{1}{4} \xi_{\mu\nu} \sigma_{\mu\nu}, \]
and
\[ S^+ = 1 - \frac{1}{4} \xi_{\mu\nu} \sigma^+_{\mu\nu} = 1 - \frac{1}{4} \xi_{\mu\nu} \sigma_{\mu\nu} \]
\[ = 1 - \frac{1}{4} \xi_{\mu\nu} \beta \sigma_{\mu\nu} \beta = \beta S^{-1} \beta. \]
Since $\beta^2 = 1$, this relation holds also for the full transformations which are obtained by exponentiation. It also is valid for the reflection transformation.

As will become evident in a moment, it is very convenient to deal with a slightly different quantity that the hermitian adjoint of $\psi$. We introduce the Dirac adjoint $\bar{\psi} = \psi^\dagger$.

All observable quantities in quantum mechanics are constructed from bilinear forms in $\psi^+$ and $\psi$. Due to the simple relation between $\psi^+$ and $\bar{\psi}$, these quantities can be constructed from $\bar{\psi}$ and $\psi$. Now a very important question is how do these bilinear forms transform under a Lorentz transformation. Considering only bilinear forms which do not involve derivatives, there are but 16 of them, which may be arranged in the following groups:

\[ \bar{\psi} \psi \]

\[ \bar{\psi} \gamma_\mu \psi \]
\[ \bar{\psi} \xi_{\mu\nu} \psi = \bar{\psi} \frac{1}{2i} \left( \gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu \right) \psi \]
\[ \bar{\psi} \xi_{\mu\nu} \gamma_\lambda \psi \quad \mu \neq \nu \neq \lambda \neq \mu \]
\[ \bar{\psi} \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho \psi \quad \text{all 4 subscripts different.} \]

Let us now find out how, for example, $\bar{\psi} \psi$ transforms.
\[ \bar{\psi}^+ \psi = \psi^+ \psi \quad \text{and} \quad \psi^+ S^+ \beta S \psi = \psi^+ \beta \psi = \bar{\psi} \psi. \]

Thus, $\bar{\psi} \psi$ is a scalar quantity. Similarly, let us look at $\bar{\psi} \gamma_\mu \psi$.\]
\[ \bar{\psi}^\dagger \gamma_\mu \psi = \bar{\psi} S^\dagger \beta \gamma_\mu \gamma_5 \gamma_\mu \psi = \bar{\psi} S^{-1} \gamma_\mu \gamma_5 \gamma_\mu \psi = L_{\mu \nu} \bar{\psi} \gamma_\nu \psi \]

Therefore, \( \bar{\psi} \gamma_\mu \psi \) transforms like a four-vector. In exactly the same way, it can be verified that \( \bar{\psi} \sigma_{\mu \nu} \gamma_5 \gamma_\mu \gamma_\nu \psi \) transforms like an antisymmetric tensor, that \( \bar{\psi} \frac{1}{4 !} \gamma_\mu \gamma_5 \gamma_\nu \gamma_\lambda \gamma_\rho \gamma_\sigma \gamma_\tau \ldots \) (all indices different) is a completely antisymmetric tensor of the third rank, and that \( \bar{\psi} \gamma_\mu \gamma_5 \gamma_\nu \gamma_\rho \gamma_\lambda \gamma_\sigma \gamma_\tau \ldots \) (all indices different) is a completely antisymmetric tensor of fourth rank. Now, it is well-known that a completely antisymmetric tensor of fourth rank transforms like a pseudoscalar, and that a completely antisymmetric tensor of third rank is equivalent to a pseudovector.

If we introduce the Levi-Civita tensor density \( \varepsilon_{\mu \nu \lambda \rho} \), so defined that
\[ \varepsilon_{\mu \nu \lambda \rho} = +1 \text{ if } (\mu, \nu, \lambda, \rho) \text{ is an even permutation of } (1, 2, 3, 4), \]
\[ \varepsilon_{\mu \nu \lambda \rho} = -1 \text{ if } (\mu, \nu, \lambda, \rho) \text{ is an odd permutation of } (1, 2, 3, 4), \]
\[ \varepsilon_{\mu \nu \lambda \rho} = 0 \text{ otherwise,} \]
then
\[ \frac{1}{4 !} \bar{\psi} \varepsilon_{\mu \nu \lambda \rho} \gamma_\mu \gamma_5 \gamma_\nu \gamma_\rho \gamma_\lambda \gamma_\sigma \gamma_\tau \ldots \gamma_\zeta \ldots \gamma_{\delta} \ldots = \bar{\psi} \gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\lambda \gamma_\sigma \gamma_\tau \ldots \gamma_\zeta \ldots \gamma_{\delta} \ldots \]

(where \( \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 \)) transforms like a pseudoscalar. Finally, we may note that
\[ \frac{1}{13 !} \bar{\psi} \varepsilon_{\mu \nu \lambda \rho} \gamma_\mu \gamma_5 \gamma_\nu \gamma_\rho \gamma_\lambda \gamma_\sigma \gamma_\tau \ldots \gamma_\zeta \ldots \gamma_{\delta} \ldots = \bar{\psi} \frac{1}{13 !} \gamma_\mu \gamma_5 \gamma_\nu \gamma_\rho \gamma_\lambda \gamma_\sigma \gamma_\tau \ldots \gamma_\zeta \ldots \gamma_{\delta} \ldots \]
transforms like a pseudovector.

\( \gamma_5 \) is an interesting and important matrix. One should remember that its square is unity, and that it anticommutes with \( \gamma_1, \gamma_2, \gamma_3, \) and \( \gamma_4 \).
\[ \gamma_5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \]

Current-Density Four Vector

To introduce the coupling with electromagnetism, we want to define a charge current density for fermions. This quantity is a four vector. We can form a four vector by taking gradients, or by using the set of matrices \( \gamma_\mu \).

The choice between these is made by looking for a conserved four vector, that is, one which satisfies the equation of continuity. Evidently \( \bar{\psi} \gamma_\mu \psi \) does not fit the bill, for \( \gamma_\mu \gamma_5 \gamma_\nu \gamma_\rho \gamma_\lambda \gamma_\sigma \gamma_\tau \ldots \gamma_\zeta \ldots \gamma_{\delta} \ldots \gamma_\mu \gamma_5 \gamma_\nu \gamma_\rho \gamma_\lambda \gamma_\sigma \gamma_\tau \ldots \gamma_\zeta \ldots \gamma_{\delta} \ldots \psi \neq 0 \), in general. However, if the equation for the Dirac adjoint \( \bar{\psi} \) is constructed:

\[ (\gamma_\mu \gamma_5 + m)\psi = 0 \]
\[ \gamma_\mu \gamma_5 \gamma_\nu \gamma_\rho \gamma_\lambda \gamma_\sigma \gamma_\tau \ldots \gamma_\zeta \ldots \gamma_{\delta} \ldots \psi = 0 \]
\[ \gamma_\mu \gamma_5 \gamma_\nu \gamma_\rho \gamma_\lambda \gamma_\sigma \gamma_\tau \ldots \gamma_\zeta \ldots \gamma_{\delta} \ldots \psi \neq 0 \]

it can be seen easily that \( \gamma_\mu \psi \) forms a conserved four vector:
\[ \delta^{\mu} \left( \bar{\psi} \Gamma_{\mu} \psi \right) = m \bar{\psi} \psi - \bar{\psi} m \psi = 0. \]

Therefore, the current-density four vector is taken to be \( \mathbf{j} = i e \bar{\psi} \gamma \psi \).

In the three-dimensional notation

\[ \rho = e \bar{\psi} \gamma \psi \]
\[ \mathbf{j} = e \bar{\psi} \gamma \mathbf{A} \psi. \]

The same result can be seen in an easier way, if we just take the Dirac equation and follow the familiar prescription for inserting the electromagnetic coupling. That is, just replace \( \delta^{\mu} \) by \( \delta^{\mu} - i e A^{\mu} \). Then the Dirac equation becomes

\[ (\gamma^{\mu} \delta_{\mu} - i e \gamma^{\mu} A^{\mu} + m) \psi = 0. \]

Thus we identify the current-density operator with \( i e \gamma^{\mu} \).

We can now go ahead and solve problems involving the interaction of an electron with a classical electromagnetic field. The two most famous are the problem of an electron moving slowly in an external magnetic field and the problem of an electron in a coulomb field. In the first of these, it will become apparent why the electron has a magnetic moment of 1 Bohr magneton instead of 1/2 as is expected classically. When Dirac solved the second of these, he found the energy levels of the hydrogen atom. Both of these agreed perfectly with experiment, and thus people were very impressed by the theory despite the fact that the negative frequency solutions weren't handled too well. Actually Dirac immediately came up with the idea that the negative frequency solutions would be associated with anti-particles, although he said this in a rather complicated way involving "seas" and "holes", and so on. He predicted therefore that there should be an anti-particle with the opposite charge but the same mass. However, he didn't go so far as to say that because it seemed so absurd to predict the existence of another particle. There were already enough mysterious particles around, namely the proton and the electron! And it was certainly going too far to suppose that there would be still more! So he didn't say anything like that, instead he tried to interpret the proton as being this anti-particle, arranging in some way for the difference in mass and the fact that protons and electrons don't annihilate each other. After a year or so, when Anderson here discovered the positron, it was no longer necessary to go through this mysterious finagling, and everybody was completely satisfied with Dirac's equation.
For convenience, I wish to introduce the symbol $\not{\alpha} = \sum_{\mu} A_{\mu} \gamma^{\mu}$. (The components $A_{\mu}$ commute with the $\gamma^{\mu}$ matrices). Then, in this shorthand notation, the Dirac equation is simply:

$$(\not{\alpha} + m)\psi = 0.$$ 

Likewise, the equation satisfied by the Dirac adjoint is easily expressed as:

$$\bar{\psi}(\not{\alpha} - m) = 0.$$ 

There are many interesting algebraic relations between such quantities, which we shall employ from time to time. Perhaps the most important is:

$$\not{\alpha} \not{\beta} = \hat{\alpha} \cdot \hat{\beta} + \frac{i}{2} \sigma^{\mu \nu} (A_{\mu} B_{\nu} - A_{\nu} B_{\mu}).$$

Using this relation, which says $\not{\alpha} \not{\beta} = \hat{\alpha} \cdot \hat{\beta} = \vert J \cdot J \vert^2$, it is trivial to pass to the squared equation:

$$(\not{\alpha} - m)(\not{\alpha} + m)\psi = (\vert J \cdot J \vert^2 - m^2)\psi = 0.$$ 

I now want to show why the normalization $u^+ u = E/m$ was adopted in these notes. If you recall, it was shown that $\bar{\psi} \psi$ was an invariant under Lorentz transformations, while $\bar{\psi} \psi$ transforms like the fourth component of a four vector. Consider now the equations satisfied by the momentum spinors for electrons:

$$(\not{\gamma} - im) u_{\alpha} = 0; \quad \text{and} \quad \bar{u}_{\alpha} (\not{\gamma} + im) = 0.$$ 

$$\text{where } k = (k, iE).$$

Employing these, the following relation is obtained:

$$\bar{u}_{\alpha} (\gamma_{\mu} \not{\gamma}^{\mu} + \not{\gamma}^{\mu} \gamma_{\mu}) u_{\alpha} = 2im \bar{u}_{\alpha} \gamma_{\mu} u_{\alpha}.$$ 

But, $$\gamma_{\mu} \not{\gamma}^{\mu} + \not{\gamma}^{\mu} \gamma_{\mu} = 2k \cdot \mu.$$ 

Therefore, $$\bar{u}_{\alpha} \gamma_{\mu} u_{\alpha} = -\frac{k \cdot \mu}{m} \bar{u}_{\alpha} u_{\alpha},$$ and as a special case:

$$u^+ u = \bar{u} \gamma_\mu u = -\frac{ik \cdot \mu}{m} \bar{u} u = \frac{E}{m} \bar{u} u.$$ 

Now we desire that the spinor $u_{\alpha}^k$ may be obtained from the "rest" spinor $u_0$ by an appropriate Lorentz transformation. Since we follow the almost universal custom of normalizing the rest spinors such that $\bar{u}_0 u_0 = 1$, we normalize such that $u_{\alpha}^k u_{\alpha}^k = E/m$.

Let us now carry out the process of obtaining $u_{\alpha}^k$ from $u_0$ by the appropriate Lorentz transformation. We know the answer from our earlier work. The electron spinor is given by:
Now to get an electron with momentum \( \vec{k} \) from one at rest, we have to transform to a new system of coordinates which is moving with velocity \( -\vec{k}/E \) relative to the rest system. Therefore, the transformation matrix \( S \) is given by

\[
S = \cosh \frac{\eta}{2} - \frac{\vec{k} \cdot \vec{a}}{k} \sinh \frac{\eta}{2}
\]

(see page 80);

where

\[
\cosh \frac{\eta}{2} = \sqrt{\cosh^2 \frac{\eta}{2} + 1} = \sqrt{\frac{E + m}{2m}}
\]

and

\[
\sinh \frac{\eta}{2} = -\sqrt{\frac{E - m}{2m}} = \frac{-\vec{k}}{\sqrt{2m(E + m)}}.
\]

Hence,

\[
S \hat{u}_0 = \frac{E + m + \vec{a} \cdot \vec{k}}{\sqrt{2m(E + m)}} \hat{u}_0 = \hat{u}_k,
\]

which demonstrates explicitly how these transformations work. As an exercise, you may like to work out the same thing for the negative energy spinors for momentum \( \vec{k} \): \( \nu_k \). There is only one subtlety. The negative energy spinors for momentum \( \vec{k} \) are associated with positrons of momentum \( -\vec{k} \). So we must transform to a system moving in the opposite direction!

**Projection Operators**

In the course of many problems we have a problem of finding the sum of the absolute square of a matrix element over the initial and final spin states of the fermion. In the non-relativistic Pauli spin theory, this could be done quite easily by just taking the trace of the product of the spin operator and its hermitian adjoint. It is not possible to do this in the same way in the Dirac theory, for we want to sum, for example, over states of positive energy only and not over all four spinor states. It is possible to accomplish the summation, however, by employing appropriate projection operators.

First, let us take note of the fact that

\[(\bar{\psi} \theta \psi)^* = (\bar{\phi} \theta \psi) \quad \text{where} \quad \bar{\theta} = \theta^\dagger \beta.
\]

Now our problem is to compute

\[A = \sum_{\text{spins}} (\bar{\nu}_{k'} \bar{\theta} u_k)(\bar{u}_k \theta \nu_{k'})\]

where, for example, the \( u \)'s are the two positive energy spinors. Recall the Dirac equation

\[(\vec{a} \cdot \vec{k} + \beta m) \nu_k = \pm E_k \nu_k.
\]

Hence, by the use of projection operators, we may write the sums as being over
all four spinors $\psi_k$ for a given momentum:

$$A = \sum \psi_k^+ \beta \theta \left( \frac{E_k + i\alpha \cdot \vec{k} + \beta m}{2E_k} \right) \psi_k^+ \beta \theta \left( \frac{E_k + i\alpha \cdot \vec{k} + \beta m}{2E_k} \right) \psi_k.$$

But, since $\psi_k$ spinors are orthogonal to each other

$$\Sigma \psi_k^+ \psi_k = \frac{E_k}{m} I.$$

Therefore,

$$A = \text{Trace} \left( \frac{E_k + i\alpha \cdot \vec{k} + \beta m}{2m} \beta \theta \frac{E_k + i\alpha \cdot \vec{k} + \beta m}{2m} \beta \theta \right);$$

$$A = \text{Trace} \left( \frac{m - i\vec{k}'}{2m} \beta \theta \frac{m - i\vec{k}}{2m} \beta \theta \right).$$

In general, we see that

$$\Sigma \psi_k^+ \psi_k = \frac{m - \beta \theta}{2m}.$$ 

It is left as an exercise for the reader to verify that

$$\Sigma \psi_k^+ \psi_k = \frac{m + \beta \theta}{2m},$$

where $\vec{k} = (k, iE)$.

**Dirac Magnetic Moment**

To find what magnetic moment is predicted by the Dirac equation, we examine the non-relativistic approximation to the Dirac equation when it is coupled to an electromagnetic field. To facilitate the notation, let us define $\pi_\mu = p_\mu - eA_\mu$. And let us recall the definition of the electromagnetic field tensor $F_{\mu\nu}$:

$$F_{\mu\nu} = \delta_{\mu} A_{\nu} - \delta_{\nu} A_{\mu} = \begin{pmatrix} 0 & B_y & -B_z & -iE_x \\ -B_y & 0 & B_x & -iE_y \\ B_z & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}$$

Then the Dirac equation becomes simply

$$(\gamma^\mu - i\gamma^5)\psi = 0.$$

It is easier to get the non-relativistic approximation from a squared equation

$$(\gamma^\mu + i\gamma^5) (\gamma^\mu - i\gamma^5) \psi = 0;$$

$$\gamma^2 + m^2 + \frac{1}{2} \sigma_{\mu\nu} [\pi_\mu, \pi_\nu] \psi = 0 \quad \text{(see page 84).}$$
But 

\[ [\pi_\mu, \pi_\nu] = i e F_{\mu\nu}, \text{ so that } \]

\[ (\pi^2 + m^2 - \frac{e^2}{2} \sigma_{\mu\nu} F_{\mu\nu})\psi = 0. \]

Let us now consider a time independent problem, using the non-relativistic approximations \( E = m = W, \) \( m >> W, eA, e\varphi. \)

Then:

\[ W\psi \approx (e\varphi + \frac{1}{2m} \pi^2 - \frac{e}{4m} \sigma_{\mu\nu} F_{\mu\nu})\psi. \]

Let us rewrite this in terms of the field strengths:

\[ W\psi \approx (e\varphi + \frac{1}{2m} \pi^2 - \frac{e}{2m} (\hat{E} \cdot \hat{B} - i \hat{a} \cdot \hat{E}))\psi. \]

The term in \( \hat{a} \cdot \hat{E} \) should also be dropped because it is of order \( v/c \) compared to the term in \( \sigma \cdot \hat{B} \), since \( a \) connects the small and large components of \( \psi. \)

What is left is just a Schrödinger equation for a charged particle with a magnetic moment of 1 Bohr magneton. This was the first triumph for the Dirac theory, and persuaded Dirac that he was on the right track.

**Electron in a Coulomb Field**

The second thing that encouraged Dirac was the fact that he got the Sommerfeld fine structure formula for the hydrogen atom, just as Schrödinger was convinced that he was on the right track when he obtained the Balmer formula. So the next thing we are going to do is look, very briefly, at the Coulomb field problem. I will just skim over it; if you want to look at it in more detail, you might look in Dirac’s book or in any of the other texts on quantum mechanics.

The Dirac equation is:

\[ E\psi = (\hat{a} \cdot \hat{V} + \beta m - \frac{Z e^2}{4\pi r})\psi. \]

We proceed by looking for the constants of the motion. It is clear that the Hamiltonian is invariant under the group of spatial rotations, and also under the inversion of space. Therefore the angular momentum is a constant of the motion, as is the parity. The angular momentum operator is

\[ J = L + \frac{1}{2} \Sigma; \]

and the parity operator is: \( P = \beta R, \) where \( R \) is an operator which changes the sign of the spatial coordinates \( x, y, z. \) These two operators commute with each other so that the energy eigenstates of the Coulomb field problem can be taken to be eigenstates of \( J^2, J_z, \) and \( P. \)
The quantum number \( L^2 \) is not a constant of the motion, because it is not possible to make a four component spinor which is an eigenfunction of \( J^2 \), \( P \) and \( L^2 \). This is so because \( L^2 \) determines the behavior of a wave function under the operation \( R \). But because of the \( \beta \) in the \( P \) operator, the upper and lower components of the spinor must behave oppositely under \( R \). If we agree to speak only of the upper (or of the lower in the case of positrons) components of the spinor, we can, however, talk about the spatial angular momentum of the electron in an approximate way. This we shall find most convenient to do. In this mode of description, we can say that a positron of a given angular momentum has the opposite parity as a similar electron.

From the non-relativistic Pauli spin theory, we know that there are just two states for each given value of the radial quantum number, \( J^2 \) and \( J_z \). In the non-relativistic theory, these correspond to \( \ell = j \pm 1/2 \). This duality is taken into account exactly by the two parity possibilities in the relativistic enumeration of the bound states. Thus, in the hydrogen atom, we can expect to find the following states for each value of some radial quantum number:

\[
\begin{align*}
\ell^P &= 1/2^+, \ 1/2^-, \ 3/2^+, \ 3/2^-, \ 5/2^+, \ldots ; \\
\text{or, if we like to talk about orbital angular momentum, these correspond to} \\
S_{1/2}, \ P_{1/2}, \ D_{3/2}, \ P_{3/2}, \ D_{5/2}, \ldots 
\end{align*}
\]

For each of these states, we know what the spinors must look like from our considerations of parity and angular momentum. For example, the spinor for any state with \( \ell^P = 1/2^+ \) must be:

\[
\psi_{1/2^+} = \begin{pmatrix} F_{nj}^+(r) \\ S_{1/2}^m \end{pmatrix}
\]

where the \( S_{1/2}^m \) is a non-relativistic two-component spinor, so composed that

\[
(L + \frac{1}{2} \sigma) \cdot S_{1/2}^m = \frac{3}{4} \ S_{1/2}^m, \quad \text{and} \quad (L_z + \frac{1}{2} \sigma_z) \cdot S_{1/2}^m = m \ S_{1/2}^m.
\]

We are familiar with these things from our general discussion of angular momentum last year. As another example, the \( 3/2^- \) spinor must be:

\[
\psi_{3/2^-} = \begin{pmatrix} G_{n \frac{3}{2}^+}^-(r) \\ F_{n \frac{3}{2}^+}^- \end{pmatrix}
\]

The problem now is reduced to that of finding a two-component "spinor"
\[ \begin{pmatrix} F_{nj}^p(r) \\ G_{nj}^p(r) \end{pmatrix}, \] which remains after the angular dependence has been removed.

We will just outline the procedure. It is conventional to make use of another operator \( K \) instead of \( P \). Consider the operator
\[
\mathbf{\hat{\mathbf{\nabla}}} \cdot \mathbf{\hat{L}} + 1 = \frac{J^2}{\hat{\mathbf{\nabla}}} - \frac{L^2}{\hat{\mathbf{\nabla}}} - \frac{1}{4} \frac{\hat{\mathbf{\nabla}}^2}{\hat{\mathbf{\nabla}}} + 1
\]
\[
= j(j+1) - l(l+1) + \frac{1}{4}
\]
\[
= \begin{cases} j + 1/2 & \text{for } l = j - 1/2 \\ -(j + 1/2) & \text{for } l = j + 1/2 \end{cases}
\]
\[
= (-)^l + 1/2 - j (j + 1/2).
\]

But recall the \( R = (-)^l \); thus we can define a new operator \( \hat{\mathbf{\nabla}} \) which is a constant of the motion:
\[
\hat{\mathbf{\nabla}} = \hat{\mathbf{\nabla}} \cdot \mathbf{\hat{L}} + 1 = \hat{\mathbf{\nabla}} R(-)^l = \frac{1}{2} \left( j + \frac{1}{2} \right) = P (-)^{j-1/2}(j+1/2).
\]

\( K \) thus replaces \( P \) in our enumeration of the states. Note that it is an integer which can take all values except zero. It is positive for the smaller of the two possible \( \hat{\mathbf{\nabla}} \) values for a given \( j \).

\[
J^P \quad 1/2^+, \quad 1/2^-, \quad 3/2^+, \quad 3/2^-, \quad 5/2^+
\]

The standard procedure is to eliminate one of the functions by forming second order equations, and then to solve these by power series methods or otherwise. (There is a certain select group of individuals who are familiar with coulombic wave functions, and who know these functions to be hypergeometric functions.) However, we will just accept the results of somebody else's labor. The
functions are normalizable to unity only if the energy has certain values.
The eigenvalues are:
\[ E_n^m = \left\{ 1 + \frac{\alpha^2}{n + \sqrt{K^2 - \frac{Z^2}{\alpha^2}}} \right\}^{-1/2} \]
where the radial quantum number \( n = 0, 1, 2, 3, \ldots \) for positive \( K \), and
\( n = 1, 2, 3, 4, \ldots \) for negative \( K \).
Since \( K^2 \) appears as the energy formula, there is a complete degeneracy
between levels of \(+K\) and \(-K\) according to the Dirac theory. For example, the
\( S_{1/2} \) and \( P_{1/2}, P_{3/2} \) and \( D_{3/2} \), etc. levels should have the same energy. But
this is an accidental degeneracy; it is not due to any fundamental symmetry
property. Hence, we would expect that the degeneracy would be broken by some
correction which has not been included so far in the theory of the hydrogen
atom. In fact, the degeneracy is destroyed by radiative corrections due to
the vacuum polarization and other virtual photon effects.

If the total quantum number \( N = n + |K| \) is defined, the formula can be
expanded in powers of \( \alpha \), thus yielding the familiar Balmer formula and the
fine structure corrections:
\[ E_n^m = 1 - \frac{\alpha^2}{2N^2} + \frac{\alpha^4}{N^3} \left( \frac{3}{8N} - \frac{1}{2|K|} \right) - \ldots \]

One final interesting point. The functions \( F(r) \) and \( G(r) \) start out
like \( r^{-2\alpha} \) for \( S \)-electrons. This is OK as long as \( 2\alpha < 1 \), for although the
solution is infinite, the functions are still normalizable. If \( Z > 137 \), the
particle falls in toward the origin until it reaches the nucleus where the
field behaves quite differently from \( 1/r \). For the Klein-Gordon equation, a
similar situation occurs when \( Z > 137/2 = 68.5 \). This \( Z \) actually occurs in
nature, and in fact, for heavy nuclei, the \( n^- \) is confined within the nucleus
before it is eaten by the nucleons!
Charge Conjugation

We asserted some time ago that, outside of minor shuffling of the components, the complex conjugates of the negative frequency solutions of the Dirac equation were the positive frequency solutions for positrons. Let's check this out.

It is a trivial fact that the complex conjugates of negative frequency wave functions are positive frequency wave functions.

In general, we define the positron wave function as:

$$\psi'(p) = C\psi^*$$

where the $C$ is the charge conjugation matrix for the given representation of the $\gamma$ matrices. We wish to choose $C$ in such a way that if

$$[\gamma_\mu (a_\mu - ia_\mu) + m] \psi = 0;$$

then $\psi'(p)$ satisfies the same equation except the sign of $e$ is reversed:

$$[\gamma_\mu (a_\mu + ia_\mu) + m] \psi'(p) = 0.$$ 

To cut down on the wide range of possibilities for $C$, and for convenience, we stipulate that

$$C C^* = 1.$$ 

(Remember * denotes complex conjugate; + indicates the hermitian adjoint.)

Multiplying the complex conjugate of the Dirac equation for $\psi$ by $C$, and writing $\psi^*$ as $C^*\psi(p)$, we obtain the following equation for $\psi'(p)$:

$$[C\gamma_\mu (a_\mu + ia_\mu) + m] \psi'(p) = 0.$$

In order that this be identical with the Dirac equation for a particle of charge $-e$, it is necessary and sufficient that:

$$\gamma^*_2 C^* = \gamma^*$$ and $$\gamma^*_4 C^* = -\gamma_4$$

or

$$\gamma^*_2 = \gamma C$$ and $$\gamma^*_4 = -\gamma_4 C.$$

In the standard representation, the elements of $\gamma_2$ and $\gamma_4$ are real, while those of $\gamma_1$ and $\gamma_3$ are imaginary. It is evident that $\gamma_2$ fits the bill of particulars for $C$ in the standard representation

$$C = \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad \text{(SR)}.$$
In dealing with problems involving charge conjugation, it would be very convenient to work in a representation of the $\gamma$'s such that $C$ can be taken as $1$. In such a representation the elements of $\gamma_1, \gamma_2, \gamma_3$ must all be real and those of $\gamma_4$ must be pure imaginary. Majorana constructed such a representation; it is simply:

$$
\gamma_1 = \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix}, \quad 
\gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad 
\gamma_3 = \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}, \quad 
\gamma_4 = \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}
$$

No. 7. Field Theory of Fermions

We now have the Klein-Gordon equation and the Dirac equation on the same footing; in each case we encountered the existence of negative frequency solutions, which could be interpreted as complex conjugates of solutions for anti-particles. At this point we can quantize the Dirac theory just the way we set up the field theory of the Klein-Gordon equation. There are a couple of differences in the procedure, but these are simple and obvious.

The first difference is that in the case of the Klein-Gordon equation which describes particles of spin 0, we could consider with physical application two cases: one in which the particle and antiparticle are distinct, and the other in which the particle and the antiparticle are identical. There are even some ambiguous cases, like that of the K's, where from the point of view of their production it is convenient to treat the antiparticle as being distinct from the particle, but when studying decay it is wise to consider linear combinations

$$
\frac{|K\rangle + |\bar{K}\rangle}{\sqrt{2}} \quad \text{and} \quad \frac{|K\rangle - |\bar{K}\rangle}{\sqrt{2}}
$$

which are their own antiparticles. But in the case of fermions, nothing like this has ever arisen. We do not know of any fermions for which the particle is the same as the antiparticle. For all the fermions, the particle and anti-particle are not only distinct, but also prevented from turning into each other by an absolute conservation law, of which no violation has been observed. All the fermions that we know are either baryons, such as the proton and neutron, or leptons, like the electron and muon. The corresponding absolute conservation laws are the conservation of baryons, and the conservation of leptons. Thus we don't have to deal with the two cases; we deal only with the case analogous to the charged scalar particle, in which we quantize with two degrees of freedom. In other words, we use the complex field formalism.
The major difference is in the statistics which we build into our field theory. The Fermi particles, all those with half-integral spin, obey the Pauli exclusion principle, whereas the bosons obey the anti-exclusion principle. The proof that the fermions obey Fermi-Dirac statistics, like the proof that bosons obey Bose-Einstein statistics, is twofold: (1) experimentally all the fermions that we know do in fact obey the exclusion principle; and (2) if one tries to write the field formalism with the wrong statistics, one winds up with something haywire like negative energies or negative probabilities! I hope we will have time later on to go into this point. It was made first by Pauli in 1940, who quantized the wrong way and found negative energies. Seven or eight years later Feynman tried to do the same thing a little differently, and he wound up with negative probabilities. One is presumably as bad as the other. Thus, we have to construct our field theory for fermions embodying the Fermi-Dirac statistics.

Let us recall the major steps in setting up our complex scalar field theory. We had a set of levels with wave functions $\varphi_n(x)$. Next the creation and destruction operators for the particle and antiparticle corresponding to each level, $a_n^+$, $a_n$ and $b_n^+$, $b_n$ were introduced in such a way that the occupation number operators for each level could be taken to be $a_n^+ a_n$ and $b_n^+ b_n$. The commutation rules for these operators were taken to be:

$$[a_n^+, a^-_n] = 1 = [b_n^+, b^-_n] \quad \text{ (all others commute)}$$

which guaranteed that the occupation number for any level could be any non-negative integer. The natural choice for the Hamiltonian operator was

$$H_{\text{free}} = \sum_n E_n a_n^+ a_n + b_n^+ b_n.$$  

The equation of motion of any operator in the Heisenberg representation could be obtained by employing the relation:

$$\dot{\hat{\varphi}} = i [\hat{\varphi}, \hat{H}].$$

Furthermore, it was shown that if the field operator $\hat{\varphi}$ and its hermitian adjoint $\hat{\varphi}^+$ were defined as:

$$\hat{\varphi} = \sum_n a_n^+ \varphi_n(x) + b_n^+ \varphi_n^*(x),$$

$$\hat{\varphi}^+ = \sum_n a_n \varphi_n^*(x) + b_n \varphi_n(x),$$

one could calculate the wave function is totally antisymmetric.

\* W. Pauli, Phys. Rev. 58, 716 (1940).
a Lagrangian density could be constructed from these two operators, from which one could get a conserved SME tensor. Also the equations of motion could be obtained for $\phi$, and it behaved in many respects like a classical field.

Let us now set up the fermion field theory by analogy. The only change we must make is in the statistics, which state that the wave function of two particles must be antisymmetric. Consequently, the occupation number for each level can assume only two values, 0 or 1. As we well know, the spectrum of eigenvalues of $a_n^+ a_n$ for bosons comes from the commutation rule $[a_n^+, a_n^+] = 1$, and the total symmetry of the two particle wave function is assured by the fact that everything else commutes. Let us examine this latter point. In momentum space, the wave function of a two particle state $|\Psi(2)\rangle$ is given by:

$$\varphi(k_1, k_2) = \langle \text{vac} | a_{k_1}^+ a_{k_2}^+ | \Psi(2) \rangle.$$ 

Since the two $a$'s commute the wave function is totally symmetric:

$$\varphi(k_2, k_1) = \varphi(k_1, k_2).$$

Now we don't want either of these two consequences of the commutation rules for the fermion field theory. In the 1920's, it was a popular puzzle how to quantize the fermion field so as to build in the exclusion principle. The quantization of the electromagnetic field had been carried out using Bose-Einstein statistics, and when the Dirac equation came out, the problem was to quantize a system using Fermi-Dirac statistics. The problem was solved by Jordan and Wigner, who saw that it was possible to write down an exactly analogous set of anticommutation rules which would have the desired consequences. That is, for fermions the fundamental relations are:

$$\{ a_n^+, a_n^+ \} = a_n^+ a_n^+ + a_n^+ a_n^+ = 1$$

and everything else anticommutates. It is evident immediately that these rules guarantee that the wave function will be completely antisymmetric. And, they guarantee that $N_n$ can take on only the values 0 and 1. For, by the rule $\{ a_n^+, a_n^+ \} = 0$, we have $a_n^+ a_n = 0$. So we also have

$$N_n (N_n - 1) = N_n^2 - N_n = a_n^+ a_n a_n^+ a_n - a_n^+ a_n = a_n^+ (1 - a_n^+ a_n) a_n - a_n^+ a_n = 0$$

Thus there are only two states of each level. If we only had to consider one level, we could take the representatives of $a$ and $a^+$ to be:

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
However, this is not the whole story; Jordan and Wigner had another tricky problem in setting these operator representations up. The matrix elements have to be defined in such a way that, for example, $\hat{a}_n$ and $\hat{a}_m$ ($m \neq n$) anticommute. Jordan and Wigner did this by a simple trick. First, by agreement the levels for particle and antiparticle are ordered, so that we can correlate a positive integer uniquely to one of the particle or antiparticle levels. Then one simply defines the matrix elements as in the following example.

\[
\langle \psi(N'_1 N'_2 \ldots) | \hat{a}_{m} | \psi(N'_1 N'_2 \ldots) \rangle = (-)^{\sum_{j=1}^{m-1} N_j} \delta_{N'_1 0} \delta_{N'_2 1} \delta_{N'_3 2} \ldots
\]

where $\chi = \sum_{j=1}^{m-1} N_j$.

Let's show explicitly how this works for one example. Suppose $m > n$. Then each of the following matrix elements is zero or they both have the values shown.

\[
\langle \psi(N'_1 N'_2 \ldots) | \hat{a}_{m} \hat{a}_{n} | \psi(N'_1 N'_2 \ldots) \rangle = (-)^{\sum_{j=1}^{m-1} N_j + \sum_{j=1}^{n-1} N_j}
\]

\[
\langle \psi(N'_1 N'_2 \ldots) | \hat{a}_{m} \hat{a}_{n} | \psi(N'_1 N'_2 \ldots) \rangle = (-)^{\sum_{j=1}^{m-1} N_j + \sum_{j=1}^{n-1} N_j}
\]

Hence, the two $\hat{a}$'s anticommute.

So there exists a mathematical trick for finding a representation of the destruction and creation operators. Having once proved this, we can forget about the representation, for it is not necessary to worry about a detailed representation. In the old days, people used to make use of this representation, but I don't think we will have any use for it. You can always use just the algebraic properties of the $a$'s and $b$'s to calculate what you want.

Of course, it is desired that the Hamiltonian should take the customary form

\[
\hat{H} = \sum_{k \mathbf{s}} E_k (\hat{a}_{\mathbf{k}\mathbf{s}}^+ \hat{a}_{\mathbf{k}\mathbf{s}} + \hat{b}_{\mathbf{k}\mathbf{s}}^+ \hat{b}_{\mathbf{k}\mathbf{s}})
\]

If the Dirac field operator $\psi$ is now defined as

\[
\psi = \sum_{k \mathbf{s}} \sqrt{\frac{m}{\mathbf{k}}} \left( \hat{a}_{\mathbf{k}\mathbf{s}}^+ \varphi_{\mathbf{k}\mathbf{s}} + \hat{b}_{\mathbf{k}\mathbf{s}}^+ \varphi_{-\mathbf{k}\mathbf{s}} \right)
\]

where $\varphi_{\mathbf{k}\mathbf{s}} e^{-iEt} = \frac{\varphi_{\mathbf{k}\mathbf{s}}}{\sqrt{\mathbf{V}}} e^{-i(k \cdot \mathbf{x} - Et)}$, and $\varphi_{\mathbf{k}\mathbf{s}} e^{-iEt} = \frac{\varphi_{\mathbf{k}\mathbf{s}}}{\sqrt{\mathbf{V}}} e^{-i(k \cdot \mathbf{x} + Et)}$ when
momentum eigenstates are chosen as the basis, the anticommutation rules take
the form:

\[ \{ \psi_m(x), \psi_n^\dagger(\xi) \} = \delta_m^n \delta(x-\xi), \]

\[ \{ \psi_m(x), \psi_n^\dagger(\xi) \} = \delta_m^n \delta(x-\xi). \]

One should bear in mind that \( \psi \) is an operator with four spinor components \( \psi_m(x) \).

In the case of the scalar boson field, we could set up a Lagrangian and
a Hamiltonian formalism, which was directly analogous to the classical formulation of field theory, and from which one could get the equations of motion of the field operators and also define naturally a conserved S-M-E tensor \( \Theta^\mu_\nu \)
whose 4x4 component was the Hamiltonian up to a constant. Then, using the
Lagrangian or Hamiltonian the coupling with other fields could be put simply
in such a way that the conservation laws associated with the interaction were
clearly evident. This can be carried out similarly for the fermion field theory,
and we shall proceed according to it.

For the complex scalar field, the free Lagrangian density can be taken
to be

\[ \mathcal{L} = - (\mu^2 + D \phi^+ + \lambda^2 \phi^2 + \phi^2). \]

(It should be kept in mind that whenever we set up the operator equations of
motion, we are working implicitly in a Heisenberg representation.) The
Lagrangian is, of course, the spatial integral of the Lagrangian density

\[ \mathcal{L} = \int L \, d^3x. \]

The classical Lagrangian form of the equations of motion are

\[ \frac{\partial}{\partial \phi} \left( \frac{\partial \mathcal{L}}{\partial (\phi^+)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \]

which yields the result that \( \phi \) and \( \phi^+ \) must satisfy the Klein-Gordan equation:

\[ (D^2 - \mu^2) \phi = 0 = (D^2 - \mu^2) \phi^+. \]

The Lagrangian formalism has the advantage that it is covariant all the way.
Nevertheless, most of quantum mechanics is built upon the non-covariant
Hamiltonian formalism. The latter is constructed by defining canonically
conjugate momentum densities:

\[ \pi^\dagger = \frac{\delta L}{\delta \phi^\dagger} = \phi^+ \quad \text{and} \quad \pi = \frac{\delta L}{\delta \phi} = \phi. \]

The Hamiltonian density is then

\[ \mathcal{H} = - \mathcal{L} + \pi^\dagger \phi + \pi \phi^+ = \mu^2 \phi^+ \phi + (\phi^* \phi) \cdot (\phi \phi^*) + \pi \pi^\dagger. \]
and the Hamiltonian is \( \hat{H} = \int d^3x \hat{H} + \text{const.} \) The equations of motion are obtained by employing the commutation rules and the familiar quantum mechanical prescription for the time derivative of any operator in the Heisenberg representation. We know the results for the case of the scalar complex field, for example:

\[
\dot{\phi} = i [\hat{H}, \phi] = \pi^+ \\
\dot{\pi} = i [\hat{H}, \pi^+] = (V^2 - \mu^2) \phi.
\]

The final field equations are identical with those obtained on the classical Lagrangian formalism.

In the case of the fermion field, the free Lagrangian density can be taken as

\[
\mathcal{L} = - \bar{\psi} (\not{\partial} + m) \psi.
\]

With this Lagrangian there is only one non-zero canonical momentum density, namely,

\[
\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i \psi^+.
\]

and thus the Hamiltonian density is

\[
\mathcal{H} = \mathcal{L} + \pi \dot{\psi} = -i \pi (\partial \cdot \not{\partial} - \beta m) \psi = \psi^+(\partial \cdot \not{\partial} + \beta m) \psi.
\]

\[
\hat{H} = \int d^3x \mathcal{H} + \text{const.}
\]

As with the boson field, the operator equations of motion are obtained by taking commutators with the Hamiltonian. However, the fundamental quantum mechanical relations are expressed in terms of anticommutators:

\[
\{\psi, \psi\} = 0, \quad \{\psi^+, \psi^+\} = 0, \quad \{\psi, \psi^+\} = \delta(x - \xi).
\]

Nevertheless, by making use of the handy relation

\[
\]

we can evaluate commutators and find the equation of motion for \(\psi\):

\[
\dot{\psi} = - [\hat{H}, \psi] = (\alpha \cdot \not{\partial} + \beta m) \psi.
\]

This is just the Dirac equation for the field operator \(\psi\). Again we get the same equation for \(\bar{\psi}\) as we would have obtained classically using the Euler-Lagrange equation. So in quantum mechanics, anticommutator relations work just as well as commutator relations. This is somewhat remarkable, and leads one to wonder whether all the possible ways of setting up quantum mechanical systems have been exhausted. Certainly there should be no new kinds of statistics. But the fact that there is more than one way of getting back the equations of motion may indicate that, within a given type of statistics, more subtle procedures for getting the equations of motion exist. The classical quantization procedure of replacing the fundamental Poisson brackets by
It doesn't lead to any harm at all. Maybe it can be broken in still more serious ways!

No. 8: Relativistic Nucleon-Meson Field Theory

This is the basis of the field theory of a Fermi-Dirac particle. All that remains to be done in setting up a general field theory of a fermion interaction with a boson is to write the Lagrangian density as the sum of the free Lagrangian densities for the fermion field and the boson field plus a coupling term. The commutation and anticommutation rules are carried along and we need only specify that the boson field operators commute with the fermion field operators. The question arises as to which field theory we should develop here: quantum electrodynamics, a relativistic meson theory, or a general field theory. The latter would be very nice, but there are too many special difficulties in some types of fields to make a general treatment profitable. The great advantage of quantum electrodynamics is that the expansion parameter, 1/137, is quite small, and so calculations can be made with the theory. These agree perfectly with the experiments today, and thus quantum electrodynamics is the most admired branch of theoretical physics at the present. Thus we will want to do many of the calculations of quantum electrodynamics which can be compared with experiment. However for deriving the rules of calculation, I don't want to use quantum electrodynamics as the example. It has a few minor difficulties which are not at all general features of field theories. That is to say, the initial equations are very simple for quantum electrodynamics, and the final Feynman rules for perturbation calculations are also very simple, but in the intermediate steps of deriving those rules there are some peculiar difficulties which I don't want to go into. The relativistic meson theories do not have these special difficulties. Therefore we shall deal with a meson theory and derive the Feynman rules for calculating. We shall see how they reduce to our old rules in the non-relativistic approximation for nucleons. Then we shall pass over to quantum electrodynamics to perform our calculations.

In discussing the pion-nucleon field theory, it is convenient to regard the operator $\hat{V}$ as having eight components: the first four are the four spinor components of the proton field; the last four being those of the neutron field. The Lagrangian density for an uncoupled system of pion, proton, and nucleon fields is:

$$\mathcal{L} = -\frac{1}{2}(\mu_0^2 \vec{v} \cdot \vec{\phi}^+ + \vec{v} \cdot \vec{\phi}^+ \cdot \vec{v} \cdot \vec{\phi}^+) - \bar{\Psi} (\vec{p} + M_0) \Psi.$$
If this is regarded as the complete Lagrangian, the canonical formalism leads to the Dirac equation for the nucleon fields, and the Klein-Gordon equation for the meson fields. The problem is to put a coupling term into the Lagrangian density which reduces to the same coupling that was successful in the static theory. Therefore, we wish to insert a pseudoscalar coupling term. We can construct a suitable coupling term using gradients as in the static theory. However, gradients have the obnoxious feature in relativistic field theories that they lead to higher order divergences which are not removed completely by the various renormalization procedures. An obvious way to introduce a pseudoscalar coupling term is to employ the matrix
\[ \gamma_5 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \]
in the term \( \bar{\psi} \gamma_5 \psi \) which we saw behaves like a pseudoscalar under Lorentz transformations. If we use a coupling which does not involve gradients, we get the pleasant consequence that all the terms in the perturbation series are finite after the renormalizations have been performed. Thus we have a special motivation for considering such a theory. Since the Hamiltonian is to be hermitian, an \( i \) has to be put into the coupling Lagrangian density:
\[ \mathcal{L}_{\text{coup}} = -g_0 \bar{\psi} \gamma_5 \psi. \]
This complete Lagrangian density is the basis for our consideration of relativistic pion-nucleon interactions.

It is appropriate to show explicitly how this relativistic coupling Lagrangian density reduces to the coupling put in on the static model. For this, we regard \( \psi \) as a one-particle spinor wave function and pass to the non-relativistic limit. The coupling Hamiltonian density is:
\[ \mathcal{H}_{\text{coup}} = g_0 \bar{\psi} \gamma_5 \psi. \]
Non-relativistically, the four component spinor \( \psi \) describing a fermion with momentum \( k \) can be constructed from a two-component spinor \( \chi \) in the following way:
\[ \psi = \begin{pmatrix} \chi \\ (g \cdot k / 2m) \chi \end{pmatrix} \quad \text{(NR)}. \]
In terms of the two component spinor \( \chi \) the NR limit of the coupling Hamiltonian can be written as
Thus the $\gamma_5$ coupling assumed above does indeed reduce to the coupling of the static model, and the relation between the expansion parameters is:

$$\frac{g^2}{4\pi} \approx \left(\frac{2M}{|\mu|}\right)^2 \approx (13.5)^2 (0.08) \approx 15.$$ 

So the coupling constant $g^2 / 4\pi$ which is appropriate to the relativistic Hamiltonian is about 180 times as large as the expansion parameter in the non-relativistic static theory. For consideration of P-wave scattering, the expansion parameter has a factor $(\mu/2M)^2$ in addition to $g^2 / 4\pi$ and so the work we went through is a good approximation to the content of the relativistic theory at low energies. However, when one considers other phenomena such as S-wave scattering, the $g^2 / 4\pi$ comes in with its full force of 15 and renders a perturbation treatment of this theory impossible. But before we see that it is impossible, we shall develop the Feynman rules for perturbation calculation for this Lagrangian. Then we shall pass over to quantum electrodynamics and perform some calculations using the Feynman rules.

December 2, 1958

In the last lecture it was shown that the field theory of nucleons and pions generated from the Lagrangian density

$$\mathcal{L} = -\frac{1}{2} \left( \mu_0^2 + \varphi_1 \varphi_1 + \varphi_{\lambda_1 \mu} \varphi_{\lambda_1 \mu} \right) - \varphi \left( \beta + M_0 \right) \psi - ig_0 \varphi \gamma_5 \tau_1 \psi \varphi_1$$

is a relativistic generalization of the model of nucleon-pion interactions which has been successful in predicting some low energy phenomena. By the standard procedures, the Hamiltonian density

$$H = \frac{1}{2} (\mu_0^2 \varphi_1 \varphi_1 + (\varphi_1)^2 + \varphi_{\lambda_1 \mu} \varphi_{\lambda_1 \mu}) + \varphi \left( \beta + M_0 \right) \psi + g_0 \varphi \gamma_5 \tau_1 \psi \varphi_1$$

can be obtained from the Lagrangian density. It is possible to carry out the mesic charge renormalization in such a way that the relation between the renormalized coupling constants is

$$g_1^2 / 4\pi = (2M/\mu)^2 f_1^2 / 4\pi,$$

which is identical to the relation between the "bare" constants that has been
deduced. The theory is quantized by the application of the following rules:

At a given time, 
\[
\{\hat{\psi}(x), \hat{\psi}(\xi)\} = \delta(x-\xi), \quad [\hat{\phi}, \hat{\phi}^\dagger] = i\delta(x-\xi),
\]

any other product of two fermion field operators anticommutes, while all other products of field operators commute. In the Heisenberg representation, the equations of motion of the operators are obtained by calculating the commutator of the operators with the Hamiltonian, which is

\[
\hat{H} = \int d^3x \text{ constant.}
\]

The constant is to adjust for the zero point on the energy scale. The quantum mechanical equations of motion are the same as those obtained classically from the Euler-Lagrange equations:

\[
(\Box - \mu^2)\hat{\phi}^\dagger = \frac{i}{\hbar} \left[ E_0 \gamma_5 \tau_1 \frac{\hat{\phi}}{\hbar}, \hat{\phi}^\dagger \right]
\]

The physical meaning of these equations is evident. The Dirac equation holds for nucleons except for a term which permits the nucleon to emit or absorb a meson. The meson is coupled through \(\gamma_5\), which reduces to \(\gamma_5\gamma/2M\) in the non-relativistic limit, and through the proper isospin matrix \(\tau_1\). On the other hand, the Klein-Gordon equation governs the meson field except for a nucleon source term. These equations are much richer than the corresponding equations for the static theory because the motion of the nucleons is taken into consideration, and because the possibility of nucleon pair production is included in the theory. If it were just a question of putting in recoil, the Chew theory would not be appreciably changed. Indeed, this can be done by truncating the relativistic theory and looking only at the positive frequency solutions. This procedure leads to another theory which, although it is not perfectly consistent because of causality difficulties, can be examined, and which looks very much like the Chew theory. Recoil is included, and so there is a little mixing of the pion angular momentum states. There is a little S- and D-wave scattering, but this arises only on account of recoil considerations. And, also due to recoil, there is an automatic cutoff at around \(\omega = M\). So the explicit cutoff which we had to put into the Chew theory need not be inserted in such a fashion into the truncated theory. However, this truncated theory is not completely relativistic, in that it is not fully causal or local, and nobody would ever take it too seriously, because the possibility of pair production is so important in determining the nature of the solutions. The possibility of making
anti-nucleons leads to a tremendous change, -- such a large change that nobody knows how to solve the equations in the presence of pair production with the coupling constant as large as it is. However, for the derivation of formal methods of solution it doesn't matter whether the coupling constant is large or small.

**Representations**

In quantum mechanics there are three different classes of representations which are commonly used. These are the classes of Schroedinger, Heisenberg, and Interaction representations. Each of them has certain features which makes it particularly well suited for some of the problems of quantum mechanics. The Schroedinger representation is usually employed in the discussion of eigenvalues and eigenfunctions. The Heisenberg representation contains time-dependent operators whose equations of motion are very similar to the equations of classical mechanics, and thus the Heisenberg representation is frequently used in setting up a quantum mechanical system. And thirdly, the interaction representation finds application in problems dealing with transitions, scattering, and the like. Therefore, it is common to pass from one type of representation to another as the nature of the specific problem at hand varies. Up to now, no explicit discussion of these changes in representation has been given, but it is necessary to go into these matters in order to present the derivation of the S-matrix.

The Schroedinger representation is familiar to all from the study of non-relativistic quantum mechanics. A fixed orthogonal basis is chosen, and the state of a system is described by a state vector \( \psi_\text{S}(t) \) rotating as time progresses, whose components describe the expansion of the vector in terms of the basis vectors. The dynamical variables, such as the position, momentum, current, energy, etc. are all represented by time-independent operators in the Schroedinger representation, except if some interaction with a classical oscillating field is put into the Lagrangian as is done in semi-classical radiation theory. The fundamental equation is the Schroedinger equation

\[
\frac{i}{\hbar} \frac{d}{dt} \psi_\text{S}(t) = H_\text{S} \psi_\text{S}(t),
\]

where, non-relativistically, the Hamiltonian is a function of the position and momentum operators:

\[
H_\text{S} = H_\text{S}(x_\text{S}, p_\text{S}),
\]

and, relativistically, it is a function of the field operators:

\[
H_\text{S} = \int \mathcal{H}_\text{S}(\phi_\text{S}, \pi_\text{S}) \, d^3x + \text{constant}.
\]
A formal solution of the Schroedinger equation exists in the form
\[ \psi_s(t) = \exp(-iH_{\omega s}(t-t_0)) \psi_s(t_0), \]
and the major problem is to diagonalize \( H_{\omega s} \) so that this solution may be used.

Suppose one considers a time-dependent change of basis which is given by the unitary transformation \( T = \exp(iH_{\omega s}(t-t_0)) \). The description by means of this new basis is called a Heisenberg representation. The new state vectors do not change with the time, for
\[ \psi_H(t) = T\psi_s(t) = \psi_s(t_0). \]
In other words, we have chosen to describe a state by means of a basis which rotates along with the state vectors. In the Heisenberg representation, the things that change with time are the operators, for
\[ \dot{\psi}_H(t) = T\dot{\psi}_s(t) = \dot{\psi}_s(t_0). \]

Clearly, for any operator \( \theta_s \) which commutes with the Hamiltonian, \( \theta_H = \theta_s \). The time derivative of any operator in the Heisenberg representation, which is time-independent in the Schroedinger representation, is given by
\[ \dot{\theta}_H = i[H, \theta_H]. \]

Using the commutation (or anticommutation) rules, one can set up all the equations of motion for the operators. These equations are very similar to the equations of classical mechanics, and therefore people usually construct quantum mechanical theories by working in the Heisenberg representation. Unless all the equations are linear, however, it is a much more difficult task to solve these equations compared to the Schroedinger equation. For an example of such a case where the Heisenberg representation yields solvable operator equations, let us consider a problem in the quantum mechanical theory of the linear harmonic oscillator.

(The subscript \( H \) will be dropped, and \( t_0 \) will be taken to be 0.) The Hamiltonian is
\[ \hat{H} = (\hat{p}^2 + m^2 \omega^2 \hat{x}^2)/2m. \]
The commutation rule \([x, \hat{p}] = i\), is used to obtain
\[ \dot{x} = i[H, x] = p/m; \quad \dot{p} = i[H, p] = -m \omega^2 x. \]
These equations are identical in form to the classical equations, and because they are linear, they may be solved as if the operators were classical variables.

Setting \( x = x(t), p = p(t) \), the solution is simply:
\[ x = X \cos \omega t + P \frac{\sin \omega t}{m \omega}; \]
\[ p = P \cos \omega t - m \omega X \sin \omega t. \]

Suppose it is desired to ask for the amplitude that a particle at \( x' \) at \( t=0 \) is at \( x'' \) at \( t \). In the Schrödinger representation it is quite easy to write the answer formally:

\[ \langle x'' | e^{-iHt} | x' \rangle = \sum_n \varphi_n(x'') e^{-i\epsilon_n t} \varphi_n^*(x'), \]

but it is also a fairly involved task to perform the sum, which is over the eigenfunctions of the harmonic oscillator and thus includes products of Hermite polynomials. However, in the Heisenberg representation it is quite easy to get the answer up to some factor which is a function of \( t \) only. What is the formal answer in the Heisenberg representation? Well, define the state \( |X\rangle \) such that \( x(t)|X\rangle = x''|X\rangle \), and the answer is

\[ \langle x|x'\rangle. \]

The practical problem is to find an explicit formula for the answer. Consider the desired matrix element as a function of \( x' \). Let us call it \( \chi(x') \). Then, making use of the defining property of \( |X\rangle \),

\[ x''\chi(x') = x' \cos \omega t \chi(x') + i \frac{\sin \omega t}{m \omega} \frac{d}{dx'} \chi(x'). \]

The solution of this linear differential equation which fits the boundary condition must be unchanged under the product of these three operations:

1. interchange \( x'' \) and \( x' \),
2. complex conjugation,
3. reverse the sign of \( t \).

These conditions determine \( \chi(x') \) up to some factor \( f(t) \):

\[ \chi(x') = f(t) \exp \left\{ i \frac{m \omega}{2 \sin \omega t} (x''^2 \cos \omega t - 2x'x'' + x'^2 \cos \omega t) \right\}. \]

Notice that in the limit of \( t \to 0 \), this reduces to the free particle Green's function

\[ g(t) e^{im(x''-x')^2/2t} \]

and also in the limit:

\[ \langle x|x'\rangle \to f(t) e^{iS}, \]

where \( S \) is the classical action. (In this special case of the harmonic oscillator this correspondence with the classical action holds for finite time intervals also.)

Enough of this digression! We now return to the consideration of problems in relativistic field theory. The theory, if you like, is specified by the Heisenberg operator equations of motion, and the commutation relations
at a given time \( t \). This is all that is necessary to fix the theory. That the theory is derivable from a Lagrangian assures us that it will possess the desired invariants and conservation laws, and enables us to construct a conserved S-M-E tensor. The problem is to solve the equations of motion. We can do this only formally because the equations are not linear in the operators. However, in a series in \( g \), these equations can certainly be handled. Therefore, we are going to set up a formal solution which leads directly to a series solution.

For the time being, the problem will be simplified somewhat in that renormalization questions will be ignored. Later on some corrections for renormalization will be put in. From our work on the simpler field theory, we are familiar with the nature of these corrections, so they shouldn't cause any trouble.

In the consideration of any scattering problem, the essential step is to divide the Hamiltonian into two parts

\[
H = H^O + H^I = H^O_H + H^I_H(t) + H^I_S(t),
\]

where \( H^O \) is called the "free" Hamiltonian, and \( H^I \) is the interaction Hamiltonian. If \( H \) were the complete Hamiltonian, then the fields would obey the "free" field equations. The presence of \( H^I \) induces transitions. In order to construct the S-matrix, we are interested in comparing the results of the complete equations with the results of the "free" equations. The S-matrix description of scattering tells us how the complete equation takes us from one solution of the free equations to another solution of the free equations. Therefore, it is desired to compare the time dependence of the Heisenberg operators with the time dependence of the free operators which act as if there were no coupling.

To do this, it is very convenient to introduce a new type of representation called the Interaction Representation. Essentially, we change our basis continually in time so as to eliminate that part of the rotation of the state vectors due to the free Hamiltonian \( H^O \). Starting from the Schrödinger representation, the Interaction Representation is reached by means of the unitary transformation \( T' \):

\[
T' = e^{-i\int H_S^O(t-t_0)}.
\]

The operators undergo the customary transformation:

\[
\hat{\Theta}_{int} = T' \hat{\Theta}_S T'^{-1} = e^{i H_S^O(t-t_0)} e^{-i H_S^O(t-t_0)}.
\]

At \( t = t_0 \), \( \hat{\Theta}_{int} = \hat{\Theta}_S = \hat{\Theta}_H \). Also for all operators \( \hat{\Theta}_S \) which commute with \( H_S^O \), \( \hat{\Theta}_{int} = \hat{\Theta}_S \). The operators \( \hat{\Theta}_{int} \) change with time like free field operators.
Since they are related by unitary transformations to the Schroedinger operators and Heisenberg operators, the commutation (or anticommutation) relations are of identical form in all three representations. The unitary transformation which carries you from the Heisenberg representation to the interaction representation is called $U(t, t_0)$; it is simply:

$$U(t, t_0) = T' T^{-1} = e^{-i\mathcal{H}(t-t_0)} e^{i\mathcal{H}(t-t_0)}.$$  

It relates the rotating state vector in the interaction representation to the stationary state vector in the Heisenberg representation:

$$\begin{aligned}
\left|\psi_{\text{int}} (t)\right> &= U(t, t_0) \left|\psi_{\text{H}} (t_0)\right>, \\
\left|\psi_{\text{H}} (t_0)\right> &= U^{-1}(t, t_0) \left|\psi_{\text{int}} (t)\right>, \\
\end{aligned}$$

and also the operators

$$\Theta_{\text{int}} (t) = U(t, t_0) \Theta_{\text{H}} (t) U^{-1}(t, t_0).$$

Several properties of $U(t, t_0)$ are worth noting. It is unitary, and

$$U^{-1}(t, t_0) = U(t_0, t),$$

and $U(t_0, t_0) = 1$.

A differential equation satisfied by the transformation operator $U(t, t_0)$ is easily obtained from its definition:

$$i \frac{\partial}{\partial t} U(t, t_0) = H_{\text{int}} (t) U(t, t_0).$$

This does not immediately write a formal solution for $U(t, t_0)$ in terms of an exponential series, because $H_{\text{int}} (t')$ does not in general commute with $H_{\text{int}} (t'')$. But this difficulty can be bypassed formally by making use of the "P-bracket" -- the time-ordering operator. This operator arranges the terms of a product so that those associated with later times appear on the left. Then we may write the solution as:

$$U(t, t_0) = P(\exp - i \int_{t_0}^{t} H_{\text{int}} (t') dt').$$

The trouble with field theory is that exceedingly little is known on the properties of such functions. The only way it may be used practically is as a series, which is acceptable in electrodynamics, but most unfortunate in mesodynamics.

$$U(t, t_0) = 1 - i \int_{t_0}^{t} H_{\text{int}} (t') dt' - \frac{1}{2} \int_{t_0}^{t} \int_{t_0}^{t} P(H_{\text{int}} (t'), H_{\text{int}} (t'')) dt'' dt' + \ldots .$$

$$U(t, t_0) = 1 - i \int_{t_0}^{t} H_{\text{int}} (t') dt' \int_{t_0}^{t} H_{\text{int}} (t') H_{\text{int}} (t'') dt'' dt' + \ldots .$$
The essence of scattering is the following. Suppose at a time $t_0$ in the remote past, the state of a system $|\psi_{int}(t_0)\rangle$ is such that it is governed essentially by the "free" Hamiltonian $H$. Then, as time progresses, the state vector in the interaction representation rotates under the influence of $H'$. After a sufficiently late time $t'$, the state vector is again assumed to be governed only by the "free" Hamiltonian. The relation between the two state vectors is given by the $U$ operator:

$$|\psi_{int}(t')\rangle = U(t',t_0)|\psi_{int}(t_0)\rangle,$$

since $|\psi_{H}(t')\rangle = |\psi_{H}(t_0)\rangle = |\psi_{int}(t_0)\rangle$. Now, suppose we let $t_0$ tend to $-\infty$ and $t'$ to $+\infty$. In doing so, however, we wish to pass to the limits in such a way that we smear out $t_0$ and $t'$ a little to get rid of any transient behavior. Then we have the solution to our problem, the $S$-matrix, which is the operator which tells us how a given state of the free Hamiltonian becomes transformed into other states of the free Hamiltonian due to some interaction.

Thus the formula for the $S$-matrix is simply:

$$S = U(\infty, -\infty) = P(\exp -i \int_{-\infty}^{\infty} H'_{int} dt ).$$

This type of expression is quite satisfactory, for $H'$ is a spatial integral of essentially a scalar Hamiltonian interaction density, and thus the formula for the $S$-matrix is a manifestly covariant expression.

In this alternate approach we take the Heisenberg representation, $\psi_n(t)$, and we assume that the set of many conditions can be correlated in a one-to-one fashion with the set of eigenstates of $H^0$. This implies that sufficient mass renormalization has been carried out in the definition of $H^0$ so that the spectrum of $H^0$ matches that of $H$. The eigenstates of the complete Hamiltonian $H$ will be designated by $|\psi_n\rangle$ and their corresponding eigenstates of $H^0$ by $|\phi_n\rangle$. (This correlation cannot always be accomplished, however, if there are genuine bound states. The existence of bound states requires a substantial modification of this procedure which has not yet been completely worked out. Thus we shall assume that we are working with a system which has no genuine bound states, such as quantum electrodynamics.)

In working with a complete set of $|\psi\rangle$ we have at our disposal the choice of the boundary conditions. For example, a set of eigenstates of $H$ exists which represent incoming plane waves plus outgoing spherical waves, and another set of incoming spherical waves plus outgoing plane waves also exists. To
We have sketched a derivation of a formula for the S-matrix by introducing an interaction representation in which all the operators $\phi_{\text{int}}(t)$ obey free field equations, and in which the rotation of the state vectors is induced by only a portion of the total Hamiltonian, $H'_{\text{int}}(t)$. In such a representation, at equal times the operators obey the normal commutation rules, and $H^0_{\text{int}}(t)$ together with all those operators which commute with it are time-independent.

We did not take up the question of how the limiting processes were to be carried out in the transition from $U(t,t_0)$ to $U(\infty, -\infty)$. Now there is another approach to the S-matrix theory which embodies a particular method of passing to the limits. Also this alternate approach permits us to verify easily the identity of the manifestly covariant time-dependent formula for the S-matrix.

$$S = \mathcal{P}\left\{ \exp -i \int_{-\infty}^{\infty} H'_{\text{int}}(t) \, dt \right\},$$

with the customary time-independent formula,

$$S_{mn} = \langle \psi^+_m | \psi^+_n \rangle = \delta_{mn} - 2\pi i \delta(E_m - E_n) \langle \phi^+_m | H'_{\text{int}} | \psi^+_n \rangle,$$

where

$$\psi^+_m = \phi^+_m + \eta \lim_{\eta \to 0} \frac{1}{E^+_m - i\eta - H^0} H'_{\text{int}} \psi^+_m.$$
obtain an orthogonal set of eigenstates, it is desired that a set be chosen with the same boundary condition throughout the set. Furthermore, to obtain the non-covariant expression for the S-matrix with which we are accustomed, it is expedient to work with the states containing outgoing scattered stuff.

In this calculation, it is convenient to define two new functions, \( \delta_+(x) \) and \( \delta_-(x) \), which are respectively the positive and negative frequency parts of \( \delta(x) \).

\[
\delta_+(x) = \frac{1}{2\pi} \int_0^\infty \text{e}^{-iwx} \, dw = \lim_{\eta \to 0} \frac{1}{2\pi i(x - i\eta)} = \frac{1}{2} \delta(x) + P \frac{1}{2\pi i x}. \\
\delta_-(x) = (\delta_+(x))^* 
\]

These functions have a useful property: \( \lim_{k \to \infty} \text{e}^{-iKx} \delta_+(x) = 0 \), and the complex conjugate relation \( \lim_{k \to -\infty} \text{e}^{-iKx} \delta_-(x) = 0 \).

To demonstrate the validity of the property, we apply \( \text{e}^{-iKx} \delta_+(x) \) to a function \( f(x) \) and integrate over the range of \( x \). It is assumed that the function \( f(x) \) is bounded or smooth enough so that the integral of its Fourier transform \( F(w) \) converges.

\[
\lim_{k \to \infty} \{ (2\pi)^{-1} \int_0^\infty \text{e}^{-iwx} \text{e}^{-ikx} f(x) \, dx \, dw = \int_0^\infty \int_0^\infty \text{e}^{-i(w+k)x} F(w) \, dw \} = 0.
\]

With the aid of \( \delta_+(x) \), a compact expression for \( \langle \Phi_m | \Psi_n^+ \rangle \) can now be written:

\[
\langle \Phi_m | \Psi_n^+ \rangle = \delta_{mn} - 2\pi i \delta_+ (E_m - E_n) R_{mn}, \quad \text{where, of course}
\]

\[
R_{mn} = \langle \Phi_m | H^+ | \Psi_n^+ \rangle.
\]

What we are trying to do is to introduce a time-dependent formalism so that we may get a manifestly covariant formula which agrees with the usual formula for the S-matrix. For this purpose, we have to define two operators, \( \mathcal{Q} \) and \( U(t) \). \( \mathcal{Q} \) is defined as the operator which transforms the state vector \( |\phi_n\rangle \) into \( |\psi_n^+\rangle \). It is a unitary operator provided that the \( |\psi_n^+\rangle \) have been normalized. Relative to the basis of \( \Phi \) the operator \( \mathcal{Q} \) has the matrix elements

\[
\mathcal{Q}_{mn} = \delta_{mn} - 2\pi i \delta_+ (E_m - E_n) R_{mn}.
\]

Then \( \mathcal{Q} \) has the vital property that it transforms \( H^0 \) into \( H \), that is,

\[
\mathcal{Q} \ H^0 \mathcal{Q}^{-1} = H.
\]

Now, the Heisenberg operators are related to the Schrödinger operators just by the transformation \( e^{i\mathcal{Q}t} \), that is,
\[ \theta_H(t) = e^{iHt} \theta e^{-iHt} \quad (\theta = \theta_s = \theta_H(0)). \]

We wish to define "in" operators with a time-dependence which differs from the Heisenberg operators. These are defined as follows. The operator \( U(t) \) is introduced as

\[ U(t) = e^{iHt} \Omega e^{-iHt}, \]

which is analogous to the \( U(t, -\infty) \) in the previous formalism. Roughly speaking, the transformation \( U(t) \) changes the time dependence of the fields so that it involves \( H^0 \) instead of \( H \).

\[ \theta_{in}(t) = U(t) \theta_H(t) U^{-1}(t) = e^{iHt} \Omega \theta \Omega^{-1} e^{-iHt}. \]

To find a formal expression for \( U(\infty) \) which is manifestly covariant, one can compute the time derivative of \( U(t) \):

\[ i \dot{U}(t) = e^{iHt} [\Omega, H] e^{-iHt} = e^{iHt} \Omega H' e^{-iHt} = H_{in}^1(t) U(t). \]

With this behavior of \( U(t) \), it is quite evident that the "in" operators obey free field equations, for

\[ i \theta_{in} = [H_{in}'(t), \theta_{in}] - [H_{in}, \theta_{in}] = [\theta_{in}', H_{in}^0]. \]

The equation for \( U(t) \) is the same as the one we obtained for \( U(t, -\infty) \) in our previous derivation. To complete this derivation of the S-matrix, we have only to show that \( U(-\infty) = 1 \), and that

\[ U(\infty) = P(\exp - i \int_{-\infty}^{\infty} H_{in}(t) \, dt) = S. \]

For this demonstration, we need only to calculate the matrix elements of \( U(t) \). The unusual feature of this formalism is that the matrix elements of \( U(t) \) are to be taken between the exact states \( |\Psi_m> \). Hence, we compute

\[ \langle \Psi_n' | U(t) | \Psi_m' \rangle = e^{i(E_m - E_n)t} \quad \langle \Psi_n' | \Omega | \Psi_m' \rangle = e^{i(E_m - E_n)t} \langle \Phi_m | \Psi_n' \rangle; \]

\[ i(E_m - E_n)t = e^{i(E_m - E_n)t} (\delta_{mn} - 2\pi \delta_+ (E_m - E_n) R_{mn}). \]

Recalling the theorem concerning \( \delta_+ (x) \), and the fact that \( \delta_+(x) = \delta(x) - \delta_-(x) \), it becomes evident that

\[ U_{mn}(-\infty) = \delta_{mn}, \quad \text{and} \quad U_{mn}(+\infty) = S_{mn}. \]

We thus have the same result as before in which the S-matrix is expressed covariantly in terms of operators which obey the free-field equations. The peculiar thing about this approach is that the S-matrix elements or U-matrix...
elements are taken between exact Heisenberg states. We can understand this peculiarity if we ask what the unperturbed Hamiltonian is in the "in" representation. Since the time derivative of operators in the "in" representation is given by the commutator of the operators with \( H^0_{\text{in}} \):

\[
H^0_{\text{in}}(t) = H^0_{\text{in}}(0) = \Omega H^0 \Omega^{-1} = H.
\]

Thus the eigenstates of \( H^0_{\text{in}} \) are indeed just the eigenstates of \( H \). That is why the initial and final states in the S-matrix formalism just presented are taken to be the exact Heisenberg states - the true particle states. In the "in" language, the creation and destruction operators produce or eliminate real particles, while the Heisenberg creation and destruction operators work between the "bare" states - the fake particles. The exact states \( |\Psi\rangle \) can be expanded in terms of the states \( |\Phi\rangle \), e.g. a physical nucleon can be expanded in terms of the \( |\Phi\rangle \) states representing a bare nucleon, a bare nucleon plus a bare meson, a bare nucleon plus two bare mesons, etc. Those \( |\Phi\rangle \) states correspond to creation and destruction by the Heisenberg operators in this way of doing things, whereas the \( |\Psi\rangle \) states correspond to creation and destruction by the "in" operators, \( a_{\text{in}}^+ \) and \( a_{\text{in}} \).

The two approaches are the same mathematically; the difference lies in the physical interpretation of the formalism. The common feature of the two approaches is the same answer for the S-matrix. In either approach the operator \( S \) is evaluated between eigenstates of \( H^0_{\text{in}} \) in the representation in which one is working.

Substantial difficulties arise in carrying through this "in" formalism when there are bound states, because the spectra of \( H \) and \( H^0_{\text{in}} \) no longer coincide. Numerous papers have recently appeared on this problem, and currently the experts are trying to handle it by introducing a new "in" field for each bound state, i.e. a separate field for the deuteron, one for the ground state of \( O^{12} \), \( O^{16} \), etc. The problem then is to calculate a connection between the "in" fields and the Heisenberg fields. For example, roughly, the "in" field of the ground state of \( O^{16} \) might be the product of 16 Heisenberg \( \phi(x_i) \)'s times some function of the 16 relative \( x_i \)'s, which would be analogous to the wave function of \( O^{16} \); this relationship is, however, extremely complicated. This idea is all right, but the point is that the bound states are intrinsically different from other things in that their energies and strengths are calculable. When we introduce the pion in the Yukawa way by a separate field describing particles of mass \( \mu \).
and with a coupling strength $g$, $\mu$ and $g$ are specified at will. However, the deuteron is different from that; the weight of the deuteron and the strength of its coupling are calculable from the theory of nuclear forces involving the nucleon-meson interactions. It has not yet been made clear by the experts on bound states how one is to determine the masses and coupling strengths of these "in" fields for bound states, except by going back to the other formalism, the Schroedinger formalism, calculating these parameters, and then plugging them into the "in" formalism. It has already been shown how one would express these separate "in" fields in terms of the Heisenberg fields, but it has not yet been demonstrated how to determine the parameters of these fields in a satisfactory way.

This is completely a formal question, for we know how to solve problems involving bound states in the Schroedinger representation. The interest that is attached to this question is concerned with the fact that certain formalisms may suggest new theoretical ideas, and people hope that maybe in the process of working out the theory of bound states in the "in" formalism a parallelism may arise between the deuteron and the pion, so that possibly the pion may be treated as a bound state of a nucleon and an anti-nucleon. It would then be possible, in principle, to calculate the mass of the pions and the coupling strength between the pion and nucleon fields. Nobody has any idea whether this is possible, but people hope that such new ideas may be stimulated by working with formalism.

References:
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2. Lehmann, "\textquotedblright\textquotedblright\textquotedblright\textquotedblright\textquotedblright\textquotedblright, Nuovo Cimento, 5, 319, (1957)

From either approach, one obtains the same covariant expression for the $S$-matrix. Our problem now is to show how Feynman's rules for calculating can be obtained by using the free field equations, the ordinary commutation rules, and a specific form of $H'$ in the perturbation expansion of the $S$-matrix. We shall go through an example of the reduction procedure here. The general theory of the reduction is treated in a paper by G. C. Wick (Phys. Rev. 80, 268 (1950)), in which careful attention is given to all the logical points.
For the example, let us consider the lowest order $g^2$ scattering of a meson by a nucleon. The second order term in the $S$-matrix is

$$\frac{1}{2} \int \int F(\gamma'_{\text{int}}(x,x'), \gamma'_{\text{int}}(y,y')) \, d^4x \, d^4y.$$  

However, this expression is a little misleading. It is certainly true that the $H$ coupling is expressible as a spatial integral of a Hamiltonian density $\gamma'$, but $H'_{\text{coup}}$ is not $H'$. Various renormalization terms have to be introduced into $H'$ in order to make the energy spectra of the free and complete Hamiltonians identical. The renormalization terms must then be subtracted out of $H'_{\text{coup}}$. The following three terms are necessary:

$$H''_{\text{coup}} = H'_{\text{coup}} + \Delta E_{\text{vac}} + \Delta M \int \frac{\psi^* \psi}{\lambda} \, d^3x + \frac{1}{2} \Delta \mu \frac{\phi^* \phi}{\lambda} \, d^3x.$$  

In a theory where pair production is possible, the vacuum acquires an enormous self-energy. It is desired, of course, that the energy zero point be chosen so that the energy of the vacuum be zero. We recall that in the static theory we had to put in a constant to make the energy of the vacuum zero. Here, however, the possibility of pair production makes the situation a little more complicated. The point here is that the vacuum of the free fields and the vacuum of the coupled fields seem to have different lowest eigenvalues and thus we must introduce a term $\Delta E_{\text{vac}}$ into $H''_{\text{coup}}$. The next thing is that every fermion also acquires a self-energy when the coupling is introduced. This term is just like the term in the static theory; $\int \frac{\psi^* \psi}{\lambda} \, d^3x$ just counts the number of spin 1/2 particles. And, since the pion can turn into a nucleon-antinucleon pair, it also has a self-energy which has to be added into $H''_{\text{coup}}$ and removed from $H''_{\text{coup}}$. As a result of all this manipulation, $H''_{\text{coup}}$ is just like $H'_{\text{free}}$, only the bare masses are replaced by the physical masses, and the energy of the vacuum is adjusted so that it is zero.

Thus $H'_{\text{coup}} = H''_{\text{coup}} - (\text{several self energy terms})$, such that every time true self energy diagrams appear in scattering, for example, they are cancelled by $\Delta E_{\text{vac}}$, $\Delta M$, or $\Delta \mu^2$ terms.

The $g^2$ contribution to the $S$-matrix coming from $H'_{\text{coup}}$ is:

$$\frac{1}{2} \int \int \langle k'' p'' | \{ \gamma'_{\text{coup}}(x), \gamma'_{\text{coup}}(y) \} | k' p' \rangle \, d^4x \, d^4y =$$

$$= \frac{g^2}{2} \int \int \langle k'' | \{ \phi^*_{\text{coup}}(x), \phi_{\text{coup}}(y) \} | k' \rangle \langle p'' | \{ \psi^*_{\text{coup}}(x), \psi_{\text{coup}}(y) \} | p' \rangle \, d^4x \, d^4y.$$  

Now we have an initial meson and a final meson. Thus if one of the $\phi$'s creates a meson, the other $\phi$ has to annihilate a meson. Now the $P$-bracket is perfectly
symmetrical in $x$ and $y$. We can therefore evaluate the expression by assuming that a meson is created at $x$ and a meson is destroyed at $y$ providing we multiply the resulting expression by a factor of 2. Notice that the factor of 2 just cancels the factor of $2!$ in the perturbation expansion. This is a very general property; indeed, in a $n^{th}$ order term, there will be a $1/n!$, which will be cancelled by the $n!$ ways the specific creation and destruction functions of the meson field operators can be assigned to the $n$ variables $x, y, z, \ldots$. There is one exception to this rule which arises in the consideration of the vacuum self-energy where a $1/n$ remains; we shall meet this later.

Consider now the meson matrix element above. If there really is scattering, i.e., the final meson state is different from the initial meson state, then the $P$-bracket cuts no ice because $\phi_{i,k'}$ commutes with $\phi^+_{i',k''}$, and the meson matrix element can be evaluated immediately. ($i$ designates the charge state of the initial meson, and $f$ that of the final meson.)

$$\langle k'' | P(\phi_{i}(x), \phi_{j}(y)) | k' \rangle = \delta_{i1} \delta_{jf} e^{i(k' \cdot y - k'' \cdot x)}/\sqrt{4V_{k'k''}}.$$  

Often terms of the form $\bar{\psi}(x)\Gamma \psi(x)$ are encountered. Expressions of this form are interpreted to mean that the antiparticle created by $\bar{\psi}(x)$ can not be annihilated by $\bar{\psi}(x)$. In other words, in the terms of the bilinear form $\bar{\psi}(x)\Gamma \psi(x)$, all the creation and destruction operators for the same state are ordered by commuting them so that the creation operator stands to the left of the destruction operator. The $c$-numbers which arise from the commutation are disregarded. This ordering, which is sometimes called the normal product, guarantees that the expectation value of $\bar{\psi}(x)\Gamma \psi(x)$ in the vacuum state vanishes. ($\Gamma$ represents a Dirac matrix.)

$$\langle \text{vac} | \bar{\psi}(x)\Gamma \psi(x) | \text{vac} \rangle = 0.$$  

This ordering is necessary in order that the vacuum be such that its mesic charge, electromagnetic current-density, and any other kind of charge is zero. We are free to do this ordering because it is not specified in the classical Lagrangian. We have not said until now what the ordering was going to be in the quantum mechanics which is the only place the ordering is important.

With this ordering restriction, there are now four possible cases of what the $\psi$ and $\bar{\psi}$ operators can do. These can be represented by diagrams in which the time ordering of the internal fermion lines is important.
I. $\Psi(y)$ destroys initial nucleon; $\bar{\Psi}(y)$ creates intermediate nucleon which is destroyed by $\Psi(x)$; $\bar{\Psi}(x)$ creates final nucleon.

II. $\bar{\Psi}(x)$ creates final nucleon; $\Psi(x)$ creates intermediate antinucleon which is destroyed by $\bar{\Psi}(y)$; $\Psi(y)$ destroys initial nucleon.

III. $\Psi(x)$ destroys initial nucleon; $\bar{\Psi}(x)$ creates intermediate nucleon which is destroyed by $\bar{\Psi}(y)$; $\Psi(y)$ creates final nucleon.

IV. $\bar{\Psi}(y)$ creates final nucleon; $\Psi(y)$ creates intermediate antinucleon which is destroyed by $\bar{\Psi}(x)$; $\Psi(x)$ destroys initial nucleon.

It is apparent that III and IV are just the corresponding crossed diagrams for I and II.

The great simplicity in the Feynman rules is that I and II are both incorporated into a single matrix element and hence can be represented by one diagram. In such Feynman diagrams no meaning is given to the time ordering of the internal nucleon lines. In the Feynman method, one has many less diagrams to consider, since all the various time orderings within a diagram are included.
in one analytic matrix element. We shall now write down the five possible diagrams for the complete matrix element under consideration, paying no heed to the time ordering of the points x, y.

The last three of these are connected only with the amplitude for forward scattering. In the lowest order matrix elements, they are cancelled exactly by the self energy terms of order $g^2$. C is cancelled by the $g^2$ term in $\Delta E_{\text{vac}}$. D is eliminated by the lowest order term in the nucleon self energy, and E is annihilated by a corresponding term in the meson self energy calculation.

In the calculation of the analytic matrix element corresponding to a given diagram, certain of the $\bar{\Psi}$ and $\bar{\Psi}$ operators are doomed in that we know that only a single fourier component is going to be effective in the computation of the matrix element. In the rearranging of these doomed operators we do not have to worry about the fact that a single destruction or creation operator does not exactly anticommute with a fermion field operator which is not "doomed", because the difference amounts to changing the value of a function at a small number of points which is going to be integrated over a large region. Therefore, when rearranging these doomed operators, only the number of changes of sign due to commuting these things needs to be taken into account. The other fermion field operators which are not doomed are quite different. In rearranging them, the non-vanishing of the anticommutators can not be ignored, because of the infinite number of degrees of freedom involved.

It is now possible to evaluate the fermion part of the matrix element for diagram A. We shall write out everything completely in terms of the spinor components (greek subscripts) and isospin components (latin subscripts) this one time to make perfectly clear what is involved. When $x_0 > y_0$, the matrix element is that for diagram I, namely,
The minus sign appears because three permutations involving "doomed" field operators are carried out in order to evaluate the matrix element. Thus if we define a new function which is a Dirac matrix by

\[ \delta_{ab} S_{\alpha\beta}(x-y) = i \varepsilon(x-y) \langle \text{vac} | P(\gamma_{\alpha\alpha}(x) \gamma_{\beta\beta}(y)) | \text{vac} \rangle \]

where \( \varepsilon(x-y) = \begin{cases} +1 & \text{if } x_0 > y_0 \\ -1 & \text{if } x_0 < y_0 \end{cases} \)

the fermion matrix element for diagram A can be expressed simply as:

\[ \langle \text{vac} | \gamma_5 \tau_f S_{\alpha\beta}(x-y) \gamma_5 \tau_i u_{\alpha\beta} | \text{vac} \rangle \]

In a similar way another function \( D_{\alpha\beta}(x-y) \) will arise when one computes the meson part of matrix elements for diagrams in which internal mesons are present in addition to external ones. This function contains no \( \varepsilon(x-y) \) in its definition, because there are no sign changes when "doomed" meson field operators are rearranged. It is

\[ \delta_{ij} D_{\alpha\beta}(x-y) = i \langle \text{vac} | P(\phi_i(x), \phi_j(y)) | \text{vac} \rangle \]

These are the two fundamental functions in terms of which all matrix elements will be expressed. A convenient way to think of how the analytic matrix element arises is the following. For the initial meson we put in its wave function. At \( y \) an interaction takes place, and \( ig\gamma_5 \tau_i \) is put in (together with the meson wave function). A fermion propagates from \( y \) to \( x \) under the influence of a "propagator" \(-iS_{\alpha\beta}(x-y)\). At \( x \) another interaction takes place and \( ig\gamma_5 \tau_f \) is put in along with the final meson wave function. One then integrates over all possible interaction points. In this procedure we follow the fermion line and no attention is paid to the internal time ordering, i.e., whether the fermion is a particle or antiparticle. Thus, the complete matrix element for diagram A is:
If the Fourier transform of $S^F(z)$ is defined as

$$S^F(p) = \int e^{-i \mathbf{p} \cdot \mathbf{z}} S^F(z) \, d^4z,$$

then the above matrix element is just:

$$\frac{g^2 M}{2V^2 \sqrt{E_p E_p} k' k'' o o} \left\{ \bar{u}_{p''} \gamma_5 \tau_f \left[ \int e^{i (k' \cdot y + p' \cdot y - k'' \cdot x - p'' \cdot x)} (-i S^F(x,y)) \, d^4x \, d^4y \right] \gamma_5 \tau_1 u_{p'} \right\}.$$

If we do everything in momentum space we can write down a rather simple answer quickly, realizing that the Fourier component of the propagator is chosen which gives momentum and energy conservation at every vertex. All one has to know are the Fourier transforms of the fermion and boson propagators in order to calculate the matrix element for a given diagram. For example, using the rules, the matrix element for the crossed diagram B is:
December 9, 1958

We have illustrated in a simple example how the Feynman rules for perturbation calculations come out of the field theory formalism. The essence of the Feynman method is that the matrix element for any process in a given order of perturbation theory can be obtained by writing down the relevant diagrams and then substituting the two functions: \( S^F(x-y) \) and \( D^F(x-y) \), which are essentially the vacuum expectation values of the ordered product of two field operators.

As we have defined them, these two functions are Green's functions for the Dirac and Klein-Gordon equations, respectively.

\[
(\gamma + M) S^F = \delta^4(x-y) \equiv \delta^3(x-y) \delta(t_x-t_y)
\]

\[
(\mu^2 - \Box^2) D^F = \delta^4(x-y).
\]

Let us first prove these assertions, and then it will be quite easy to find the Fourier transforms of these functions, which were seen to be the important entities in the actual calculation of scattering matrix elements.

If \( S^F(x-y) = i \langle \text{vac} | \mathcal{P}(\psi(x), \bar{\psi}(y)) | \text{vac} \rangle \) is substituted into the Dirac equation, 0 is obtained except for the discontinuity at \( t_x = t_y = 0 \) where a contribution arises.

\[
(\gamma + M) S^F = -\frac{i\beta}{\mu} \Delta S^F \delta(t_x-t_y) = \beta \langle \{\psi(x), \bar{\psi}(y)\} \rangle \delta(t_x-t_y)
\]

\[
= \delta^3(x-y) \delta(t_x-t_y) = \delta^4(x-y).
\]

Similarly, \( D^F(x-y) = i \langle \text{vac} | \mathcal{P}(\phi(x), \phi(y)) | \text{vac} \rangle \) clearly obeys the K-G equation except for a discontinuity at \( t_x = t_y = 0 \).

\[
(\mu^2 - \Box^2) D^F = \delta(t_x-t_y) \Delta \frac{\delta D^F}{\delta t_x} = i \delta(t_x-t_y) \langle \frac{\delta \phi(x)}{\delta t_x}, \phi(y) \rangle = \delta^4(x-y).
\]

Thus these two functions are indeed Green's functions. They are Green's functions, however, with rather peculiar boundary conditions.

Consider the set of Green's functions for the Dirac equation. There are an infinite number of these with different boundary conditions. All of them have the property that

\[
\Delta S(x-y) = i \beta \delta^3(x-y) = i \sum_n \psi_n(x) \bar{\psi}_n(y).
\]

Of the unlimited number of these things, only a few are of particular interest. For example, if we were dealing classically with the equation

\[
(\gamma + M) \psi(x) = -i \gamma_5 \tau \psi(x).
\]
we would choose the retarded Green's function
\[
S_{\text{ret}}(x-y) = \begin{cases} 
\frac{1}{\hbar} \psi_n(x) \bar{\psi}_n(y) & \text{for } t_x - t_y > 0 \\
0 & \text{for } t_x - t_y < 0 
\end{cases}
\]
in setting up an integral equation for \(\psi(x)\) from which a perturbation solution may be obtained.

\[
\psi(x) = \psi_{\text{in}}(x) + \int S_{\text{ret}}(x-y) \left\{ -i g \gamma_5 \tau_x \bar{\psi}_n(y) \right\} \psi(y) \, dy.
\]

In quantum mechanics, when we solve the Heisenberg equations of motion, precisely the same type of integral equation may be used to generate the perturbation solution. But the lack of commutation in the quantum mechanics requires that a different Green's function be used in place of the retarded one. In quantum mechanics it is required that \(S^F(x-y)\) be used. The formula for \(S^F\) may be deduced quite easily from its definition as a vacuum expectation value. If we designate by \(n^+ (n^-)\) a positive (negative) frequency solution of the Dirac equation,

\[
S^F(x-y) = i \sum \psi_n(x) \bar{\psi}_n(y) \text{ for } t_x > t_y
\]
and
\[
S^F(x-y) = -i \sum \psi_n(x) \bar{\psi}_n(y) \text{ for } t_x < t_y.
\]

\(S^F(x-y)\) is retarded for the positive frequency solutions of the Dirac equation, but advanced for the negative frequencies. Thus the positive frequency solutions are required to propagate forward in time, while the negative frequency solutions must propagate backward in time. Feynman developed an interpretation of this in which the particle is said to be going forward in time, and the antiparticle is said to be a particle traveling backwards in time.

The point of view just advanced helps to clarify why the two cases can be treated as a single Feynman diagram:
In the language of field theory, the second time-ordered diagram would be described as follows: The vacuum first creates the final meson and a nucleon-antinucleon pair. The antinucleon then propagates forward in time until it meets up with the incoming meson and nucleon, where everything is annihilated. On the other hand, Feynman would describe it by saying that the nucleon and meson interact and the nucleon is scattered so that it travels backward in time. Finally the nucleon reverses its direction, emits the outgoing meson, and again travels forward in time. From Feynman's point of view the two processes are quite alike, and thus one expects only a single matrix element for their sum.

A similarly simple evaluation of the vacuum expectation value yields the result:

\[ D^F(x-y) = i \sum_n \varphi_n(x) \phi_n(y) \quad \text{for } t_x > t_y \]

and

\[ D^F(x-y) = i \sum_n \varphi_n(x) \phi_n(y) \quad \text{for } t_x < t_y. \]

The similarity of the quantum mechanical solution with the classical iterative solution of an integral equation is very useful for keeping track of certain factors embodied in the Feynman Rules. The actual development of those rules can be found in the article by G. C. Wick on normal products. But a simple trick for getting the right factors — it is not a derivation of the rules, of course — is to think of solving a classical problem with the Feynman Green's function. This mnemonic device gives the right number of i's, g's, etc. in a \( n \)-th order matrix element, and also indicates there are no factors of \( n \) or \( n! \). (When we get to closed loop diagrams, there will be a few \( n \)'s, but otherwise there are no \( n \)'s floating around.) For example, up to second order:

\[
\psi(x) = \psi_{in}(x) + \int S^F(x-y) d^4y \{- i g \gamma_5 \tau_i \varphi_i(y)\} \psi_{in}(y) \\
+ \int S^F(x-y) d^4y \{- i g \gamma_5 \tau_j \varphi_j(y)\} S^F(y-z) d^4z \{- i g \gamma_5 \tau_j \varphi_j(z)\} \psi_{in}(z).
\]

In this expansion, if \( \varphi_i(y) \) and \( \varphi_j(z) \) are associated with the creation and absorption of a virtual meson, they are replaced by \( \delta_{ij} (-i D^F(y-z)) \), the meson propagation function.

Our major interest is the form of these propagators in momentum space. Since they are Green's functions, we can assert immediately that:

\[
S^F(x) = \frac{1}{(2\pi)^4} \int \frac{d^4p}{i^2 + M^2} \quad \text{and} \quad D^F(x) = \frac{1}{(2\pi)^4} \int \frac{d^4q}{i^2 + q^2}.
\]

All Green's functions differ only by the prescription at the poles where \( p_\sigma^2 = \vec{p}^2 + M^2 \), and \( q_\sigma^2 = \vec{q}^2 + \mu^2 \). The pole prescription which yields \( D^F \) and \( S^F \) is the relativistically invariant procedure of regarding the masses involved as having a slightly negative imaginary part. This prescription can be made self-evident in the following way.

A glance at the Fourier decomposition of these Green's functions as exhibited in terms of the \( \psi_n(x) \) and \( \varphi_n(x) \) reveals that the characteristic feature of \( S^F \) and \( D^F \) is that for positive \( t = t_x \) only positive frequency components are included, while for negative \( t \) only negative frequencies occur. Now the time dependence of any component, \( e^{ip't} \), of these functions is obtained from an integral \( \int e^{-ip_\sigma t} f(p_\sigma) \, dp_\sigma \), where \( f(p_\sigma) \) has two simple poles, one on the positive real axis and the other on the negative real axis. Since the integrals may be evaluated by closing the contour by a loop in the lower (upper) half plane for positive (negative) \( t \), it is clearly desired that the contour simply pass above the pole on the positive real axis, and below the pole on the negative real axis as shown:

![Contour Diagram]

This route is guaranteed if the mass is regarded as having a small negative imaginary part. Thus:

\[
S^F(p) = \frac{1}{i \vec{p} + M - i\epsilon}, \quad \text{and} \quad D^F(q) = \frac{1}{\mu^2 + q^2 - i\epsilon}.
\]

All the factors are now available for the explicit calculation of \( S \)-matrix elements in momentum space corresponding to a given Feynman diagram.

Let us now consider a few examples of meson-nucleon scattering matrix elements before we pass over to electrodynamics.

Example 1:

![Feynman Diagram]
The common R-matrix element for this process is just:

\[ R_{\text{fi}} = \frac{1}{\sqrt{2E_{p''}}} \frac{1}{\sqrt{2k''}} \frac{1}{\sqrt{2k'}} \frac{1}{\sqrt{2E_{p'}}} \frac{1}{(\not{p'} + \not{k'}) + M - i\epsilon} \frac{1}{(\not{p''} + \not{k''}) + M + i\epsilon} \frac{1}{(\not{p'} + \not{k'}) + M} \gamma_5 \tau_1 u_{p'} \]

If one takes the absolute square of the R-matrix element and multiplies it by \(2\pi\) times the density of final states, one gets a transition rate, and if one divides that by the incoming flux of particles, a cross section is obtained.

For unpolarized nucleons, one then has to sum the cross sections over the final nucleon spin states and average it over the initial nucleon spin states.

Example 2:

The corresponding R-matrix element is:

\[ R_{\text{fi}} = \frac{1}{\sqrt{2E_{p''}}} \frac{1}{\sqrt{2k''}} \frac{1}{\sqrt{2k'}} \frac{1}{\sqrt{2E_{p'}}} \frac{1}{(\not{p'} - \not{k'}) + M} \frac{1}{(\not{p'} + \not{k'}) + M} \gamma_5 \tau_1 u_{p'} \]

Example 3:

The corresponding R-matrix element is the same as the one obtained by interchanging \(\tau_1\) and \(\tau_f\), and replacing \(k'\) by \(-k''\). Therefore, there is also a crossing theorem in the calculation of the R-matrix in the relativistic theory.
These are some of the diagrams involved in the relativistic calculation of meson-nucleon scattering. Renormalizations must be performed here as in the static theory, and we still must investigate how these go through. We now have most of the Feynman Rules; the only tricks we haven't got are the Pauli principle, which comes into play when we consider diagrams with two fermions, and the rules for closed loops, which are essentially similar but for which a little care will indicate a factor of \( n \) comes into play, and finally the renormalizations. In these relativistic theories, the renormalizations remove all dependence upon cutoffs. This property of a theory is called "renormalizability".

"Renormalizability" means that if one performs the necessary energy and mass renormalizations, and then, for convenience, renormalizes the coupling constant, and perhaps one or two other things, then, in the limit of high cutoffs, the resulting theory is independent of the cutoff. Electrodynamics has this property as does this \( \gamma_5 \) meson theory. This property is not in any way a required property of a correct theory, if you believe there really is a cutoff, as everybody does. However, in electrodynamics the results are independent of a cutoff, and electrodynamics stands today as a correct theory. The results of the \( \gamma_5 \) theory are also independent of cutoffs, but we have no evidence of their correctness. Thus I wish to warn you against believing that renormalizability must be a necessary property of a correct theory.

The point of interest now is concerned with a technical problem which arises when one derives Feynman's Rules for the FS-FV theory from the field-theoretic formalism; it is independent of the lack of "renormalizability" of this theory. The coupling Lagrangian for the FS-FV theory contains a gradient, which is not serious, but also for the fourth component a time derivative of the meson field operator. This implies that the momentum conjugate to \( \phi_4 \) is not just \( \partial_4 \), but rather

\[
\pi_4 = \partial_4 - \frac{\partial}{\partial \tau} \gamma_3 \gamma_4 \gamma_4 \gamma_4 \phi_4.
\]

Thus when the coupling Hamiltonian density is computed, it turns out to be not the negative of the coupling Lagrangian density, but rather has an extra term which cancels a term in the coupling Lagrangian density:

\[
\mathcal{H}_{\text{coup.}} = \frac{1}{\hbar} \left\{ \gamma_3 \gamma_4 \gamma_4 \gamma_4 \phi_4 - \gamma_3 \gamma_4 \gamma_4 \gamma_4 \phi_4 \right\}
\]

We must use both of these terms in calculating the S-matrix according to the conventional Hamiltonian formulation of quantum mechanics.
This final lecture of the term was devoted wholly to various topics raised by questions from students.

We have indicated how the Feynman Rules can be derived from the field theory of a pseudoscalar meson with pseudoscalar coupling to nucleons, where the coupling term in the Lagrangian density is:

$$\mathcal{L}_{\text{coup.}} = -ig \bar{\psi} \gamma_5 \tau_i \psi_1.$$  \hspace{1cm} (PS-PS theory)

Equally well, we could have concerned ourselves with pseudovector meson-nucleon coupling in which

$$\mathcal{L}_{\text{coup.}} = -i \frac{e}{\mu} \bar{\psi} \gamma_5 \gamma_\nu \tau_i \psi_1 \partial_\nu \varphi_1$$  \hspace{1cm} (PS-PV theory),

for both of these yield the same coupling in the static limit. People always use the PS-PS theory when they calculate, because after the renormalizations are performed, the results are independent of a cutoff in the limit of a high cutoff, whereas the PS-PV theory has gradients in it, which introduce extra powers of the momenta of virtual mesons into calculations, so that the results depend heavily upon the cutoff, even after renormalization. The introduction of a cutoff as a physical parameter is somewhat of a nuisance, and thus most of the attention has been directed toward the PS-PS \pi meson theory. This is not, however, an argument that the PS-PS theory is more correct than the PS-PV theory, as many people allege.

The point of interest now is concerned with a technical problem which arises when one derives Feynman's Rules for the PS-PV theory from the field-theoretic formalism; it is independent of the lack of "renormalizability" of this theory. The coupling Lagrangian for the PS-PV theory contains a gradient, which is not serious, but also for the fourth component a time derivative of the meson field operator. This implies that the momentum conjugate to \varphi_1 is not just \dot{\varphi}_1, but rather

$$\pi_1 = \dot{\varphi}_1 - i \frac{e}{\mu} \bar{\psi} \gamma_5 \gamma_4 \tau_1 \psi.$$  

Thus when the coupling Hamiltonian density is computed, it turns out to be not the negative of the coupling Lagrangian density, but rather has an extra term which cancels a term in the coupling Lagrangian density;

$$\mathcal{H}_{\text{coup.}} = i \frac{e}{\mu} \left( \bar{\psi} \gamma_5 \gamma_\nu \tau_1 \psi \partial_\nu \varphi_1 - \bar{\psi} \gamma_5 \gamma_4 \tau_1 \psi \partial_4 \varphi_1 \right).$$

We must use both of these terms in calculating the S-matrix according to the conventional Hamiltonian formulation of quantum mechanics.
But Feynman's Rules for calculating the S-matrix are stated only in terms of the interaction Lagrangian density, and contain no special provisions for cases where the interaction Lagrangian contains a time derivative. However, it turns out that there are two effects which exactly cancel each other in the derivation of the rules for perturbation calculations in such a theory. One of them is the extra term in the coupling Hamiltonian density, and the other is that when one encounters expressions of the form

$$\langle \text{vac} | F(\partial_t \phi(x), \partial_y \phi(y)) | \text{vac} \rangle,$$

these cannot be equated to

$$-\frac{\delta^2}{\delta t^2} D^F(x-y),$$

because of the discontinuous definition of $D^F$. Not only in this theory, but also in any theory you write down, it turns out that the effects of all terms in the coupling Hamiltonian density beyond the negative of the interaction Lagrangian density are cancelled exactly in a similar fashion. This is a rather disturbing result, for it shows that a rather poor formalism is being used to derive a simple answer.

In regard to these cancelling terms, the original situation in the literature was somewhat more involved, for people introduced general wiggly time-like hypersurfaces in four dimensions, instead of slicing space-time perpendicular to the time axis in a fixed reference system as we have done. This procedure is very general and leads to the same results, but introduces into intermediate steps terms dependent on the curvature of the surface, etc.

The PS-FV theory has been somewhat neglected, because its results depend heavily upon a cutoff, even after the usual renormalizations have been performed. If one believes that there really is new physics above 1 or 2 BEV/c, i.e. there is a cutoff, then that is no reason for preferring the PS-FS theory over the PS-FV theory. Both of them are incorrect for high energies, but the question remains as to which one explains low energy phenomena more completely. Although in the static limit they reduce to the same theory, they are really different, as is evident from a perturbation expansion.

Methods of Calculation

The strong coupling calculations are the only calculations which are not really perturbation calculations. All the rest of the methods are "glorified"
perturbation calculations in that one looks at the perturbation series and
sums what appear to be the most important diagrams for a certain phenomenon.
We gave some examples of this in the "magic formula" and the Tamm-Dancoff
methods. The dispersion equations are totally independent of perturbation
theory. However, the dispersion relations do not determine the answers for a
given problem.

In discussing available methods of calculation in field theories, we
must distinguish between static theories and relativistic theories. In the
static theories, a wide variety of things have been tried. First of all, there
is ordinary perturbation theory. Also the Tamm-Dancoff approximation has seen
wide use. The "magic formula" is the latest improvement in perturbation cal-
culations. Then there is strong coupling, which is an expansion in $1/g$. If
one puts enough restrictions on the dispersion theory equations, the answers
are then determined. Low's one-meson approximation is an example of such a
"restricted" dispersion theory. Tomonaga has developed an intermediate
coupling method to bridge the gap between the perturbation calculations of
weak coupling and strong coupling.

If one looks at a static theory, say at one we examined this term, then
the "glorified" perturbation methods indicate the existence of resonances,
such as the $3\cdot3$ resonance. If one looks at strong coupling then it tells us
that there is a set of bound states between the nucleon and nucleon + pion
energies, the number of which increases as $g^2$ increases. Such bound states
would be called true isobars of the nucleon. As far as the meson field is
concerned, these states would be absolutely stable; if they existed in nature,
they would decay by photon emission. And sure enough, the lowest of these is
$3/2, 3/2$. The next one is $1/2, 1/2$. A comparison of the extremes makes it
evident that as the coupling constant increases, the virtual states which start
out with energy + $\infty$ come down and down until they become true bound states.
The method of Tomonaga provides a mathematical bridge, a sort of interpolation
formula, which enables one to see the continuous change of behavior as the
coupling constant goes from one extreme to the other. If one does it right,
such a method will give rigorously both limits correctly and provide a formula
for in between.

---

The physical basis of Tomonaga's method is that in the cloud around a nucleon, in either very weak or very strong coupling, the following situation obtains. There is, of course, in the weak coupling approximation, only a small amplitude for having one meson in the cloud, and a still smaller amplitude for having two mesons in the cloud, etc. But that's not a feature that holds in strong coupling, so we forget it. We look for a common feature. Such a common feature is that the total number of meson wave functions, independent meson states, that contribute to the cloud is very small. In the first order of perturbation theory for weak coupling only states with one meson contribute and thus there is only one meson wave function involved. In strong coupling, there is a large amplitude for states with many mesons, 2, 3, 6, 12, 15, etc., but the total number of independent meson wave functions in the cloud is quite small. So Tomonaga's idea was to assume that there are just a few meson states which are populated. However, they may be populated by any number of mesons. Thus, he was led to define new creation operators:

\[ a^+_i = \int a^+_k f_i(k) \, d^3k. \]

In our static theory one needs about four of these, i.e. \( i = 1,2,3,4 \). In simpler theories, one needs less of them. In the charged scalar theory only two are required, one for positive mesons and one for negative mesons. Anyway, one now uses a variational method, and attempts to find a minimum of

\[ \langle \Phi | H | \Phi \rangle \]

where \( |\Phi\rangle \) is allowed to be formed from the bare nucleon state by allowing any number of these new creation operators to act upon it with any coefficients, and with the \( f_i(k) \) to be varied. This may sound like a lot, but actually it's not very much, for actually one is working with only four of the infinite number of field degrees of freedom. The problem is thus tremendously simplified and soluble. For the charged scalar theory, one can solve it in five minutes, while for the FS-PV symmetric static theory one obtains differential equations which are solved with a computing machine. One can then calculate, in weak or strong coupling, what this variational principle tells about the state vector of a nucleon, and one finds that the weak and strong coupling limits are both given rigorously by this method. It also gives something in between, which can be considered as an approximation to the correct answer. Also if this method is applied to meson-nucleon scattering, fairly good agreement with experiment is obtained.
The trouble is that all of these methods run into difficulty when one tries to use them on a relativistic theory. The only hopeful method now in sight consists in trying to restrict the relativistic dispersion equations so that answers may be obtained. This suffers from certain difficulties in that the relativistic dispersion theory requires certain subtractions, that is to say, there are some constants in the dispersion equations which are not given by dispersion theory, they have to be fed in from some other source. However, since nobody knows what to do with the constants, this method is likewise awkward. Thus we are left with zero as far as methods of calculation in relativistic meson theory. The dispersion theory, however, with its subtractions — not the restricted dispersion theory, but the actual dispersion equations with the real part of the amplitude on one side and the imaginary part on the other — have enjoyed tremendous success, and this is essentially all we have. It provides the only information on the relativistic meson-nucleon theories which is really reliable.

The Lee Model

A few years ago Lee invented an abridged field theory of three particles which is so simplified that problems can be solved exactly. It is chopped up so much that the Dancoff approximation is rigorous, and thus all the results of the Dancoff approximation are the exact results of the theory. Having solved the mathematical parts of the theory, one can examine the qualitative results and speculate as to whether the qualitative features are characteristics of common field theories.

Lee's theory designates three particles as V, N, and θ, and allows the single reaction: \( V \leftrightarrow N + \theta \). This differs from usual field theories in that ordinarily the θ would have an antiparticle, perhaps itself, and that the inverse reaction \( V + \bar{\theta} \leftrightarrow N \) would also be allowed, because any common field theory associates positive and negative frequencies together. If you leave the θ out, this means that you are leaving out the negative frequencies. Now a field containing only positive frequencies cannot be local or casual, because the operation of coupling only positive frequencies involves performing an integral over all time. The positive frequency part of a function is

\[
f_+(t) = \int \delta_+(t - t') f(t') \, dt'.
\]

So in the Lee model, the V and N at a given time t is coupled to the θ field at all times. The interaction is tremendously non-local, and in this sense is the
artifact of our ordinary field theoretic interactions.

A more familiar notation would be that of \( p, n, \) and \( \pi^+ \). Because of the limitation to just these three particles, the static meson-nucleon theory is easily worked with. Consider the construction of the physical nucleon states on the Lee model. The bare neutron is the complete neutron for the \( n \) can do nothing by itself. The physical proton has a structure due to the diagram

![Diagram of the nucleon states](image)

but the only other proton self-energy diagrams are simply iterations of the above like

![Diagram of proton self-energy iterations](image)

Thus a physical proton is part of the time a bare proton, and part of the time a bare neutron plus a single meson in some definite state. Because of this obvious simplicity, the theory can be solved exactly.

This theory has some rather remarkable properties, and controversies have raged concerning whether the properties are characteristics of field theories in general. They involve renormalization results. In this model there are four arbitrary parameters: the \( (p,n) \) mass difference, the meson mass, the bare coupling constant \( g_0 \), and the cutoff. From the theory it is possible to calculate the renormalized coupling constant in terms of the initial parameters, and the answer is something like

\[
\frac{g_1^2}{g_0} = \frac{g_0^2}{1 + \chi^2 g_0^2}
\]

where, say, \( \chi \) is proportional to the log of the cutoff. Thus if we plot the result,

![Graph of \( g_1^2 \) vs. \( g_0^2 \)](image)
and the important result is that for any given value of the cutoff, there is a maximum value of the renormalized coupling constant.

Now in a physical situation, one observes the $g_1^2$ and adjusts the cutoff to give agreement with experiment, and lets the $g_0^2$ fall where it may. The Lee Model raises the following interesting question. What if, for the required $g_1^2$ and cutoff, there exists no value of $g_0^2$? Many people believe that this is a property of the static theory we have investigated this semester, and that thus this theory is internally inconsistent. They believe that the cutoff around the nucleon mass is so large that no possible value of the bare coupling constant gives the required renormalized coupling constant. They assert, therefore, that the static theory is rubbish! And then they go on to present a more trenchant criticism of relativistic field theories. In relativistic theories, having renormalized, we remove the cutoff entirely, keeping the renormalized coupling constants at a given value. For example, in electro-dynamics, $g_1^2$ is kept at 1/137. If relativistic electrodynamics has the renormalization property of the Lee Model, this process of removing the cutoff to $+\infty$ is total rubbish.

However, one may ask where the trouble is introduced as the cutoff is removed in electrodynamics. If one does this, it is clear that introducing a cutoff at about that point won't disturb anything very much. Since the divergences in electrodynamics are logarithmic, a cutoff is necessary at about

$$\Lambda \approx m \exp(4\pi/e^2) \approx 10^{50}\text{ Mev}.$$ 

Nobody would quarrel with a cutoff up there! Thus if the Lee Model property does obtain in electrodynamics, things are easily fixed up. But if one looks at relativistic meson theories, the problem is much more serious. There the cutoff would have to be at 1/2 BEV. So the criticism of relativistic meson theory is a quite serious one, if it's valid.

Now nobody knows whether this property of the Lee Model is at all a general property of field theories. Therefore, these criticisms may be wrong or may be right. It would be nice if they were right, for that would provide the necessary impetus for dumping field theory as we know it, and looking for some radical new idea. As long as we don't know, the field theory may be the correct formalism, and it is difficult to discard something that has had some successes and could have some more. On the other hand, we do know that probably these theories must be thrown away. Otherwise, how are we to introduce some new physics, say at $10^{-14}\text{ cm.}$, which will lead to the existence of 31 or
more particles? The properties of the particles must be determined by some new physical principle. It is, of course, possible that a new principle, when added to the present formalism, will lead to an understanding of the particles. That is, you write down a field theory with 31 or so fields, and this principle just tells you what masses, coupling constants, and spins to put in. But this is very unlikely! It seems to many of us that there is new physics at 0.1 fermis which replaces field theory by something else, and that the new theory will contain enough in it to determine some of the things which are now put into field theory ad hoc.

There is a second thing that people have looked at in connection with the Lee Model, which is to follow the curves down to negative values of $g_0^2$:

Thus, if you are willing to take a negative value of $g_0^2$, you could get any value of $g_1$ you desire with a given cutoff. But $g_0^2$ negative means that the coupling Hamiltonian is antihermitian, the S-matrix is not unitary, and probabilities are either not conserved or not positive-definite. People have tried all sorts of schemes to make sense out of this region, but so far sense appears to come out only at the price of microscopic causality. But until somebody really succeeds with one of these schemes, the only thing we can say is that field theory is in great trouble if the property of the Lee Model is general.

**Nature of the Perturbation Series**

It seems clear that the perturbation series of, for example, quantum electrodynamics are not convergent. The following proof, due to Dyson, while not absolutely rigorous, is fairly convincing. Suppose we calculate something to all orders, getting a series $f(e^2)$. Then if the series is convergent for some positive value of $e^2$, it must converge in some circle of the complex $z = e^2$ plane centered at the origin. Thus the series must converge on some interval of the negative real axis.

Consider now the calculation of the polarization of the vacuum, in a power series in $e^2$, where diagrams such as
arise. If the series converges for positive $e^2$, it must also do so for negative $e^2$, and so it must be possible to describe, in a convergent way, the series of physical processes which would go on if electrons attracted each other and repelled positrons. Now a physical argument is presented to show that what happens for negative $e^2$ is not a convergent phenomenon, that is, the contribution of higher order diagrams to the series is much more important than that of lower order diagrams. If $e^2$ were negative, the vacuum would break down, for states in which there are large numbers of positron-electron pairs, with the positrons clustered together in a region of space, and the negatrons clustered together in another region of space far removed, would be of lower energy than the vacuum if the number of pairs were large enough. The vacuum state would thus be unstable relative to such states; the more particles in the state, the more stable it would be relative to the vacuum. This increasing importance of many particle-states for negative $e^2$ means that the series could not be convergent.

Although the series probably does not converge, for positive $e^2$ it may be an asymptotic series in that the first few terms give a better and better approximation to the true result, and if you go out far enough in the series, say at the 137th term, the series begins to diverge. This is entirely possible and gives us a warning as to what may happen in relativistic meson theory expansions. There the series might start to diverge in order $1/15$th.
In going over to electrodynamics to perform calculations, we may use the following "dictionary".

<table>
<thead>
<tr>
<th>Meson Theory</th>
<th>Electrodynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>e</td>
</tr>
<tr>
<td>π</td>
<td>γ</td>
</tr>
<tr>
<td>τ</td>
<td></td>
</tr>
<tr>
<td>γ5</td>
<td>γν (summed over for virtual photons). We work in a Lorentz gauge where k•e=0, e being the polarization four vector. For real photons, e₄=0, i.e. &quot;transverse&quot; gauge.</td>
</tr>
<tr>
<td>μ²</td>
<td>0</td>
</tr>
<tr>
<td>ε²/4π = 15</td>
<td>e₂/4π = a = 1/137</td>
</tr>
<tr>
<td>M</td>
<td>m</td>
</tr>
<tr>
<td>-1 ε Y5 τ₁ ε₁</td>
<td>Aν / 1 e μ</td>
</tr>
</tbody>
</table>

Using this dictionary, we shall give some examples of perturbation calculations in electrodynamics. This facet of relativistic field theory is very satisfactory, for one starts with a well-defined theory, calculates something physical, and finds perfect agreement between the answer and the experimental result.

First we shall consider the interaction of an electron with an external field. The external field is taken to be weak; in Feynman's language, the electron interacts only once with it. Although the external field is considered only in the first order of perturbation theory, we still have to expand in powers of the radiation field.

In the lowest order of perturbation theory, for true scattering, only this diagram...
contributes, and the first order R-matrix element is:

\[
\frac{\sqrt{E_1 E_2}}{m} R_{11}(1) = -i e (\gamma_\mu (p_2-p_1)) (\bar{u}_{p_2} \gamma_\mu u_{p_1}).
\]

In the third order, the following diagrams are involved.

(a)

(b)

(c)

(d)

(e)

Diagram (e) gives no contribution, since it is cancelled by a vacuum self-energy term. Diagram (d) is essentially the same no matter what transition the external field induces. It is connected with vacuum polarization, and charge renormalization. In the vacuum polarization terms, there should be contributions from virtual pair formation of nucleons, mesons, etc., but these are quite small compared to electron pair effects because of the high masses involved.

A careful treatment of renormalizations yields the prescription that one should take only half of diagram (b) plus half of diagram (c) in the calculation of the R-matrix. For our present purposes, we may just take only one of these, say (b), and forget the 1/2.

Whenever \( \gamma_1 \) or \( \gamma_2 \) occurs in one of our matrix elements, using the commutation rules we can slide them over to the right or left side, respectively, of the group of Dirac matrices, so that they operate immediately on corresponding free spinors. There they yield just the factor \( i\mu \). With this in mind, consider the possible final form for the matrix element under discussion summed to all orders in the radiative corrections. Factoring out the \( A_\mu \), the result must be a vector, and not, for example, a pseudo vector or pseudoscalar.
The only invariant in the problem is \((p_2 - p_1)^2\) and the only vector matrix elements are those of:

\[
(p_2 \pm p_1)_\mu, \quad \gamma_\mu, \quad \text{and} \quad \sigma_{\mu\nu}(p_2 \pm p_1)_\nu.
\]

Next consider the matrix element corresponding to some diagram like

\[
\begin{array}{c}
  \text{Diagram} \\
  \text{with } p_2 \text{ and } p_1
\end{array}
\]

for such a matrix element in the sum there is also a corresponding matrix element in the sum obtained by reversing the order of the \(\gamma\) matrices and interchanging \(p_1\) and \(p_2\). It belongs to a "mirror" diagram like

\[
\begin{array}{c}
  \text{Mirror diagram} \\
  \text{with } p_2 \text{ and } p_1
\end{array}
\]

Therefore, the complete sum must be unchanged under those two operations. Hence, the only vector matrix elements which will appear are:

\[
(p_1 + p_2)_\mu, \quad \gamma_\mu, \quad \text{and} \quad \sigma_{\mu\nu}(p_2 - p_1)_\nu.
\]

Finally, consider the following identities:

\[
\begin{align*}
\bar{u}_2 \sigma_{\mu\nu}(p_2 + p_1)_\nu u_1 &= \frac{1}{2i} \bar{u}_2 (\gamma_\mu (p_2 + p_1) - (p_2 + p_1) \gamma_\mu) u_1 = \bar{u}_2 \frac{(p_2 - p_1)_\mu}{i} u_1 \\
\bar{u}_2 \sigma_{\mu\nu}(p_2 - p_1)_\nu u_1 &= \frac{1}{2i} \bar{u}_2 (\gamma_\mu (p_2 - p_1) - (p_2 - p_1) \gamma_\mu) u_1 = \bar{u}_2 \left(\frac{(p_2 - p_1)_\mu}{i} - 2m \gamma_\mu\right) u_1.
\end{align*}
\]

As a result of these identities, the final matrix element summed to include all radiative corrections must be of the form:

\[
R_{I_1} = \left\{-ie \frac{A_\mu}{\sqrt{\varepsilon_1 \varepsilon_2}} \bar{u}_2 \gamma_\mu u_1 \Gamma((p_2 - p_1)^2) + \frac{ie}{2m} \frac{A_\mu}{\sqrt{\varepsilon_1 \varepsilon_2}} \bar{u}_2 \sigma_{\mu\nu} u_1 (p_2 - p_1)_\nu G((p_2 - p_1)^2) \right\} \left(\frac{m}{\sqrt{\varepsilon_1 \varepsilon_2}}\right)
\]

From a non-field theoretic approach, we can say that the electron thus acts as if its charge were

\[
e F((p_2 - p_1)^2),
\]

and as if it had an anomalous magnetic moment of

\[
\left(\frac{e}{2m}\right) G((p_2 - p_1)^2).
\]
The functions $F$ and $G$ of the square of the invariant momentum transfer are called respectively the charge and magnetic moment form factors of the electron. Corresponding entities are of considerable interest in the study of nucleons. The charge form factor $F(q^2)$ is a very weak form factor for the electron; it starts out at 1 when $q^2=0$, and its frequency variations are of order $1/137$ over a large range. For the nucleon on the other hand, $F(q^2)$ and $G(q^2)$ vary quite a bit over a few hundred MEV due to the strong meson effects. $G(0)$ for the electron is the Pauli anomalous moment due to virtual photons. $G(q^2)$ is, of course, strictly of order $1/137$, and thus its frequency variation is not drowned out by a zero order term.

We shall now calculate the anomalous moment of the electron to order $1/137$. Apart from the effects of charge renormalization, a contribution arises only from diagram (a):

\[
\begin{align*}
\mathbf{R}_{\text{m}}(3)(a) &= \frac{e^3}{\mu} \frac{4}{(2\pi)^4} \frac{d^4k}{ik^2} \left\{ \bar{u}_2 \gamma_\nu \frac{1}{1(p_2-k)+m} \gamma_\mu \frac{1}{1(p_1-k)+m} \gamma_\nu u_1 \right\}.
\end{align*}
\]

Rationalizing the denominators, the matrix element becomes:

\[
\begin{align*}
1 \frac{e^3}{\mu} \frac{4}{(2\pi)^4} \frac{d^4k}{ik^2} \left\{ \bar{u}_2 \gamma_\nu \frac{(-1)}{k^2 - 2p_2 \cdot k} \gamma_\mu \frac{(-1)}{k^2 - 2p_1 \cdot k} \gamma_\nu u_1 \right\}.
\end{align*}
\]

In the evaluation of such matrix elements, certain identities are of considerable use.

\[
\begin{align*}
\gamma_\mu + \gamma_\nu &= 2a \cdot b \\
\gamma_\mu \gamma_\nu &= 4 \\
\gamma_\mu \gamma_\nu &= -2\delta \\
\gamma_\mu \gamma_\nu &= 4a \cdot b \\
\gamma_\mu \gamma_\nu &= -2\delta / \delta
\end{align*}
\]

The verification of these identities is left as an exercise for the reader. Employing these and also the facts

\[
\begin{align*}
\bar{p}_1 u_1 &= i m u_1, \\
\bar{u}_2 \bar{p}_2 &= \bar{u}_2 i m,
\end{align*}
\]

one can reduce the matrix element as follows:
\[ \frac{\text{Weizsäcker}(3)(a)}{m_{p_1}} = \int \frac{1}{(2\pi)^4} \frac{d^4 k}{k^2 - 2p_2 \cdot k (k^2 - 2p_1 \cdot k)} \{ \bar{u}_2 (+i\gamma_5 k - 2ip_2 \nu) \gamma_\mu (+i\gamma_5 \gamma_\nu - 2ip_1 \nu) u_1 \} \]

\[ = \int \frac{3}{(2\pi)^4} \frac{d^4 k}{k^2 - 2p_2 \cdot k (k^2 - 2p_1 \cdot k)} \{ \bar{u}_2 (2\gamma_\mu k - 4p_2 \cdot p_1 \gamma_\mu + 2\gamma_\mu k p_2 + 2p_1 \nu \gamma_\mu ) u_1 \} \]

\[ = \int \frac{3}{(2\pi)^4} \frac{d^4 k \bar{u}_2 \{ -2k^2 - 4p_2 \cdot p_1 + 4k^* (p_2 + p_1) \} \gamma_\mu + 4imk + 4k (k - p_2 - p_1) \} u_1 }{k^2 - 2p_2 \cdot k (k^2 - 2p_1 \cdot k)} \]

For the evaluation of such integrals, Feynman developed a technique which is based upon the following identities. These also should be checked by the reader as an exercise.

\[ \frac{1}{a_1 a_2 \cdots a_n} = (n-1)! \int \frac{dx_1}{\circ} \frac{dx_2}{\circ} \cdots \frac{dx_{n-2}}{\circ} \frac{dx_n}{\circ} . [a_1(1-x_1)+a_2(x_1-x_2)^+ + \cdots + a_{n-1}(x_{n-2}-x_{n-1})+a_n x_{n-1}]^n \]

\[ \int \frac{d^4 k}{[k^2 - i\epsilon - 2k \cdot r]^3} = \frac{\pi^2}{2i\epsilon^2} \quad \int \frac{d^4 k}{k^2 - i\epsilon - 2k \cdot r} = \frac{\pi}{2i\epsilon^2} \]

From this point on we will keep only those terms in the integral which give a contribution to the moment form factor \( G(q^2) \); thus we drop the terms which give a \( \gamma_\mu \) matrix element. Using the first Feynman trick, the part of the matrix element of diagram (a) under consideration can be written as:

\[ \frac{e^3}{2\pi^4} \int \frac{1}{d\tau} \frac{d}{dx} \frac{d}{dy} \bar{u}_2 \{ i m k \gamma_\mu + (k - p_2 - p_1) \} u_1 \]

where \( r = p_2 (x-y) + p_1 y \). If we drop that part of \( \gamma_\mu \) which yields a \( \gamma_\mu \) term upon integration, the remaining quadratures can be carried out by applying the given formulae, and there results:

\[ \frac{e^3}{2\pi^4} \int \frac{1}{d\tau} \frac{d}{dx} \frac{d}{dy} \bar{u}_2 \{ i m k \gamma_\mu + \gamma_\mu (r - p_2 - p_1) \} u_1 \frac{\pi^2}{2i\epsilon^2} \]

Inserting the value of \( r \), and defining the four-vector \( \Delta \) by

\[ m\Delta = p_2 - p_1 = q, \]

one can put the expression in this form:
Further reduction is expedited by noting that \( \tilde{u}_2 \nu \tilde{u}_1 = \tilde{u}_2 \nu \tilde{u}_1 = \tilde{u}_2 \nu \tilde{u}_1 = \tilde{u}_2 \nu \tilde{u}_1 \).

Hence, we get

\[ -\varepsilon_2^A \left\{ \frac{1}{2} \int \int_0^\infty \frac{p_2(x-y) - p_1(x-y)}{x^2 + \Delta^2 y} \, dy \, dx \right\} \tilde{u}_2 \nu \tilde{u}_1 = \tilde{u}_2 \nu \tilde{u}_1 = \tilde{u}_2 \nu \tilde{u}_1 \]

The breakthrough in the evaluation of the integrals is made by exploiting fully the symmetry under the change of variables \( y \leftrightarrow x-y \). Judicious use of this device produces the following form:

\[ + \frac{\varepsilon_2^A}{4\pi m} \left\{ \tilde{u}_2 \nu \left( p_2 + p_1 \right)_\mu \tilde{u}_1 \right\} \left( \frac{1}{2} \int \int \frac{x \, y \, dy \, dx}{x^2 + \Delta^2 y} \right) \]

Let us call the number generated by the above double integral, which is a function of \( \Delta^2 \), \( g(\Delta^2) \). It is quite easy to see that \( g(0) = +1/4 \). If one makes the substitution \( y = tx \), then

\[ g(z) = \frac{1}{4} \int \int_0^\infty \frac{t \, dt}{1 + z \, t(1-t)} \, dx = \frac{\sinh^{-1} \sqrt{\frac{z}{4}}}{\sqrt{\frac{z}{4}}}. \]

Remembering that the matrix element of \( \left( p_2 - p_1 \right)_\mu \) was shown to be equivalent to the matrix element of \( 2 \tilde{i} \nu \gamma_\mu + i \gamma_\mu \nu \tilde{p}_2 \nu \tilde{p}_1 \nu \),

we see that the radiative corrections gives rise to an anomalous Pauli moment for the electron, and that to the lowest order in \( e^2 \),

\[ G(q^2) = \frac{a}{2\pi} 4g(\Delta^2). \]

Thus, to lowest order, the Pauli anomalous moment, \( G(0) \), is \( a/2\pi \) Bohr magnetons.

It would be nice if we could calculate the nucleon magnetic moments similarly, but, of course, the magnitude of the relativistic coupling constant makes this impossible. But this is supposed to be the physical basis also of the structure of the nucleon; it emits virtual mesons, and if you were to sum all orders in the meson coupling, you hope that you would get the nucleon magnetic moments and their shapes.

Due to the minuteness of the variation in the magnetic form factor over the range of momentum transfers observed experimentally up to now, this shape factor has not been checked. In fact, the behavior of quantum electrodynamics...
for momentum transfers higher than a few MEV has never been checked. This is a
shame, for electrodynamics is the only quantum mechanical field theory in which
exact calculations to a desired accuracy are available, and thus one could tell
from experiment exactly where the modifications apparently necessary begin to
show up. For this reason the most exciting experiment that has been proposed
recently is the colliding beam arrangement of O'Neill and Panofsky, which may
probe small enough distances to find a discrepancy with quantum electrodynamics.

January 8, 1959

In the last lecture, the discussion of the scattering of an electron by
a weak external field was commenced, and it was shown that the matrix element
for this process including all radiative corrections must be of the form:

\[ R_{fi} = \{-i e_1 A_\mu \{ \bar{u}_2 \gamma_\mu u_1 \} F(q^2) + \frac{i e_1 A_\mu}{2m} \{ \bar{u}_2 \sigma_\mu\nu u_1 \} q_\mu G(q^2) \} \frac{m}{\sqrt{E_1E_2}} \]

where \( q = p_2 - p_1 \). The magnetic part of the interaction comes, in second order,
only from diagram (a), and it was possible to compute \( G(q^2) \) to that order. The
existence of such a term in the matrix element can be described by assigning to
the electron an anomalous magnetic moment, \( G(0) = \alpha/2\pi \), together with a shape
for the magnetic interaction, \( G(q^2)/G(0) \). Similarly, the existence of \( F(q^2) \) is
described by saying that the electron has a charge, \( F(0) \), together with a shape
for the charge interaction, \( F(q^2)/F(0) \). Unlike \( G(0) \), \( F(0) \) is not a consequence
of the theory. The real charge \( e_1 \) is a constant which is fed into the theory,
and thus \( F(0) = 1 \).

To continue the discussion, we shall investigate the contribution to \( F(q^2) \)
in second order from diagrams (a), (b), and (c), temporarily omitting the effect
of the vacuum polarization diagram (which gives rise to the actual charge re-
normalization in electrodynamics).

Whereas in the calculation of the magnetic moment, there are no divergences, and
thus the existence of an anomalous magnetic moment might have been predicted
many years before the advent of the renormalization program, the computation of
the charge form factor involves many terms which are strongly dependent upon a cutoff. However, even this might have been done many years ago, except that the arithmetic was so hard in the days before Feynman developed his methods, and thus people used to drop terms which should have cancelled out many of the divergences. The improvement in the methods of calculation has really been a great boon for quantum electrodynamics.

When we look at the effects of diagrams (a), (b), and (c), it will be found that the divergences in each diagram cancel each other. Thus there is no reason to renormalize the charge due to the fermion self-energy and vertex corrections (in second, and in all orders.) Unfortunately, in the calculation, divergences appear in various steps at the infrared end of the photon spectrum, which are due to the vanishing rest mass of the photon. These divergences, however, are not serious in the theory, since if one adds up all the higher order terms in the series, the divergences will cancel. Also, in any practical calculation, one has to compute a total cross-section which includes the possibility of soft photon bremsstrahlung in addition to elastic scattering, and to a given order in $\varepsilon$, the infrared divergences in both these cross sections cancel each other. As a result, a practical calculation of a radiative correction will result in an answer which is logarithmically dependent on the minimum detectable photon energy. So we shall not worry henceforth about the divergences in the infrared.

Let us consider now the effect of a self-energy correction such as appears in diagrams (b) and (c). The effect of such a bubble on an internal fermion line is a correction to the fermion propagator $S_F(p) = (i\gamma + m)^{-1}$. If one considers the effect of all self-energy corrections to an internal line

![Diagram](image)

It is apparent that a new propagator $S^0(p) = (i\gamma + m + \Sigma^*(p))^{-1}$ should be inserted for $S(p)$. To the second order,

$$S^0(p) \approx S(p) - S(p)\Sigma(p)S(p)$$

where $\Sigma(p) = \Sigma^*(2)(p)$. Analytically,
The Dirac matrix function $\Sigma(p)$ is a scalar analytic function of the scalar variable $i\vec{p} + m$. Therefore, it is possible to write

$$\Sigma(p) = \text{constant} + A(i\vec{p} + m) + (i\vec{p} + m)^2 \Sigma_f(p).$$

It is important to realize that the $2\Delta m(2)$ is chosen so that the constant term in this expansion vanishes. That is, $\Delta m$ is defined implicitly by the relation $\Sigma^*(p) = 0$ when $\vec{p} = \vec{m}$. Thus, to second order,

$$S^0(p) = (1 - A) S(p) + \Sigma_f(p).$$

Now if one calculates the values of $\Delta m$, $A$, and $\Sigma_f(p)$, it will be found that $\Delta m$ and $A$ are logarithmically dependent upon a cutoff, whereas $\Sigma_f(p)$ is independent of a cutoff. Therefore, in the absence of knowledge of what happens at high momenta, we must reformulate the theory so that $A$ and $\Delta m$ do not appear in the answers. This is, of course, the renormalization program. $\Delta m$ is just the electron mass renormalization, and the factor $(1 - A) \approx Z_2$ is associated with the wave function renormalization. The wave function renormalization is carried through by defining a new propagator $S'(p)$ by the relation $S'^0(p) = Z_2 S'(p)$, and by absorbing the factor of $Z_2$ in the charge renormalization. In this way, the remaining propagator, $S'(p)$, turns out to be independent of a cutoff in the limit of a high cutoff.

We must now pass over to the vertex correction from diagram (a). Again, let us first consider the effect of all possible vertex corrections

\[
\begin{align*}
\Gamma^0(p', p) &= \gamma^\mu + A(p', p).
\end{align*}
\]
Analytically,
\[ \Lambda_\mu(p',p) = \frac{i e^2}{(2\pi)^4} \int \frac{d^4k}{k^2} \gamma_\nu \frac{1}{i(p'-k) + m} \gamma_\mu \frac{1}{i(p-k) + m} \gamma_\nu. \]

On inspection, it is apparent that \( \Lambda_\mu(p',p) \) is logarithmically dependent on the cutoff. Therefore, it will have to be modified so that it does not appear explicitly in calculations. We can do this by writing
\[ \Lambda_\mu(p',p) = \Lambda_\mu(p,p) + \{ \Lambda_\mu(p',p) - \Lambda_\mu(p,p) \}. \]

The first term is logarithmically divergent and can be absorbed in the charge renormalization, and the second term converges. Let us look at the term which now contains the divergence. \( \Lambda_\mu(p,p) = B \gamma_\mu + \Lambda^f_\mu(p,p)(ip' + m) \). The \( \Lambda^f_\mu \) term is convergent, and thus to second order,
\[ \Gamma^0_\mu(p',p) \sim (1 + B) \gamma_\mu + \text{(terms independent of the cutoff)}. \]

One now factors out the term in \( \Gamma^0_\mu(p',p) \) which is logarithmically dependent on the cutoff, and defines in this way the renormalized vertex, \( \Gamma^0_\mu(p',p) = Z_1^{-1} \Gamma^1_\mu(p',p) \), and also absorbs the \( 1/Z_1 \) in the charge renormalization. However, in electrodynamics, a rather unexpected thing happens. To all orders, \( Z_2 = Z_1 \) and the contributions to charge renormalization from the wave function corrections and from vertex corrections exactly cancel one another. (There is still charge renormalization from the vacuum polarization diagrams.)

We can easily verify this fortuitous cancellation of the divergent terms in the second order. To this order, \( 1/Z_1 = 1 + B \). Now
\[ \Lambda_\mu(p,p) = \frac{i e^2}{(2\pi)^4} \int \frac{d^4k}{k^2} \gamma_\nu \frac{1}{i(p'-k) + m} \gamma_\mu \frac{1}{i(p-k) + m} \gamma_\nu. \]

From this identity, we see that \( A = B \) in electrodynamics, and thus there is no charge renormalization in lowest order from these corrections. Indeed, it is not hard to see that the cancellation of divergents is complete to any order, for it is clear that actually
\[ \Lambda^g_\mu(p,p) = \gamma_\mu \frac{d}{d(ip' + m)} \Sigma(p), \]

and thus that since
\[ \Gamma^0_\mu(p,p) = \gamma_\mu \left( 1 + \frac{d}{d(ip' + m)} \right), \]
and

\[ S^\circ (p) = \frac{1}{(i\gamma^\mu + m)(1 + \frac{dZ(p)}{d(i\gamma^\mu + m)})}, \]

the cancellation occurs at a vertex with no four-momentum transfer. This cancellation is due directly to the vector nature of the electromagnetic interaction, which gives rise to the identity between the vertex and wave function corrections.

Considerable simplification occurs in the actual problem at hand, that of computing the second order radiative corrections to scattering by a weak external field, because there are free particle spinors at the ends of the matrix elements and thus \((i\gamma^\mu + m)\) can be set equal to zero whenever it occurs at an end of a group of Dirac matrices. Thus, in the present problem, \(Z_\mu^p(p)\) and \(\Lambda^\mu_\nu(p,p)\) do not give any contribution. Therefore, the total effect of diagrams (a), (b), and (c) is

![Diagram](image)

This correction obviously has the value zero when \(q = p_2 - p_1 = 0\), but it does have a non-zero value when \(q^2 \neq 0\). Consequently, there is a "shape" associated with the electron's charge, in addition to a magnetic moment. This charge shape or form factor, is not too difficult to compute, and plays a crucial role in the Lamb shift.

**Problem 7:** We have computed the \(G_\mu^\nu\) (magnetic) part of this second order correction in order to get the electron's anomalous magnetic moment \(G(0)\) and the magnetic form factor \(G(q^2)\). Complete the calculation by finding \(F(q^2)\) to this order, neglecting again the charge renormalization effect from vacuum polarization.

At the beginning of this discussion, it was stated that the proper procedure was to take only half of diagrams (b) and (c). It will be evident that this gets rid of an extra factor of \(A\), which, entering from every electron line, would foul up the charge renormalization. There is no really clean way to justify this, but it must be done to get the correct answer. In the field theory of the static model, we got rid of the extra term by discussing the K-matrix, and a similar maneuver could be undertaken also in electrodynamics. However, there is perhaps a more convincing plausibility argument for this procedure. A factor \(Z_2\) arises from each contraction of two spinor operators. But when only one spinor operator is effective, such as at free ends, only a factor of \(\sqrt{Z_2}\) should be
introduced. But to lowest order $\sqrt{Z_2} = 1 - \frac{1}{2}B$, and thus in this order, the prescription is to take half of the end bubbles.

We have considered the consequences of diagrams (a), (b), and (c), which are finite, and thus are predictable from the theory of quantum electrodynamics.

The theory also says that there is an electromagnetic mass $\Delta m$, which is heavily dependent on the physics of high energies, or small distances, but is unable to give its value other than in terms of an arbitrary cutoff parameter $\lambda$. Nevertheless, let us make an approximate calculation of the dependence of $\Delta m$ upon the cutoff point. Now, $\Delta m$ is determined by the condition: $\Sigma^*(p) = 0$, when $(i\not{p} + m) = 0$. Therefore,

$$\Delta m(2) = \frac{i\epsilon^2}{(2\pi)^3} \int d^4k \frac{d^4P}{k^2} \{ \bar{u}_p \gamma_v \frac{i\not{p} - i\not{k} - m}{(p-k)^2 + m^2} \gamma_v u_p \}$$

$$= -2i\epsilon^2 \int d^4k \frac{1}{(2\pi)^3} \left\{ \bar{u}_p (2m + i\not{p} - i\not{k}) u_p \right\}$$

$$= -2i\epsilon^2 \int d^4k \int_0^\infty \frac{\bar{u}_p (m - i\not{k}) u_p}{(k^2 - m^2 + i\epsilon)^2}.$$ 

At this point in the evaluation of the integral, it appears that the right step is to change the variable of integration from $k$ to $k' = k - px$. However, the cutoff should be a function of the invariant $k^2$ and not $k^2$, and thus one cannot just blindly shift the origin of the integration variable, when the integral diverges. A careful manipulation of the integrand yields the result that the variable can be shifted in a linearly divergent integral provided a constant is added to the resulting integral. In the case of a logarithmically divergent integral, it is not necessary to tack on a constant when the variable's origin is shifted. Having stated that there is a little hitch in changing variables, we shall proceed, nevertheless, as if there were not. Our result, therefore, will be correct only to within a constant independent of a cutoff, i.e. we shall get only the dominant term in $\Delta m(2)$ in the limit of a high cutoff.

$$\Delta m(2) \approx \frac{\Delta m}{(2\pi)^4} \int_0^\infty dx \int d^4k' \frac{(1 + x)}{i\epsilon^2 [ k'^2 + m^2 (1 + x)^2 ]^2}$$

since the term in $k'$ vanishes upon symmetrical integration, and $\not{p} = im$ between the spinors, and $\bar{u}_p u_p = 1$. The problem now is to assign a value to an integral of the type

$$\int d^4k' [k'^2 + m^2 - i\epsilon]^2$$

by incorporating an invariant cutoff in the integrand. There are many ways to
formulate the cutoff, and it is not particularly worth while to worry about actually carrying out the integral in a particular way. A rough result is that, if the integration is carried out only over the region $|k^2| \leq \lambda^2$, the integral is $i \pi^2 \log \lambda^2/r^2$. We can check this answer partly by remembering that

$$\int d^4k (k^2 + r^2 - i\epsilon)^{-3} = \frac{\pi^2}{2r^2},$$

and noticing that the new integral formula gives this result:

$$\frac{\pi^2}{r^2} = -\frac{d}{dr^2} \int d^4k (k^2 + r^2 - i\epsilon)^{-2} = 2 \int d^4k (k^2 + r^2 - i\epsilon)^{-3}.$$

The rest of the computation is quite simple.

$$\Delta m^2 \approx \frac{3am}{2\pi} \int_0^1 dx \left(1 + x\right) \left\{ \log \frac{\lambda^2}{m^2} - 2 \log x \right\}$$

$$\approx \left\{ \frac{3am}{2\pi} \log \frac{\lambda}{m} \right\}^2$$

(We drop the constant term here also since we have neglected such terms many times during the calculation.) Thus in electrodynamics, the self mass is positive, proportional to m, and logarithmically dependent on the cutoff. The proportionality factor, $3a/2\pi$, is quite small, being about $1/300$. The higher order corrections yield a series in $(a \log \lambda/m)$, which has only been partially summed. The partial sum resulted in a small power of $a \log \lambda/m$ times the mass m. At any rate, it is apparent that if one should want the electromagnetic mass to account for all the mass, a cutoff would have to be really high -- about $e^{300}$ MEV. Actually, the theory of quantum electrodynamics has been verified up to momentum transfers of about 10 MEV/c, and it appears quite likely that field theory requires modification around 1000 MEV/c. Thus, the attributing of the electronic mass to electromagnetic interactions exclusively is a somewhat doubtful proposition.

Problem 8: As an exercise in computational techniques, calculate the value of the wave function renormalization constant A, the only other divergent quantity we have yet encountered.
January 13 1959

To complete the discussion of the radiative corrections to scattering by a weak external field, the effect of the vacuum polarization diagram (d) will now be investigated.

It is evident that the essential part of this radiative correction is independent of the specific scattering process being described, i.e., it is the same no matter if it were an electron, muon, or proton, etc. that was being scattered by the electromagnetic field. This correction consists of a modification of the photon propagator.

The total matrix element for scattering by a weak external field is a product of the square roots of the complete fermion propagators applied to the external lines, the complete vertex, and the complete photon propagator applied to the external photon being absorbed. Diagramatically, the complete matrix element takes this form:

while analytically, it becomes:

$$ R_{ij} = - \frac{1}{2} (\bar{u}_2) \sqrt{Z_2} e_0 \sqrt{Z_3} Z_1^{-1} \frac{A_\mu}{1 + q^* \eta(q)} Z_1^{-1} \Gamma_{\mu} (P_2, P_1) \sqrt{Z_2} u_1 \frac{m}{\sqrt{E_2 E_1}} V. $$

The blobs on the fermion lines and at the vertex were discussed previously, and it was indicated why the renormalization factors $Z_2$ and $Z_1$ were necessary to make the theory yield results independent of a cutoff. Our task now is to see why the third renormalization constant $Z_3$ must be introduced, and in so doing, the effect of vacuum polarization diagrams such as (d).
A weak external field can be thought of as a virtual photon, i.e., an external field, treated in first order, interacts in every way like a photon, except that it is not transverse, and doesn't have $q^2=0$ like a real photon. In the consideration of an external field, we, therefore, are led to investigate the propagator of photons, which is the propagator of any electromagnetic disturbance in the vacuum. Were Maxwell's equations completely exact, that propagator would be just $1/q^2$ in momentum space. However, the fact that a photon can create a positron-electron pair leads to a modification of the propagator, and thereby to a modification of Maxwell's laws.

At this stage, it becomes necessary to look at the gauge invariance of the theory and the Lorentz condition. In the theory of a vector meson field, it is correct that

$$i\langle P(\varphi^\text{int}_{\lambda}(x), \varphi^\text{int}_{\lambda}(y))\rangle_0 = \delta_{\nu\mu} D_\mu(x-y).$$

However, in the case of the electromagnetic field, the application of the Lorentz condition, $\partial_\mu A_\mu = 0$, requires that the propagator $\delta_{\nu\mu}/q^2$ be replaced by

$$\delta_{\nu\mu} - \frac{q_\mu q_\nu}{q^2}.$$

Now since this modification has been hitherto neglected, the question arises as to what changes are necessary in our previous work. The answer is that no changes are required, as we shall now proceed to demonstrate.

As an example of the effect of the change, consider the simplest diagram in the scattering of an electron by a weak external field.

The $R$-matrix element for this diagram, using the modified propagator, is proportional to $\bar{u}_2 i_{\nu}(q^2 \delta_{\nu\mu} - q_\nu q_\mu) A_\mu u_1$. The change in the matrix element is proportional to $\bar{u}_2 A_\mu u_1$. This matrix element clearly vanishes, for $\bar{u}_2 A_\mu u_1 = \bar{u}_2 (\nu_2 - \nu_1) u_1 = 0$. (In the Lorentz gauge, it also vanishes because $q_\mu A_\mu = 0$).
Similarly, if one considers the sum of all diagrams of a given order, one will find that the extra terms in the photon propagators give a zero contribution to the matrix element. This general result must have some symmetry consideration as its basis. The symmetry is the invariance under gauge transformations, which is the invariance of answers under the transformation of the photon momentum wave functions from $A_{\mu}(q)$ to $A_{\mu}(q) + q_{\mu}\chi(q)$, where $\chi(q)$ is an arbitrary scalar function. The replacement of $\delta_{\nu\mu} - q_{\nu}q_{\mu}/q^2$ by $\delta_{\nu\mu}$ amounts to the neglect of a term involving two factors of $\phi$ in a matrix element.

But such a term corresponds to a change in the matrix element for another physical process when a gauge transformation is made. For example, the neglected term for this diagram

![Diagram](image)

is

$$\gamma_{\nu}k_{\nu}\lambda_{\alpha} \frac{1}{(p'_{2} - k) + m} \gamma_{\lambda} \frac{1}{(p'_{1} - k) + m}$$

or equivalently,

$$k'(i(p'_{2} - k) + m)^{-1} A'(i(p'_{1} - k) + m)^{-1} \gamma_{\lambda} \gamma_{\nu} k_{\nu}$$

But such a term is just like the extra term arising when the added external fields $A(k)$ in this diagram

![Diagram](image)

undergo a gauge transformation $A(k) \rightarrow A(k) + k\chi$. Since the theory is gauge invariant, this term yields no contribution. Another way to understand the result is that the Lorentz condition need not be forced explicitly by using $\delta_{\nu\mu} - q_{\nu}q_{\mu}/q^2$ because all the sources of the electromagnetic potentials $A_{\mu}$ obey the Lorentz condition, i.e. the current $j_{\mu}$ obeys the equation of continuity, $\partial_{\mu}j_{\mu} = 0$.

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* R. P. Feynman, Phys. Rev. 76, 769 (1949); see pp. 780-1.
Consider now the complete photon propagator \( D^0_{\nu\mu}(k) \). It is a tensor, made from the four vector \( k \). Thus it must consist of a linear combination of the two possible tensors \( k_{\nu} k_{\mu} \) and \( \delta_{\nu\mu} \) with coefficients that are functions of the invariant \( k^2 \). Since it must be gauge invariant, the two coefficients are not independent, and thus we can write \( D^0_{\nu\mu}(k) \) in terms of one unknown function \( \Pi^*(k) \):

\[
D^0_{\nu\mu}(k) = \frac{(\delta_{\nu\mu} - k_{\nu} k_{\mu}/k^2)}{k^2 + \Pi^*(k)}
\]

It is this change of the form of the photon propagator, i.e. from \( 1/k^2 \) to \( 1/(k^2 + \Pi^*(k)) \), which modifies Maxwell’s laws. For static charges, therefore, Coulomb’s \( 1/r \) law is not exactly valid for the field of virtual photons is modified at small distances by the fact that they can dissociate into negaton-positon pairs which are polarized by the field. This polarization manifests itself at large distances by changing the value of the charge from \( e_0 \) to \( e_1 \), and at small distances by actually changing the spatial dependence of the potential. At distances of the order of 386 fermies, the electron’s Compton wavelength, there will be a correction of order \( 1/137 \) from electron pairs, as the distance decreases to 1 or 2 fermies, the correction from muons and pions will also rise to order \( 1/137 \), and so on for the heavier charged particles.

Let us now calculate this propagator. Diagrammatically,

\[
D^0_{\nu\mu}(k) = \frac{\partial}{\partial \mu} + \frac{\partial}{\partial \nu} + \cdots + \{ \frac{\partial}{\partial \mu} + \cdots \}
\]

and to lowest order,

\[
D^0_{\nu\mu}(k)(2) = \frac{\partial}{\partial \mu} + \frac{\partial}{\partial \nu} + \{ \frac{\partial}{\partial \mu} + \cdots \} = (\delta_{\nu\mu} - k_{\nu} k_{\mu}/k^2) \left[ \frac{1}{k^2} - \frac{1}{k^2} \Pi(k) \frac{1}{k^2} \right].
\]

Since the photon has a rest mass of zero, \( \Pi^*(0) \) must have the value zero. The question arises as to how to arrange this, and the answer is that two ways are available.

The most obvious way to wipe out any constant term in \( \Pi^*(k) \) is to do it in the same way that was employed to wipe out the constant term in \( \Sigma^*(p) \). That was done by saying that the particle has a bare mass, which is different from the observed mass, but which appears in the Lagrangian, so that when we
set up a scattering formalism using the real mass, a term $\Delta \mu^2$ will come in so that it just cancels the self-energy bubble of a real particle. We can do the same for the photon. We can say that the photon has a bare mass $\mu_0$, and that the interaction of photons with the charged matter fields produces a change in the square of the mass $\Delta \mu^2$ such that the observed mass is zero:

$$\mu_0^2 + \Delta \mu^2 + \mu^2 = 0.$$ 

This is, in a way, a sensible procedure. Many people, however, do not follow it, and say that it is wrong. And perhaps it is. What they do, though, sounds a little peculiar also. Namely, they get the value of $\Pi^\Phi(0)$ in terms of an integral, and claim that it is zero, even though it looks like it equals a constant times the square of a cutoff. Nevertheless, they claim that the integral is obviously zero. If you look at it very hard, and add to it, subtract from it, and interchange orders of integration, etc., you can put it into a form so that it's zero! I offer you your choice between these two ways of proceeding; the point has never been settled.

Some people claim that theory ought to be gauge invariant from the beginning, which means that the bare mass as well as the observed mass has to be zero. Therefore, they insist that this integral must be zero. A much more rational argument for saying that this integral really ought to be zero, in other words, that the formalism should be set up in such a way that the integral is zero, is the following. In no other case in nature do we have a situation in which it is the real mass that is simple, and the bare mass which is complicated. Zero is an extremely simple number for the real mass! In all other cases, it appears that the bare masses are simple but that the interactions give rise to complicated mass spectra. For example, charge independence tells us that the neutron and proton bare masses are exactly equal, and that's simple. The observed masses are then unequal due to other interactions, presumably electromagnetic in this case. The same holds for the pions, and the other charged multiplets. In the case of the neutrino, since the neutrino is apparently always longitudinal, the bare mass as well as the real mass is zero. You just can't tolerate any kind of a mass for an object which is polarized 100 percent. Therefore, it would be very peculiar if the photon was considered to have a bare mass, which was corrected precisely back to zero by the interactions. For this reason, I agree with the people who say that the formalism ought to be set up as if that integral actually is zero. The way to do this is to force gauge-invariance at every step of a calculation which
involves quantities dependent on some sort of high-energy cutoff. Therefore, wherever ambiguities arise, we will impose gauge invariance in order to get rid of certain terms.

Now we can look at $\Pi(k)$. Analytically,

$$
\Pi(k) \left( \delta_{\nu\mu} - k_{\nu} k_{\mu}/k^2 \right) = + \frac{ie^2}{(2\pi)^4} \text{Tr} \left\{ \int d^4p \gamma_{\nu} \frac{1(p+k) - m}{(p+k)^2 + m^2} \gamma_{\mu} \frac{ip - m}{p^2 + m^2} \right\}
$$

$$
\approx \frac{e^2}{2(2\pi)^4} \left( \delta_{\nu\mu} - k_{\nu} k_{\mu}/k^2 \right)
$$

which is the contribution from this closed loop:

This integral is quadratically divergent in appearance. The easiest way to evaluate $\Pi(k)$ is to contract the indices $\nu$ and $\mu$.

$$
\Pi(k)(3) = \frac{ie^2}{(2\pi)^4} \text{Tr} \left\{ \int d^4p \gamma_{\mu} \frac{1(p+k) - m}{(p+k)^2 + m^2} \gamma_{\nu} \frac{ip - m}{p^2 + m^2} \right\}
$$

Whenever one encounters closed loops, one has to evaluate the traces of several products of Dirac gamma matrices. There are a couple of useful rules which greatly facilitate this job. We leave their proofs as exercises for the reader.

Tr $\{1\} = 4$

Tr $\{\gamma^\nu\} = 4 \ a \ b$

Tr $\{\gamma^\nu\gamma^\mu\} = 4(\ a \ b \ c \ d - a \ c \ b \ d + a \ d \ b \ c)$

and, in general,

Tr $\{\gamma^{\nu_1} \gamma^{\nu_2} \ldots \gamma^{\nu_n}\} = 0$, if $n$ is odd;

Tr $\{\gamma^{\nu_1} \gamma^{\nu_2} \ldots \gamma^{\nu_n}\} = 4 \sum_{p \ P} \delta_{P_1 p_2 p_3 p_4 \ldots p_n} a_1 a_2 a_3 a_4 \ldots a_{n-1} a_n$, if $n = 2m$,

where the sum is taken over all different ways one can dot the vectors into each other, subject to the restriction that $p_1 < p_2$, $p_3 < p_4$, \ldots, and $\delta_P = \pm 1$ depending on whether $p_1 p_2 \ldots p_n$ is an even or odd permutation of $1, 2, \ldots, n$. (There are $(2m)!/2^m m!$ terms in this sum.)

Using these rules and remembering that $\gamma_{\mu} \gamma_{\nu} = -2\delta_{\mu\nu}$, $\gamma_{\mu} \gamma_{\nu} = 4$, one finds that:

$$
\Pi(k) \left( \delta_{\nu\mu} - k_{\nu} k_{\mu}/k^2 \right) = - \frac{e^2}{6\pi^4} \int d^4p \frac{2m^2 + p^2 + p \cdot k}{(p^2 + m^2)(p^2 + m^2 + 2p \cdot k + k^2)}
$$
Now assuming a suitable cutoff, we can expand $\Pi(k)$ as a power series in the variable $k^2$. By gauge invariance, or by mass renormalization, $\Pi(0) = 0$, so:

$$\Pi(k) = 0 + C k^2 + k^4 \Pi^f(k).$$

The $C$ term is logarithmically dependent on the cutoff, and the $\Pi^f(k)$ term is independent of a cutoff in the limit of a high cutoff. A similar expansion can be carried out for $\Pi^*(k)$:

$$\Pi^*(k) = 0 + C^* k^2 + k^4 \Pi^{0*}(k),$$

where, however, the $\Pi^{0*}(k)$ does contain terms dependent on the cutoff. One then has the complete photon propagator in the form

$$D_0^0(k) = \frac{1}{(1+C)k^2 + k^4 \Pi^{0*}(k)}.$$

By defining a renormalized propagator $D^1(k)$ through an equation

$$D_0^0(k) = \frac{1}{1+C^*} D^1(k) = Z_3 D^1(k),$$

one creates a new propagator with the rather nice feature that it is independent of a cutoff in the limit of a high cutoff. To remove the renormalization constants from the theory, one defines the renormalized charge

$$e_1 = \sqrt{Z_3} \frac{Z_2}{Z_1} e_0$$

and identifies it with the real charge.

The renormalized photon propagator $D^1(k) = 1/(k^2 + k^4 \Pi^{0*}(k))$ must be inserted everywhere the bare propagator would come in, and the charge is to be identified with the renormalized value. For a real photon the only effect is that the charge is renormalized since $k^2 = 0$. For virtual photons, however, in addition vacuum polarization effects manifest themselves because of the additional factor $1/(1+k^2 \Pi^{0*}(k))$. These effects have been observed in the study of the Lamb shift, where they account for 27 of the 1060 megacycle shift.

Let us now calculate the first order contribution to $Z_3$ in terms of the cutoff we used for the calculation of the self-energy of the electron. We desire now the coefficient of $k^2$ in the expansion of

$$\Pi(k) = -\frac{e^2}{6\pi^4} \int d^4p \frac{[-(2m^2+p^2)(2p\cdot k+p^2) + p\cdot k(p^2+m^2)]}{(p^2+m^2)^2(p^2+m^2+2p\cdot k+k^2)}$$

$$\Pi(k) = -\frac{ie^2}{6\pi^4} \int d^4p \frac{p\cdot k(3m^2+p^2) + k^2(p^2+m^2)}{(p^2+m^2)^2(p^2+m^2+2p\cdot k+k^2)}.$$
At first glance, this integral appears to be linearly divergent; however, the postulate of symmetrical integration renders the integral only logarithmically divergent. Because the coefficient of $k^2$ is divergent, we shall keep only the dominant part in evaluating it. The cutoff dependent part of $C$ is

$$C \approx -\frac{ie^2}{6n^4} \int d^4p \left\{ \frac{1}{(p^2+\pi^2)^2} - \frac{2(p \cdot k)^2}{k^2 (p^2+\pi^2)^3} \right\}$$

The second part of the integrand can be replaced by

$$-\frac{p^2}{2(p^2+\pi^2)^3}$$

when it is noticed that

$$\int f(p^2) p^2 \, d^4p = \frac{1}{4} \delta_{ab} \int f(p^2) p^2 \, d^4p.$$

Using this trick, and assigning to an integral like

$$\int d^4p (p^2+\pi^2)^{-2} \approx \frac{1}{p^2} \log (\lambda^2/\pi^2)$$

a cutoff dependent value, one obtains the following expression for $C$:

$$C \approx \frac{e^2}{12 \pi^2} \log \frac{\lambda^2}{\pi^2},$$

To this order, $Z_3 \approx 1 - C \approx 1 - \frac{e^2}{12 \pi^2} \log \frac{\lambda^2}{\pi^2}.$

By incorporating the $Z_3$ into a renormalization of the charge, one simply modifies the interaction so that instead of the product of the two bare charges $Q_1^0$, $Q_2^0$, it is the product of the renormalized charges $Q_1^1$, $Q_2^1$, which enters. Any bare charges suffer a reduction by this amount, and, in fact, in any diagram $e^2$ should be replaced by $e_1^2$, since in any photon line, which gives rise to a factor $e^2$, one can insert the bubbles of fermion pairs which give the renormalization of the charge. The second order result we have obtained indicates that $e_2^2$ is less than $e_1^2$.

The reason why the real charge should be less than the bare charge is clear physically. Suppose one starts with a bare charge $Q_0$, assumed positive.
for definiteness. Virtual pairs are always being produced in the vacuum, and normally they annihilate again and contribute to nothing but an unobservable vacuum self-energy. But in the presence of this charge, the positrons are repelled to infinity, and the virtual negatons are drawn in close to the charge. So when one includes vacuum polarization, the bare charge $Q_0$ is surrounded by a very weak cloud of virtual negatons, and the virtual positrons just go away.

Thus, what one actually sees at large but finite distances is a charge $Q_0$ minus the charge of the cloud. This is $Q_1$, and clearly should be less than $Q_0$. As you go in closer and closer, though -- as you do experiments which probe at smaller and smaller distances -- you should begin to see the charge at the center. As you do experiments involving higher and higher frequencies, or momentum transfers, in the limit of infinite frequencies, one will measure the bare charge. (This can be shown mathematically, and next time we shall state the mathematical basis for this assertion.) At low frequencies, on the other hand, which is where the charge is defined -- by Gauss' law at large distances -- one sees only the charge plus cloud, or the renormalized charge.

This argument makes it extremely reasonable that $Z_3$ should be less than one. Of course, in the limit of large cutoffs, this second order term makes it appear that $e_0^2$ is less than zero. Such a result indicates strongly that a cutoff is necessary in the theory before $e_0^2$ goes negative. However, it is not clear that this is true though, for the higher terms in the series might invalidate that conclusion about the sign of $e_0^2$. This point has never been settled; it is the same question as is raised by the Lee Model.

The place where such a cutoff would necessarily have to be introduced to circumvent such a conclusion is at a very high energy, $e^{300}$ Mev. Physically therefore, this is not an interesting question, because we know that a theory developed for low and intermediate energies must surely go wrong far below such fantastically high energies. For the meson theory the question is interesting, for the place where such a cutoff might be necessary could be of order $1$ BEV, and thus the physics of the cutoff would play a decisive role.
We can now go on and compute the cutoff independent terms of $\Pi^f(k)$, in other words, $\Pi^f(k)$. This is what really leads to a change in the nature of electromagnetic interactions, i.e. to a change in Maxwell's laws. For a slowly varying external field, we would just want to continue this expansion in powers of $k^2$; if we have a rapidly varying electromagnetic field and we want to calculate the finite effects of vacuum polarization, we would have to calculate the entire function $\Pi^f(k)$.

Let us now calculate $\Pi^f(0)$.

$$\Pi^f(0) = \frac{-ie^2}{6\pi^4} \int d^4p \left\{ \frac{-(p^2 + 2m^2)}{(p^2 + m^2)^4} + \frac{\lambda(p \cdot k)^2(p^2 + 2m^2)}{k^2(p^2 + m^2)^5} \right. $$

$$\left. - \frac{\delta(p \cdot k)^2(p^2 + 3m^2)}{k^4(p^2 + m^2)^6} + \frac{\lambda(p \cdot k)^2(p^2 + 3m^2)}{k^2(p^2 + m^2)^5} \right\}. $$

Now employing these two consequences of symmetrical integration:

$$\int f(p^2) p^a p^b d^4p = \frac{1}{4} \delta_{a \beta} \int f(p^2) p^2 d^4p,$$

$$\int f(p^2) p^a \delta_{\beta \gamma} p^\gamma d^4p = \frac{1}{2m} (\delta_{a \beta} \delta_{\gamma \nu} + \delta_{a \gamma} \delta_{\beta \nu} + \delta_{a \nu} \delta_{\beta \gamma}) \int f(p^2) p^4 d^4p,$$

that integrand can be reduced to the following form

$$\Pi^f(0) = \frac{-ie^2}{6\pi^4} \int d^4p \frac{2m^6}{(p^2 + m^2)^6}. $$

Integrals of this nature are quite easy to evaluate by differentiating

$$\int d^4p \frac{(p^2 + m^2)^{-3}}{(p^2 + m^2)^{1/2}} = \frac{2m^2}{3m^2},$$

with respect to the parameter $m^2$. Doing this, one finds that

$$\int d^4p \frac{(p^2 + m^2)^{-6}}{(p^2 + m^2)^{1/2}} = \frac{4m^2}{5m^2}. $$

Théoréme, $\Pi^f(0) = -\frac{e^2}{60 \pi^2 m^2}$.

We shall stop at this point, and pass on to another topic in relativistic field theory. It is strongly urged, however, that the student consult the literature for a more complete discussion of problems in electrodynamics. In particular, for an excellent discussion of the Lamb shift, one should consult Dyson's notes.* More information on renormalization questions can be found in Jauch and Rohrlich, and this text as well as Akhiezer and Berestetsky are good sources of standard calculations in electrodynamics.

**PROBLEM 9:** Compute the cross section for Compton scattering to lowest order, and compare your answer with the Klein-Nishina formula. Also verify the gauge invariance of the second order R-matrix element explicitly.

* See also Fried and Yennie, Phys. Rev. 112, 1391, (1958).
It is possible to make some general statements concerning the complete propagators, which limit their possible forms. In setting up the mathematical formalism for such a discussion, one also can see the reasons for the assertions that the renormalization constants should be less than unity and that in the limit of high frequencies (momentum transfers) it is the bare coupling constant that comes into play rather than the renormalized coupling constant. We will set up the formalism for meson fields to avoid complications due to gauge invariance, etc., but the major results can either be carried over directly for other fields, or have counterparts for other fields.

In the interaction or "in" representation, where the fields obey the free-field equations, the propagator $D_F(x-y)$ is simply

$$-i D_F(x-y) = \langle P(\phi_{\text{int}}(x), \phi_{\text{int}}(y)) \rangle_0 = x \rightarrow y.$$  

Similarly the complete propagator $D_F^0(x-y)$ is directly related to the Heisenberg fields:

$$-i D_F^0(x-y) = \langle P(\phi_H(x), \phi_H(y)) \rangle_0 = x \rightarrow y + z \rightarrow y + \ldots$$

Let us now start to evaluate the propagator, by introducing a complete set of states between the two field operators. For $D_F(x-y)$ we could take the intermediate states to be eigenstates of the free Hamiltonian and of the momentum. (For definiteness, the case $x_0 > y_0$ will be taken.)

$$-i D_F(x-y) = \sum_{n_0} \langle 0 | \phi_{\text{int}}(x) | n_0 \rangle \langle n_0 | \phi_{\text{int}}(y) | 0 \rangle$$

where

$$H_0 | n_0 \rangle = E_0 | n_0 \rangle, \quad P_0 | n_0 \rangle = \vec{P}_0 | n_0 \rangle.$$  

Since the four-vector momentum operator $P_{\text{int}}^{\mu}$ is the generator of infinitesimal space-time displacements, the following relation must hold for any operator in the interaction representation, and in particular for $\phi_{\text{int}}(x)$:

$$-i [P_{\text{int}}^{\mu}, \phi_{\text{int}}(x)] = \frac{\partial \phi_{\text{int}}}{\partial x^{\mu}}$$

And since the states $| n_0 \rangle$ are taken to be eigenstates of $P_{\text{int}}^{\mu}$, as is, of course, the vacuum, the relation
-i \langle 0 | \left[ P_{0\mu}, \phi_{\text{int}}(x) \right] | n_o \rangle = i P_{n_\mu} \langle 0 | \phi_{\text{int}}(x) | n_o \rangle = \frac{3}{8\pi} \langle 0 | \phi_{\text{int}}(x) | n_o \rangle \]

holds. One can integrate this simple differential equation, and obtains:

\[
\langle 0 | \phi_{\text{int}}(x) | n_o \rangle = e^{IP_{\cdot x}} \langle 0 | \phi_{\text{int}}(0) | n_o \rangle.
\]

In the calculation of \( D_F(x-y) \), only the intermediate states of one bare meson make a contribution, because the only thing that can communicate with the vacuum by \( \phi_{\text{int}} \) are the single bare meson states. Thus the effective intermediate states are specified by one parameter, the momentum of the bare meson. Hence, the propagator for a bare meson can be written simply as:

\[
-1 D_F(x-y) = \sum_k | \langle 0 | \phi_{\text{int}}(0) | k \rangle |^2 e^{ik \cdot (x-y)}.
\]

Now, these matrix elements are between states which are Lorentz transforms of each other, i.e., the vacuum \( |0\rangle \) goes into itself under a Lorentz transformation, and by a suitable change of frame the state \( |k\rangle \) can be made to go into a state \( |0\rangle \) representing a bare meson whose momentum vanishes. Under any such transformation, the field operator \( \phi_{\text{int}}(0) \) goes into itself, because it is a pseudoscalar. Therefore, the coefficient of \( e^{ik \cdot (x-y)} \) is determined entirely by Lorentz invariance; it depends only on the square of the rest mass, \( \mu^2 \), and the momentum \( k \), in a manner specified completely by Lorentz invariance.

Thus the propagator can be written so that it is a sum over states of different momenta and energies representing the same physical object, the dependence on momentum being given by Lorentz invariance, and the sum is definitely identifiable as one known function, the Feynman propagator for a boson with given rest mass \( \mu^2 \). On the basis of this analysis, we can calculate an expression for the complete propagator \( D_F^C(x-y) \). But in this case, the intermediate states are taken to be eigenstates of the complete four-momentum operator \( P \). Then the propagator assumes this form:

\[
-1 D_F^C(x-y) = \sum_n | \langle 0 | \phi^H_n | n \rangle |^2 e^{iP_n \cdot (x-y)}
\]

For the interaction operators, we knew that the intermediate states were only one bare meson states. However, now the intermediate states which contribute can be almost anything since the Heisenberg field operator can connect the vacuum to any state with the same symmetry quantum numbers. For example, states of one meson, several mesons, pairs of nucleons in addition to mesons,
all give contributions to the sum. Nevertheless, we are going to be able to get a somewhat similar formula for the complete propagator.

Let us now classify all the intermediate states by dividing them up into groups of states which are Lorentz transforms of one another. For instance, among all the states with fourteen pairs and eleven mesons, we can distinguish subclasses in which one can go from one state to another by performing a Lorentz transformation. In other words, we distinguish the internal variables of a state which have nothing to do with translations, from the variables describing the translation in a fixed frame. The intermediate states are now specified by two indices: \( \alpha \), which really distinguishes one state from another, and a momentum \( k \), which just tells you how fast the system is moving past some observer. Each of these states \( \alpha \) has a mass \( m_\alpha \) which is independent of \( k \), for it is defined to be the energy when \( k \) is zero. Then for each \( \alpha \) we can perform the sum over \( k \), and since the dependence on \( k \) of the coefficient of \( e^{ik \cdot (x-y)} \) is determined completely by Lorentz invariance, that sum over \( k \) will yield just a multiple of the Feynman propagator for a mass \( m_\alpha ^2 \).

\[
-i \, D_\alpha ^0 (x-y) = \sum_\alpha \sum_k \left| \left< \mathbf{0} \right| \phi^-_\alpha (0) | \alpha \mathbf{k} \right> \left| e^{ik \cdot (x-y)} \right| e^{ik \cdot (x-y)} = -i \sum_\alpha p_\alpha \, D_\alpha ^0 (x-y).
\]

In fact, by writing
\[
\sum_\alpha \sum_k \left| \left< \mathbf{0} \right| \phi^-_\alpha (0) | \alpha \mathbf{k} \right> \left| e^{ik \cdot (x-y)} \right| e^{ik \cdot (x-y)}
\]
as
\[
\sum_\alpha \left( 2m_\alpha V \right) \left| \left< \mathbf{0} \right| \phi^-_\alpha (0) | \alpha \mathbf{Q} \right> \left| e^{i(\mathbf{Q} \cdot (x-y))} \right|^2 \left| e^{i(\mathbf{Q} \cdot (x-y))} \right| \frac{1}{2m_\alpha} \frac{\left| \left< \mathbf{0} \right| \phi^-_\alpha (0) | \alpha \mathbf{k} \right> \left| e^{i(\mathbf{k} \cdot (x-y))} \right|^2}{\left| \left< \mathbf{0} \right| \phi^-_\alpha (0) | \alpha \mathbf{k} \right> \left| e^{i(\mathbf{k} \cdot (x-y))} \right|^2}
\]
one can identify \( p_\alpha \) and see that it is a positive number:
\[
p_\alpha = 2m_\alpha V \left| \left< \mathbf{0} \right| \phi^-_\alpha (0) | \alpha \mathbf{Q} \right> \left| e^{i(\mathbf{Q} \cdot (x-y))} \right|^2.
\]

So the exact propagator \( D_\alpha ^0 (x-y) \) is simply a sum involving probabilities \( p_\alpha \) of Feynman propagators for particles with various masses \( m_\alpha \), where these masses are the masses of the various configurations that can be connected to the vacuum by the Heisenberg field operator. One of these masses is the mass of a single meson itself. Then, in pseudoscalar meson theory, there are the masses of all the possible three meson states, representing the various configurations of three mesons relative to one another. Then come all the rest of the possible masses for states with pairs, and so on, in various
states of relative motion. Each one of these states has a certain energy when the total momentum of the configuration is zero -- that's \( m_a \) -- and a certain probability that the field operator \( \phi_H \) will connect it to the vacuum -- that's \( p_a \).

That these really can be called probabilities follows from the fact that \( \Sigma p_a = 1 \). Even though none of these \( p_a \) can be computed individually in meson theory it is quite easy to prove this relation. The time derivative of \( D^p_{\phi}(x-y) = i \langle P(\phi_{\text{int}}(x), \phi_{\text{int}}(y)) \rangle_0 \) has a certain discontinuity at \( t_x - t_y = 0 \), which is independent of the mass \( m_a \). Also the time derivative of \( D^p_{\phi}(x-y) = i \langle P(\phi_H(x), \phi_H(y)) \rangle_0 \) has a discontinuity at \( t_x - t_y = 0 \). In both cases the discontinuity is determined by the commutation rules. But since the Heisenberg fields are related to the interaction or "in" fields by a unitary transformation, the commutation rules are the same at equal times. Therefore,

\[
\frac{\Delta}{\Delta t} D^p_{\phi}(x) = \frac{\Delta}{\Delta t} D^c_{\phi}(x) \quad \text{at } t = 0,
\]

and thus, \( l = \Sigma p_a \). So these are really probabilities. They are the probabilities that a mesonic disturbance in the vacuum leads to the various states. There is a certain probability that it leads to one meson, a certain probability that it leads to three in some internal configuration, a certain probability that it leads to 15 mesons and 27 pairs, in a given configuration, and so forth.

In meson theory, this spectral representation of the complete propagator has one isolated point at \( m^2 = \mu^2 \). There is no other contribution until \( m^2 \) gets up to \( \sqrt{3} \mu^2 \), which represents a state with three mesons at rest relative to each other. (The pseudoscalarity of the meson field prohibits a contribution from two meson states.) At \( \sqrt{3} \mu^2 \) there begins a continuum of contributions from states representing various possible configurations of the motion of three mesons relative to each other. The contribution of the states with pairs and more mesons starts at still higher values of \( m^2 \), but still belongs to the continuum. Therefore, we can write the complete meson propagator as:

\[
D^p_{\phi}(x-y) = P_1 \text{ meson } \frac{D^\mu_2_{\phi}(x-y)}{\mu^2} + \int^\infty P(m^2) \frac{D^\mu_{\phi}(x-y)}{\mu^2} \ d(m^2),
\]

where

\[
P_1 \text{ meson } + \int^\infty P(m^2) \ d(m^2) = 1, \text{ and the } P's \text{ are positive. Or in momentum space, we can write}
\]

\[
D^p_{\phi}(k) = \frac{P_1 \text{ meson}}{k^2 + \mu^2} + \int^\infty \frac{P(m^2)}{\mu^2 k^2 + m^2} \ d(m^2).
\]
This is called the parametric or spectral representation of the complete meson propagator.

For electrodynamics, one can obtain a similar result:

$$D_F^0(k) = \frac{P_1 \text{photon}}{k^2} + \int_0^\infty \frac{P(m^2)}{k^2 + m^2} d(m^2)$$

only since the photon is massless, the continuum begins at $m^2 = 0$. (The $P(m^2)$'s are different, of course, for meson theory and electrodynamics.) In electrodynamics, we have given the number $P_1 \text{photon}$ a name already; that is, we have defined $D_F^0(k) = Z_3 D_F^1(k)$, where $D_F^1(k)$ contains the term $1/k^2$ in its expansion in $k^2$. Therefore

$$P_1 \text{photon} = Z_3$$

and, if we define $\rho(m^2) = P(m^2)/P_1 \text{photon}$,

$$D_F^0(k) = Z_3 \left( \frac{1}{k^2} + \int_0^\infty \frac{\rho(m^2)}{k^2 + m^2} d(m^2) \right),$$

and

$$1/Z_3 = 1 + \int_0^\infty \rho(m^2) d(m^2).$$

Instead of using $D_F^0(k)$, however, one normally uses $D_F(k)$, and incorporates the factor $Z_3$ into the product of the renormalized charges, one on each side of the propagator. So in a matrix element wherever something like

$$Q_1^u D_F^0(k) Q_1^v$$

appears, we replace it by $Q_1^u D_F(k) Q_1^v$, where $D_F(k)$ is independent of a high cutoff. However, for $D_F(k)$ to be really convergent, the same thing must be done to all the products of charges which occur in the computation of $D_F^1(k)$; every time an $e^2$ occurs, it occurs multiplied with $Z_3$, and this factor of $Z_3$ must be incorporated into $e^2$, making it $e^2_1$. This can be seen in the spectral representation; we had to renormalize the probabilities of going to the various states by employing $\rho(m^2)$ in the expansion of $D_F^0(k)$ rather than $P(m^2)$.

In this mathematical formalism, $Z_3$ is interpreted as the probability that an electromagnetic disturbance in the vacuum makes one photon. Thus, it is evidently less than unity, and thus, the renormalized charge is less than the bare charge, which is what we demonstrated physically in the last lecture. Also, this mathematical formalism gives substance to the allegation that at high momentum transfers, it is the bare charge rather than the renormalized
charge that becomes effective. For

\[ \lim_{k^2 \to \infty} D_F^r(k) = \frac{1}{k^2} (1 + \int_0^{\infty} \rho(m^2) \, d(m^2)) = \frac{Z_3^{-1}}{k^2}, \]

and therefore, \[ e_1^2 D_F^r(k) \to e_0^2 / k^2. \] Thus instead of the renormalized charges, one sees the unrenormalized charges if he explores very closely. If you have an external charge -- a classical body with a charge on it sitting in a vacuum -- and measure the effective charge distribution by examining the Laplacian of the potential, then at large distances one will see \( Q_1 \) times a delta function of positions, while if one goes in very close -- to distances of the order \( e^{-137} \) times the electron's Compton wavelength -- it is \( Q_0 \) which one sees. This integral representation of the propagator is, of course, useful in itself besides being useful to demonstrate such properties of the renormalized theory.

It is a very considerable restriction on the form of the propagator. In meson theory where we can't calculate, and hence have to talk exclusively on the basis of general properties, any such relation is quite valuable. It is just one example of a relativistic dispersion relation (remember that \( m^2 \) is regarded as having an infinitesimal negative imaginary part), so that the real and imaginary parts are related to one another.

To put this spectral representation into the usual form of a dispersion relation, let us look at the imaginary part of \( D_F^r(k) \). Since \( \rho(M^2) \) is real,

\[ \text{Imag} \, D_F^r(k) = \pi \left\{ \delta(k^2) + \rho(-k^2) \right\} \quad \text{for} \quad k^2 \leq 0; \]

\[ = 0 \quad \text{for} \quad k^2 > 0. \]

Thus the imaginary part of the propagator is directly proportional to the probability of making a real system of particles by an electromagnetic disturbance in the vacuum. When \( k^2 \), that is the wave number squared minus the frequency squared, is greater than zero, the disturbance cannot create a real system of particles. But when the frequency squared is greater than or equal to the wave number squared, i.e., when the virtual photon has a real mass, then there is an imaginary part and the disturbance has a real probability of making a physical system with that mass.

We notice now that the whole propagator can be expressed as an integral over the imaginary part

\[ D_F^r(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\text{Imag} \, D_F^r(k')}{{k^2 - k'^2 - i\epsilon}} \, d(k'^2), \]
which is a standard form for a dispersion relation. This is the first discovered example of a dispersion relation in relativistic field theory. Actually not only the propagator of the photon, but also the propagator of the electron, and not only propagators but also vertex functions, and in fact any of these functions of field theory obey dispersion relations somewhat like the above. These dispersion relations are interesting for two reasons: (1) they are exact consequences of field theory, and so if we ever find that they are violated experimentally, field theory will either have to be discarded or modified substantially; and (2) they may lead to new methods of approximation which are needed to work with the strong couplings. However, they are not equations which determine solutions to a given problem. All kinds of solutions to all kinds of problems satisfy similar dispersion relations, so additional constraints are needed in addition to the dispersion equations. So far, such additional constraints have not been discovered.

Consider now the lowest order approximation to the spectral representation of the renormalized photon propagator. In second order the only states contributing to the continuum will be those of a positron-neutron pair in various configurations. Therefore, in second order the continuum will start at $M^2 = 4m^2$.

$$\frac{D_A^{(2)}(k)}{k^2} = \frac{1}{k^2} + \int_{4m^2}^{\infty} \frac{\rho^{(2)}(M^2)}{k^2 + M^2} d(M^2).$$

PROBLEM 10: Show that

$$\rho^{(2)}(M^2) = \frac{-e^2}{12\pi^2 M^2} \left(1 + \frac{2m^2}{M^2}\right) \sqrt{1 - 4m^2/M^2}.$$

For this problem, Feynman's parametric integral technique will be found very useful.

On the basis of the result of this problem, and the relation

$$1/Z_3 = 1 + \int_0^{\infty} \rho(M^2) d(M^2),$$

it is quite easy to see that to second order, keeping only the leading term,

$$Z_3 \approx 1 - \frac{e^2}{12\pi^2} \log\frac{\lambda}{m^2},$$
The spectral representations can be obtained for electrons and nucleons also. They are slightly more complicated, but can be proved in much the same way the parametric representation of the complete meson propagator was proved. For the fermions, the Fourier transform of the complete unremormalized propagator takes the form

$$\mathcal{S}_F(p) = \frac{1}{i \gamma^2 + i e} + \int \frac{g(M) dM}{i \gamma^2 - i e} + \int \frac{h(M) dM}{i \gamma^2 - i e}$$

The normalization in this propagator, that the sum of all the probabilities is unity, comes from the fact that, at equal times, the anticommutation rules for the Heisenberg field operators are identical with those of the interaction or "in" operators. The major difference between the boson propagator representation and the fermion propagator representation is that extra integral containing $h(M)$. These integrands $g(M)$ and $h(M)$ are directly proportional to the probabilities that an electronic or nucleonic disturbance in the vacuum will make a real system. Now the $g(M)$ terms arise from intermediate states which contain an odd number of nucleons or electrons, while the contribution to $h(M)$ arises from intermediate states containing an odd number of antinucleons or positrons. For example, in the lowest order of meson theory, a contribution to $g(M)$ comes from this diagram,

while a contribution to $h(M)$ comes from this diagram.

In mesodynamics, the $\rho(m^2)$, $g(m)$, and $h(m)$ are always positives, and thus we can really call them probabilities. In electrodynamics they are not; we still call them probabilities, although sometimes they're negative. The reason lies in a very peculiar feature of electrodynamics, which we have not gone through, because it is such a mess and not too important. If all four polarizations of the photon are treated on an equal footing, as we have done, then there are really negative probabilities in the theory. The fourth, or time component of the polarization gives a negative probability, while the transverse and longitudinal components of the polarization give positive
probabilities. In this way of doing things, the difference between the time like polarization effects and the longitudinal polarization effects is what gives rise to the coulomb field. This peculiarity of a negative sign is the reason why in electrodynamics, unlike all other theories, two like particles repel each other.

From the spectral representations, one can get not only the Z's, but also the mass shifts. From the form of the propagator, clearly

$$\frac{1}{Z_2} = 1 + \int_{-\infty}^{\infty} g(M) \, dM + \int_{-\infty}^{\infty} h(M) \, dM.$$  

It can be shown that the mechanical mass $m_0$, the "bare" mass, is given by

$$m_0 = \frac{m + \int_{-\infty}^{\infty} g(M) \, M \, dM + \int_{-\infty}^{\infty} h(M) \, (-M) \, dM}{1 + \int_{-\infty}^{\infty} g(M) \, dM + \int_{-\infty}^{\infty} h(M) \, dM}.$$  

This relation states that the mean mass over the probability distribution given by $g(M)$ and $h(M)$ is equal to the bare mass. Thus the mean mass is another adiabatic invariant of the theory, like the sum of the unrenormalized probabilities. In other words, these two quantities do not change as the magnitude of the coupling constant is changed. The same thing is true in meson theory for the mean of the squared meson mass:

$$\mu^2 = \frac{\int_{-\infty}^{\infty} \rho(M^2) \, M^2 \, dM^2}{1 + \int_{-\infty}^{\infty} \rho(M^2) \, dM^2}.$$  

Such a relation cannot be proved for any theory, but it can be proved when the theory does not involve high powers of gradients in the coupling. In mesodynamics, $\rho(m^2)$ is always positive, so that this relation also tells us that the bare meson mass is greater than the real meson mass; in other words, that $\Delta \mu^2$ is negative.

The corresponding relation for the photon reveals the source of the fake photon "bare" mass, which one keeps getting if one just calculates in a straightforward fashion. Gauge invariance prohibits such a thing, nevertheless one keeps running into it! The usual prescription is to calculate in a gauge invariant fashion, putting in enough of the $k \cdot k / k^2$ everywhere, and it will vanish, because it cannot exist. However, it keeps cropping up, and we can identify in this formalism just what keeps appearing.
It is \( Z_3 \int_0^\infty \rho(M^2) M^2 \, d(M^2) \). In the lowest order, we can drop the \( Z_3 \) and since \( \rho(M^2) \) goes like \( 1/M^2 \) at large \( M^2 \), the fake "bare photon mass" will be a positive constant times the log of some cutoff squared. It must not be there, and has to be zero, and so on, but if necessary we can look and see what it is! This is a peculiar thing, and it is satisfying that we can identify just what tries to foul up the gauge invariance. The theory keeps trying to preserve the condition that the mean of the squared mass is an adiabatic invariant, but in electrodynamics gauge invariance doesn't allow it to do so. It is actually not very difficult to prove the equality of the mean mass with the bare mass, so let's do so. Again to simplify matters we will look at mesons. Consider the equation of motion of the Heisenberg field operator \( \phi_\Lambda^H \):
\[
(\Box^2 - \mu_o^2) \phi_\Lambda^H = ig \overline{\psi}_\Lambda^H \gamma_5 \psi_\Lambda^H.
\]
Let us look at the time derivative of \( (\Box^2 - \mu_o^2) D_F^O(x) \):
\[
\frac{\partial}{\partial t} (\Box^2 - \mu_o^2) D_F^O(x) = i \frac{\partial}{\partial t} \langle \overline{F(\psi_\Lambda^H(x) + \gamma_5 \psi_\Lambda^H(x)), \phi_\Lambda^H(0)} \rangle_0 - \frac{2}{\pi} \delta^4(x).
\]
At \( t=0 \), there is no discontinuity in this time derivative, besides that of the delta function, since at equal times the commutation rules of the Heisenberg operators are the same as those of the "in" operators, and thus at \( t = t_x = 0 \), \( \phi_\Lambda^H(0) \) commutes with \( \psi_\Lambda^H(x) \), \( \overline{\psi}_\Lambda^H(x) \), and \( \frac{\partial}{\partial t} \psi_\Lambda^H(x) \), and \( \frac{\partial}{\partial t} \overline{\psi}_\Lambda^H(x) \). But since
\[
(\Box^2 - \mu_o^2) D_F^O(x) = \mu_o^2 D_F^O(x) - \delta^4(x),
\]
we must have:
\[
(\Box^2 - \mu_o^2) D_F^O(x) = Z_3 \left\{ -\delta^4(x) + \delta^4(x) \rho(m^2) d(m^2) \right\} + \left( \mu_o^2 - \mu_o^2 \right) D_F^O(x) + \int (m^2 - \mu_o^2) \rho(m^2) D_F^O(x) \, d(m^2) \right\}.
\]
Therefore, there can be no discontinuity in the time derivative of
\[
Z_3 \left\{ (\mu_o^2 - \mu_o^2) D_F^O(x) + \int (m^2 - \mu_o^2) \rho(m^2) D_F^O(x) \, d(m^2) \right\},
\]
and since the magnitude of the discontinuity of the time derivative of \( D_F^O(x) \) is independent of the value of \( m^2 \), it follows that
\[
(\mu_o^2 - \mu_o^2) + \int (m^2 - \mu_o^2) \rho(m^2) d(m^2)
\]
must be zero.
Therefore
\[ \mu_o^2 = \mu^2 \pm \int \rho(m^2) \frac{m^2 \, d(m^2)}{1 + \int \rho(m^2) \, d(m^2)} \].

Such a relation cannot be used to demonstrate the sign of the self-energy of fermions, as it does for bosons, since the sign of the contribution from \( h(M) \) is opposite to that from \( g(M) \), and we don't know the relative magnitude of \( h(M) \) and \( g(M) \). The sign of the fermion self-energy can thus be either positive or negative, depending on the interaction; for instance, in second order, the fermion self-energy is positive in electrodynamics, in scalar meson theory it's negative, in pseudoscalar meson theory it's positive, and so on.

January 27, 1959

**No. 11. NUCLEAR FORCES**

Before we consider nuclear forces, we can well mention how the customary electromagnetic interaction arises from the theory of quantum electrodynamics. For this purpose, let us consider the lowest order diagram in the scattering of an electron and a muon.

![Diagram](image)

The common \( R \)-matrix element corresponding to such a diagram is:

\[ R_{fi} = \frac{e^2}{q^2} \frac{\bar{u}_f \gamma_\mu \gamma_\nu u_i^\dagger}{2} \frac{m_f m_i}{\gamma^2 |E_i E_i'|} \left( \gamma^\dagger \right)^2 - (\Delta E)^2 \]

If the \( \beta \)'s from the adjoint are taken out explicitly, then this matrix element becomes

\[ \frac{e^2}{q^2} \frac{u_f^+ \gamma_{\mu\nu} u_i^\dagger (1 - \vec{a} \cdot \vec{a'}) u_i^\dagger}{(\Delta p)^2 - (\Delta E)^2} \]

which is exactly what one expects to arise in the first Born approximation on the basis of the classical electromagnetic interaction between charged particles. For, if the particles were moving slowly, one would expect just the Fourier transform of the coulomb potential \( e^2/\hbar v \), which is \( e^2/(\Delta p)^2 \).
When they become relativistic, two corrections become important: the magnetic interactions, and the effect of retardation. In classical electrodynamics the magnetic interactions are accounted for by a factor \((1 - \nu \nu'/c^2)\), whose quantum mechanical analogue is clearly \((1 - \vec{a} \cdot \vec{a}')\). The retardation correction in momentum space amounts to replacing the square of the spatial momentum transfer by the square of the four-momentum transfer. Thus the expression obtained from quantum electrodynamics is just the Fourier transform of the relativistic transformation of the coulomb interaction. If we are interested in two slowly moving particles for which we want to use the Schroedinger equation, some kind of an adiabatic or static potential must be employed. That potential must obviously be such that in the Born approximation calculation of scattering it yields the non-relativistic limit of the field theory expression. Following this prescription one can construct the adiabatic potential by calculating the scattering amplitude from relativistic field theory.

If you wish to use the relativistic matrix element, and put it into the equivalent of a Schroedinger equation, in other words, iterate it, you must put it into something more sophisticated than the Schroedinger equation. For, obviously, if you want to include relativistic corrections to the potential, you must also include the relativistic corrections for the dynamics to the same order in \(v/c\). This you may do if you like; such a relativistic iteration procedure has been developed, and the resulting equation is called the Bethe-Salpeter equation. In the case of the coulomb force, you should notice though that the non-relativistic approximation to the dynamics is connected to the expansion of the potential in powers of \(e^2\) for a bound state. For example, in a hydrogen atom, the velocity is around \(e^2/\hbar m c\), so expanding in \(v/c\) and expanding in \(e^2\) really amounts to expanding in the same parameter, and one must use considerable care to include every term of a given order.

If you use something like the Bethe-Salpeter equation to generate the dynamical part of an equation good to a certain order in \(v/c\), you must include the higher order terms in the potential also. However, if you don't have a bound state problem, but rather a high energy scattering problem, then there is no need to compute a potential to plug into an equation, for you just have to compute the sequence of Born approximations directly from the field theory expressions.

Now if you are going to use the field theory scattering formalism to compute the adiabatic potential for electron-electron or nucleon-nucleon
interactions, you ought to remember the rule for what to do when you have Feynman diagrams with two identical particles. The rule can be obtained very easily as follows. If you have identical particles, you must use completely antisymmetrical wave functions. Thus, instead of computing this matrix element, \[ \langle \Phi_f(1,2) | R \Phi_i(1,2) \rangle \]

it is necessary to compute

\[ \left( \frac{\Phi_f(1,2) - \Phi_f(2,1)}{\sqrt{2}} \right) \left| R \right| \left( \frac{\Phi_i(1,2) - \Phi_i(2,1)}{\sqrt{2}} \right) \]

which, since the 1 and 2 are dummy labels, is just equal to

\[ \langle \Phi_f(1,2) - \Phi_f(2,1) | R | \Phi_i(1,2) \rangle \]

Thus the rule is just to compute the matrix element for scattering of two labeled particles, and subtract the matrix element with the two final states interchanged. The physical basis for this is that if two identical particles scatter, you cannot tell which of the two incident particles is going out in a particular direction.

Now we are in a position to look at the nuclear force problem. In doing so we shall shift back and forth between the relativistic PS-PS theory and the static model. There is no obvious a priori reason why these two theories should agree, but we will look sometimes at one and sometimes at the other and just see what can be done in the way of describing experiments.

First of all, we shall consider the second order, and it will be found that the two theories give the same result in second order. In higher order, however, there are complications which we shall have to discuss in detail.

Relativistically, the second order contribution to nucleon-nucleon scattering comes from this diagram

![Diagram]

and is

\[ \frac{g^2 \gamma_5 \gamma_5 \gamma_5 \tau \cdot \tau}{q^2 + \mu^2} \cdot \frac{1}{q^2 + \mu^2} \]

When we pass to the non-relativistic limit, the matrix elements of \( \gamma_5 \) approach those of \( \frac{\tilde{\sigma} \cdot (P_f - P_i)}{2M} \), (see page 100), and since \( dE = \frac{P}{E} dp \approx \frac{\gamma}{c} dp \), \( q^2 \rightarrow (\Delta p)^2 \). The non-relativistic limit of this diagram is thus
This is to be the Fourier transform of the adiabatic potential, which therefore must be

$$-\frac{g^2}{(2M)^2} \int \frac{e^{ik \cdot r}}{2\omega_k} \frac{k \cdot \sigma \cdot \sigma_1 \cdot r \cdot r_1}{k^2 + \mu^2} \frac{d^3k}{(2\pi)^3}.$$  

Consider now the static model. If we want the adiabatic potential, we must compute the energy of two fixed nucleons at the positions $0$ and $r$ minus their self-energies. There are two sources of the meson field, each of them independently has an energy $M_0$ plus a self energy, but together there are additional terms coming from the exchange of mesons; sum up all the additional energy and you get a potential. If the two sources are then allowed to move, one puts that potential into the Schrödinger equation. This procedure is called the adiabatic method.

Let's do that! There are now two diagrams, since the time ordering of internal events is important.

Each of these two diagrams yields the same contribution to the energy, which is

$$\left( \frac{2}{\mu} \right)^2 \int \frac{d^3k}{(2\pi)^3} \frac{v^2(k)}{2\omega_k} \frac{e^{ik \cdot r}}{2\omega_k} \frac{k \cdot \sigma \cdot \sigma_1 \cdot k \cdot r \cdot r_1}{k^2 + \mu^2} e^{ik \cdot r}.$$  

The factor $e^{ik \cdot r}$ arises because we have to put in the value of the meson field at the point where the nucleon is located. This never arose before, because we dealt with one-nucleon problems, in which the nucleon was fixed at the origin. The two diagrams give the same term, because one can be obtained from the other by reversing the sign of $r$. But since the integrand is even in $k$, we can just reverse the sign of $k$ also, and thus nothing is changed.

The only difference between the static potential and that obtained from the relativistic theory in second order is the presence of the cutoff factor $v^2(k)$. This cutoff is important even though the integral converges without it. It should be realized that there is also an effective cutoff in the relativistic formula, even though it doesn't appear explicitly. That implicit cutoff is due to recoil effects, and is hidden because we have calculated in an absolutely adiabatic limit. Therefore the validity of the relativistically derived formula is destroyed as soon as the momentum transfer becomes anywhere near
the magnitude of the mass. This adiabatic calculation breaks down at distances smaller than the Compton wavelength of a nucleon. Hence, it is not so stupid that a cutoff appears explicitly in the static model calculation, even though the integral can be calculated without one.

So we have from both theories, effectively the same second order nucleon-nucleon potential. It is substantially the Yukawa potential, modified by the fact that the pions are pseudoscalar, and by the inclusion of isotopic spin. If one defines

\[ Y_s(r) = (2\pi)^{-3} \int v^2(k) (k^2 + \mu^2)^{-1} e^{i k \cdot r} d^3 k, \]

(the "s" stands for the "smearing out" of the potential caused by the cutoff \( v^2(k) \)), then the potential is

\[ v^{(2)}\text{\_nucleon} = \left( \frac{\mu}{\mu} \right)^2 \frac{r}{S} \cdot \frac{\tau}{\tau} \sigma \cdot \nabla \sigma' \cdot \nabla Y_s(r). \]

(By the way, we remember that we must equate \( f/\mu \) and \( e/2M \) so that the two theories agree in the static limit of the relativistic theory.) \( Y_s(r) \) clearly satisfies the following equation:

\[ (\mu^2 - \nu^2) Y_s(r) = \frac{1}{(2\pi)^3} \int e^{i k \cdot r} v^2(k) d^3 k \equiv \delta_s(r), \]

where \( \delta_s(r) \) is like a delta function, only that it is spread out in space by a nucleon's Compton wavelength. In the limit of no cutoff, i.e., \( v^2(k) = 1 \), \( Y_s(r) \) tends, of course, to the Yukawa function

\[ Y(r) = Y_0 \frac{e^{-\mu r}}{4\pi r}. \]

Let us now discuss the physics of such a potential. Obviously, the second order potential is not enough to describe nuclear forces, but it does contain several interesting features which are characteristic of nuclear forces. First of all, we must evaluate

\[ \sigma \cdot \nabla \sigma' \cdot \nabla Y_s(r). \]

Introducing the unit vector in the radial direction \( \hat{r} \), one finds that

\[ \sigma \cdot \nabla \sigma' \cdot \nabla Y_s(r) = \sigma \cdot \nabla \sigma' \cdot \hat{r} Y_s = \sigma' \cdot \hat{r} Y_s = \sigma' \cdot \hat{r} (Y_s - \frac{Y_0}{r}). \]

It is customary to define a new operator \( S_{12} = 3 \sigma' \cdot \hat{r} - \sigma \cdot \sigma' \) and to write the nuclear potential in terms of \( \sigma \cdot \sigma' \) and \( S_{12} \):

\[ Y^{(2)} = \left( \frac{\mu}{\mu} \right)^2 \frac{r}{3} \{ \sigma \cdot \sigma' \left( \frac{2Y^1_s}{r} + Y^2_s \right) + S_{12} \left( Y^2_s - \frac{Y_0}{r} \right) \}. \]
The operator $S_{12}$ has a pure tensor character with respect to $\mathbf{r}$ or with respect to $\sigma$. The operator $S_{12}$ therefore carries two units of spin angular momentum and two units of orbital angular momentum. Since $S_{12}$ commutes with the total angular momentum $J$, it therefore must transfer two units of angular momentum from the spatial part of the wave function to the spin part of the wave function. For example, acting upon a $^3S_1$ state, $S_{12}$ will produce a $^3D_1$ state. $S_{12}$ is called the tensor force operator.

**Problem 11**: Show explicitly that $[J, S_{12}] = [S^2, S_{12}] = 0$, where $S = \frac{1}{2} (\sigma + \sigma')$, and that $[P, S_{12}] = 0$, where $P$ is the parity operator.

Neglecting the concept of isotopic spin, the Pauli principle simply states that two identical fermions must have a completely anti-symmetric wave function. When the concept of isospin is introduced, we can generalize the Pauli principle by requiring that the wave function be completely antisymmetric under the interchange of spatial coordinates, spin, and isospin. This means that we are treating the neutron and proton as two states of one fermion, the nucleon, and that in field theory, the neutron part of the nucleon field anticommutes with the proton part.

In addition to the requirement that isotopic triplet states have antisymmetric space-spin functions, which follows from the ordinary Pauli principle, the generalized Pauli principle also requires that all isotopic singlets have symmetric space-spin wave functions. Actually, the generalized Pauli principle does not restrict the state of a neutron-proton system, if the neutron and proton are regarded as distinct particles, although it may appear to do so. For then, the generalized Pauli principle just tells you the total isotopic spin corresponding to the symmetric and antisymmetric parts of the space-spin function.

Since $Y_0^0(r)$, $\sigma \cdot \sigma'$, $\mathbf{\tau} \cdot \mathbf{\tau}'$, and $S_{12}$ all commute with $S^2$, and $I^2$, effectively, we have to consider four potentials: one for $I=0$, $S=0$; another for $I=1$, $S=0$; a third for $I=0$, $S=1$; and one for $I=1$, $S=1$ states. Given the spin wave function therefore, there is a definite potential for each value of the parity, since, by the generalized Pauli principle, the parity and the character of the spin function determines the isotopic spin of the state.

We consider first the potential in the singlet even states. Since $S^2$ and $I^2$ commute with $S^2$, and $I^2$, respectively, we have to consider four potentials: one for $I=0$, $S=0$; another for $I=1$, $S=0$; a third for $I=0$, $S=1$; and one for $I=1$, $S=1$ states. Given the spin wave function therefore, there is a definite potential for each value of the parity, since, by the generalized Pauli principle, the parity and the character of the spin function determines the isotopic spin of the state.
and consequently, $S_{12} = 0$. (It is perhaps just as easy to see that $S_{12} = 0$ when applied to a singlet state, if one remembers that $S_{12}$ carries two units of spin angular momentum, but doesn't change $S^2$, and that you can't add two units of angular momentum to a singlet state and still have a singlet state.)

\[ V^{(2)}_{1^+} = -\left(\frac{f}{\mu}\right)^2 \left(\frac{2Y^1}{r} \right) = -\left(\frac{f}{\mu}\right)^2 \sqrt{\gamma} Y_0(x) \]

\[ = -f^2 Y_0(x) + \frac{f^2}{\mu^2} \delta_0^2(x). \]

If the delta function in this potential weren't spread out, it would be quite embarrassing, for then the potential would have too strong a singularity for the solution of the Schrödinger equation. This is a general feature of all the potentials in such a theory without a cutoff. However, the presence of the cutoff in the static theory, or of recoil effects in the relativistic theory leads to a smoothing out of the potential. This means that the nature of the cutoff is a quite important feature for the determination of the solutions of the Schrödinger equation.

This delta function always used to be discarded, on the basis that it was meaningless and led to an unsolvable Schrödinger equation. It was first noticed by Levy about nine years ago that if one included recoil effects in some approximate way in a relativistic calculation, then the delta function would effectively be spread out, as in the static model. And so it was not meaningless, but should be retained. What it represents is a strong repulsive force of very short range, which is called the repulsive core or the hard core of the nucleon. At about the same time it was noticed phenomenologically that such a thing would help quite a bit in the understanding of certain scattering experiments. So the hard core is both theoretically and experimentally a useful thing to have, and is presumably actually there physically. There may, however, be other reasons for the hard core in addition to this simple calculation. It is essentially something which depends on the physics of small distances, of which at present nobody has any knowledge.

This second order potential, using $f^2/\hbar\gamma = .08$, is by no means sufficient to account for the scattering in the singlet even states. Experimentally this potential should be almost strong enough to produce a bound state,
and to explain such a condition one needs either a stronger potential, or one that is much more singular at the origin. The higher order corrections go in the right direction to give agreement with experiment, and, in fact, the second and fourth order potentials give good agreement. Why the corrections beyond the fourth order don't seem to be necessary is not very clear. But when we calculate the fourth order we will see that the attraction in the singlet even states is immensely increased, just about by enough to give agreement with experiment.

For the singlet odd states, we are considering an isotopic singlet, \( I=0 \), so that the only change is that \( \tau \cdot \tau' = -3 \), and thus
\[
\nu^{(2)}_1 = -3 \nu^{(2)}_{1^+}.
\]
The hard core now becomes an attractive hole, and the attractive Yukawa potential changes into a repulsive one three times as strong. The predictions of meson theory for these states are still not easily checked by experiment, for the data on the odd states necessarily must come from higher energy scattering, and the analysis there is much more difficult.

As we go on to the triplet states, the evaluation of the potential becomes considerably more involved because of the presence of the tensor force. Since \( S_{12} \) commutes with \( J \) and \( P \), each eigenstate of character triplet even must be constructed only from states in one of the following groups:
\[
(3S_1, 3D_1); \ (3D_2); \ (3D_3, 3G_3); \ (3G_4); \ (3G_5, 3I_5); \ (3I_6); \ ...
\]
For these states \( \sigma \cdot \sigma' = 1 \), and \( \tau \cdot \tau' = -3 \), since we are dealing with an isotopic singlet. But now we have to look at \( S_{12} \). \( S_{12} \) is not a number usually, but a matrix, which connects the states in the groups listed above. Since the evaluation of the matrices for each group is not an entirely trivial task, and since the experimental data is good only for states of low orbital angular momentum, we shall construct only the first of these matrices.

First we look at \( \langle 3S_1 | S_{12} | 3S_1 \rangle \). The expectation value of \( r_i r_j / r^2 \) in a spherically symmetric \( S \) state is just \( \frac{1}{3} \delta_{ij} \), and so
\[
\langle 3S_1 | S_{12} | 3S_1 \rangle = \langle 3S_1 | 3 \sigma_a \sigma' \frac{1}{3} \delta_{ab} - \sigma \cdot \sigma' | 3S_1 \rangle = 0
\]
Now since the matrix involves only two states, it must be that
\[
a \ S_{12} | 3S_1 \rangle = | 3D_1 \rangle,
\]
where \( a \) can be chosen to be real by adjusting the phase of \( | 3D_1 \rangle \). \( a \) can be
computed by means of the normalization condition:

$$1 = \langle 3S_1^2 | 3D_1^2 \rangle = a^2 \langle 3S_1^2 | S_{12}^2 \rangle \langle 3S_1^2 | S_{12}^2 \rangle.$$ 

But

$$\langle 3S_1 | S_{12}^2 | 3S_1 \rangle = \langle 3S_1 | \left( \sigma \cdot \sigma \right)^2 (\sigma' \cdot \sigma')^2 - 3(\sigma \cdot \sigma') \sigma \cdot \sigma' + \sigma \cdot \sigma' \sigma \cdot \sigma' + (\sigma \cdot \sigma')^2 \rangle | 3S_1 \rangle$$

$$= 9 - 6\left(\frac{1}{3}\right) + 1 = 8.$$

and therefore,

$$\langle 3S_1 | S_{12} | 3D_1 \rangle = a \langle 3S_1 | S_{12}^2 | 3S_1 \rangle = 1/a = 2\sqrt{2}.$$

Finally, we have to compute

$$\langle 3D_1 | S_{12} | 3D_1 \rangle = a^2 \langle 3S_1 | S_{12}^2 | 3S_1 \rangle.$$ 

This one is a little more involved!

$$\langle 3S_1 | S_{12}^3 | 3S_1 \rangle = \langle 3S_1 | 27(\sigma \cdot \sigma') \left( \sigma' \cdot \sigma \right)^3 - 9(\sigma \cdot \sigma')^2 (\sigma' \cdot \sigma) \sigma \cdot \sigma'$$

$$- 9 (\sigma \cdot \sigma') (\sigma' \cdot \sigma) \sigma \cdot \sigma' + 3 \sigma \cdot \sigma' \sigma \cdot \sigma' (\sigma \cdot \sigma')^2 + 3 \sigma \cdot \sigma' \sigma \cdot \sigma' \sigma \cdot \sigma'$$

$$+ 3 (\sigma \cdot \sigma')^2 \sigma \cdot \sigma' \sigma \cdot \sigma' - (\sigma \cdot \sigma')^3 \rangle | 3S_1 \rangle$$

$$= 27\left(\frac{1}{3}\right) - 9 - 9 + 3\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right) - 1$$

$$- 9 \langle 3S_1 | (\sigma \cdot \sigma') \sigma \cdot \sigma' \sigma \cdot \sigma' | 3S_1 \rangle$$

$$= - 7 - \langle 3S_1 | (S_{12} \sigma \cdot \sigma') \sigma \cdot \sigma' (S_{12} \sigma \cdot \sigma') | 3S_1 \rangle$$

$$= - 7 - 9 = -16.$$ 

Therefore, \( \langle 3D_1 | S_{12} | 3D_1 \rangle = -2 \), and for the first group of states:

$$s_{12} = 3S_1 \left( \begin{array}{c} 3S_1 \\ 3D_1 \end{array} \right) \left( \begin{array}{cc} 0 & 2\sqrt{2} \\ 2\sqrt{2} & -2 \end{array} \right)$$

If one wants to investigate states of higher angular momentum, then the matrices have to be computed which involve the states in question. There is a definite matrix for each value of the total angular momentum \( J \), and the parity \( P \), which we may call \( s_{12}(J,P) \). Accordingly, the investigation of the triplet states is considerably more complex than for the singlet states.
Since again the experimental situation is such that the data for states of high angular momenta is hard to get accurately, it would not pay us to look now at the potentials for the higher triplet even states, or the potentials for the triplet odd states. But in general, we can write the potential for the triplet even states as:

\[ V_{\text{even}}^{(2)} = -\frac{f^2}{3} \left\{ s(r) - \frac{\delta_s(r)}{\mu^2} + s_{12}(J,+) (s(r) - \frac{\delta_s(r)}{\mu^2} - \frac{3Y_1^1(r)}{\mu^2 r}) \right\}, \]

and the potential for the triplet odd states as:

\[ V_{\text{odd}}^{(2)} = \frac{f^2}{3} \left\{ s(r) - \frac{\delta_s(r)}{\mu^2} + s_{12}(J,-) (s(r) - \frac{\delta_s(r)}{\mu^2} - \frac{3Y_1^1(r)}{\mu^2 r}) \right\}. \]

If we consider those triplet even states with \( J=1 \), we see that we have a central force potential, and a tensor force potential. The tensor force potential acts as a transfer potential between the \( ^3S_1 \) and the \( ^3D_1 \) states, and gives a potential in the \( ^3D_1 \) state. The tensor potential is the same as the central potential except for an additional term

\[ \frac{f^2}{3} \frac{2Y_1^1(r)}{\mu^2 r}, \]

which is quite singular in the absence of a cutoff, and thus which depends quite heavily upon the nature of the cutoff. In fact, without a cutoff, the additional term is

\[ -\frac{3\mu^2}{r} \left\{ \frac{e^{-\mu r}}{(\mu r)^2} + \frac{e^{-\mu r}}{(\mu r)^3} \right\}, \]

which is so singular as to make it impossible to find solutions to the Schrödinger equation. The cutoff chops off the highly singular potential at a distance of the order of 0.2 fermies. The character of the tensor force as given by the second order potential is apparently consistent with what is known experimentally. The fourth order corrections do not change this character very much; they affect mainly the description of the singlet potentials.
January 29, 1959

Let's look once more at the generalized Pauli principle, this time from the point of view of the charge independence of the nuclear forces. We start with a situation in which there are two different particles, the neutron and the proton, and the neutrons separately obey the Pauli principle, as do the protons. We then notice that for the description of the forces between nucleons it is very convenient to introduce the concept of isotopic spin. And the question arises as to what to do about the Pauli principle, because when the isotopic spin is introduced, the neutron and proton become just two different states of one particle, the nucleon. The answer is really very simple. Previously we had just the following possibilities:

\[
\begin{align*}
pp & \quad \text{with antisymmetric space-spin wave function;} \\
\{ \text{pn, mn} & \quad \text{with symmetric space-spin wave function.}
\end{align*}
\]

From the point of view of charge independence, we observe that the three systems pp, pn, and mn in an antisymmetric space-spin configuration are exactly the same, i.e., neglecting the charge, all the properties of the three states are identical; for example, the energy, phase shifts, etc. are all the same. So upon introducing the isotopic spin, one would say that these three systems form a genuine isotopic triplet. And one would similarly want to designate a np state with a symmetric space-spin configuration as an isotopic singlet. Since the isotopic triplet has symmetrical isotopic wave functions, and the isotopic singlet state has an antisymmetrical isotopic wave function, we see that we can take care of the Pauli principle requirements upon the introduction of the isotopic spin just by the statement that the total wave function is antisymmetric under the interchange of position, spin, and isospin between the two particles. Thus the generalized Pauli principle doesn't tell us anything new, but is just a consequence of the introduction of isospin. There is no new physics in it; it simply regulates the mathematics of isospin. It is, of course, a great convenience after the introduction of the isotopic spin formalism to regard the two distinct particles as being actually two states of one particle which obeys the generalized Pauli principle.

When people first started to consider baryons which obeyed the Pauli principle, but to which one would like to assign an integral isotopic spin, there was a question of whether one could again consider the distinct
particles of an integral isotopic multiplet to be just different states of a single particle by generalizing the Pauli principle in the same way. Clearly, this can be done for isotopic singlets of which the $\Lambda$ particle is an example. For, a combination of two $I=0$ states again yields a symmetric $I=0$ state, and the generalized Pauli principle just tells you that the space-spin configuration of the two $\Lambda$'s must be antisymmetric, which is exactly what the Pauli principle requires. As a simple exercise, the reader may verify that the same thing works also for the $\Sigma$ isotopic triplet. So the generalized Pauli principle introduces nothing, but just allows us to consider a group of particles as different states of a single particle.

In the last lecture we derived the second-order nuclear potential, which is typical of all orders of a nuclear potential obtained the way we got the lowest order. We did it by two ways and got the same answer: we took the Born approximation for nucleon scattering in the relativistic PS-PS theory and reduced it to an equivalent potential in the adiabatic limit; and we considered 2 fixed nucleons in the static theory and calculated their interaction energy. A general characteristic of the nuclear forces was exhibited when it was shown that there are four different forces, one for each of the following types of states specified by the spin character and the parity: singlet even, singlet odd, triplet even, and triplet odd. By virtue of the generalized Pauli principle, two of these occur only for isotopic triplets, and the other two only for isotopic singlets. The triplet even and triplet odd forces are of two origins, a central potential, and a tensor potential, which connects $S$ states to $D$ states, or $P$ to $F$, etc. and also gives a contribution in a single angular momentum state if the angular momentum is not zero. So in a triplet nuclear system, there is a limited mixing of orbital angular momentum states. Altogether we need six potentials, a central potential for each of the four types of states, and a tensor potential for each of the two kinds of triplet states. All of this applies only to the calculation of an adiabatic potential.

If we wish to consider higher energy nuclear phenomena, we have to modify this notion of an adiabatic potential. Many relativistic effects must be accounted for in so doing. First of all, the equation would no longer be the Schroedinger equation, but would have to involve kinematical $v/c$ corrections. But besides this, in the dynamics you would have to put in velocity dependent terms as corrections to the potentials. In particular the first non-adiabatic correction would be a term linear in the velocity,
the simplest form this correction could take would be a term like \( S \cdot L \, V(r) \), a spin-orbit potential. And in fact, a generalization of an adiabatic potential including such a spin-orbit term has helped greatly in the phenomenological understanding of nuclear forces up to fairly high energies. But if we leave out such a term and all other non-adiabatic velocity-dependent corrections, we must restrict ourselves only to low energy phenomena.

The fourth, sixth, and higher order terms in an adiabatic calculation just correct the six potentials obtained in second order; they do not change the general structure of the forces. If we were to calculate to higher order, with what finally would we compare the answers; what kind of experimental parameters are available to test such a calculation? Restricting ourselves to quite low energies so that non-adiabatic effects do not enter in, there are only a limited number of experimental parameters. The "classical" experimental results that have always been used to discuss the low energy nuclear forces are the triplet and singlet scattering lengths and effective ranges for the even states; these are concerned almost exclusively with \( S \) states. The odd state behavior is still poorly known experimentally, except within a framework of analysis which involves scattering at fairly high energies, and which thus involves non-adiabatic effects. So we can compare only the even state adiabatic potentials with experiment.

The triplet states include a bound state, the deuteron, whose properties can also be examined to yield checks on the calculation. The most obvious parameter of the deuteron is its binding energy. However, it is so small that the combination of scattering length, effective range, and binding energy are not independent parameters. So the binding energy of the deuteron doesn't tell us anything new. However, there are other properties of the deuteron which come into an analysis of the triplet state, its quadrupole moment and magnetic moment. If the deuteron were a pure \( ^3S_1 \) state, it would have no quadrupole moment. But the tensor forces introduce an admixture of \( ^3D_1 \) into the deuteron state, and thus it can have a quadrupole moment. The amount of admixture is quite closely related to the quadrupole moment, although you do have to know something about the radial wave function to get the relation. Thus the quadrupole moment of the deuteron is an additional triplet even parameter. The magnetic moment of the deuteron is known very accurately, but is nearly equal to the sum of the proton and neutron magnetic moments. A theory of nuclear forces such as is being con-
sidered now would determine only the difference between this sum and the observed moment. But because this difference is quite small, the effect of relativistic non-adiabatic corrections is almost as important as the effect of the static potential, with the result that the deuteron magnetic moment is not a very good test of adiabatic nuclear potentials.

In the early days a great deal was done trying to fit these parameters with phenomenological potentials. The earliest attempts involved the use of three square wells to describe the even states: one for the singlet potential, another for the triplet central potential, and a third square well for the triplet tensor potential. Each of these has a depth and a range, and the six parameters were fitted to agree with experiment. As of a few years ago, the most sophisticated thing that was done was to calculate as well as possible a nuclear force from meson theory, in particular the sum of the second and fourth order meson theory potentials, with certain arguments here and there as to what you should take and what you should leave out, which we shall go through. Having got such a potential, one wiggled the relative strength of the potentials by a little bit, presumably to simulate the effect of the higher orders. Then accurate agreement with all the experimental parameters was obtained. The potential obtained in this manner is called the Gartenhaus potential.* This paper is presumably the closest anybody has yet come to the description of the adiabatic limit of nuclear forces from meson theory.

The more recent work, which is of a non-phenomenological nature, is concerned with the dispersion theory approach to the problem, and in particular with the fact that the diagram

\[ \text{Diagram} \]

which we considered last time, has a pole when \((\Delta E)^2 = (\Delta p)^2 + \mu^2\), and that it is the only diagram in all orders which contributes such a pole at that point. Any diagram in the higher orders which gives a pole there merely contributes to the renormalization of the coupling constant, and thus it has already been included by using \(g_1^2\). So if you could extrapolate the

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*See S. Gartenhaus, Phys. Rev. 100, 900, (1955)
experiments to the neighborhood of the pole, which can never occur in the physical range, then the first Born approximation would absolutely dominate everything else. As you go to higher and higher energies, the pole moves closer and closer to the physical domain, and so you begin to see the effects of the pole at small forward and backward scattering angles. And by extrapolating to this pole, it is possible to measure its strength, which is the square of the coupling constant. It comes out accurately the same as the result from the two other methods, pion scattering, and photoproduction.

The most modern analysis of the two nucleon problem proceeds along the following lines. The scattering amplitude is first expressed as an expansion in phase shifts plus the Born approximation. This makes it unnecessary to include in the expansion phase shifts for very high angular momenta, because most of the contributions to such phase shifts arise just from the Born approximation term. The number of phase shifts to be determined is thus greatly reduced. From a complicated analysis of polarization experiments, double and triple scattering experiments, etc., these phase shifts can really be determined as functions of energy. Then these results are compared with the predictions of a slightly fudged second plus fourth order adiabatic meson theory potential to which some sort of a phenomenological spin-orbit potential has been appended. By adjusting the spin-orbit potential, it is found that the scattering can be fitted up to fairly high energies quite well.* At the present this fit can be made up to 150 MEV, and there is some hope that the data may be understood up to 300 MEV without the use of non-adiabatic correction terms depending on a higher order of v/c.**

The outstanding problems include a refinement of the phase shift analysis to remove any ambiguities in their determination. Also a halfway decent determination of the adiabatic potential is needed, which doesn't depend on taking just the second and fourth order, since at the present we have no justification for the hope that higher order corrections cause just a slight jiggling of the potentials, as is being assumed. This may possibly be done with the aid of dispersion theory. And thirdly, the

* Signell and Marshak, Phys. Rev. 109, 1229 (1958)

** Signell, Zinn, and Marshak, Phys. Rev. Lett. 1, 416 (1958)
most challenging problem is to find the physics of the spin-orbit potential, i.e. to derive the spin-orbit term from meson theory the way that the adiabatic potentials have been derived. This last problem is really quite a challenge since the spin-orbit term is so important. A spin-orbit potential determines to a huge extent the structure of nuclei, and it is now fairly well agreed that the effective spin-orbit potential in nuclei arises largely from the true spin-orbit potential between two nucleons. For many years this was a subject of confusion because an effective spin-orbit potential can be induced by the tensor force; a tensor force taken among several particles and iterated a few times can lead to an effective spin-orbit potential. Thus there was considerable controversy over the origin of the effective spin-orbit potential in nuclei until a two-nucleon spin-orbit potential was found to be so useful in the phenomenological understanding of scattering. The nucleon-nucleon spin-orbit potential appears to be of about the right size and of the right sign to give rise to the nuclear term, and thus many now feel that it probably does account for the lion's share of the nuclear effective spin-orbit coupling.

**Fourth-Order Nuclear Potentials**

Now that we have an idea of where things fit in, let us examine the work of Gartenhaus, by calculating a fourth-order adiabatic nucleon-nucleon potential, and looking up how it was necessary to fudge such a second plus fourth order potential in order to get agreement with experiment. We ought to again calculate the fourth order in both ways: by passing to the adiabatic limit from the relativistic theory, and by using the static theory. These ought to agree in fourth order as they do in the second order, if the physics is reasonable for each of them. But they don't agree at all! And therefore I must digress to describe what it is that must be done to the relativistic theory in order that it may agree with the non-relativistic theory or with experiment, and whether that is reasonable or whether it simply means that the relativistic theory is wrong. If you're calculating with the relativistic theory in perturbation theory, you must change it radically in order that it looks like the static model or like experiment. The higher order effects in the relativistic theory may do this, so what you are doing thereby is merely to simulate the higher orders, or the relativistic theory may not do this, in which case what you're doing is throwing away the relativistic
theory and replacing it by something quite different.

We are interested in the low-energy limit of the relativistic theory, and thus let us examine the non-relativistic limit of the PS-PS theory in the same way that we examined the non-relativistic limit of the Dirac equation with electromagnetic coupling. The nucleon field operator satisfies the equation \((\not{\partial} + M + ig \gamma_5 \not{\tau} \not{\phi})\psi = 0\), which we multiply on the left by \((\not{\partial} - M + ig \gamma_5 \not{\tau} \not{\phi})\), in order to get a squared equation

\[
\left\{ \mathcal{L}^2 + \not{\partial} \right\} \psi = 0
\]

from which the passage to the non-relativistic limit is quite easy:

\[
\frac{\partial \psi}{\partial t} = \left\{ M - \frac{g^2}{2M} + \frac{ig \gamma_5 \not{\tau} \not{\phi}}{2M} + \frac{g^2 \not{\phi}^2}{\not{\phi}} \right\} \psi.
\]

To complete the derivation, we must realize that in the non-relativistic limit:

\[
\gamma_5 \not{\tau} \not{\phi} \longrightarrow -1 \not{\sigma},
\]

and

\[
\gamma_5 \not{\gamma}_4 \longrightarrow 0.
\]

Thus the PS-PS theory in the NR limit looks like a static theory with the following coupling Hamiltonian density:

\[
\mathcal{H}_{\text{coup}} \simeq \left[ \frac{g^2}{2M} \sigma \cdot \not{\tau} \not{\phi} + \frac{g^2}{2M} \not{\phi}^2 \right] \rho(x).
\]

There are two distinct terms in this density. One of them is the familiar static model term, giving a P-wave interaction with a coefficient \(g/2M = f/\mu\). The second term, however, is fantastically embarrassing, because it leads to an extremely strong short range repulsion between S-wave pions and nucleons which is independent of isotopic spin. In fact, the term indicates a delta function potential, but we know that this really has to be spread out a little. Well, what do we do with this term? Is there such a thing in practice? Certainly no particularly large S-wave scattering is noticed in the pion nucleon interaction; in the first few hundred MEV the S-wave scattering is small and strongly isotopic spin dependent. But does such a very strong short range repulsive potential give rise to a lot of scattering? Although in first Born approximation it does, in the higher approximations the wave function is pushed out of the potential region, and therefore the scattering becomes smaller and smaller as the potential becomes infinitely strong. Thus the \(\not{\phi}^2\) term in Born approximation produces
tremendous isospin independent S-wave pion-nucleon scattering, but altogether very little — so little that it might be just a small piece of what's actually observed. You can see this damping by actually doing the problem of scattering by a potential due to

\[ \mathcal{L}_{\text{coup}} = \frac{g^2}{2M} \phi^2 (x) \]

In the static theory this problem is an exactly soluble problem.*

So we come to the interesting conclusion that the PS-PS theory gives this tremendous bilinear S-wave interaction in the NR limit between pions and nucleons, and that all Born approximations which involve it are highly suspect, because if you were to sum all orders in the static model you would find that it actually gives almost nothing. People therefore say that they hope that if you could consider all orders in the relativistic theory, the effect of this term would again be damped out. This is not clear because whereas you can solve things exactly in the static theory and verify the hope, the recoil in the relativistic theory couples S-waves to P-waves, P-waves to D-waves, etc., and makes everything unsolvable. Therefore, it is only a supposition that the relativistic theory behaves the way the non-relativistic static theory does. Suppose we accept this hope that the \( \phi^2 \) term wherever it occurs damps itself out. Then what shall we do? We should examine any higher order calculation of the relativistic theory and pluck out those pieces which come effectively from the \( \phi^2 \) and discard them. Then we will get agreement with the static model and rough agreement with experiment. Nobody knows, however, whether this is a way of simulating the higher order corrections of the theory, or whether it is just a way of completely altering the theory so that it looks like experiment.

In regard to the S-wave scattering, another point deserves mention. There is another parameter in the relativistic meson theory besides the coupling constant and the masses of the particles, and it comes about in the following way. When the renormalization program was first carried through for quantum electrodynamics, it was shown that after renormalization of the mass and the charge, which were the only parameters originally present in the theory, everything was finite in perturbation theory. It was a tremendous advance to understand this. Immediately people began to look at all other

---

*See G. Wentzel, Phys. Rev. 86, 802, (1952)
theories of the couplings of particles, to find out whether they possessed the same remarkable property. It was soon discovered that almost all theories were violently unrenormalizable. Almost anything one writes down with non-linear coupling, gradients, or something fancy, doesn't work. Even some things that look particularly simple also do not work, such as a point contact coupling of two fermions, which is used nowadays to describe the weak interactions. Only a very limited number of theories worked, and people began to examine these. It was found that the only other things that came close to being renormalizable and were physically interesting were the coupling of scalar and pseudoscalar particles to the electromagnetic field, and a coupling of scalar and pseudoscalar mesons to nucleons.

This is of some interest because there are pseudoscalar mesons coupled to nucleons and to the electromagnetic field. It therefore looks as if nature might have chosen only couplings which fall into the renormalizable class. But then it was discovered that each of these theories after the usual renormalization still possessed one primitive divergence. This divergence arose from a diagram appearing in the scattering of two mesons by each other, both in the electrodynamics of pseudoscalar particles and the nucleonics.

The same divergence almost appears in the case of photons as well, but gauge invariance rushes in at the last moment and saves everything!

This divergence means that the meson theory at one juncture has to include a cutoff dependent term explicitly. However, the enthusiasts of the renormalization program argue in the following way, which could well lead to a correct result. They say that the divergence is an indication that originally some of this meson-meson scattering should have been introduced into the Lagrangian density. If that was done, then the divergence would just be another renormalization, and then after the renormalization of the meson-meson scattering parameter, the whole theory would be finite in the perturbation expansion. Therefore, in setting up the PS-PS theory people introduce not only the terms we have used, but also a term $\lambda_0 \Phi^4$. Then the
divergence can be lumped into a renormalized $\lambda, \lambda_1$. If there were fifteen of these things, or a constantly increasing number of them as you go to higher and higher orders, this would be a ridiculous point of view. But since there is only one, there might be something in it. And therefore it is extremely popular to say that there may be a fundamental interaction of this nature, which is renormalized, and that this $\lambda_1$ is a parameter of physical interest.

Nobody has been able to test this hypothesis so far, since nobody has been able to scatter a pion from a pion. This is not necessarily a permanent state of affairs. By means of dispersion theory, it is possible with experiments on reasonable particles to test the scattering of pions by pions, at least to some extent. For example, a contribution to the production of pions by pions comes from diagrams like

which have the structure of pion-pion scattering in them. This blob can be measured by looking at the case where the relativistic four-momentum transfer squared equals $-\mu^2$. Again an extrapolation out of the physical domain is necessary, but at high energies this is not too bad. The coefficient of the pole as a function of the other variables tells you all about the blob. And so pion-pion scattering is not inaccessible to experiment, and is now being studied.

Pion-pion scattering also makes an important contribution to many other things, not in a way that can be separated experimentally, but possibly in a way that can be separated theoretically. There are important diagrams in the S-wave scattering which look like
The meson-nucleon scattering occurs through the incoming meson being scattered by a meson in the cloud, through meson-meson scattering. The meson-meson scattering arises in part from the fundamental meson-meson interaction, and in part through the nucleon-antinucleon loops. In particular, the $\phi^4$ term contributes to S-wave scattering, and in computing, therefore, the S-wave scattering not only the $\phi^2$ term from the $\gamma_5$ has to be included with its possible damping, but also the $\phi^4$ term, and its possible damping. So the problem is considerably more involved than it first appeared.

In electrodynamics, the coefficient of the coupling is $e$ and $e$ is a dimensionless number, i.e., you don’t have to specify a length to establish $e$. The same is true of PS-PS meson theory, $g$ and $\lambda$ are both dimensionless. It is interesting that these theories require only dimensionless coupling. This is not true of any theory; for example, the PS-PV requires a length to define a coupling, as does a scalar meson theory in a $\phi^3$ interaction term necessary for the removal of diagrams like...

by renormalization. Maybe something is at work which restricts the fundamental interactions to those with dimensionless coupling! It may not be accidental.
February 3, 1959

In the last lecture, we began to investigate the fourth-order nucleon-nucleon adiabatic potential. In the second order, the result of the static theory is essentially the same as that of the relativistic PS-PS theory, but this does not hold true as we consider higher orders. To force the relativistic theory to yield results like the static theory, which gives fairly good agreement with experiment at low energies, it is necessary to chop it in half. This can be done by going to a second order equation for the nucleon field:

\[
(\nabla^2 - M^2 - g^2 \phi^2 - i\gamma_5 \tau \cdot (\vec{\sigma} \phi))\Psi = 0
\]

and eliminating all diagrams containing a \(\phi^2\) interaction. Experimentally all the effects of the \(\phi^2\) term are absent, which can be for two reasons: Either (1) the relativistic PS-PS theory is completely wrong, or (2) the \(\phi^2\) damps itself out if all higher orders are accounted for, just as it does when it is the only interaction present in the static theory.

If just the pseudovector interaction is kept, i.e., if one adopts the equation

\[
(\nabla^2 - M^2 - i\gamma_5 \tau \cdot (\vec{\sigma} \phi))\Psi = 0
\]

to calculate low energy phenomena, then one gets something very much like the static theory, only the relativistic kinematics are included in an unambiguous way. Notice that although the interaction appears pseudovector, this chopped up theory does not have the serious divergence difficulties which plague the PS-PV theory. The reason is, of course, that the nucleon propagator one would employ here is \(1/(p^2 + M^2)\), and not \(1/(ipM)\) as in the PS-PV theory. All the divergences are, therefore, only logarithmic as in the PS-PS theory. However, it is suspected that they may not be removable from the theory by a simple renormalization program. Nevertheless, the theory is valuable for treating low-energy problems with some relativistic kinematics.

Let us now calculate the fourth-order potential using the static model; if we have time we may also actually use this chopped up relativistic theory to calculate the fourth order. Proceeding in a straightforward fashion, we would expect that the fourth-order adiabatic potential is just the energy of two sources at \(0\) and \(r\) minus the two masses \(M\), calculated in fourth order. We will have some reason to challenge this statement later, but it is a good place to begin.

The formula for fourth-order perturbation theory can be derived quite
easily in the following way (see page 30). Let $E$ be the total energy of the complete state with two sources with specified spins and isospins at $Q$ and $L$. Then we define

$$H^O = H_{\text{free}}^O + E,$$

and $H' = H - H^O = H_{\text{coup.}} - (E - 2M)$. This definition assures us that $\langle \Phi | H' | \Psi \rangle = 0$, where $\Phi$ is that state of two bare sources, and $\Psi$ is the complete state. (It should be realized that $E$ is a function of the total spin $S$, and isospin $I$ of the state.) Since

$$(E - H^O - H') | \Psi \rangle = 0,$$

the following equation can be written for $\Psi$, which leads easily to an iteration scheme:

$$| \Psi \rangle = Z | \Phi \rangle + \frac{1 - P}{E - H^O} H' | \Psi \rangle$$

$(P = | \Phi \rangle \langle \Phi |)$

Any order in the Rayleigh-Schroedinger perturbation theory can be obtained by substituting the iteration solution for $\Psi$ into the formula

$$\frac{1}{2} \langle \Phi | H' | \Psi \rangle = 0$$

which gives:

$$\Delta E = \langle \Phi | H_c | \Phi \rangle + \langle \Phi | H_c \frac{1 - P}{E - H^O} H_c | \Phi \rangle + \langle \Phi | H_c \frac{1 - P}{E - H^O} H' \frac{1 - P}{E - H^O} H_c | \Phi \rangle + \ldots$$

In particular, since $H_c$ creates or destroys a meson, all the terms are of an even order, and thus:

$$\langle \Delta E \rangle (4) = \langle \Phi | H_c \frac{1 - P}{H_c} H_c \frac{1 - P}{H_c} H_c \frac{1 - P}{H_c} H_c | \Phi \rangle - \langle \Phi | H_c \frac{1 - P}{H_c} (\Delta E) (2) \frac{1 - P}{H_c} H_c | \Phi \rangle.$$ 

This derivation shows that the Rayleigh-Schroedinger perturbation series consists of the usual terms where you have a matrix element of $H_c$, then an energy denominator, matrix element, energy denominator --- the energy denominators always forbidden to be zero --, matrix element, etc. plus some peculiar terms. If the first order perturbation is zero, then the peculiar terms start in fourth order; if not, they begin at the third order. Now we will be quite concerned with the distinction between the fourth-order terms arising from the first part of the formula, and those from the $(\Delta E)(2)$ part. Therefore, we will be careful not to mix them up.

Let us consider now the computation of

$$- \langle \Phi | H_c \frac{1 - P}{H_c} H_c \frac{1 - P}{H_c} H_c \frac{1 - P}{H_c} H_c | \Phi \rangle.$$
There are 32 diagrams for this part of the fourth order. To simplify matters, we shall consider that the first meson is emitted by the unprimed source. Near the end of the computation, the remaining diagrams can be included by adding the contribution obtained by interchanging the primed and unprimed operators. The 16 diagrams are:

(Note) The diagrams with an intermediate state equal to the initial state are omitted from the perturbation theory. They are:
We are interested only in $\Delta E^{(4)} - 2\Delta M^{(4)}$. Therefore, we need not compute diagrams 13 through 16, which give contributions only to $\Delta M^{(4)}$. Diagrams 9 through 12 give wave function renormalization corrections to the second-order potentials, and diagrams 5 through 8 yield vertex corrections correspondingly. Both of these sets of diagrams are sets of reducible diagrams. Their major effect is a renormalization of the mesic charge, which we include from the start by employing $f^{2}_{1}$ instead of $f^{2}_{0}$. Even after the charge renormalization, there are contributions from 5 through 12; these are called fourth-order radiative corrections to the second-order potentials. It is alleged by everybody that these are quite small, and thus we shall not bother to compute them.

The analytic terms from diagrams 2 through 4 are:

$$
- \left( \frac{\alpha}{\mu} \right)^{4} \frac{1}{(2\pi)^{6}} \int \frac{d^{3}k}{\omega} \frac{d^{3}k^{'}}{\omega^{'}} \ e^{i(k+k^{'}) \cdot r} \ v^{2}(k) v^{2}(k^{'}) \left[ \tau_{j} \tau_{i} \tau_{i} \tau_{j} \right] \left[ \sigma \cdot k \sigma \cdot k \sigma \cdot k \sigma \cdot k \right] \\
\times \left[ \frac{2}{\omega^{'}} + \frac{1}{\omega^{'}} \right] \\
$$

Now we have to do a little "spinology!" Since $\tau_{i}^{2} = 1$, and $\tau_{j} \tau_{j} = i \tau_{j} (i,j,k$ in cyclic order), it is clear that

$$\tau_{j} \tau_{i} \tau_{i} \tau_{j} = 3 + 2 \tau \cdot \tau^{'},$$

and

$$\tau_{j} \tau_{i} \tau_{i} \tau_{j} = 3 - 2 \tau \cdot \tau^{'},$$

Also, we know that

$$\sigma \cdot k \sigma \cdot k = k \cdot k + i \sigma \cdot (k \times k).$$

Thus the spin and isospin part of the above integral is

$$(3 + 2 \tau \cdot \tau^{'}) \left\{ (k \cdot k)^{2} + (k \cdot k)(\sigma \cdot \sigma^{'}) \cdot (k \cdot k) + \sigma \cdot (k \times k) \cdot \sigma^{' \cdot} (k \times k) \right\}. $$

Remembering that we must add to this the corresponding expression with the primed and unprimed nucleon operators interchanged, ($\sigma \rightarrow \sigma^{\prime}$, $\tau \rightarrow \tau^{\prime}$) we get the following contribution to the nucleon-nucleon potential:

$$
- \left( \frac{\alpha}{\mu} \right)^{4} \frac{1}{(2\pi)^{6}} \int \frac{d^{3}k}{\omega} \frac{d^{3}k^{'}}{\omega^{'}} \ e^{i(k+k^{'}) \cdot r} \ v^{2}(k) v^{2}(k^{'}) \left[ \tau_{j} \tau_{i} \tau_{i} \tau_{j} \right] \left[ \sigma \cdot k \sigma \cdot k \sigma \cdot k \sigma \cdot k \right] \\
\times \left[ \frac{2}{\omega^{'}} + \frac{1}{\omega^{'}} \right] \\
$$

The term from diagram 1 is

$$
- \left( \frac{\alpha}{\mu} \right)^{4} \frac{1}{(2\pi)^{6}} \int \frac{d^{3}k}{\omega} \frac{d^{3}k^{'}}{\omega^{'}} \ e^{i(k+k^{'}) \cdot r} \ v^{2}(k) v^{2}(k^{'}) \left[ \tau_{j} \tau_{i} \tau_{i} \tau_{j} \right] \left[ \sigma \cdot k \sigma \cdot k \sigma \cdot k \sigma \cdot k \right] \\
\times \left( \frac{1}{\omega^{'2}} \right),
$$

It is now quite easy to compute this extra term, being the subtraction for $E^{(2)}$. 

\textbf{Ph 234}\hfill \textbf{Page 191}
It should be noted that the integrand is unchanged by the interchange of \( k' \) and \( k \), which fact we shall employ to eliminate a term in the following reduction. Using the same tricks, the spin and isospin parts become:

\[
(3 - 2\tau \cdot \tau') \left[ (k' \cdot k)^2 + ik' \cdot k (\sigma \cdot \sigma') \cdot (k' \times k) - \sigma' \cdot (k' \times k) \sigma' \cdot (k' \times k) \right].
\]

Dropping the term odd under the interchange of \( k' \) and \( k \), and adding the contribution from the interchange of the nucleon primed and unprimed operators, we obtain:

\[
- \left( \frac{3 - 2\tau \cdot \tau'}{2(2\pi)} \right) \int \frac{d^3k d^3k'}{\omega^2 \omega'^2} \left[ \frac{(k' \cdot k)^2 - \sigma' \cdot (k' \times k) \sigma' \cdot (k' \times k)}{\omega + \omega'} \right].
\]

If we make use of the identity:

\[
\frac{1}{\omega + \omega'} - \frac{\omega'}{\omega^2} = \frac{1}{\omega} - \frac{\omega'}{\omega (\omega + \omega')}
\]

then the net result of our computation is:

\[
- \left( \frac{3 - 2\tau \cdot \tau'}{2(2\pi)} \right) \int \frac{d^3k d^3k'}{\omega^3 \omega'} e^{i(k' \cdot k) \cdot r} v^2(k) v^2(k') \left[ \frac{(k' \cdot k)^2 - \sigma' \cdot (k' \times k) \sigma' \cdot (k' \times k)}{\omega + \omega'} \right] \times \left\{ \frac{3(k' \cdot k)^2 + 2\tau \cdot \tau' \cdot k' \cdot k \cdot \sigma' \cdot k' \times k}{\omega^2} + \frac{2\tau \cdot \tau' (k' \cdot k)^2}{\omega + \omega'} \right\}
\]

Strangely enough, this is the Gartenhaus potential, (except for the slight amount of fudging necessary to get perfect agreement with low energy nuclear physics experiments). Yet a sizable term has not been included! So what we have to look at now is the extra term, and then we will consider the arguments people have frantically concocted purporting to show that perhaps it should be left out. If the extra term is put in, the potential gives results which disagree rather badly with experiment. So obviously, the "challenge" was to invent arguments as to why the second term should not be used.

The left-over term is

\[
- \left\langle \phi | H_C \frac{\Delta E^{(2)}}{H_C^2} \frac{H_C}{\pi} | \phi \right\rangle
\]

and is essentially a wave-function renormalization effect. Now there seems to be some confusion even on how to compute this term. According to the time-independent perturbation theory, which we have been using, \( \Delta E^{(2)} \) is a number depending only upon the specification of the state of the two static nucleons. Treating it as such, we can remove it from the interior of the matrix element:

\[
- \left\langle \phi | H_C \frac{\Delta E^{(2)}}{H_C^2} \frac{H_C}{\pi} | \phi \right\rangle = - \Delta E^{(2)} \left\langle \phi | H_C \frac{1}{H_C^2} \frac{H_C}{\pi} | \phi \right\rangle.
\]

It is now quite easy to compute this extra term. Using the expression for \( \Delta E^{(2)} \)
which we obtained on page 170, we find the term to be:

\[
\frac{1}{(2\pi)^6} \left( \frac{\hbar}{\mu} \right) \int \frac{d^3 k}{\omega^2} \left( \langle \phi | \tau' \cdot \tau | \sigma_k' \sigma_k | \phi \rangle \right) e^{ik \cdot r} \left[ -\frac{d^3 k}{\omega} \langle \phi | \tau' \cdot \sigma \cdot k \sigma_k' | \phi \rangle e^{ik \cdot r} \right]
\]

We choose the states \( \phi \) to be eigenstates of the isospin, in such a representation \( \tau' \cdot \tau \) is diagonal, and we can put the term in this form:

\[
\frac{1}{(2\pi)^6} \left( \frac{\hbar}{\mu} \right) \int \frac{d^3 k}{\omega^2} \frac{d^3 k}{\omega} \langle \phi | \sigma_k' \sigma_k' | \phi \rangle \langle \phi | \sigma_k' \sigma_k | \phi \rangle e^{i(k \cdot k')} \cdot r
\]

This is not the term which Brueckner and Watson, Henley and Ruderman allege to be the additional contribution. Theirs is of the form:

\[
\int \frac{d^3 k}{\omega} \cdot k' \cdot k \cdot \sigma_k' \sigma_k' \cdot \sigma_k \sigma_k' \cdot k
\]

which, however, is precisely equal to ours if one forgets to treat the \( \sigma_k' \cdot \sigma_k' \cdot k \) as a number defined with respect to the initial state. If \( \sigma_k' \cdot \sigma_k' \cdot k \) were a diagonal matrix then our term would be identical to Brueckner and Watson's.

We shall see, in the next lecture, that the additional term arises from a recoil correction to the iterated static second order potential. When a correct interpretation of the pieces in a nucleon-nucleon scattering matrix element is made, one finds an extra term arises of the form

\[+V(2) \frac{1}{4\pi} V(2)\]

A sensibly defined potential must give agreement with the R-matrix calculation of the scattering amplitude. In this case, \( V(2) \) is to be treated as a matrix, and thus it would appear that, dynamically, the \( \sigma_k' \cdot \sigma_k' \cdot k \) should be treated as do Brueckner and Watson.

Brueckner and Watson claim that further investigation of their term shows that it gives a very strong repulsive central force in the triplet even states, and leads to rather unacceptable results for the deuteron ground state.

Now that we see the magnitude of this term and how it radically modifies the potential, the question arises as to why people argue that it should be excluded. The first answer is obvious: if you put it in the agreement with experiment is destroyed. If it is left out, Gartenhaus shows in his paper that it works very nicely. A couple of years earlier, Brueckner and Watson (Phys. Rev. 92, 1023 (1953)) showed that a similar potential can give the deuteron binding energy, quadrupole moment, singlet and triplet scattering lengths and effective ranges quite well. The difference between Brueckner and
and Watson in 1953 and Gartenhaus in 1955 is only that Gartenhaus took seriously the $v^2(k)$ $v^2(k')$ cutoff factors. Recall that in the second order the $v^2(k)$ spread out the delta function, and made it into a repulsive core. Also it smooths out any highly singular behavior in the tensor potential. Well, it does the same thing in fourth order, and thus potentials are obtained which can be put into a Schroedinger equation. Brueckner and Watson were forced to say that they believed their potentials only down to a certain distance, at which point they put in a boundary condition, which was essentially a repulsive core boundary condition. So the only thing new in the work of Gartenhaus is that he noticed that the cutoff, if taken seriously, would automatically put in a repulsive core.

In any case it is clear that the second and fourth order potentials should not be trusted below a certain distance. Higher order corrections must be included at smaller distances where they become appreciable. Also when you get in close enough the effects of the cutoff become crucial, and we certainly have no idea of the details of pair-formation effects, recoil corrections, etc. which are treated by a phenomenological cutoff.

The question still remains, however, whether a good reason exists why the term which fouls up the experimental agreement should not be included.

February 5, 1959

Let us now examine the arguments advanced by those who think that the $\Delta E^{(2)}$ term appearing in the energy of two sources should not be included in the adiabatic potential. They argue that one is interested primarily in nucleon-nucleon scattering or in the solution of the Schroedinger equation for the deuteron, and not interested in what has been calculated, that is, the energy of two fixed sources of the meson field. So it is worthwhile to look directly at a scattering problem, and see how these two parts of the energy appear in a calculation of the scattering in the fourth order. It will be found that although both parts appear definitely, they do so in somewhat different roles, and the difference in how they arise will be used to justify (?) the omission of the second part.

Consider now nucleon-nucleon scattering. The second order term in the scattering comes from the second order potential in first-Born approximation, or from these two diagrams: (The time-ordering of internal lines is dis-
The fourth order scattering comes from two sources, however. The first is the fourth order potential in first-Born approximation, represented by diagrams like:

\[
\begin{array}{c}
\text{Diagram 3} \\
\text{...}
\end{array}
\]

and the second source is the second order potential in the second-Born approximation, which appears graphically as:

\[
\begin{array}{c}
\text{Diagram 4} \\
\text{...}
\end{array}
\]

The scattering diagrams associated with the fourth-order potential in the first-Born approximation have the property that there is never an intermediate state of two bare nucleons, whereas the iteration of the second order potential is correlated with those diagrams containing an intermediate state with no mesons.

Let us now designate the potentials to be used in these calculations of the scattering as \( V_2 \) and \( V_4 \). Then the scattering complete to fourth order would result from the matrix element of

\[
\left( V_2 + V_4 \right) \left( 1 + \frac{1}{0(p^2/M)} \right) V_2,
\]

where the energy denominator of order \( (p^2/M) \) is that energy of the two bare nucleon intermediate state, which is only a few MEV.

If the energy of interaction of two fixed sources of the meson field complete to fourth order is called

\[
U_2^{(2)} + U_1^{(4)} + U_2^{(4)},
\]

where the \( U_2^{(4)} \) is that part in fourth order arising from the \( \Delta E^{(2)} \), then...
upon looking at the diagrams, one would identify $V^{(2)}$ with $u^{(2)}$, and $V^{(4)}$ with $U^{(4)}_1$, neglecting the recoil term in $V^{(2)}$ and $V^{(4)}$. Thus one might readily conclude that $U^{(4)}_2$ does not appear in an actual problem, and hence its inclusion into the adiabatic potential is wrong.

However, in the computation of the $V$'s and in particular $V^{(2)}$, an energy denominator appears, which is not $1/\omega$ but instead $\frac{1}{\omega + O(p^2/M)}$, when the non-relativistic nucleon recoil is included. If the energy denominator is expanded, one finds that

$$V^{(2)} = u^{(2)} + \frac{1}{\omega} u^{(2)} O(p^2/M).$$

A similar consideration holds for $V^{(4)}$, etc. Now these recoil corrections to the $V$'s cut no ice, except in the iteration of the potentials, where the smallness of the intermediate state energy denominators cancels the smallness of the correction to the $V$'s, leaving a sizable term. Therefore the scattering complete to fourth order comes from a matrix element of the form

$$u^{(2)} + U^{(4)}_1 + \frac{1}{\omega} u^{(2)} + u^{(2)} - \frac{1}{\omega} u^{(2)} O(p^2/M),$$

and it is clear that the term of the form

$$\frac{1}{\omega} u^{(2)}$$

is what is called $U^{(4)}_2$. Thus we have finally identified where $U^{(4)}_2$ appears in the scattering, and thus we have shown that it is a part of the adiabatic potential. $U^{(4)}_2$ comes from taking the non-static part of the second order potential, iterating it, and then noticing that the non-static piece can be used to cancel an energy denominator, and thereby giving a static fourth-order contribution to the potential.

Everything is consistent with the result of the static treatment, namely that both pieces should be included in the fourth order adiabatic potential, except that looking at the problem in this way, we see that the two pieces have slightly different roles. And therefore it is possible, (maybe), to weasel out of including the second piece by arguing that the approximation that is being made should be stated differently. Thus they prescribe that, in order to calculate the potentials, all the scattering diagrams should be added up with the recoil corrections to the mesonic energy denominators completely neglected. Following this prescription, it will be found that all the terms coming from $\Delta W$'s on the static model have been
thrown away.

Brueckner and Watson, and Gartenhaus throw away the \( U_2^{(1)} \) term on the basis that it arises from a calculation of iterated non-static effects, and that this calculation is poor. They, of course, believe that the iterated non-static effects have to be included eventually, but they believe that calculating them on the basis of perturbation theory is very bad, and thus they would rather employ the estimate of zero for these effects in preference to the perturbation estimate. They propose to calculate just the static part, which is \( U^{(2)} + U_1^{(1)} \), and in sixth order a lot of other diagrams are discarded, and they say that they believe that this expansion is good in perturbation theory. Brueckner and Watson claim that the reason that the perturbation treatment is so terrible for the non-static effects is that the largeness of the correction \( U_2^{(1)} \) is due to the fact that \( v^{(2)} \) is so singular that it brings into the wave function very high momentum components, and that these high momentum components would not be nearly as important if we used the correct behavior of \( V \) at small distances. Whereas the \( V^{(2)} \) diverges like \( 1/r^3 \), the correct potential is by no means that singular. They claim to have calculated an analogous quantity using a modification of a \( v^{(2)} \) due to non-static effects, together with a complete smoothed-out potential, and in this case the result for \( U_2^{(1)} \) is much smaller.

The trouble with such a claim is that Gartenhaus showed that the potentials are not nearly as singular as claimed in low orders, so that the \( U_2^{(1)} \) should not be overemphasized by tremendously high momentum components of \( v^{(2)} \). So Brueckner and Watson's argument is not nearly so impressive as it might be.

In the paper following that of Brueckner and Watson, Henley and Ruderman refute this approximation by examining the neutral scalar theory. They show that if the argument is carried over intact, nonsense results. The neutral scalar theory does not contain highly singular potentials in any order; however, neither does the static theory with the cutoff. In the neutral scalar theory the potential between two particles can be computed exactly to all orders, and it is found that the second order potential between two particles is the complete potential. There are no higher order potentials. The argument of Brueckner and Watson then begins to look very foolish when applied to this theory, because \( U_2^{(1)} = -U_1^{(1)} \), etc. If you
now claim that these two parts of $V^{(4)}$ arise from quite different effects, and claim that perturbation theory is good for $U^{(4)}_1$, but not for $U^{(4)}_2$, and for this reason drop $U^{(4)}_2$, and similarly in the higher orders, you wind up with a potential which has huge contributions in the higher orders, whereas the actual one has none. So that makes the argument look very suspet, and any defense of the procedure has to rest on the fact that in the particular coupling theory one is working with, the $V^{(2)}$ is really completely unrepresentative of the complete potential, and drags in effective singularities despite the cutoff which invalidate the perturbation treatment of the non-static effects, while retaining the validity of the strictly static part of the potential.

No. 12. THE STRONG INTERACTIONS

Global Coupling of $\pi$'s to Baryons

We have no idea of what the strong couplings among the strange particles are. There exist several types of baryons: the nucleon doublet, $\Lambda$ singlet, $\Sigma$ triplet, and the $\Xi$ doublet, plus perhaps others which have not been discovered; and two types of mesons are known: the $\pi$ triplet, and the $K$ doublet. The pions couple particles of the same strangeness, whereas the four $K$'s carry a unit of strangeness, and hence couple particles which differ in their strangeness quantum numbers. Now it is a simple thing to imagine the coupling pattern of all of these baryons with all of these mesons follows the Yukawa picture. This may, of course, be completely false, but since the Yukawa picture works for the pion-nucleon coupling pretty well, there is no reason why you shouldn't try to generalize the scheme to include the whole system of strong interactions.

Thus, apart from intrinsic meson-meson interactions which seems to be necessary in a pion-nucleon system and may be necessary here also, we try a Yukawa coupling scheme, which means that the interactions are of this form:

\[
\begin{align*}
E_1 & & N & N & \pi \\
E_2 & & A & \Sigma & \pi + H.A. \\
E_3 & & \Xi & \Xi & \pi \\
E_4 & & \Xi & \Xi & \pi
\end{align*}
\]

and
These are the only couplings linear in the meson fields which conserve strangeness and isotopic spin. So if a Yukawa theory is tried, there are eight coupling constants which can come in.

The question arises as to what inkling can we get as to these constants and the forms of the couplings from the experimental information. The forms of the couplings are not obviously all the same, because the relative parities of these particles could be different; for instance, the \( \pi \) is known to be pseudoscalar from the \( \overline{N}N\pi \) coupling, but the \( \pi \) could be coupled in a scalar fashion to \( \overline{\Lambda} \Sigma \) if the \( \Lambda \) and \( \Sigma \) have opposite intrinsic parity. For the \( K \) meson there are various possibilities. If the \( \overline{N}\Lambda K \) interaction is pseudoscalar, we could arbitrarily assign a negative relative parity to the \( \Lambda \) and \( N \). It would be only for this purpose, for in the decay of the \( \Lambda \), which is the only other place you can detect the relative parity of the \( \Lambda \) and \( N \), parity isn't conserved! The only meaning to the relative parity of the \( \Lambda \) and \( N \) is precisely in the strong interactions, and the \( K \) can be either pseudoscalar or scalar. If we would continue to look at the other interactions, we would find that there are all sorts of possible complicated couplings. The global coupling scheme is one suggestion as to how they might go, and it is a suggestion which is tied up somewhat with the experiments.

It is known that \( \Lambda \) particles bind to nuclei, to make hyperfragments, which are stable except for the disintegration of the \( \Lambda \), which takes \( 10^{-10} \) sec. The binding energies are somewhat weaker for the light nuclei than that of regular nucleons. For the heavy nuclei, this is undoubtedly not true, for the \( \Lambda \) is not excluded by the Pauli principle from cascading down to the lowest energy level in the nucleus. However, heavy hyperfragments have never been studied experimentally, because the recoil is insufficient to separate the fragment decays from the stars in which they are produced. But for the light hyperfragments, the pattern of binding energies has been extensively studied, and it indicates a force slightly weaker than the nucleon-nucleon force and considerably spin-dependent, and, so far as anybody can tell, charge independent. So there exists an attractive \( \Lambda-N \) potential.

What gives rise to such a potential? Certainly, there are a tremendous number of things. On the Yukawa scheme, we might consider two different
contributions to the binding due to π exchange and K exchange, in the low orders. The simplest examples we may consider are:

So if we look at the situation from the Yukawa point of view, we would say that one of these or the other or a combination of both of them is responsible for the strong Λ-N attraction. What information do we have on the K particle interaction? Well, no one really knows yet for sure whether the K is scalar or pseudoscalar. However, regardless of whether it's scalar or pseudoscalar, the relativistic coupling constant does not appear to be fantastically large, because of the synchrotron experiments here and at Cornell, which investigate the low energy photoproduction of the K particles. It appears pretty much that for either scalar or pseudoscalar K's, a coupling constant around 1.5 or 2 is appropriate. If the K's are scalar we should probably be able to calculate the S-wave scattering of the K's on nucleons, but the theoretical estimate doesn't look anything like the actual S-wave scattering. It therefore seems reasonable to suppose at the moment that the coupling is pseudoscalar, and I will assume so. (This, however, is a very weak conclusion.) If the K's are pseudoscalar, then some fairly substantial conclusions can be drawn.

The K particle force then appears too weak by an order of magnitude to account for the strong attraction. (If the K's are scalar, the potential is repulsive.) Therefore, it looks strongly that non-vanishing Yukawa couplings of the form $\bar{\Lambda} \Lambda N$ exist, and if one inquires as to the magnitude of $g_2$ necessary to account for the observed binding, one finds $g_2^2/\mu_N \approx 15$. So one begins to speculate as to whether there might be a lot of symmetry in the $\pi$ couplings, and whether the couplings might all be of the same form, like perhaps

$$g_1 \bar{N} \gamma_5 N + g_2 \bar{\Lambda} \gamma_5 \Sigma + g_3 \bar{\Sigma} \gamma_5 \Sigma + g_4 \bar{\Xi} \gamma_5 \Xi + H.A.$$ 

with the same g's. But what exactly does "the same" mean in this case? For example, how do we know whether to put a plus or minus sign in the front of each g? The minus signs are not entirely vacuous. Whether you attach a minus sign to $g_1$ and/or $g_2$ is of no importance at all, because that amounts just to
a change in the definition of the sign of the pion and Λ fields. However, once the signs of \( g_1 \) and \( g_2 \) have been chosen, it then matters what the signs of \( g_3 \) and \( g_4 \) are taken to be. Also, how do we know that all the constants should be of relative magnitude one? Maybe the symmetrical thing to do is to make some of them of relative magnitude equal to \( \sqrt{2} \)? Exactly what pattern corresponds to physical symmetry?

Well, there's a physical criterion which suggests itself. All the baryons have similar masses. They're not as similar as the masses of the neutron and proton, or the masses of the members of any other isospin multiplet, the differences of which are attributed to the electromagnetic interactions which are comparatively rather weak. Suppose though that we have an analogous situation. We have the \( W \) forces, the \( K \) forces, and the electromagnetic forces. The strong forces together give isotopic spin multiplets, which are broken up by the weak electromagnetic forces, with splittings of the order of a few MeV. Maybe the pion interactions give still greater symmetry. If we could ignore the \( K \) forces and the electromagnetic forces, perhaps all the baryons would have the same mass. The splittings among the multiplets, of order 100 MeV, might be due to the \( K \) particles.

This is in essence the theory. It states that when the couplings constants for the pions are chosen, they should be taken so that all the baryons remain degenerate, and that the \( K \) couplings, which are weaker, are responsible for the splitting of the isospin multiplets. This is a very restrictive criterion for the four \( g \)'s. There are only a couple of solutions, which amount to
\[
|g_1| = |g_2| = |g_3| = |g_4|; \quad g_2 = g_3
\]
by a convention as to the relative sign of the \( Λ \) field. There remain several possibilities as to the relative signs. It looks, however, that all four are the same in sign as well as strength, and this is the hypothesis of global symmetry. (This can be changed at will, however, without changing the basic symmetry assumed.) Under this hypothesis, it is essential that the \( K \) couplings be asymmetrical, otherwise nothing would split the multiplets.

It is, of course, possible that interactions are not responsible for the splittings. In the case of the nucleon, we like to believe that the photons are responsible for the neutron being heavier than the proton. It is conceivable that in this case there is no interaction which completely accounts for the splitting; they just differ.

If this interpretation of the mass differences as being due to \( K \) inter-
actions is correct, then to the lowest order a sum rule can be found which states:

$$\frac{M_N + M_S}{2} = \frac{M_\Lambda + 3M_\Sigma}{4}.$$

This is not true experimentally, but is not very far from being true. Experimentally,

$$\frac{M_N + M_S}{2} = \frac{(940 + 1320)/2 = 1130 \text{ MEV}},$$

and

$$\frac{M_\Lambda + 3M_\Sigma}{4} = (115 + 3\cdot1190)/4 = 1170 \text{ MEV}.$$

It is therefore not completely idiotic to suppose that this formula is succeeding, and that the difference in the mean masses is due to higher order effects in the K couplings, which are neglected in the derivation of the sum rule. On the other hand, you can't conclude from this that the theory works. The only important things about such a hypothesis are the experimental tests, otherwise it is just a pure speculation. The first experimental test, which helped in the formulation of the hypothesis, but which is nevertheless a test of the idea because the idea has a simplicity apart from the argument which led to it, is the \(\Lambda-N\) force; that is, is that \(g_2\) really is the same as \(g_1\)? This is something that can be answered experimentally, and the agreement is fairly good. Now that a symmetry principle is available, you don't have to rely just on the fourth order estimate, but rather you can say that the \(\Lambda-N\) force due to \(\pi\)'s is the mean of the \(n-p\) force and the \(p-p\) force. The second-order potential disappears in that average, and only the fourth, and higher orders are left. But you can make that statement in terms of the potentials themselves, so that any phenomenological potential can be used to check the hypothesis.

In the absence of K particle effects, it is possible to predict the magnetic moments of the hyperons due to the symmetry, and the conclusions are that: (1) the magnetic moments of \(\Lambda\) and \(\Sigma^0\) vanish, but there is a transition magnetic moment from \(\Lambda\) to \(\Sigma^0\) or back, which equals the neutron moment;

(2) the moment of \(\Sigma^+\) is equal to that of the proton;

(3) the moment of \(\Sigma^-\) is equal to minus that of the proton;

and so forth. These predictions, however, cannot be checked so easily.
The most attractive hypothesis for the $K$ couplings at the moment is that the second order splitting of the $\Lambda$ and $\Sigma$ is zero. That can be arranged by making $g_{N\Lambda} = -g_{N\Sigma}$, and $g_{\Lambda\Sigma} = -g_{\Sigma\Sigma}$. Then the $\Lambda$ and $\Sigma$ split first in order $g_{\mu\nu} \frac{2}{2}$, while the $\Xi$ and $N$ split in order $g_{K}^{2}$ if $|g_{N\Lambda}| \neq |g_{\Sigma\Lambda}|$. But again, this is only a speculation.

The trouble so far is that the predictions which are fairly reliable have not been able to be checked. Nobody has measured the magnetic moments of the hyperons. And the other predictions having to do with $K^{-}$ absorption have not been tested thoroughly. Another conclusion is that the pion-hyperon system has a resonance in it like the pion-nucleon system at about the same energy, and with the same angular momentum, but different isotopic spin. Now a pion and a hyperon cannot be scattered directly from each other, but in $K^{-}$ absorption, a pion and a hyperon are made, and so the investigation of it should reveal whether that resonance is there or not.
February 10, 1959

Application of Invariance Properties to the Determination of Selection Rules

Let's examine some of the conditions under which it is possible to conclude that the matrix element for a given process vanishes to all orders. These statements are based on invariance properties of the Lagrangian density and upon the possibility of constructing various tensors from the parameters specifying the initial and final states. In the following, whenever we discuss parity, or charge independence, or something else, it will be assumed that we are working with a Lagrangian which has the corresponding invariance property. Since there are so many considerations which can in general be applied, we shall only illustrate the techniques by specific examples, rather than construct a long list of them.

Suppose we have the following process:

\[ \phi P_3 \rightarrow \phi P_1 \]

where the \( \phi \) is a scalar field, and the \( \pi \) is a pseudoscalar field. This process can never occur! Why? Well, the matrix element for this process is the Fourier transform of

\[ \langle P(\phi_1(x), \pi_1(y), \phi_2(z)) \rangle \]

which is a pseudoscalar. Now there are only two independent parameters in the problem, which can be taken to be \( p_1 \) and \( p_2 \), the two four-vectors specifying the initial momenta. But a pseudoscalar can be constructed from four-vectors only if there are at least four of them available; i.e.

\[ \varepsilon_{\mu \nu \kappa \lambda} p_\mu q_\nu r_\kappa s_\lambda \]

is the simplest relativistic pseudoscalar that can be made from four-vectors, and the four-vectors \( p, q, r, \) and \( s \) must be linearly independent. Since it is not possible to construct the required pseudoscalar from \( p_1 \) and \( p_2 \), that matrix element must vanish.

Similarly, the matrix element for this process

\[ \phi P_3 \rightarrow A_\mu P_1 \]

must vanish, where \( A_\mu \) is a vector field. The matrix element must be a pseudovector, which is impossible to construct with only two four-vectors. The simplest pseudovector, made from four-vectors, is \( \varepsilon_{\mu \nu \kappa \lambda} q_\nu r_\kappa s_\lambda \), where
again \( q, r, \) and \( s \) must be linearly independent. Assuming that the \( \phi \) and \( \pi \) fields are real (i.e. particle = antiparticle), and that the vector field \( A_\mu \) is the electromagnetic field, then this kernel vanishes for still another reason. Any non-vanishing kernel like this must be even under charge conjugation \( C \). But the real spinless fields are even under \( C \), while the electromagnetic field is odd under \( C \), (i.e., \( C A_\mu(x) C^{-1} = -A_\mu(x) \)), and thus this matrix element is identically zero.

Since the \( \pi^0 \) field is real, any process involving \( \pi^0 \gamma \gamma \) is even under charge conjugation and therefore allowed. It is also allowed, (as it must be, since the process \( \pi^0 \rightarrow \gamma + \gamma \) occurs rapidly!) by relativistic invariance considerations. The kernel is a pseudotensor, which is \( \varepsilon_{\mu\nu\lambda\rho} p_\mu q_\nu \) if \( p \) and \( q \) are the two independent four-momenta involved. However, it should now be clear that the invariance of the theory under charge conjugation forbids the process involving \( \pi^0 \gamma \gamma \), \( \pi^0 \pi^0 \gamma \), and indeed anything involving a kernel like

\[
\begin{array}{c}
\begin{array}{c}
\varepsilon_{\mu\nu\lambda\rho} p_\mu q_\nu \\
\end{array}
\end{array}
\]

with only an odd number of photon lines emerging from a blob. This last assertion is called Furry's theorem.

The processes symbolized by \( \pi \pi \pi \) do not occur, because the matrix element must be a pseudoscalar, which cannot be constructed from two independent four-momenta, but those represented by \( \pi \pi \pi \) are allowed by the invariances we have considered. Likewise, \( \pi \pi \pi \pi \) would appear to be allowed, since there are now four independent momenta involved, but in fact it is forbidden -- by \( G \)!

Invariance under \( G \) is nothing new; it is merely a result of invariance under charge conjugation and under rotations in isospace. If we designate a rotation in isospace about the 2nd axis by the angle \( \pi \) as \( I_2(\pi) \), then

\[ G = I_2(\pi) C. \]

Now \( I_2(\pi) \) transforms \( \pi_x \rightarrow -\pi_x \) and \( \pi_z \rightarrow -\pi_z \); and \( n \) into \( p \), and \( p \) into \(-n\).

Or we can state that \( I_2(\pi) \) is the charge symmetry transformation, under which

\[
\begin{align*}
\pi^- &\rightarrow -\pi^+, \\
\pi^0 &\rightarrow -\pi^0, \\
\pi^+ &\rightarrow -\pi^-,
\end{align*}
\]

and

\[
\begin{align*}
n &\rightarrow p, \\
p &\rightarrow -n.
\end{align*}
\]

Now, if we assume we are working in the Majorana representation to avoid the shuffling of spinor components, under \( C \)
Therefore, under the operation $G = I_2(n) C$

\[
p ightarrow -p; \quad n ightarrow -n; \quad \text{and} \quad p ightarrow -n.
\]

Therefore, any process involving only the strong interactions, which has a diagram containing such a structure

\[\text{---}~\text{---}~\text{---}~\text{---}~\text{---}\]

with only an odd number of pions emerging from the blob, is forbidden. This $G$ is a very useful operation in dealing with problems involving pions.

You can consider the $G$ operation, if you want, as a reflection in isotopic spin space. This is the only meaning that anyone has yet been able to assign to a reflection in isospin space. The pion is then a regular vector in isospace, because under rotations it transforms like a vector, and under reflection it changes sign. $N \rightarrow -N$ is likewise a vector, so that the interaction $N \cdot Tl$ is a scalar in isospace.

You may argue even, although it might be stretching the point a bit, that this is a more natural interpretation. The reason that charge conjugation invariance is a property of the strong interactions is simply that the nuclear forces are invariant under both reflections and rotations in isospace.

From this view, $C = I_2(-n) G$. It is rather mysterious why $C$ should be conserved. The experience with the weak interactions has taught us that the real spatial reflection operation under which all of nature is symmetrical is not $P$ but $CP$. Nature is completely left-right invariant, and the manifestation of that symmetry is the invariance of all interactions under $CP$.

Why then do the strong interactions have this invariance under $C$ and $P$ separately? They also have the peculiar symmetry of charge independence.

From the point of view that $G$ is the isospin space reflection operator, these two questions reduce to the following. Why are the strong interactions invariant under the full group of orthonormal transformations in isospace?

Given that this supercharge independence is a property of the nuclear forces, it then follows that one has $C$ invariance, and from $CP$ invariance, $P$ invariance follows. The flaw in this point of view as to the origin of separate $C$ and $P$ invariance would seem to be the fact that for those particles
which have no strong interactions, the electron and muon, there is still electromagnetism to consider, which does conserve C and P separately. However, this may not really be a flaw. Apparently, the rule for constructing electromagnetism, i.e. arrange everything else and then replace \( p \) by \( p - eA \), automatically insures separate invariance under C and P. In fact, it is extremely difficult, and essentially impossible, to find any scheme for electromagnetism which is invariant under CP, but not under C and P individually.

Thus the conservation of P in the strong and electromagnetic interactions might be looked at as the consequence of two things: (1) the strong interactions are characterized by super charge independence; and (2) when electromagnetism is turned on, C is not disturbed. Out of the complete group of rotations and reflections in isospin space, only certain symmetry operations are unaffected by the electromagnetic coupling, which are rotations about the z-axis and C. The next thing that is remarked is that the free Lagrangians for the electron and muon are evidently invariant under charge conjugation and parity, and that again when you turn on electromagnetism, this situation is not altered. This may actually be the way to look at the invariance under P, but we'll never know until we understand more about the charge independence of the strong interactions.

Consider the following process:

Do the symmetries we have explored so far forbid it? Well, it is not forbidden by relativistic invariance considerations, because the matrix element must be a pseudovector, and this we can form from the three independent four-vectors involved. Indeed, the kernel must be of the form

\[
\varepsilon_{\mu\nu\kappa\lambda} \, p^-_\nu \, p^+_\kappa \, p^0_\lambda \, F(p^-, p^+, p^0) \]

where \( F \) is a scalar. An inspection of the behavior of the kernel under charge conjugation yields the information that

\[
F(p^-, p^+, p^0) = F(p^+, p^-, p^0),
\]

in other words, the \( \pi^+ \) and \( \pi^- \) must be in a state of odd relative angular momentum, or the final state is odd under the interchange of \( \pi^+ \) and \( \pi^- \).
If we look at the isotopic spin dependence of this process, we can deduce a little more information about $F$. Because the photon interacts through $l$ and $I_z^{\prime}$, the photon brings either isospin 0 or 1 into the system each time it acts. Thus to order $1/137$, the final state of three pions must have isospin 0 and/or 1, and to all orders $I_z = 0$. Now it is possible to show that in the final state, $I = 0$. For, consider the matrix element in its general form

$$\sum_p \epsilon_{\mu\nu\kappa\lambda} p_\mu^1 p_\mu^2 p_\mu^3 F(p_1^1, p_1^2, p_1^3) \chi^0_{I}(l, 2, 3),$$

where $\chi^0_I$ are the isotopic spin functions for $I_z = 0$, and $I = 0$ or 1. There are three of these for $I = 1$, which can be taken to be:

$$\frac{1}{\sqrt{60}} \left\{ 3(40^-) + 3(0+-) + 3(0-+)+ 3(-04) - 2(+-0) - 2(-04) - h(000) \right\}$$

$$\frac{1}{2} \left\{ (40^-) - (0+-) - (0-+) + (-0+) \right\}$$

$$\frac{1}{\sqrt{3}} \left\{ (+0) - (000) + (-0) \right\}$$

and one for $I = 0$:

$$\frac{1}{\sqrt{6}} \left\{ (40^-) - (0+-) + (-0) - (+0) - (0-4) - (+0) - (-04) \right\}.$$

Now we have deduced that the amplitude must be odd under charge conjugation. A little inspection will serve to confirm the fact that all the isospin wave functions for $I = 1$ and $I_z = 0$ are even under charge conjugation and thus cannot appear in the kernel since they are linearly independent. On the other hand, the isotopic wave function for $I = 0$ is odd under charge conjugation, and hence the process is not forbidden to occur for the $I = 0$ final state.

You will notice that the isotopic wave function for $I = 0$ is completely antisymmetric. The generalized Pauli principle for bosons therefore requires that the spatial wave function be also completely antisymmetric. In this way we may conclude that $F(p_1^1, p_2^2, p_3^3)$ is completely symmetric under the interchange of $1, 2, 3$, and thus that: $F(p_1^-, p_2^+, p_3^0)$ is a completely symmetric function of the three four-momenta.

Because the photon brings in only $I = 0$ for this process, if we were to try to compute it, we would only have to compute the isotopic scalar part of the electromagnetic interaction.
Now we ought to compute an estimate for this process, because some question has arisen whether there might not be a good experiment that could be done at the synchrotron, which involves this diagram. It would be very nice to have some information on this, because this diagram is suspected of making the major contribution to the isotopic scalar part of the nucleon's electromagnetic form factors. In the symbolism of dispersion theory, this diagram would contribute in the following way:

\[ \text{Diagram} \]

The isotopic scalar part of the magnetic moment is not understood, and it would be very nice if we could find out this part of it. There are lots of barriers to calculating it; we have to calculate this part

\[ \text{Hard calculation} \]

which is very hard, or impossible; and compute also this process,

\[ \text{Other process} \]

whose attributes are similar! To do an experiment which tells you the value of the \( \gamma \mu \mu \) diagram at one or two points will not be very useful to get the form factors. However, such an experiment may be useful because if somebody does make a crude computation of this diagram, it would be nice to have an experiment which would test the theoretical estimate. The second diagram also could be estimated and compared with a measurement of the production of pions by pions

\[ \text{Production of pions} \]

to which it is intimately related.

The way in which the \( \gamma \mu \mu \) diagram might be examined experimentally is by the popular technique of looking at poles. One searches for a point, which must lie outside the physical region, where all the pions are real. At this point the amplitude will have a pole. You hope that by working as near the pole as possible, you can observe the cross section heading off to infinity. By extrapolation, one then tries to measure the residue at the pole, which has a simple relation to the desired amplitude. The experiment which has been suggested for investigating the \( \gamma \mu \mu \) diagram is the
following:
$$\gamma + p \rightarrow p + \pi^+ + \pi^-.$$  
There will be a simple pole in the amplitude due to the following diagram:

![Diagram](attachment:image.png)

at an energy where the $n^0$ is real. This diagram can be split in half; the upper half is the $\gamma \pi \pi$ diagram, while the lower diagram is just the process used to define the renormalized coupling constant. The value of that renormalized vertex is $(\bar{u}_f g \gamma_5 u_i)$. So we can calculate the bottom half rigorously, knowing the renormalized coupling constant which has been measured in many experiments. We are assured that there will be no other isolated pole, since the other singularities due to the exchange of several particles lead to branch lines.

The question the experimental physicists are asking before going ahead with such an experiment is how likely is it that the contribution from the pole will be a dominant feature in the cross section in the physical region of energies and angles. Thus they would like an estimate of the cross section for the process due only to the diagrams containing the pole. If one can compute this and compare it with their estimate of the background, one would have some idea of whether the experiment would be worthwhile.

Unfortunately there is no way to compute the $\gamma \pi \pi$ diagram that is anything like reliable. The only thing we could try is perturbation theory, whose deficiencies are well known. Furthermore, there is the additional contribution that not only a nucleon can occur in a closed loop, but also some of the hyperons. So even in the perturbation theory there are four terms, of which we know only one, the other three we can only guess. So it is very unclear how to estimate this process, but perhaps we can say the following. Suppose we use perturbation theory in lowest order, and make the most favorable assumption about the behavior of the hyperons, then it is quite likely that we shall overestimate the cross section. The corrections to the perturbation theory, it seems, would be very unlikely to make it larger, because the perturbation series is in the huge $g$'s, which are unrealistically large for most processes - at best they are a correct estimate. Once in a long while, such as the $3/2$, $3/2$ resonance, something occurs which raises the physical cross section above the perturbation
estimate, but this must be a peculiar phenomenon and is probably unlikely to occur in such a complicated process as γγπππ.

Regarding the other baryons, it turns out that the most favorable assumption is that of global symmetry, which makes the amplitudes add constructively. Let's examine that scheme in more detail, so that we can see why the various baryons would give contributions which add. The most general pseudoscalar π coupling scheme one can set up among the baryons which is charge independent would be of the form, with the \( i\nu \) suppressed:

\[
\begin{align*}
g_1 & \left( \overline{p} \ p \ o - \overline{n} \ n \ o + \sqrt{2} \ \overline{p} \ n \ p + \sqrt{2} \ \overline{n} \ p \ n^- \right) \\
+ g_2 & \left( \overline{\Sigma}^+ \ \Lambda \ o + \overline{\Sigma}^- \ \Lambda \ o + \overline{\Sigma}^0 \ \Lambda \ o + \overline{\Lambda} \ \Sigma^- \ p + \overline{\Lambda} \ \Sigma^- \ p^- \right) \\
+ g_3 & \left[ \left( \overline{\Sigma}^+ \ \Sigma^- \ n^0 + \overline{\Sigma}^- \ \Sigma^+ \ n^0 \right) \ p + \overline{\Sigma}^0 \ \Sigma^- \ n^- + \overline{\Sigma}^0 \ \Sigma^- \ n^0 \right] \\
+ g_4 & \left( \overline{\Sigma}^0 \ \Sigma^0 \ o^0 - \overline{\Sigma}^- \ \Sigma^- \ o^0 + \sqrt{2} \ \overline{\Sigma}^0 \ \Sigma^- \ n^0 + \sqrt{2} \ \overline{\Sigma}^0 \ \Sigma^- \ n^- \right).
\end{align*}
\]

The global scheme consisted of choosing the four \( \nu \)'s in such a way that if no other interactions are introduced, the \( \nu \) coupling would leave the eight baryons degenerate if they were originally degenerate. It was stated that the way to do this was to have \( |g_1| = |g_2| = |g_3| = |g_4| \) and by a convention as to the sign of the \( \Lambda \) field, \( g_2 = g_3 \).

If \( g_2 = g_3 \), we can put the couplings in a form that is quite symmetrical, and which indicates immediately the complete degeneracy of the baryons if the coupling strengths are the same. For this purpose, it is convenient to define:

\[
\begin{align*}
y^0 &= \frac{1}{\sqrt{2}} (\Lambda - \Sigma^0) \quad \text{and} \quad z^0 = \frac{1}{\sqrt{2}} (\Lambda + \Sigma^0).
\end{align*}
\]

Then the above coupling scheme becomes:

\[
\begin{align*}
g_1 & \left( \overline{p} \ p \ o^0 - \overline{n} \ n \ o^0 + \sqrt{2} \ \overline{p} \ n \ p + \sqrt{2} \ \overline{n} \ p \ n^- \right) \\
+ g_2 & \left( \overline{\Sigma}^+ \ \Sigma^- \ n^0 + \overline{\Sigma}^- \ \Sigma^+ \ n^0 \right) \ o + \sqrt{2} \ \overline{\Sigma}^- \ \Sigma^+ \ n^- + \sqrt{2} \ \overline{\Sigma}^- \ \Sigma^+ \ n^0 \right) \\
+ g_3 & \left( \overline{\Sigma}^0 \ \Sigma^- \ o^0 - \overline{\Sigma}^- \ \Sigma^- \ o^0 \right) \ n^0 + \sqrt{2} \ \overline{\Sigma}^- \ \Sigma^- \ n^- + \sqrt{2} \ \overline{\Sigma}^- \ \Sigma^- \ n^0 \right) \\
+ g_4 & \left( \overline{\Sigma}^0 \ \Sigma^- \ o^0 + \sqrt{2} \ \overline{\Sigma}^0 \ \Sigma^- \ n^- + \sqrt{2} \ \overline{\Sigma}^0 \ \Sigma^- \ n^0 \right).
\end{align*}
\]

It is evident with \( |g_1| = |g_2| = |g_3| \) that the baryons will remain degenerate.

The signs of \( g_1, g_4 \) relative to \( g_2 \) are not determined, but the specific hypothesis of global symmetry assumes them to be both positive.

The latter form of the \( \nu \)-couplings on the global scheme is much more convenient for certain problems. In our present problem, that of computing
this process

\[ \begin{array}{c}
\pi^+ \\
| \quad | \\
\pi^0 \\
\end{array} \]

it is clear that we need only multiply the matrix element for the case in which the photon interacts with the proton by a factor of four to account for the other baryons. For, in the latter two couplings, the change of sign due to the opposite charge of the baryon in the photon absorption is compensated by the change of sign in the interaction which gives the final \( n^0 \).

We shall now produce a crude estimate of this process by using perturbation theory in the lowest order for the \( N-n \) interaction, and then multiplying the matrix element by four to include in a most favorable way the effects due to the other baryons. Next this estimate will be combined with the \( n^0 \) absorption diagram to produce the cross section for the proposed experiment. By comparison with the background, some idea may be obtained as to whether the experiment is likely to be fruitful.

Consider the kernel for this loop:

\[ \begin{array}{c}
p^+ \\
| \quad | \\
p^0 \\
| \quad | \\
p^- \\
\end{array} \]

It is an axial vector and must be of the following form, since the particles involved are all real, and \( k = p^+ + p^0 + p^- \):

\[ \varepsilon_{\mu
u\lambda\sigma} p^+_{\mu} p^0_{\nu} p^-_{\lambda} F(k p^0, k p^+) . \]

The computation of \( F \) is quite a chore, and since the approximations made so far are crude, it would not pay us to do it. The momenta involved are of the order of a fraction of a BEV, and we shall treat them only in first order. In other words, we will compute only \( F(0;0) \).

The matrix element for the loop is:

\[ \frac{-2e^2 g^3}{(2\pi)^4} \int d^4 p \, \text{Tr} \left\{ \gamma_\mu \frac{1}{p^2 + m^2} i \gamma_5 \frac{1}{(p^- + p^-)^2 + m^2} \gamma_5 \frac{1}{(p^- + p^-)^2 + m^2} \gamma_5 \frac{1}{(p^- + p^-)^2 + m^2} \right\} \]

Treating \( p^+ \), \( p^0 \), and \( p^- \) as small quantities, and keeping only the first order terms, the matrix element becomes:

\[ \frac{-2e^2 g^3}{(2\pi)^4} \int d^4 p \, \text{Tr} \left\{ \gamma_\mu \gamma_5 \frac{1}{p^2 + m^2} i \gamma_5 \frac{1}{p^2 + m^2} i(p^- + p^-) \gamma_5 \frac{1}{p^2 + m^2} i(p^- + p^-) \right\} \]
Collecting terms, and noticing that since $\gamma_5$ is the product of four $\gamma$ matrices and the trace of an odd number of $\gamma$ matrices is zero, the $\gamma$ term doesn't contribute, and we get:

$$2ieg^3M \int d^4p \frac{\text{Tr} \left\{ \gamma_5 \gamma_{\mu} \gamma_{\nu} \gamma_{\sigma} \left( p_{\mu} + \gamma_{\nu} p_{\nu} + \gamma_{\sigma} p_{\sigma} \right) \right\}}{(p^2)^4}.$$ 

But since we have already seen that

$$\int \frac{d^4p}{(p^2)^3} = \frac{i\pi^2}{2M^2},$$

by differentiation with respect to $M^2$, we can obtain the following integral:

$$\int \frac{d^4p}{(p^2)^4} = \frac{i\pi^2}{6M^4}.$$ 

$\text{Tr} \left\{ \gamma_5 \gamma_{\mu} \gamma_{\nu} \gamma_{\sigma} \right\}$ is a pseudovector, and thus must equal

(constant) $e^{\mu\nu\lambda\sigma} p^\nu p^\lambda p^\sigma.$

By inspection of the case where $\mu = 1$, $p^+ = (0,1,0,0)$, $p^o = (0,0,1,0)$, $p^- = (0,0,0,1)$, it is obvious that the constant is 4.

Therefore, $F(0;0) = -eg^3/12\pi^2 M^3$, and to lowest order in the momenta:

$$\omega^+ \ldots \omega^- = \frac{-eg^3}{12\pi^2 M^3} e^{\mu\nu\lambda\sigma} p^\nu p^\lambda p^\sigma.$$ 

We have to include the other two possibilities:

Both of these give the same result as above, since in each of them there are two changes of sign which cancel: (1) a minus sign from rearranging the terms in constructing the pseudovector; and (2) a minus sign from the fact that a proton emits the $\pi^0$, rather than a neutron.

Multiplying the result by 4 to account for the other baryons in the global scheme, one obtains

$$\frac{-eg^3}{M^3} e^{\mu\nu\lambda\sigma} p^\nu p^\lambda p^\sigma$$

for the estimate of the $\gamma\pi\pi\pi$ part of the diagram.
Using the estimate
\[-e \varepsilon g^3 \frac{1}{2M^2} \sigma_{\mu \nu} P_{\lambda} P_{\sigma} - \frac{e \varepsilon g^3}{n^2 M^3} \sigma_{\mu \nu} P_{\lambda} P_{\sigma}\]
for the loop, it is clear that the contribution to the common R-matrix element from this diagram

is:

\[i \frac{M}{\sqrt{E_1 E_2}} \frac{1}{\sqrt{8k \omega + \omega^2}} \left( \bar{u}_f G \gamma_5 u_i \right) \frac{-i}{p_0^2 + \mu^2} \frac{-e \varepsilon g^3}{n^2 M^3} \varepsilon_{\mu \nu \lambda \sigma} P_{\nu} P_{\lambda} P_{\sigma}.\]

Using this estimate for \( R_{fi} \), the differential cross-section for the photo production of two pions from a nucleon at rest, i.e., \( p_1 = (0,0,0,IM) \), is:

\[d\sigma = \frac{d^3 p_f}{(2\pi)^3} \int |E|^2 \delta_4(p_+ + p^- + k - p_1) d^3 p^- d^3 p^+\]

\[\frac{d\sigma}{d\Omega_f dE_f} = \frac{1}{(2\pi)^3} \int \Sigma R_{12} \delta(E - k - M) d|p^+| d\Omega^+\]

where \( p^0 = p_1^0 - p_1^0 = k - p^+ - p^- \), and \( E = \omega^+ + \omega^- + E_f \) = total energy, and is considered as a fnctn of \(|p^+|\) and the angle \( \theta^+ \) between \( p^+ \) and \( k - p_f \). \( \phi^+ \) will denote the angle of \( p^+ \) measured around \( k - p_f \). It is simplest if we use the \( \delta \)-fnctn to perform \( \int d \cos \theta^+ \) and then do the \( \int d|p^+| \), rather than do the integrals in the other order. We are also to sum over the final spin states of the nucleon, and average over its initial spin states and the polarizations of the photon. The sum over the nucleon spins amounts simply to calculating the following trace:

\[\Sigma_{\text{spins}} \left| \bar{u}_f G \gamma_5 u_i \right|^2 = -\frac{1}{2} \text{Tr} \left\{ \gamma_5 \frac{M - i \gamma_5 \tau_f}{2M} \gamma_5 \frac{M - i \gamma_5 \tau_i}{2M} \right\}\]

\[= \frac{-g^2}{2M^2} \left\{ M^2 + \tau_f \tau_i \right\} = \frac{-g^2}{2M^2} \left( M^2 + p_f \cdot p_i \right) = \frac{g^2}{2M} E_f - M.\]

Collecting the various pieces and introducing the two polarization vectors \( e^+ \) for the photon, one can express the cross section in the following form:

\[\frac{d\sigma}{d\Omega_f dE_f} = \frac{e^2}{2M} \frac{g}{M} \left| p_f^\lambda \right| (E_f - M) \frac{1}{[(p_f^2 + \mu^2 - (E_f - M)^2]^2} \times\]

\[\int \frac{d^2 \phi^+}{\omega^+ + \omega^-} \int d\cos \theta^+ \frac{1}{2} \frac{1}{s^2} \varepsilon_{\lambda \sigma} e^+ \varepsilon \sigma \left| P_{\nu}(p_f - p_i) \right| \frac{dE}{d\theta^+}\]
\( \omega^- \) is the only term in \( E \) which depends on \( \cos \theta^+ \), and since

\[
(\omega^-)^2 = \mu^2 + \vec{p}_f^+ + (\vec{p}_f' - \vec{p}_f)^2 - 2|\vec{p}_f'| |\vec{p}_f' - \vec{p}_f| \cos \theta^+ ,
\]

Thus:

\[
\frac{1}{\omega^-} \left| \frac{d \cos \theta^+}{d E} \right| = \frac{1}{|\vec{p}_f'| |\vec{p}_f' - \vec{p}_f|}.
\]

Thus:

\[
\frac{d \sigma}{d \Omega d E_f} = \frac{\frac{e^2 g^8}{\sqrt{\hbar \mu}}}{(4\pi)^5} \frac{|\vec{p}_f'| (E_f - M)}{k |\vec{p}_f' - \vec{p}_f| [\mu^2 + \vec{p}_f^2 - (E_f - M)^2]^2} \times
\]

\[
\int \frac{d|\vec{p}_f'|}{\omega^+} \frac{d|\vec{p}^+|}{d\phi^+} \frac{1}{2} \sum_{s=1}^3 \vert e^s_\alpha \lambda \sigma e_\alpha^s p_v^+ (p_f'^- - p_f^l) \lambda p^- \sigma \vert^2 .
\]

The range of integration is determined by the requirement that \( \theta^+ \) must be a physical angle. In terms of the relation for \( \omega^- \) given above this requires

\[
[\mu^2 + (|\vec{p}_f' - \vec{p}_f|^2 + |\vec{p}_f|^2)]^{1/2} \leq \omega^- \leq [\mu^2 + (|\vec{p}_f' - \vec{p}_f|^2 + |\vec{p}_f|^2)]^{1/2} ;
\]

thus

\[
[\mu^2 + (|\vec{p}_f' - \vec{p}_f|^2 + |\vec{p}_f|^2)]^{1/2} \leq k + M - E_f - [\mu^2 + \vec{p}_f^2]^{1/2} \leq [\mu^2 + (|\vec{p}_f' - \vec{p}_f|^2 + |\vec{p}_f|^2)]^{1/2} .
\]

This clearly gives no restriction on \( \phi^+ \).

To actually evaluate this integral is not an easy task, and in view of the negative result as to the relative magnitudes of the cross sections and the experimental background, which we shall anticipate now, it would certainly not be profitable to get involved in such a messy computation. Since we are looking for the pole of \( [\mu^2 + \vec{p}_f^2 - (E_f - M)^2]^{-2} \), we are interested in \( \vec{p}_f^2 \) small (to be close to \( \vec{p}_f^2 \approx - \mu^2 \)). For \( k = M \), let us put an upper bound on

\[
\sum_{s=1}^3 \vert e_\alpha \lambda \sigma e_\alpha^s p_v^+ (p_f'^- - p_f^l) \lambda p^- \sigma \vert^2 = \sum_{s=1}^3 \vert e_\alpha \lambda \sigma e_\alpha^s p_v^+ (p_f'^- - p_f^l) \lambda p^- \sigma \vert^2 .
\]

The first and third terms are linear in \( \vert \vec{p}_f' \vert \), whereas the second term contains

\[
(E_f - M) \approx \vec{p}_f^2/2M \text{ and will be neglected. For an upper bound take } \vert \vec{p}_f' \vert \text{ and } \vert \vec{p}_f \vert \text{ large and approximate } \omega^+ \approx \vert \vec{p}_f' \vert \text{ and } \omega^- \approx \vert \vec{p}_f \vert . \text{ Thus}
\]

\[
\sum_{s=1}^3 \vert e_\alpha \lambda \sigma e_\alpha^s p_v^+ (p_f'^- - p_f^l) \lambda p^- \sigma \vert^2 \leq \vert \vec{p}_f' \vert \vert \vec{p}_f' \vert \vert \vec{p}_f \vert \vert \vec{p}_f \vert \left[ \delta^s_\alpha \hat{\alpha}_f \times (\hat{\beta} - \hat{\beta}^+) \right] , \text{ whence}
\]
Maximizing $|\mathbf{p}^+| |\mathbf{p}^-|$ subject to $|\mathbf{p}^+| + |\mathbf{p}^-| \leq k$ gives $(\frac{k}{2})^2$. The cross product part can apparently be as large as $[2]^2$, but this would require $\mathbf{p}^+$ and $\mathbf{p}^-$ in opposite directions and thus $\mathbf{p}_f = k$, which is not the region of $\mathbf{p}_f$ we are interested in. Rather $\mathbf{p}^+$ and $\mathbf{p}^-$ should be almost in the same direction, which makes the cross product part small. As a gross over-estimate, therefore, we can justifiably take

$$\sum_{s=1}^{2} |\epsilon_{\alpha \nu \lambda \sigma} e^s_{\alpha} p^+_\nu p^0_{\lambda} p^-_{\sigma}|^2 \leq \frac{2} {p \cdot k} \frac{2} {16};$$

$$\mu^2 + F^2_k - (E_f - M)^2 \leq \mu^2 + p^2 \leq \mu^2;$$

$$|\mathbf{k} - \mathbf{p}_f| \approx k \approx M;$$

$$|\mathbf{p}^+|/\omega \leq 1;$$

and the range of integration over $d|\mathbf{p}^+|$ is certainly less than $k$.

Therefore we conclude that:

$$\frac{d\sigma}{d\Omega_f dE_f} \leq \frac{1} {\pi^4} \left(\frac{e^2} {l M}\right)^4 \frac{1} {M^6} \frac{p^2_f}{2 M} \frac{1} {k^2} \frac{1} {\mu^4} 2 \pi k p^- \frac{2} {k^2} \frac{k^4} {16};$$

$$\approx \frac{1} {16 \pi^3} \frac{1} {137} (15)^4 \left(\frac{|\mathbf{p}_f|} {M}\right)^5 \left(\frac{k} {\mu}\right)^4 \frac{1} {2 \times 10^{-13} \text{cm}} \left(\frac{0.2 \times 10^{-31} \text{cm}^2} {\text{MeV}}\right)^2 \approx 0.3 \times 10^{-31} \text{cm}^2 / \text{MeV}$$

for $|\mathbf{p}_f| = 100 \text{ MEV}$.

Before we state the order of magnitude of the experimental background, let us just notice that any rise due to the pole in the denominator is obliterated in the physical region by the vanishing character of the pseudovector piece of the matrix element as the momentum of the virtual pion tends to zero. Thus, even if the background were to be lowered quite a bit below the cross section for the process, it is very unlikely that one could separate out
that part of the cross section due to the diagram we have estimated, and so the outlook is very discouraging.

The background quoted from the synchrotron is of the order of \(2 \times 10^{-31}\) cm\(^2\)/MeV, and so we would not expect the contribution from diagrams containing a pole to be observable because even our fantastically favorable estimate comes out less than the background.

One reason is that there is so much complication among the particles that it is felt that there must be some machinery which generates this complexity, concealed somewhere, say at small distances. And it's a sort of intuitive feeling that if there is some sort of machinery which governs the mass spectrum and assigns masses, it should have some visible manifestation other than simply assigning those properties to the particles; it ought also to affect cross-sections, decay rates, etc. In other words, if we have to introduce new physical ideas in order to derive the basic properties of the particles, these new physical ideas should also influence the formulas somehow. However, although this sounds convincing, it is obviously not an argument free of loophole.

The other reason that people feel that field theory breaks down, is that they believe it doesn't mean anything because it is internally inconsistent. These people in turn fall into two groups: some believe that the theories are and because the self-energies are infinite, while the others, who are willing to forget the necessary renormalizations, claim that the theories, even with renormalizations, are inconsistent. It is in the last set of questions that this lecture will be aimed.

Although it is not necessary to do so, these questions will be discussed with reference to a particular quantity in a particular renormalizable theory. For convenience, let us consider quantum electrodynamics, and in particular, the relation between the bare coupling constant and the real coupling constant. We have shown by a physical argument, and also a mathematical argument, that \(\lambda = \frac{\alpha}{\sqrt{\pi} \alpha_0} < 1\). And if you calculate it, to the lowest order:

\[
\sum_{i=1}^{\infty} \frac{\log \alpha_0^2 + \ldots}{N^2} = \frac{1}{\alpha_0^2} + \frac{1}{\alpha_0^4} + \frac{1}{\alpha_0^6} + \ldots
\]

\[
\frac{1}{\alpha_0^2} + \frac{1}{\alpha_0^4} + \frac{1}{\alpha_0^6} + \ldots = \frac{1}{\alpha_0^2} \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \frac{1}{\alpha_0^2}
\]

Thus, the observable coupling constant is smaller than the bare coupling constant, which is consistent with the experimental data.
No. 13: The Character of Field Theories

We have often mentioned in passing some of the conjectured features of present-day field theories. In so doing, the conclusion that they are wrong physically, below some distance or above some energy, has arisen many times, although at present there is no real evidence for this conclusion. In electrodynamics, the formulae seem to agree perfectly with experiment; in meson theory, we do not know what the formulae are, and to the extent that we do, i.e., by means of the dispersion relations, there again appears to be good agreement with experiment. So we have no compelling reason to believe that field theory breaks down anywhere; it's only that we think it ought to. Why do we think that?

One reason is that there is so much complication among the particles that it is felt that there must be some machinery which generates this complexity, concealed somewhere, say at small distances. And it's a sort of intuitive feeling that if there is some sort of machinery which governs the mass spectrum and spin assignments, it should have some visible manifestation other than simply assigning these properties to the particles; it ought also to affect cross-sections, decay rates, etc. In other words, if we have to introduce new physical ideas in order to derive the basic properties of the particles, these new physical ideas should also influence the formulae somehow. However, although this sounds convincing, it is obviously not an argument free of loopholes.

The other reason that people feel that field theory breaks down, is that they believe it doesn't mean anything because it is internally inconsistent. These people in turn fall into two groups: some believe that the theories are bad because the self-energies are infinite; while the others, who are willing to forget the necessary renormalizations, claim that the theories, even with renormalizations, are inconsistent. It is to the last set of questions that this lecture will be devoted.

Although it is not necessary to do so, these questions will be discussed with reference to a particular quantity in a particular renormalizable theory. For convenience, let us consider quantum electrodynamics, and in particular, the relation between the bare coupling constant and the real coupling constant. We have shown by a physical argument, and also a mathematical argument, that $Z_j = \frac{\alpha^2}{\alpha_0^2} < 1$. And if you calculate it, to the lowest order:

$$Z_j = 1 - \frac{\alpha^2}{12 \pi^2} \log \frac{\alpha}{\pi} + \ldots .$$
To pursue this discussion, we put out of our minds any question of whether the theory agrees with experiment; we concern ourselves now only with the mathematical properties of the theory. If electrodynamics does go wrong of itself, it does at such small distances that we could never find this out experimentally, and thus we would have to find this out by a mathematical argument. (We must remember, however, that the corresponding distances in meson theories can be probed experimentally, and that the electro dynamical formulae may be modified at these distances.) We restrict ourselves to the mathematical question: What does the theory say if one takes it perfectly seriously? It is worthwhile to investigate the internal consistency of such a theory even if we believe that the theories are wrong physically because if the present theories fail to make sense logically, we will at least see something we ought to correct when we construct the next theory.

There are three possibilities for the value of \(1/ Z_3\): 

a) \(1/ Z_3 = 1 + \frac{1}{12\pi^2} \log "\omega" + \ldots = + \omega; \)
b) \(1/ Z_3 \) is finite;
c) \(1/ Z_3 \) is negative, or "the theory contains ghost states".

Let us see just what these possibilities mean. For this, we return to our spectral representation of the renormalized complete photon propagator:

\[ D(k^2) = \frac{1}{k^2 - i\epsilon - M^2} \]

where \(1/ Z_3 = 1 + \int_0^\infty \rho(M^2) d(M^2)\). Instead of working with \(D(k^2)\), let us consider

\[ R(k^2) = k^2 D(k^2) = 1 + \int_0^\infty \frac{k^2 \rho(M^2) d(M^2)}{k^2 + M^2} \]

Then \(R(0) = 1\), and \(R(\infty) = 1/ Z_3\).

Ordinarily, \(\rho(M^2)\) is computed in perturbation theory by summing all the possible diagrams to a given order. There is, however, a second way of constructing the propagator which attempts to take into account the damping. Often we split all the diagrams into irreducible diagrams like
and reducible diagrams like

If we sum up only the irreducible diagrams, we get not $R$, but a quantity $A$. However, these are very closely related: $R = 1/(1-A)$.

Now, how do people become convinced of which of the three alternatives for $1/Z_3$ is true? Any conviction that they have is, I think, based partly upon falsehoods; it is impossible to say at the moment which of these three is right.

First of all, suppose we look at the $\rho(M^2)$ in an ordinary perturbation expansion. In the lowest order

$$\rho(M^2) \underset{M \to \infty}{\longrightarrow} \frac{\alpha}{3\pi} \frac{1}{M^2},$$

and thus:

$$R(k^2) \approx 1 + \frac{\alpha}{3\pi} \log \left( \frac{k^2}{4\pi m^2} \right) + \ldots .$$

If you look at this result, and consider it as just the first term in the perturbation expansion, it appears as though the perturbation expansion is pointing to a definite case, case a): $1/Z_3 = R(\infty) = \infty$. As one probes smaller and smaller distances, the charge squared, starting at $e_1^2$, keeps increasing without limit, and at zero distance which is an infinite momentum transfer, it becomes $e_0^2$ which is infinite. Another way to express this is to say if you take two classical charges and measure the force between them, it will be $e_1^2/4\pi m^2$ at large distances, and as the distances get less and less, the force will become more and more than $e_1^2/4\pi m^2$; at zero separation, the singularity will be stronger than $1/r^2$, say like $r^{-2} \log (1/r)$. There's nothing terribly unphysical about this possibility; it simply means that the Coulomb law, including vacuum polarization, which looks like a $1/r$ potential at large distances, does not look like $1/r$ at the origin, but rather something slightly more singular, so that the coefficient of $1/r$ at the origin is $\infty$. There's no violation of the theory in this possibility; it is perfectly consistent with having real positive probabilities in the spectral density function. The theory may perfectly well be of this character. It does mean, of course, that it is not useful to talk about the bare charge; the bare charge is a figment, and we should always work from the low energy - large distance end of the theory. This is how some people become convinced of alternative a).
Now what sort of argument induces some to believe in possibility c)? Those who argue for c) do so by including the damping. For large $k^2$, in the lowest order, $A$ is obviously the same as the lowest order correction to $R(k^2)$, i.e.,

$$A \to \frac{\alpha_1}{3\pi} \log \frac{k^2}{\mu m^2}.$$ 

Thus in lowest order, this damped perturbation theory yields for $R(k^2)$:

$$R(k^2) = \frac{1}{1 - \frac{\alpha_1}{3\pi} \log \frac{k^2}{\mu m^2} + \ldots}.$$ 

For those who believe that this damped perturbation theory gives a better idea of the solution than ordinary perturbation theory, alternative c) is indicated. For unless a cutoff is introduced, $1/Z_3$ becomes negative, and there is a contradiction with the spectral representation of $1/Z_3$ as an integral of a positive probability density. If the theory is written consistently, there must be states of negative probability, with negative cross sections for producing them, etc.; these states are called "ghosts".

And finally if you want to believe alternative b), you can also construct a kind of perturbation theory which leads to conclusion b). It is quite similar to the "magic formula" analysis. To do this we look at the analytic properties of $R$ and $A$. The equation

$$R(k^2) = 1 + \int_0^\infty \frac{k^2 \rho(M^2)}{M^2 + k^2 - i\epsilon} \, d(M^2)$$ 

is a dispersion relation, describing the analytic properties of $R(k^2)$. It says that the function $R$ of the complex variable $k^2$ is 1 at the origin, and is analytic in the $k^2$ plane except for a branch line along the negative real $k^2$ axis, where there is a discontinuity in the function upon passing from the lower half plane to the upper half plane of amount: $-2\pi i k^2 \rho(-k^2)$. Also since $\rho(M^2)$ is positive, for $\text{Im} \ (k^2) \neq 0$, $\text{Im} \ R(k^2) \neq 0$, and thus $R(k^2)$ does not vanish. The sign of $\rho(M^2)$ also guarantees that $R(k^2) \neq 0$ for $k^2$ on the real positive axis. Therefore, $A = 1 - 1/R = (R-1)/R$ is analytic also in the $k^2$ plane except for the cut along the negative real axis. Hence $A$ obeys the same type of dispersion relation that $R$ does:

$$A = \int_0^\infty \frac{k^2 \rho(M^2)}{M^2 + k^2} \, d(M^2).$$ 

We can easily compute $\rho(M^2)$ in terms of known quantities, because $-2\pi i k^2 \rho(-k^2)$
is just the discontinuity in $A$ as we go across the cut, and since $A = 1 - 1/R$ and the average value of $R(k^2)$ on the real axis is real, the discontinuity is:

$$- 2\pi i \frac{k^2 \rho(-k^2)}{|R(k^2)|^2}.$$ 

Therefore,

$$\rho(M^2) = \rho(M^2) \frac{|R(M^2)|^2}{|R(-M^2)|^2}.$$ 

These relations lead to a kind of "magic formula" perturbation calculation of $Z_3$. One calculates $\rho(M^2)$ by ordinary perturbation theory, uses it to calculate $R(k^2)$, and then feeds this back into the dispersion theory relation in order to calculate an improved estimate of $R(k^2)$. If things are done in this way, then the lowest order of perturbation theory looks like case b).

The procedure is simple, and let's carry it through roughly. From our perturbation calculations, we estimate that:

$$\rho(M^2) \approx \frac{\alpha_1}{3\pi} \frac{1}{M^2} \frac{1}{(1 + \frac{\alpha_1}{3\pi} \log \frac{M^2}{\lim M^2})^2}$$

and using the dispersion relation:

$$Z_3 = 1/R(\infty) = 1 - A(\infty) \approx 1 - \frac{1}{1 + \frac{\alpha_1}{3\pi}} \approx \frac{\alpha_1}{3\pi}.$$ 

Thus, in this form of calculating, $\alpha_0 = \alpha_1/Z_3 \approx 3\pi$, a number independent of $\epsilon_1^2$.

(Note well that this is not a demonstration that the theory behaves like this, but merely an example of an approximation scheme which might indicate case b).

Even at this, it is pretty poor, for it is blatantly inconsistent. We have used the divergence of $R(k^2)$ to generate a finite $R(\infty)$; conversely, if we plug the finiteness of $R(k^2)$ into the dispersion relation, we get a divergent expression for $R(\infty)$.) This possibility had been suggested back in 1954, when it was claimed that if case b) occurred, then $\epsilon_0^2$ would be independent of $\epsilon_1^2$.

We might look very briefly at how such an argument was made. The actual argument was highly mathematical, but the physics of it goes something like this. It starts with the assumption that the theory is renormalizable, which is known to be true at least for the perturbation expansion. Physically, the renormalizability means that all the changes at very high frequencies in the theory amount simply to changes in the bare coupling constant relative to the real one, or the bare mass relative to the real mass. Therefore, all scale changes, changes in
the cutoff, in the theory at very high momenta, can be compensated by changes in the coupling constant or the mass. As applied specifically to the coupling constant, it says that the function:

\[ e_1^2 R(k^2/m^2, e_1^2), \]

(which is a sort of running coupling constant), because of the "renormalizability" of the theory, has the property that a scale change in the momenta can be made up for by a change in the coupling constant. That means that \( e_1^2 R(k^2/m^2, e_1^2) \) has the following functional form:

\[ F\left( \frac{k^2}{m^2}, \phi(e_1^2) \right). \]

Hence in case b), \( e_0^2 = R(\infty) \), \( e_1^2 = \phi(\infty) \), and is independent of \( e_1^2 \). This assertion about \( e_0^2 \), which is claimed to be a prediction of the whole theory, is verified by that calculation we performed in second order, (if one forgets the inconsistencies!).

Let us examine now more carefully where we expect significant changes in electrodynamics to be generated just by quantum electrodynamics itself. In other words, how high up in momentum transfer do we have to go before \( R(k^2) \) differs significantly from 1. Obviously no interesting singularities or big changes are going to occur until we get to a region where \( R(k^2) \) begins to get off the ground. We roughly answered this question many times. The corrections in second order of \( e_1^2 \) become of order 1 when

\[ \frac{\alpha_1}{\pi} \log \frac{k^2}{4m^2} \approx 1; \]

that is, when \( k \approx (2m)e^{600} = e^{600} \text{ MEV} \).

Now, is this a good estimate? We have to remember that electrons are not the only charged particles in the world, and that the vacuum polarization due to other types of charged matter must be reckoned for values of \( k^2 \) greater than the square of twice their rest masses. At such tremendous energies, all the charged particles should contribute: muons, pions, kaons, nucleons, the hyperons, and whatever other fundamental charged particles may exist. In addition, we must also say that we don't really know the difference between a fundamental particle and a composite too well, and it should be inquired carefully to what extent pairs of deuterons, uranium nuclei, etc., should be included in the vacuum polarization. We will someday arrive, I think, at a point of view which does separate clearly the particles which are considered fundamental and others such as deuterons, uranium nuclei, Li\(^{++}\) ions, etc., which are not, and which
will make it very plausible that the effect of the fundamental particles should be added in one by one, whereas the oxygen nuclei, etc. should be considered as minor objects arising merely from the interaction between fundamental particles, and should not be included explicitly.

However, given only the fundamental particles, at large momentum transfers the vacuum polarization is much larger than just the electronic contribution. So what might happen if we include all the known charged fundamental particles? To estimate this, let's forget the fact that some are bosons, and just compute

$$\sum n_i \frac{\alpha_i}{\pi} \log \left( \frac{k^2}{\mu_i^2} \right),$$

for $k = 500$ BEV, which is not a completely impossible energy to be obtained in a future accelerator.

Electrons: $n_1 = 1$, $k^2/\mu_1^2 = 10^{12}/4$, $\log (k^2/\mu_1^2) = 26.2$;

Muons: 1, $10^7/1.6$, 15.6;

Pions: 1, $10^7/2.9$, 15.1;

Kaons: 1, $10^6$, 14.1;

Baryons: 4, $10^5/1.6$, 11.1;

and that sum comes out to be of order:

$$\frac{1}{1200} (26 + 16 + 15 + 14 + 11) = 10\%,$$

which is four times the electronic contribution. This accounting is probably not too bad, for what we might have included in the way of more complex states (baryon pair plus five mesons, for example) is likely to be compensated by the decrease in the baryon form factors. At any rate, such a calculation gives substance to the idea that electrodynamics probably changes of itself at not too unreasonable energies.

Such a possibility makes it even more interesting to investigate what kind of a theory quantum electrodynamics is, under the assumption of case b) in which we take the theory completely seriously and suppose that it does give sensible results. If b) is true, we can make some assertions about the behavior of other quantities in electrodynamics. The following argument was first given by Källen about seven years ago, and is the first example of a dispersion theory argument.

Let's consider $\rho(M^2)$, which is the renormalized probability that an electromagnetic disturbance in the vacuum makes a real physical system with a rest mass $M$, and let us notice that $\rho(M^2) \geq \rho^{\gamma}_e(M^2)$, where $\rho^{\gamma}_e(M^2)$ is the renormalized
probability of making an electron-positron pair system of rest mass \( M \) by the electric interaction. (The interaction of a photon with a real electron pair can always be split into an electric interaction \( \gamma \) and a magnetic interaction \( \sigma_{\mu\nu} \), and in the total probabilities these do not interfere.) This latter probability density we can construct by examining the relevant graphs, which are of this form:

![Graph](image)

Recalling the form of the complete vertex, (given on page 136), one can see that

\[
\rho_e^\gamma(M^2) = \rho^{(2)}(M^2) |R(-M^2)F(-M^2)|^2,
\]

and thus that

\[
\rho(M^2) \geq \rho^{(2)}(M^2) |F(-M^2)|^2.
\]

Now, \( Z_3 = 1 - \int_0^\infty \rho(M^2) d(M^2) \), and for large \( M^2 \), \( \rho^{(2)}(M^2) \) goes like \( 1/M^2 \), so that if case b) holds,

\[
F(-M^2) \xrightarrow{M^2 \to \infty} 0
\]

sufficiently rapidly so that

\[
\int_0^\infty \frac{|F(-M^2)|^2}{M^2} d(M^2)
\]

is convergent.

This means that the assumption that case b) holds for field theory, (the same things can be shown for meaon theory as well as electrodynamics), leads to the conclusion that the form factors must go to zero at infinite momentum transfers, which means that the no-subtraction philosophy for the dispersion relations must be correct. The results of model calculations in dispersion theory show that \( F(-M^2) \) does vanish sufficiently rapidly to make that integral converge, and so it appears that a consistent picture may be possible in case b). The kind of functions obtained in these model calculations also have the property that they contain essential singularities at \( M^2 = 0 \), so that if you expand them in powers of \( e^2 \), all the coefficients diverge as functions of \( M^2 \) as \( M^2 \to \infty \), even though the sum of the series goes to zero. For example, a simple function like \( (M^2/\lambda m^2)^{-\alpha} \), which goes to zero for large \( M^2 \), has a power series expansion

\[
1 - \alpha \log(M^2/\lambda m^2) + \frac{\alpha^2}{2} (\log M^2/\lambda m^2)^2 - \ldots
\]

which has many of the features of our perturbation expansions, in that every coefficient diverges as \( M^2 \to \infty \). All of these considerations raise one's hopes that field theory may not be internally inconsistent, and that case b) actually applies.
To begin, let us first review the known particles and then we can discuss the weak interactions with respect to the whole framework of the fundamental particles. The known particles are:

- **Graviton**
- **Photon**
- **Mesons:** 
  - $K^-$
  - $\bar{K}^0$
  - $K^0$
  - $K^+$
  - two isotopic doublets
  - $\pi^-$
  - $\pi^0$
  - $\pi^+$
  - isotopic triplet
- **Baryons:** 
  - $\Xi^-$
  - $\Sigma^0$
  - $\Sigma^+$
  - $\Lambda$
  - $\eta$
  - $n$
  - $p$
  - + antibaryons
- **Leptons:** 
  - $e^-$
  - $\nu$
  - $\mu^-$
  - + antileptons.

Altogether there are 31 of these, and they fall rather nicely into families.

The first three groups are all bosons, and of these the first two are massless and neutral, one of them carrying the gravitational interaction, while the other carrying the electromagnetic interaction. The mesons carry charge, and mediate the strong interactions. The baryons are fermions with spin 1/2, are heavier than the mesons, and all of them possess strong interactions.

We observe, but do not yet understand, that the number of baryons is conserved, as is the number of leptons. Neither do we comprehend why the strong interactions should apply only to the heavy group of fermions. No fermion is observed like the Majorana neutrino, which is its own antiparticle. These so-called elementary particles all have the charge +1, 0, or -1, so there is only one unit of charge. Only the strong interactions possess the peculiar property of charge independence. The electromagnetic interactions possess the remarkable universal property that they occur through a conserved current, so that strong interactions do not disturb the equality of the charge of the baryons and leptons.

All in all, there are a large number of these simple laws of nature concealed in the framework of the particles which we know about, but do not understand why nature must be that way. The point is that this pattern of simple features should prepare us for the existence of similar things among the weak interactions. We should be prepared to find that they exist for some particles, but maybe not for all, that there is perhaps a universal law, with a bare coupling...
constant that might be the same for all the particles involved, and which may or may not be affected by the renormalizations due to strong couplings, and that the interaction might involve something analogous to charge independence, which would give it a certain symmetry.

The above list contains all the fundamental particles that are known at present. An important question is how do we know that there aren't more of them. One of the things that can be said on this point is that the known decay modes of almost all of the known particles account for all the decays up to a fraction of a percent, and thus there are no particles coupled in among the particles we observe. There could be, of course, a lot of heavy particles not interacting strongly, and these would not have been detected. The question arises as to how we would ever observe them. Well, if they were coupled to the photon, they should be produced in pairs by a high energy machine, but the evidence on pair production of heavy particles is not very good yet.

Goldhaber has suggested that there may be heavy baryons corresponding to the baryons just as the muon corresponds to the electron. The muon and the electron, unlike for example two of the known baryons, differ in nothing except their masses. They have exactly the same strong interactions, namely none, exactly the same electromagnetic interaction, and identical weak interactions as far as we know. Yet the muon is 207 times as heavy as the electron. If analogous brothers exist for the baryons which do not interact with the same photon or mesons, then we shall never see them at all. But if they do interact, then they should be made in pairs in the large machines and people should be on the lookout for them. They might share the weak interactions with the known particles, in which case they would decay into known particles. What we do know is only that there are no other particles which the known particles decay into; this is the strongest statement that can be made concerning the existence of more particles.

There also could be baryons with strangeness -3 or +1 which could easily have escaped detection. Indeed there could be many baryons with the same strangeness and strong and electromagnetic interactions. These would be metastable, like the \( Z^0 \), decaying very rapidly.

Nobody can say that there are not additional mesons, for instance, with strangeness \( \pm 2 \), which would connect nucleon to the \( Z \). And again, further metastable mesons with strangeness \( \pm 1 \), or 0 could exist, which would decay very rapidly into the \( K \)'s or \( \pi \)'s.
Further possibilities: The lepton family may not be so "lept". We have no good reason to believe that there are not heavier members of the electron-muon group. Indeed there could be many of these, so that the mass spectrum of the leptons might overlap that of the baryons. But there is an indication that there is no lepton between the muon and K. If there were, the K should decay into it, and this is not observed. Only in this limited way have we searched for more leptons. Another way to look would be by examining for pair-production. This is not terribly practical, however, at the moment; the muon pair production has been seen, but just barely. At Stanford, part of the range from 0 up to the muon has been scanned for charged leptons, and none has been found. For neutral leptons there isn't much to test. If the weak interactions couple them in, they will show up in decay schemes; otherwise, they will never be detected.

The possibilities that remain among rapidly decaying particles of all kinds are tremendous. To detect these is just a question of looking for striking angular correlations among the known produced particles which indicate sort of isobaric states. For instance, two pions might form such a state and it might be called a particle, but it would be obscure whether to call it so or not. The $J = 3/2$, $I = 3/2$ nucleon isobar might be called a short-lived particle, or just a resonance. The reason we think of it as a compound system and not a short-lived fundamental particle is that we assume that it is possible to take a theory without it, like the pion-nucleon static model we studied, and predict its existence as a compound. So there seems to be no need to put it into our theory as an entity to begin with. This is really the only criterion for separating things that we have at the moment, and it is not clear how much of a criterion it really is. There remains the possibility that there are such isobaric structures which do not come out of our equations, and then we would have the choice of calling them resonances and saying that our equations are wrong, or of calling them fundamental particles, whose existence must be incorporated into a theory from the beginning. Since the equations of field theory are so rich that we cannot solve them, it is not understood what the distinction may be of a short-lived particle from a resonance. It is a very interesting theoretical problem to find a language in which the distinction can be clearly drawn, or to set up a theory in which such a distinction doesn't matter.

The form of gravity as we understand it -- we don't understand it very well, but we have a perfectly good theory of it, Einstein's theory of general relativity -- which is so intimately tied to the equivalence between inertia and gravitation, very strongly suggests that everything gravitates in exactly
the same way, and very strongly rejects any suggestion that some of the particles don't gravitate, or that antimatter has antigravity or similar ideas. These are totally inconsistent with anything we know about gravity; they could be right, but if so they would require a total throwing out of everything we know, and replacing it by some different theory of gravity.

How do we unify gravitation with the rest of all this stuff? Einstein spent the last thirty years of his life by saying that most of the interactions did not exist, except for the electromagnetic interaction, which he included by treating photons classically, and by trying to incorporate this classical electromagnetism into the same picture he had built up for gravity, which was that the geometry was basic. He had shown that a correct theory of gravity was geometric in character, and so he concluded that all of physics ought to be geometric in character. He therefore tried to incorporate electromagnetism into a more general geometry with more dimensions, complex variables, etc. That's one way to try to unify things; however, it leaves you in the dark as to how you are going to bring in all the other interactions. Another, and probably a more fruitful, approach to the problem of unification would be to say that maybe gravitation fits into the pattern of the other interactions. One would start with a quantum mechanical graviton, write down its field equations, and then notice in the classical limit one obtains gravity, just as when one writes down the equations for the photon, one gets electromagnetism in the classical limit. One would use a boson with spin 2, since spin 0 does not give agreement with what we know about the way mass gravitates, and spin 1 gives a repulsion between like objects. If the equations for a massless spin 2 particle are constructed -- they are perfectly reasonable but a little complicated -- one then tries to couple the graviton to matter. Because it's a tensor field, it is to be coupled with a tensor, which is conserved so that the supplementary conditions can be obeyed. The obvious candidate is the symmetrized S-M-E tensor, but when the graviton is coupled in this way, you notice that the equation is nonsense, because once the coupling is put in the S-M-E tensor isn't conserved anymore. So you add in a correction term to fix it up, and it does so to one higher order. The process can be repeated to yield a suitable interaction as a series of terms more and more non-linear, and if you look at it in the classical limit, it yields the Einstein theory. This was pointed out by Gupta in the Physical Review some years ago, and is indeed a very reasonable approach to gravity.
This does not mean that we can unify gravitation with all the other things, but at least we can talk about them at the same time. Of course, gravity is different from all the other interactions, because the theory is the only one with a geometric interpretation. Once we write down the theory of gravitation, and get Einstein's equation, it is then obvious that gravitation has this geometric interpretation in terms of curved space, and so on. What happens is that there is a gauge group, like for electromagnetism, and that gauge transformations correspond to changing the coordinate system. Thus gravity retains its peculiar character, but does fit into the scheme of the other interactions.

Unfortunately, the quantum mechanical nature of gravity cannot be tested. Gravity is so weak -- $GM^2/\hbar c \approx 10^{-39}$ -- that no experiment can be designed to detect the radiation of gravitons. In fact, even the classical gravitational waves have never been detected. Since there is no hope to test the quantum mechanical nature of gravity, most people don't like to talk about it. On the other hand, it seems to me that in the absence of any knowledge, the simplest thing is to assume gravity is quantized like everything else. Then the problem is unified into that of understanding the pattern of particles and interactions, which underlies all of physics.

February 24, 1959

Last time, some of the patterns among the particles and interactions were presented -- all of them completely mysterious -- to prepare us for the notion that there may be patterns among the weak interactions. The first pattern was the universal law of gravitation. Everything gravitates with a coupling proportional to the mass. Then there is the universal coupling of electromagnetism, and there seems to be some kind of a simple Yukawa coupling yielding the strong interactions. Now what kind of patterns do the weak interactions present?

The first weak interaction to be investigated was $\beta$-decay. A neutron, free or bound in some nuclei, decays into a proton, an electron, and, it was early discovered, some other object. This other object was assumed to exist in order to fix up the conservation laws of energy, momentum, spin, and statistics. Although it had to be a spin 1/2 particle, it was not known for many years whether it was distinct from its antiparticle or not. It is known now that it is very distinct, and the particle occurring in the $\beta$-decay of the neutron is defined to be the antineutrino, so that

\[ n \rightarrow p + e^- + \bar{\nu}. \]
From general symmetries, it then follows that when a proton decays in a nucleus, a neutrino is created:

\[ p \rightarrow n + e^+ + \nu. \]

A theory of these processes was first constructed by Fermi. He had been studying quantum electrodynamics, and proposed to describe \( \beta \)-decay in a similar fashion by writing down fields for all the particles involved, and inventing a suitable coupling. Since he had electrodynamics so much in mind he wrote down a vector interaction, and thus proposed a coupling of the form:

\[ \left( \bar{\psi}_p \gamma_\lambda \psi_n \right) \left( \bar{\psi}_e \gamma_\lambda \gamma_5 \psi_\nu \right) + H.A. \]

Actually, he didn't do it quite in this way; in those days people employed the Dirac matrices \( \gamma^a \) and \( \gamma^5 \), and wrote things out in components. Fermi wrote down his interaction as a bilinear form in terms of all the field components, and in doing so he ended up with different conventions for the leptonic \( \bar{\psi}_i \)'s and for the nucleonic \( \bar{\psi}_n \)'s, so that he actually wrote down a vector term for the nucleons, and an axial vector term for the leptons:

\[ \left( \bar{\psi}_p \gamma_\lambda \psi_n \right) \left( \bar{\psi}_e \gamma_\lambda \gamma_5 \psi_\nu \right) + H.A. \]

It doesn't matter, of course, whether one has a 1 or a \( \gamma_5 \) in that position, because the neutrino wave function, representing a massless particle, satisfies the same equation as \( \gamma_5 \) times the neutrino wave function:

\[ \psi \gamma_5 = 0 \implies \psi (\gamma_5 \gamma_5) = 0. \]

Therefore one cannot tell the difference between an interaction with vector dotted into vector and an interaction with vector dotted into axial vector. But it is curious that the first interaction written down was formally parity non-conserving as well as vector.

The addition of the Lagrangian density

\[ \mathcal{L}_\beta = G \left( \bar{\psi}_p \gamma_\lambda \psi_n \right) \left( \bar{\psi}_e \gamma_\lambda \gamma_5 \psi_\nu \right) + H.A. \]

to the Lagrangian density for the rest of the world would bring in \( \beta \)-decay.

At Fermi's time, the only thing really worth comparing the theory to was the spectrum of Ra E, which fitted it very well. This was a trifle fortuitous, because subsequently the Ra E spectrum was one of the hardest to understand; it was not really resolved until just a few months ago! So it was a combination of luck and bad experiments which made the theory work. The main thing, though, was that with the Fermi theory, one gets a characteristic feature of any quantum mechanical theory, which is the density of states factor, and this is
principally what determines $\beta$-ray spectra in the lowest approximation. Fermi's theory was correct in that it contains no gradients, and thus the shape of the spectra does come mostly from the density of states. Gradients could have occurred in the interaction, in which case the spectra would be quite different. Indeed, for a while in the 1930's the experiments strongly suggested that gradients were needed, but that was due to poor measurements with thick sources, and was finally cleared up about 1949 by C. S. Wu.

Now Fermi himself was not sure whether the interaction given above was a fundamental interaction, or whether the theory might be regarded as sort of phenomenological. This question is still with us in exactly the same form. The interaction amounts to a four-fermion point interaction by means of a $\delta$-function potential, between a nucleon and a lepton. Thus the interaction is like

$$V = G \delta(r_N - r_L).$$

Essentially therefore, since one always works in first Born approximation, the theory amounts to a specification of a scattering length, or of $\int V(r) \, d^3r$. Now if you examine the theory from this point of view, it is clear that no commitment need be made as to whether $V(r)$ really is a $\delta$-function, but only that the experiments concern distances much larger than the structure of the potential. As we study the weak interactions in relation to all the other particles, in which higher momentum transfers become involved, we may find a structure of non-zero extent. Nothing yet, however, has convinced us that the potential is more elaborate than a $\delta$-function. However, we can speculate about it, and later on we shall discuss a deeper meaning for the interaction: that it may be the scattering length limit of some interaction with more structure. We do not know whether there is any meaning in this idea or not.

We can continue to work with the point contact interaction just as it was formulated originally, except for some slight variation in the form of the coupling, and use it to understand $\beta$-decay. We shall do this later on; now let us survey some of the other weak interactions.

The muon, which was thought for many years to be the Yukawa meson, was shown not to be when, after the war, it was stopped and didn't interact strongly with matter. The problem then was what the muon was and how it interacted. It was soon discovered that it had two types of interaction: 1) it decayed into an electron plus neutral objects in about 2 microseconds; and 2) in sufficiently large nuclei it would destroy itself, but in light nuclei it wouldn't, thus indicating an extremely weak absorption interaction. It
was established that the muon decays into an electron and two very light neutrals, presumably neutrinos. It is now known that one of them is a neutrino and the other an antineutrino, so that

\[ \mu^+ \rightarrow e^- + \nu + \bar{\nu}. \]

Since the absorption predominates only for large values of \( Z \) (say greater than 11 or so), the absorption interaction was clearly as weak as the decay.

\[ \mu^- + p \rightarrow n + \nu. \]

This picture was clarified around 1947-48, when about that time a lot of people, the first of which was Puppi, suggested that the three processes could be considered as being analogous, and had a number of common features. They all involve four fermions, two charged ones and two neutral ones, and if you wrote them all as Fermi interactions, the G's were all the same. We shall see this when we calculate with the theory, from the absorption rate for muons, the decay of the muon, and \( \beta \)-decay. Nowadays the question of equality is a matter of a percent or so; then it was clear only that the equality was within a factor of two or four. But when the G's differ in so many orders of magnitude from other physical parameters, it was a remarkable fact that they were the same to within say a factor of four. This is the genesis of the idea of a universal Fermi interaction, that these three processes, and perhaps others, are tied together by a common origin, which corresponded to some sort of law of a universal weak interaction, just like the universal law of electromagnetism, and the universal law of gravitation. Everything that we have learned since tends to bear this out.

Puppi proposed a triangle to describe this universal Fermi interaction:

\[
\begin{array}{c}
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\\ \downarrow
\\ \nu
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This is just an example, and doesn't necessarily work as a theory of the weak interactions, but qualitatively it is capable of explaining everything here. For example, we can understand directly the decay modes since that time the experiments have become more and more accurate, and the theory has been worked out in greater detail, and this system gets better and better all the time.

Before we start with the mathematics, let us explore briefly what the more general pattern of the weak interactions might be. Since the discovery of the strange particles, many more examples of weak interactions are known.
besides the three "classical" ones. The strange particles were called "strange" because, although they are produced in great abundance, they take a fantastically long time to decay, $10^{13}$ on the nuclear time scale. This was understood when it was realized that the strong and electromagnetic interactions would respect the strangeness number. If the weak interactions didn't, then the weak interactions would lead to strange particle decays. The strange particles would be produced two at a time by the strong interactions. This meant, however, that every single strange particle decay mode had to be explained by the weak interactions, except for $\Sigma^0 \rightarrow \Lambda + \gamma$, so that the list of weak interactions grew tremendously. For example, a few of the new ones are:

\[
\begin{align*}
K^+ &\rightarrow \mu^+ + \nu \\
K^+ &\rightarrow \mu^+ + \pi^0 + \nu \\
K^+ &\rightarrow \pi^+ + \pi^0 \\
K^+ &\rightarrow \pi^+ + \pi^- + \pi^+ \\
\Lambda &\rightarrow n + \pi^0 \\
\Lambda &\rightarrow p + \pi^- \\
\Lambda &\rightarrow p + e^- + \bar{\nu} \\
\end{align*}
\]

Could the Puppi triangle be modified so as to simply encompass all the new weak interactions? An example of how that might be done would be to convert the triangle into a tetrahedron by adding a $\Delta p$ vertex, which by the strong interactions is equivalent to a $K$ vertex, just as the $np$ vertex is equivalent to a $\pi$ vertex:

This is just an example, and doesn't necessarily work as a theory of the weak interactions, but qualitatively it is absolutely capable of explaining everything here. For example, we can understand directly the decay modes $\Lambda \rightarrow p + \mu^- + \nu$, and $\Lambda \rightarrow p + e^- + \bar{\nu}$. The $K$ decays and the $\pi$ decays can be understood as involving both a strong interaction and a weak interaction, for example:

\[
\begin{align*}
K^+_s \rightarrow p + \overline{\Lambda}_w \rightarrow \mu^+ + \nu \\
K^+_s \rightarrow p + \overline{\Lambda}_w \rightarrow p + \pi^- \rightarrow \pi^+ + \pi^0
\end{align*}
\]
\[
\pi^+ \rightarrow p + \bar{n} \rightarrow \mu^+ + \nu_w
\]
as can some of the \(\Lambda\) modes and \(\Sigma\) modes:
\[
\begin{align*}
\Lambda & \rightarrow n + p + \bar{p} \rightarrow n + \pi^0_w, \\
\Sigma^+ & \rightarrow \Lambda^+ + \pi^+ \rightarrow n + p + \bar{p} + \pi^+ \rightarrow p + \pi^0_w.
\end{align*}
\]

Thus in order to understand this tremendous number of new processes connected with the weak interactions, we do not necessarily have to violate grossly the idea of simplicity which we get from the Puppi triangle. They can possibly be interpreted in a relatively simple way. When we get to the details, though, of understanding the rates, and the angular correlations, polarizations, etc. of all of these reactions, the subject becomes much more complex and is not fully resolved. So the qualitative features of the strange particle decays can be understood in terms of the same sort of interaction as for the \(\mu\nu\) and \(e\nu\) and \(np\), but one gets into quantitative complications we will have to discuss at length.

In the meantime, let us look further at the weak interactions given by the Puppi triangle. As was mentioned, the pion decay can be understood qualitatively in this simple pattern of couplings by the mechanism:
\[
\begin{align*}
\pi^+ & \rightarrow p + \bar{n} \rightarrow \mu^+ + \nu \quad \text{p}\nonumber \\
\pi^+ & \rightarrow p + \bar{n} \rightarrow e^+ + \nu.
\end{align*}
\]

The second process should occur since the first does, and it has now been observed to do so at a branching ratio predicted by the theory. There are two difficulties in calculating such decays, however. One difficulty is that to describe the process by this kind of a diagram

is wrong because of the strong couplings. There are lots of other important diagrams also, for example:

So with the strong interactions involved, it is very hard to calculate a rate.
Secondly, there is no reason to believe that the loop could be only a np loop. There could also be a $\Sigma^+\Sigma^0$ or a $\Sigma^+\Delta$ loop involved. Insofar as these complications are forgotten, the $\pi$ decay is essentially understood. As we shall see the ratio of the $\pi^-\mu$ decay to the $\pi^-=e$ decay can be calculated since it is independent of such complications, and this ratio agrees very well now with experiment.

Another thing which is very beautiful and which pertains just to the Puppi triangle is the possibility of inverse $\beta$-decay, which led to the direct detection of the antineutrino: $\bar{\nu} + p \rightarrow n + e^+$. The antineutrino is made when a neutron decays, and in an atomic pile, quite a huge flux of antineutrinos are generated, but very few neutrinos. These antineutrinos are of such an intensity that one could hope to detect them. The cross-section comes out to be of the order $10^{-20}$ barns, but with the flux available it turns out not to be impossible to detect inverse $\beta$-decay. Reines and Cowan, working at various government installations, used a coincidence technique where the neutrons were detected via the gamma rays emitted when they are captured, and the positrons were detected by means of their annihilation gamma rays. By using tank cars full of scintillator material, they finally observed the process in about the right amount. The matter was complicated, however, by the fact that their results were off by a factor of six or so, but this can be traced to the crucial problem of correctly calculating the flux and spectrum of antineutrinos from a pile, which involves a lot of pile engineering. For this reason, such an experiment is not a very accurate quantitative test of a theory. But it does show that the neutrino is distinct from the antineutrino, for the process $\bar{\nu} + n \rightarrow p + e^-$ is not found, although the results are very poor due to the difficulty of doing the experiment.

And finally we can interchange the particles in another way to discuss the process $e^- + p \rightarrow n + \nu$, which is electron capture, which is an alternative form of $\beta$-decay occurring in some nuclei. These reactions are all part of the same problem, and are predicted by a simple coupling Lagrangian density.
We shall now take up some of the details of the weak interactions. The approach will be pseudo-historical; we shall rewrite history so that it makes sense!

After Fermi proposed his vector theory of $\beta$-decay and after the rain of theories with gradients had subsided, people considered the most general combination of Lorentz invariant, parity conserving, four-fermion point interactions, which did not involve the momenta. As we know, there are five possible types, and thus the most general interaction of that nature is:

$$\mathcal{H}_{\text{int}} = G_V \{ (\bar{\nu}_n)(\bar{\nu}_n) \} + \text{H.A.}$$

$$+ (-G_A) \{ (\bar{\nu}_n)(\bar{\nu}_n) \} + \text{H.A.}$$

$$+ G_S \{ (\bar{p} n) (\bar{e} \nu) \} + \text{H.A.}$$

$$+ G_P \{ (\bar{\nu}_n)(\bar{\nu}_n) \} + \text{H.A.}$$

$$+ \frac{1}{2} G_T \{ (\bar{p} \sigma_{\mu\nu} n)(\bar{e} \sigma_{\mu\nu} \nu) \} + \text{H.A.}$$

where we employ the convenient shorthand of $\bar{n}$ instead of $\bar{\nu}_n$, etc. Then the question was to determine these constants for the $\beta$-decay interaction, assuming the interaction is of this form.

For that purpose, we can simplify matters by making the non-relativistic approximation for the nucleons, because the nucleons in nuclei are moving with a speed which is perhaps $c/10$, and not anywhere near $c$. And so the relativistic part of the interaction as far as the nucleon is concerned can be approximately neglected, providing that the non-relativistic part of the transition matrix element does not vanish. This reduction amounts to throwing out all those parts of the interaction involving nucleonic matrices which are odd. Since $\gamma_5 = i\sigma^5$, the approximate interaction is:

$$p^{+} e^- \{ (G_V \beta' + G_S \lambda') + \sigma^\uparrow (G_A \beta' \sigma^\uparrow \lambda' + G_T \sigma^\uparrow) \} \bar{\nu} + \text{H.A.} \ldots$$

The nucleon spinors have two components, while the lepton spinors remain relativistic; the unprimed operators act on nucleon spinors, while primed operators act between lepton spinors. As far as the nucleons are concerned, the scalar and vector interactions are exactly the same in the non-relativistic limit, as are the axial vector and tensor. The difference comes only in the leptonic part of the interaction. This is the general interaction we are left with for discussing the allowed transitions.
Let us consider a nuclear transition. To do so we have to employ some sort of model for a nucleus, and we take the nucleus to be described by a many-particle wave function. Actually, as we know, a nucleus is much more complicated because of the meson cloud, but roughly we can look at it as a system of particles moving about more or less non-relativistically according to a wave function. With such a model, we wish to construct the R-matrix to describe the transition. This is fairly simple because the interaction is a point interaction, and thus the R-matrix must be:

$$\begin{align*}
\langle & \text{final nucleus; lepton state} \mid \Sigma \tau_r \left\{ \Gamma \left( G_{\gamma} \beta + G_{\gamma} \alpha \right) + \tilde{\alpha}_r \left( G_{\beta} \bar{\beta} + G_{\beta} \bar{\alpha} \right) \right\} \mid \text{initial lepton; nucleus state} \rangle \\
\end{align*}$$

where $\tau_r$ is the isotopic spin operator for the $r$th nucleon, which takes a neutron into a proton, or vice versa, depending on whether we are considering electron or positron decay. If the coulomb corrections to the final state wave functions are neglected, the R-matrix element reduces to the following form:

$$\begin{align*}
\Sigma \left[ \langle \tilde{\epsilon} | \tau_r \left( G_{\gamma} \beta + G_{\gamma} \alpha \right) \left( G_{\beta} \bar{\beta} + G_{\beta} \bar{\alpha} \right) \right] \langle \tilde{\epsilon} | \tau_r \left( G_{\beta} \bar{\beta} + G_{\beta} \bar{\alpha} \right) \left( G_{\gamma} \beta + G_{\gamma} \alpha \right) \rangle
\end{align*}$$

where $\tilde{k}$ is the total spatial momentum of the outgoing leptons. This shouldn't surprise us, for it is exactly analogous to what we deal with in the emission of light.

As in the case of light, we face the problem of what to do with the retardation factors $e^{-ik \cdot \tilde{x}}$, and the solution is the same. The factor $k\tilde{x}$ is small -- $k$ ranges up to say 5 MEV for most nuclei, and the nuclear distances are of the order $1/50$ (MEV)$^{-1}$, and so we may neglect the retardation in most cases providing that the resulting term does not vanish, just as we neglected the terms in $v/c$ providing we got something. An allowed transition is one that can go if $v/c$ and $k\tilde{x}$ are both neglected. Allowed transitions are the only ones that can be analyzed simply. The forbidden ones are the ones where powers of $v/c$ and/or $k\tilde{x}$ must be included in order to get a non-vanishing matrix element. Since $v/c$ and $k\tilde{x}$ are both roughly of the same order, people speak of $n$th order forbidden transitions when the first non-vanishing matrix element involves $n$ powers of $v/c$ and $k\tilde{x}$ in any mixture.

We shall deal with only the allowed transitions. In this case, the V and S transitions involve only the nuclear matrix element of the + or - component of the total isotopic spin. These two types of transitions are called Fermi transitions; whereas the A and T couplings are named after Gamow and
Teller, who suggested that the situation might be more complicated than just vector. The G-T nuclear matrix elements for the allowed transitions are those of $\sum \frac{1}{r} \vec{G}_T$, a more complicated operator, for which simple statements cannot be made. But you can say what its selection rules are: the operator carries a spin and isospin of 1 unit, and thus the "allowed" G-T selection rules are

$$\begin{align*}
\Delta I = 0, \pm 1 & \quad 0 \not\rightarrow 0; \\
G-T: & \quad \Delta J = 0, \pm 1 \quad 0 \not\rightarrow 0.
\end{align*}$$

No (i.e. initial nuclear parity same as final, no change).

The allowed Fermi rules are much simpler, within the accuracy of charge independence, you can go only to the "brother" state; i.e.

$$\begin{align*}
\Delta I = 0 & \quad 0 \not\rightarrow 0; \\
F: & \quad \Delta J = 0; \\
No.
\end{align*}$$

The $\beta$-decay experimentalists had the task of determining the interaction by seeing which of the four types had to be present and in what amount. (They could not detect a F interaction except by examining forbidden transitions.) They could first of all solve the problem of how much Fermi and how much Gamow-Teller, and then they could find out what mixture of $V$ and $S$ was the Fermi, and what mixture of $A$ and $T$ was the Gamow-Teller. The problem of finding how much $F$ and how much $G-T$ there is in a transition is affected by the consideration that, whereas the $F$ matrix element is very easily evaluated neglecting electromagnetic effects, the nuclear G-T matrix element isn't known because it involves the spin wave function of the nucleus, which can be very complicated. To decide how much G-T must be present, the most rational thing that could be done would be to look at the only case for which the matrix element is known exactly, the neutron. The lifetime of the neutron involves both G-T coupling constants and F coupling constants, and, in fact, is inversely proportional to $(|G_V|^2 + |G_S|^2 + 3(|G_T|^2 + |G_A|^2))$, since $V, S, A, T$ do not interfere in the total rate. The neutron decay rate thus tells us $F + 3 \ G-T$, but it doesn't tell us either one separately. On the other hand, we can easily find $F$ by looking at $J=0 \rightarrow J=0$ transitions, between brother states, for which there is no G-T contribution.
A typical example is $^{0+}_{14} \rightarrow N^{14+} + e^+ + \nu$. The situation looks like this:

$$\begin{align*}
5.1 \text{ MEV} & \quad o^+ \quad I = 1 \\
2.3 \text{ MEV} & \quad o^+ \quad I = 1
\end{align*}$$

The three members of the $I = 1$ multiplet do not have the same energy because of the electromagnetic effects. If the n-p mass difference and coulomb energy is subtracted out, the levels have the same energy. Apart from small corrections to the wave function due to electromagnetism, the evaluation of the Fermi nuclear matrix element is quite trivial:

$$\langle I = 1, I_z = 0 | \langle \alpha | I = 1, I_z = 1 \rangle \rangle = \sqrt{2}.$$  

Therefore, the decay rate comes out to be a known number times 

$$\left( |g_v|^2 + |g_S|^2 \right) = F.$$  

This $0 \rightarrow 0$ transition has the nice feature that it accounts for 99.4\% of the decays. This is a little surprising since the transition to the ground state of $N^{14}$ would appear to be an allowed G-T transition. However, the nuclear matrix element almost vanishes accidentally, a quirk which accounts for the long life of a "brother" state, the ground state of $C^{14}$, which makes $C^{14}$ useful for dating things.

Now that we see the point of studying these two particular decays, let us actually compute the transition rates. In so doing, we can examine some other features of $\beta$-decay, such as the spectrum, and the angular correlation between the electron and neutrino which appears experimentally as an angular correlation between the electron and the nucleus. First we consider $^{0+}_{14}$ as a typical case of a $0 \rightarrow 0$ allowed Fermi transition. The R-matrix element is simply:

$$\sqrt{2m \over 2E} \sqrt{2} \left| u^+_{k} (g_v \beta + g_S) u_{p}^+ \right|,$$

and we want $|R|^2$ summed over the final neutrino and electron spins, which is (see page 85-86):
2 \text{Trace}\left\{ \frac{-i\gamma^5}{2k} (g_V \beta + g_S \gamma^5) \frac{m - i\gamma^5}{2E_p} (g_V \beta + g_S \gamma^5) \right\}

since for neutrinos:

\begin{align*}
\psi_{-k}^\dagger u_k = \psi_{-k}^\dagger v_k = \frac{-i\gamma^5}{2k}.
\end{align*}

Using the rules for traces (given on page 152),

\[ \Sigma |R|^2 = 2 \left\{ (|g_V|^2 + |g_S|^2) + (|g_V|^2 - |g_S|^2)^2 \frac{\gamma^5}{k} - 2 \frac{m}{E_p} \text{Real } g_V g_S^* \right\}, \]

where \( \vec{\beta} = \vec{p}/E_p \), and \( \vec{k} = \frac{k}{\gamma} \).

This formula has two striking features. First of all there is a part which comes from the squares of the coupling constants, and then an interference part. Then there are two isotropic terms, plus a correlation term between the positron and neutrino, which gives zero when we average over neutrino directions, a procedure done explicitly in any experiment where one does not examine the positron-nucleus angular correlation. So the electron spectrum is proportional to

\[ 2 \pi^2 \left\{ (|g_V|^2 + |g_S|^2) - 2 \frac{m}{E_p} \text{Real } g_V g_S^* \right\} \rho(E_f), \]

where \( \rho(E_f) \) is the density of final states. There is a normal spectrum term plus a peculiar spectrum term which, in addition to the density of states factor, has a \( 1/E_p \) energy dependence. This peculiar term is not very important at the upper end of the spectrum in high energy transitions, but when \( E_p \) gets down close to \( m \), the term would show up. Such terms were searched for, and have never shown up. And therefore, people became convinced that \( \text{Real } g_V g_S^* \) vanished, at least to within an accuracy of 10 percent. Now here I must tell you that invariance under time reversal or CP forces the coupling constants to be real. (We shall discuss these invariance properties more fully in a couple of lectures.) Thus, the vanishing of the odd spectral term, called a Fierz term, leads to the conclusion that either \( g_S \) or \( g_V \) is zero. To find out whether the interaction is \( V \) or \( S \), the thing to do was to examine the angular correlation.

If the interaction is vector, the positron and neutrino go off in the same direction preferably, whereas they tend to go off in opposite directions if it is \( S \). Since the interaction is either \( V \) or \( S \), the angular correlation experiment immediately determines the nature of the Fermi transitions, if it is done right! Unfortunately it took some years to get this straightened out, but we're rewriting history, so what happened was that the experiments were quickly performed, and it was immediately obvious that the Fermi interaction
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We shall see when we look at the axial vector and tensor terms that precisely the same situation occurs. There is a normal spectrum involving $G_A^2 + G_T^2$, a correlation term with coefficient $G_A^2 - G_T^2$, and a Fierz term due to $G_A G_T$. Again the experiments showed the absence of a Fierz term, and thus the G-T interaction is either A or T to at least ten percent or so. The angular correlation experiment, once more rewriting history a little bit, demonstrated quickly that it is axial! In this straightforward fashion, the nature of the $\beta$-decay interaction was determined!

Having noticed the two striking facts about the spectrum and the angular correlation, and seen their implications, we can go on and compute the decay rate of $0^{11/2}$, and get $G_T^2$. The transition rate is just:

$$\Gamma = \frac{2\pi^2 G_v^2}{(2\pi)^6} \int d^3p \int d^3k \delta(\Delta - k - \sqrt{p^2 + m^2} - T_{11/2})$$

where $\Delta$ is the maximum positron energy. We shall neglect $T_{11/2}$ the nuclear recoil energy, since it is so tiny. Then

$$\Gamma = 2 \frac{G_v^2 \Delta^5}{\sqrt{2}} \int_0^1 \frac{P^2 dP}{K^2} \int_0^1 K^2 dK \delta(1 - P - K)$$

where $P = p/\Delta$, $K = k/\Delta$, and $F(m/\Delta)$ is a number which is unity when the energy of the transition is large compared to the mass of the electron, because then the mass doesn't count. On the other hand, as the mass becomes a huge fraction of the energy release, $F(m/\Delta)$ goes to zero. $F(m/\Delta)$ represents the fraction of phase space blocked out by the mass of the electron. The integrals are quite easy, and one obtains:

$$2\pi^3 \Gamma = 2 \frac{G_v^2 \Delta^5}{\sqrt{2}} F(m/\Delta).$$

The blocking factor $F(m/\Delta)$ can be found in closed form, and is:

$$F(x) = \sqrt{1 - x^2} \int_0^1 \frac{1-x}{P^2} dP \int_0^1 \frac{1-x}{K^2} dK \delta(1 - \sqrt{P^2 + x^2} - K)$$

where $x = m/\Delta$ is the half life.

We can run through this calculation for $x = 1/4$ and get:

$$F(x) = \sqrt{1 - x^2} \left(1 - \frac{\sqrt{1 - x^2}}{2} x^2 - 4 x^4\right) + \frac{15}{2} x^4 \log(\frac{1 + \sqrt{1 - x^2}}{x})$$

$$F(0) = 1; F(1/4) = .746; F(1/2) = .292; F(3/4) = .0375; F(1) = 0.$$
Near $x=0$, $F(x)=1 - 5x^2 + \ldots$, while near $x=1$, $F(x) = 6.5(1 - x)^{7/2}$.

As mentioned earlier, experiments on nuclei in which the Coulomb field is important showed this to be the case.

We have considered the positron in the final state to be a free particle, whereas actually it is subject to the coulomb field. Therefore we should have used for the positron wave function a negative frequency solution of the Dirac equation for an electron in the field of $N^{11h}$. This introduces a coulomb correction factor into the spectrum which becomes more and more important as the electron's energy gets closer to $m$. To within a few percent, the rate is proportional to $\frac{f}{\Gamma}$, where

$$f(\frac{4\pi}{2}) = \int d^3p \int d^3k \delta(\Delta - k - \sqrt{p^2 + m^2})$$

where $f(Z,F,E_p) \to 1$ as $E_p$ gets very large, or $Z \to 0$. By suitably defining $\frac{f(Z,F,\Delta/m)}{\Gamma}$, the transition rate for the decay is

$$\Gamma = 2 \frac{G^2}{2m^2} \frac{1}{20} \int d\Omega' f(Z,F,\Delta/m) F(m/\Delta) = 2 \frac{G^2}{2m^3} \frac{1}{20} f m^5$$

Now the $\beta$-ray spectroscopists quote as their result the $ft$ value for a given transition, where $f$ is as defined above, and $t = \frac{\log 2}{\Gamma}$ is the half life.

From $\frac{2m^3 \log 2}{ft m^5} = 2 \frac{G^2}{2m^2}$, and the measured $ft$ value of $3088 \pm 56$ sec. for this transition in $^0{\text{H}}$, we obtain

$$|G_p| = \frac{(1.01 \pm .01) \times 10^{-5}}{m^2 \text{proton}}$$

We can run through the same calculation for $n \rightarrow p + e^- + \bar{\nu}$:

$$\Sigma |R|^2 = \left[ (G_\nu^2 + G_s^2) + (G_\nu^2 - G_s^2) \frac{2m}{E_p} G_s G_s \right] + \text{Trace} \left[ \left( \frac{m - i\gamma \mu}{E_p} \right) (G_\nu^{1/2} G_\nu^{1/2} + G_s^{1/2} G_s^{1/2}) - \frac{i\gamma \mu}{2k} (G_\nu^{1/2} G_\nu^{1/2} + G_s^{1/2} G_s^{1/2}) \right]$$
since the Fermi and Gamow-Teller parts of the matrix elements do not interfere when we perform the sum over the nuclear spins. The only change in the Fermi part, besides the absence of a factor of 2 from the $0^{+}$ Fermi part, is the reversal of the sign of $m$, because in this case we employ the projection operator for the positive frequency states of an electron. The evaluation of the trace is a straightforward exercise for the reader, the result of which is:

$$\Sigma |R|^2 = [(G_V^2 + G_S^2) + (G_A^2 - G_T^2)\beta \cdot k + \frac{2m_p}{E_p} G_V G_S]$$

$$+ 3 \left[ (G_A^2 + G_T^2) - \frac{1}{3} (G_A^2 - G_T^2) \beta \cdot k + \frac{2m_p}{E_p} G_A G_T \right]$$

As mentioned earlier, experiments on nuclei in which the G-T is important showed no Fierz terms, so that the G-T interaction is either axial or tensor, and then by looking at the angular correlation, especially in He$^6$, the conclusion was that it is axial. Putting in this information, the rate for neutron decay is proportional to $G_V^2 + 3 G_A^2$ as compared to $2 G_V^2$ for $0^{+}$. The neutron decay rate is therefore

$$\Gamma = (G_V^2 + 3 G_A^2) \frac{1}{2m^3} \frac{A^5}{30} \int (1, \Delta/m) F(m/\Delta),$$

and the experimental ft value gives:

$$\left| \frac{G_A}{G_V} \right| = 1.19 \pm 0.04.$$

So these classical nuclear physics experiments were able to prove that the $\beta$-decay interaction was vector and axial vector, with at most small admistures of S and T, if any. (There is no sign of any admixture at all today.) They could also show that the amounts of V and A were approximately equal.

We can now look over the same material taking into account a separate discovery, which might have followed all this, although actually it didn't, namely that parity isn't conserved. This fact affects nothing of what we have just done, but introduces a set of new experiments which measure parity non-conserving processes.
March 3, 1959

**Non-Conservation of Parity**

So far, we have said nothing about parity non-conservation. What we have said is correct, but incomplete, as we shall see. As is well known, parity non-conservation was first suggested for a different aspect of the problem of the weak interactions, but its consequences for β-decay were soon recognized.

The reason why parity non-conservation did not show up in the β-ray experiments we discussed last time is that these experiments involve only scalar quantities, such as the spectrum shape and the electron-neutrino angular correlation. To demonstrate parity non-conservation, one has to measure a pseudoscalar quantity, for example, the dot product of a spin and a momentum. But in all our calculations, the spins were averaged over. In so doing, anything which depends upon interference between the parity conserving and parity non-conserving parts of an interaction vanishes, and because of the vanishing mass of the neutrino, the contribution from the parity non-conserving part of the interaction alone is exactly the same as that from the parity conserving part alone.

In fact, however, recent historical research has shown that an early experiment on β-rays did detect a consequence of parity non-conservation. In 1928, someone did look at the spins of the β-rays and found that the electrons were polarized to the extent v/c. The importance of such an observation was not recognized, and so the evidence became buried in the mass of experimental data.

The impetus for the suggestion that parity is not conserved in the weak interactions came from the decay of the K particle. Two of the decay modes of the K+ are: \( K^+ \rightarrow \pi^+ + \pi^0 \), and \( K^+ \rightarrow \pi^+ + \pi^- + \pi^+ \). The evidence shows that the K is spinless, because the three pion decay spectrum indicates an overall S state. Therefore, since the pion is pseudoscalar, the two final states are characterized by \( 0^+ \) and \( 0^- \), respectively. These two states were thus assumed to arise from two distinct particles, the \( 0^+ \) and \( 0^- \), and it was thought that all the hyperons would be parity doublets also. It was strongly expected that the lifetimes of the two members of the K parity doublet would be different, because of the widely differing set of decay modes available to each one. However, experiments performed at Berkeley soon showed that the lifetimes were the same, and thus the supposition that the two particles were
one became almost irresistible.

In the Fall of 1956 people realized that no experiments had demonstrated the conservation of parity in these weak interactions. Yang and Lee suggested a list of experiments on β-decay, and on the decays of π⁺ and μ⁺ which would uncover parity non-conserving effects. The conjecture was verified early in 1957 by C. S. Wu and collaborators, who demonstrated a correlation between the spin orientation of Co⁶⁰ and the direction of the emitted electron.

A simple way that parity non-conservation could manifest itself would be through a term \( \vec{\sigma} \cdot \vec{p} \), which correlates the spin of the electron with its direction of motion. Such a term may be obtained if the parity conserving vector interaction

\[
G_V \{ (\vec{p} \gamma_\lambda n)(\vec{e} \gamma_\lambda \nu) \} + \text{H.A.}
\]

is augmented by a parity non-conserving vector interaction

\[
G_V \{ (\vec{p} \gamma_\lambda n)(\vec{e} \gamma_\lambda \gamma_5 \nu) \} + \text{H.A.}
\]

If either of these terms were the only ones present, we could never tell which one it was, since \( \gamma_5 \nu \) satisfies the Dirac question as well as \( \nu \), because \( m_\nu \) is zero. However, the presence of both terms may give rise to asymmetries.

To investigate this, we write the general vector interaction as

\[
G_V \{ (\vec{p} \gamma_\lambda n)(\vec{e} \gamma_\lambda \frac{1 + c \gamma_5}{\sqrt{1 + |c|^2}} \nu) \} + \text{H.A.}
\]

This interaction predicts precisely the same rates and spectrums as the old parity conserving one, because any interference between the two cannot manifest itself when one sums over the spin states. The non-relativistic limit (for the nucleons) of such an interaction is:

\[
G_V \{ (p^+ e^{-i\vec{k} \cdot \vec{x}} n)(\bar{e} \beta \frac{1 + c \gamma_5}{\sqrt{1 + |c|^2}} \nu) \} + \text{H.A.}
\]

Now what we are interested in at the moment is the average over the neutrino directions, and the sum over the neutrino spins, of the lepton part of \( |R|^2 \) for a transition involving the creation of an electron and an antineutrino. That is quite easy to get. First we write the expression including the sum over the neutrino spins:

\[
\frac{m_e}{1 + |c|^2} \bar{u}_e \beta(1 + c \gamma_5) \frac{-ik}{2k} \beta(1 + c \gamma_5)u_e.
\]

Now the electron has two definite spin states: its spin can be along its
momentum vector, or opposed to it. To investigate the transition rate into one of these, we can use the projection operator for it, and then sum over both the spin states of the electron. Using this technique, the lepton part of $\Sigma |R|^2$ is:

$$\frac{1}{1 + |c|^2} \text{Tr} \left\{ \beta (1 + c^* \gamma_5) \frac{1 + \sigma \cdot \vec{P}}{2} \frac{m - ip}{2E_p} \beta (1 + c \gamma_5) \frac{-ik}{2k} \right\}$$

where $\sigma_p = \pm 1$, depending on whether the spin is along or opposed to the direction of motion; and $\hat{p} = \vec{p}/|p|$. First notice that since the trace of an odd number of $\gamma$ matrices is zero, the term linear in $m$ vanishes. Then the evaluation of the trace is easy. A little mental algebra yields:

$$\frac{1}{2} \left[ 1 + \hat{k} \cdot \hat{p} - \sigma_p \frac{2 \text{Re } c}{1 + |c|^2} \frac{1}{4} \text{Tr} \left\{ \gamma_5 \Sigma \cdot k \gamma \right\} \right]$$

where $\hat{p} = \frac{\vec{p}}{E_p}$

and $\gamma = \beta k \gamma$. But $\Sigma \cdot k = -i \beta \gamma \cdot \hat{p}$, and thus we need to know that

$$\text{Tr} \left\{ -i \beta \gamma \cdot \hat{p} \right\} = \text{Re } c (1 + \hat{k} \cdot \frac{E^2}{p^2}/p^2).$$

Putting everything together, we get:

$$\frac{1}{2} \left( 1 + \hat{k} \cdot \hat{p} - \sigma_p |\hat{p}|^2 \frac{2 \text{Re } c}{1 + |c|^2} \left( 1 + \hat{k} \cdot \frac{E^2}{p^2} \right) \right)$$

If we average over the neutrino direction, then there remains the desired correlation between the spin of the electron and its direction of motion:

$$\frac{1}{2} \left( 1 - \sigma_p |\hat{p}|^2 \frac{2 \text{Re } c}{1 + |c|^2} \right)$$

The polarization is of degree $|\hat{p}| = \sqrt{c}$, and the electron is left handed if $\text{Re } c > 0$. (The same procedure also shows that positrons from $\beta$-decay would be right handed.)

This type of coupling then produces a longitudinal polarization of $\beta$-rays, the amount of which is characterized by the parameter:

$$\frac{2 \text{Re } c}{1 + |c|^2}.$$
behind the predictions, we must take a closer look at the symmetry properties involved, and in particular, the characteristics of a theory under charge conjugation $C$, reversal of spatial coordinates $P$, and time reversal $T$. First we shall do this in a physical fashion, and in the succeeding lectures we shall look at the formal mathematical invariance properties of our theories.

All of our field theories have the property that they are invariant under the product of these three operations, i.e., under CPT. This invariance has been employed all along, and forms the basis for the Feynman point of view that an antiparticle is just a particle moving backward in time with its spatial momentum reversed. The violation of one of the three symmetries thus implies the violation of at least one more. The question of what else is violated besides $P$ in the weak interactions is equivalent to the question of the phase of the constant $c$ in the parity non-conserving part of the interaction we have been working with. There are three possibilities for the $\beta$-decay interaction:

1) $P$, $C$ violated; $CP$, $T$ invariance: $c$ is real
2) $P$, $T$ violated; $TP$, $C$ invariance; $c$ is imaginary
3) $P$, $C$, $T$, violated; $CP$, $T$ invariance: $c$ is complex.

Let us look more closely at how the invariance properties determine the phase of $c$. If $c$ has a real part, then a correlation term $\hat{\sigma} \cdot \hat{\beta}$ exists. Under time reversal, $\hat{\sigma} \cdot \hat{\beta}$ is even, but under $PT$ it is odd. By the CPT invariance, $\hat{\sigma} \cdot \hat{\beta}$ must therefore be odd under charge conjugation, as is evident from our derivation of the correlation term using the projection operators. Therefore, if C invariance is preserved, $c$ must have no real part. It has been stated, and we shall show next time, that $T$ or $CP$ invariance requires all the coupling constants to be real. Thus $c$ must have no imaginary part in case 1). Finally, if $c$ has both a real and imaginary part, then all three invariances of the theory are destroyed, leaving only the CPT invariance.

The arguments above on the conditions for the presence of a $\hat{\sigma} \cdot \hat{\beta}$ term in $\beta$-decay are valid to the extent that there are no forces on the outgoing leptons. Actually in $\beta$-decay, one has to put in the coulomb interaction, and this may give rise to a $\hat{\sigma} \cdot \hat{\beta}$ term even though C invariance were a property of the weak interactions, (which it isn't).

Landau anticipated some of the results of the experiments by applying these ideas of invariances. In particular, he recognized that if nature were invariant under only 1 and C of the group $1, C, P, T, \ldots$, then right and left
would be different. On the other hand, if nature were invariant under $l$ and CP, then right and left would be the same. The geometrical symmetry between right and left would consist of the existence of a physical state of an anti-particle with mirror image properties corresponding to every physical state of a particle. Thus he proposed that the invariance properties of the weak interactions were those of possibility 1). The realness of $c$ then gives rise to several observable effects, among which are the asymmetric distribution of $\beta$-rays from oriented nuclei, the longitudinal polarization of $\beta$-rays from unoriented nuclei, and an asymmetric distribution of $\gamma$-rays following $\beta$-decay. All of these experiments have now been performed, and the maximum effect has been found, corresponding to $|c| = 1$.

Landau anticipated even this result. With $c = \pm 1$, theories involving the neutrino would take on a particularly simple and beautiful aspect. If the neutrino had only weak interactions, and always was coupled by means of $1 \pm \gamma_5$, then that would mean that the interacting neutrinos were 100 percent polarized. For, since the neutrino is massless, $\gamma_5 v$ satisfies the Dirac equation just like $\nu$, and hence the Dirac equation just says:

$$d \cdot p(1 \pm \gamma_5)\nu = -p \cdot v \gamma_5 (1 \pm \gamma_5)\nu = \gamma_5 (1 \pm \gamma_5) \mathcal{D} \cdot p \nu = \pm p (1 \pm \gamma_5)\nu.$$ 

So, to have the neutrino field operator always multiplied by $1 \pm \gamma_5$ in any interaction is equivalent to saying that $d \cdot p = \mp 1$ for all interesting (i.e. interacting) neutrinos. This is perfectly well defined only for a massless neutrino, which moves with the speed of light; otherwise one could transform to another Lorentz frame where the polarization was reversed. Landau advanced this theory of the longitudinal neutrino between Lee and Yang's suggestion that experiments on parity non-conservation be done and the actual experiments. (A similar suggestion was proposed by A. Salam, although in a more complicated paper.) The experiments checked Landau's hypothesis, and determined the sign to be $+$, i.e., only left-handed neutrinos interact.

On applying the charge conjugation operation to the neutrino, we find that only right-handed $\bar{\nu}$ interact.
Hence we have C and P violated by the weak interactions, but CP and T remain as invariance operations.

The loss of invariance under C requires us to modify the theory of $K_1^0$ and $K_2^0$. The $K_1^0$ does decay into two pions, but the $K_2^0$ does not. This was originally assumed to be a consequence of the assignment of the charge conjugation quantum number +1 to $K_1^0$, and -1 to $K_2^0$. The systems $\pi^0 + \pi^0$ and $\pi^+ + \pi^-$ in s-states must be even under charge conjugation because of the generalized Pauli principle. Therefore, it was thought that the decays: $K_2^0 \rightarrow \pi^+ + \pi^-$, and $K_2^0 \rightarrow \pi^0 + \pi^0$ were forbidden because of the change in the charge conjugation quantum number. The non-conservation of C ruins this explanation, but Landau recognized that CP invariance was all you really needed to understand this feature of the K-decays. The final two pion states are eigenstates of CP = +1. So all one needs to do is to define that combination of $K_0^0$ and $K_0^0$, which is an eigenstate of CP = +1, to be $K_1^0$, and that combination, which is an eigenstate of CP = -1, to be $K_2^0$. Then the invariance of the weak interactions under CP forbids the two pion decay of $K_2^0$.

Just as the operator $1 + \gamma_5$ in the weak interactions leads to the production of only left-handed neutrinos, so also it produces the ν/c left-handed polarization of electrons in β-decay. For the case of very rapidly moving electrons, where the mass m is negligible compared to the energy, the electrons coupled in β-decay are all left-handed.

The complete β-decay interaction including both the Fermi and Gamow-Teller parts takes the form:

$$G_V \{(\overline{p} \gamma_\lambda n)(\overline{e} \gamma_\lambda \frac{1 + \gamma_5}{\sqrt{2}} \nu)\} + H.A. $$

$$-G_A \{(\overline{p} \gamma_\lambda \gamma_5 n)(\overline{e} \gamma_\lambda \frac{1 + \gamma_5}{\sqrt{2}} \nu)\} + H.A. $$

or, what is the same thing:

$$G_V \{(\overline{p} \gamma_\lambda [1 - \frac{G_A}{G_V} \gamma_5] n)(\overline{e} \gamma_\lambda \frac{1 + \gamma_5}{\sqrt{2}} \nu)\} + H.A. $$

The constant $-G_A/G_V$ has been measured at the Argonne in an experiment on the asymmetry of electrons in the decay of a polarized beam of neutrons and turns out to be about +1.2. The closeness of $-G_A/G_V$ to 1 is at first quite pleasing, but becomes more and more puzzling as one thinks about the complications due to the strong interactions. The strong interactions give rise to
renormalizations of the coupling constants, and so it is tempting at first to assume that in the bare coupling of the neutron, proton, electron, and neutrino \( G_V = -G_A \). However, the relative renormalization of 1 and \( \gamma_5 \) being only 1.2 is somewhat unexpected. It would not be any more surprising if the bare ratio were -1 and were renormalized to +1.2 than if the bare ratio were +1. Therefore at this stage of the game, we might speculate that the bare four-fermion interaction which gives rise to \( \beta \)-decay is:

\[
\mathcal{N}_\beta = \sqrt{2} \, G \left\{ (\bar{p} \, \gamma_\lambda \frac{(1 + \gamma_5)}{\sqrt{2}} \, n)(\bar{\nu} \, \gamma_\lambda \frac{1 + \gamma_5}{\sqrt{2}} \, \nu) \right\} + \text{H.A.}
\]

March 5, 1959

No. 15: Charge Conjugation, Parity, and Time Reversal

We shall now investigate the mathematical formalism underlying the invariance properties of field theories, which we have employed in a physical fashion heretofore. For example, we argued last time that the realness of the parameter \( c \) in the parity non-conserving \( \beta \)-decay interaction corresponded to the invariance of the interaction under the product of charge conjugation and space reversal, \( CP \), and under time reversal \( T \). And even before that, the presumed invariance of the \( \beta \)-decay interaction under \( T \) was said to imply the realness of the \( \beta \)-decay constants, \( G_V, G_S, G_A, G_T, \) and \( G_P \), which was used, together with the experimental absence of Fierz terms, in concluding that the Fermi \( \beta \)-decay coupling was either \( V \) or \( S \), and the Gamow-Teller either \( A \) or \( T \). Now we shall justify mathematically these statements.

What exactly do we mean when we assert that physics is invariant under an operation? The implication is that if we change the description of a phenomenon, such as by employing a different coordinate system, or by calling all particles antiparticles, the phenomenon itself is unaffected. As a consequence, if one has two different physical systems, which can be related to each other by an operation under which the laws of physics are unchanged, the properties of the two systems which are unaffected by the operation are identical. In quantum mechanics, we describe phenomena by means of field operators, which obey certain equations of motion, and commutation relations, with respect to a given frame of reference. If we change that frame of reference, and are able to find a new set of field operators, obeying the same
equations and commutation relations, then the physics of the new frame is the same as the physics of the original.

To see how this works out in detail, let us consider first the invariance properties of quantum electrodynamics. The theory is specified by the Lagrangian density

\[ \mathcal{L} = -N(\bar{\psi}(\gamma^\mu)\gamma^\mu \psi) + ieN(\bar{\psi} \gamma^\mu \gamma^\nu A_{\mu} - \frac{\partial A_{\mu}}{\partial x^\nu} - \frac{\partial A_{\nu}}{\partial x^\mu}), \]

(N designating the normal product), and the commutation relations

\[ \{\psi_\alpha(x,t), \psi_{\alpha'}(x',t)\} = 5_{\alpha\beta} \delta(x-x'), \quad i [\dot{A}_{\mu}(x,t), A_{\nu}(x',t)] = 5_{\mu\nu} \delta(x-x'), \]

which together generate the familiar field equations of electrodynamics:

\[ i \frac{\partial \psi}{\partial t} = \bar{\psi} \gamma^\mu A_{\mu}, \quad \frac{\partial^2 A_{\mu}}{\partial x^\mu} = \bar{\psi} \gamma^\mu \gamma^\nu \gamma_\nu A_{\mu}. \]

**Space Reflection**

Under the parity operation, the frame by which phenomena are described is changed from \((x,t)\) to \((x',t)\), where \(x' = -x\). The mathematical meaning of the invariance of quantum electrodynamics under the parity operation is that we can find new field operators \(\bar{\psi}'(x',t) = \bar{\psi}'(-x',t)\) and \(A_{\mu}'(x',t) = A_{\mu}'(-x',t)\) in terms of \(\bar{\psi}(x,t)\) and \(A_{\mu}(x,t)\), which satisfy the same equations written in terms of the coordinates \((x',t)\). The field operators \(\bar{\psi}'(x,t)\) and \(A_{\mu}'(x,t)\) give rise to the same phenomena as \(\bar{\psi}(x,t)\) and \(A_{\mu}(x,t)\) and therefore the physics of \((-x,t)\) must be the same as that of \((x,t)\). If a new set of field operators is found which obey the same equations and commutation rules, a further statement may be made. Unless somehow new degrees of freedom have been introduced, (which will not be the case in our discussion), the new set of quantities represents the same mathematical system, the same vector space of quantum mechanical states, described in a new language. Therefore, with one exception which we shall get to in a moment, a unitary transformation can be constructed which relates the new field operators to the old ones. In the case of space inversion, we can find a transformation \(P\) such that:

\[ \bar{\psi}'(x,t) = P \bar{\psi}(x,t) P^{-1}, \quad A_{\mu}'(x,t) = P A_{\mu}(x,t) P^{-1}, \]

with \(P^+ = P^{-1}\).
The converse of the assertion is clearly true also: if $\Psi$ and $A'$ are related to $\Psi$ and $A$ by such a unitary transformation, $\Psi'$ and $A'_\mu$ satisfy the equations of motion and the commutation relations.

As will be seen shortly, the fact that the transformation is unitary is not an absolute requirement, and is violated in the case of time reversal. The unitarity of the transformation is usually required by the commutation relations, but the boson commutation relation involves a time derivative, and so in the operation of time reversal we have to employ an antiunitary transformation to fix up the minus sign. In quantum mechanics, the probabilities are expressed as absolute squares of matrix elements. Unitary transformations preserve the exact value of each matrix element. But to preserve the physics involved, it is not necessary to impose so restrictive a requirement upon the allowed transformations. We can also use those transformations which turn every matrix element into its complex conjugate. This is the class of antiunitary transformations, one of which is needed to discuss the properties of quantum mechanics under time reversal.

Let us continue our discussion of $P$ by finding the set of quantities $\Psi'(x,t)$, $A'_\mu(x,t)$ in terms of the field operators at the image point, i.e. $\Psi(-x,t)$, $A'_\mu(-x,t)$, which satisfy the field equations. The simplest way to do this is just to guess the answer, and then to verify that it works. The obvious guess, in view of what we know about the properties of the unquantized Dirac equation (i.e., "unquantized" here means the absence of second quantization), and the classical electromagnetic field, is clearly:

$$\begin{align*}
A'_j(x,t) &= -A_j(-x,t) \\
A'_{\mu}(x,t) &= A_{\mu}(-x,t) \\
\Psi'(x,t) &= \beta \Psi(-x,t).
\end{align*}$$

It is indeed an easy mental task to verify that these new quantities satisfy the field equations and commutation relations. We can also find the unitary transformation $P$, the parity operator, by means of its definition:

$$\begin{align*}
\Psi'(x,t) &= P \Psi(\pm x,t) P^{-1} = \beta \Psi(-x,t) \\
A'_j(x,t) &= P A_j(\pm x,t) P^{-1} = -A_j(-x,t) \\
A'_{\mu}(x,t) &= P A_{\mu}(\pm x,t) P^{-1} = A_{\mu}(-x,t).
\end{align*}$$

Suppose we expand the field operators in the interaction representation as on page 95:
If the complete sets of functions \( \{ \varphi_{ks} \} \) and \( \{ \chi_{kp} \} \) are chosen to be the standard momentum eigenstates, then

\[
\mathcal{P} a_k \mathcal{P}^{-1} = a_{-k-s}; \quad \mathcal{P} b_k \mathcal{P}^{-1} = b_{-k-s}; \quad \mathcal{P} c_{kp} \mathcal{P}^{-1} = c_{-k-p},
\]

where the \( a, b, c \) are destruction operators for the various single particle levels. It follows that \( \mathcal{P} \) operating on some state produces an image state which is related to the original state by spatial inversion. It is clear that the \( \mathcal{P} \) just constructed has the property \( \mathcal{P} = \mathcal{P}^+ = \mathcal{P}^{-1} \). If we choose the single particle eigenstates to be those which reduce to states of a single orbital angular momentum for the electron in the non-relativistic limit, we find that the intrinsic parity of the particle is the opposite of that of the antiparticle.

The Hamiltonian of electrodynamics is an invariant under \( \mathcal{P} \), and therefore so is the \( S \)-matrix. Mathematically, the invariance of the \( S \)-matrix is expressed by \( \mathcal{P} S \mathcal{P}^{-1} = S \), from which it follows that \( \{ \mathcal{P}, S \} = 0 \). The vanishing of the commutator and the hermiticity of \( \mathcal{P} \) together imply that the \( S \)-matrix element between any two states characterized by different eigenvalues of \( \mathcal{P} \) must vanish. The parity of the final state must be the same as the parity of the initial state; the parity of a system is conserved in electrodynamics. The invariance of \( S \) under \( \mathcal{P} \) together with \( \mathcal{P}^+ \mathcal{P}^{-1} = \mathcal{P} \) tells us further that \( \mathcal{P}^+ S \mathcal{P} = S \), and thus that the \( S \)-matrix element between two states equals the corresponding \( S \)-matrix element between the spatially inverted states.

The fact that under the transformation \( \mathbf{x} \rightarrow \mathbf{x}' \), the equations remain the same if \( A_j(\mathbf{x}',t) = -A_j(\mathbf{x},t) \) and \( A_4(\mathbf{x}',t) = A_4(\mathbf{x},t) \) is the justification for the well-known statement that the electromagnetic potentials transform like a proper vector under that restricted group of Lorentz transformations for which the direction of time is unchanged. Evidently no similar statement can be made about the electron field operator, for the spinor transformation we have employed is not the only one which produces a new field operator such that the Lagrangian density and commutation relations are invariant. It does not matter in electrodynamics whether we choose \( \Psi(\mathbf{x}',t) \) as we have, or whether we choose, for example,
Depending on what is chosen, we can get a variety of parity operators. Each of these other parity operators can be regarded as products of our \( \mathcal{P} \) and another unitary operator \( \mathcal{U} \) for which \( \mathcal{U} \psi_{\mathcal{P}}(x,t) \mathcal{U}^{-1} = e^{i\alpha} \psi_{\mathcal{P}}(x,t) \), if one likes. The existence of a whole set of admissible parity operators is a consequence of the existence of other operations under which the equations are invariant. The admissible parity operators generated by the set of transformations \( \psi'_{\mathcal{P}}(x',t) = e^{i\alpha} \psi_{\mathcal{P}}(x,t) \) are due to the fact that gauge invariance of the first kind is one of the properties of quantum electrodynamics, i.e., a phase change of the electron field operator makes no difference at all.

This additional freedom in electrodynamics, that a phase change of the electron field operator makes no difference, is equivalent to the conservation of electrons; i.e., the number of electrons minus the number of positrons is a conserved quantity. The unitary operator \( \mathcal{U} \) can be taken to be:

\[
\mathcal{U} = \exp(-i\alpha \mathcal{J})
\]

where \( \mathcal{J} \) is the operator which counts the number of electrons (negatons minus positrons):

\[
\mathcal{J} = \int \mathcal{N}(\psi^+ \psi) \, d^3x.
\]

\( \mathcal{J} \) is then the generator of the group \( \mathcal{U} \), just as the angular momentum operator \( \mathcal{J} \) is the generator of the group of unitary transformations which correspond to rotations. The invariance of the Lagrangian density under the group of transformations \( \mathcal{U} \), together with the hermiticity of \( \mathcal{J} \) implies that \([\mathcal{J},S] = 0\), and consequently the number of electrons is a conserved quantity. One can thus describe the situation in electrodynamics very abstractly by saying that the conservation of the number of electrons is a consequence of the gauge invariance of the first kind, or simply by saying that this is the way we write in mathematical notation the fact that the number of electrons is conserved.

Because this invariance under a phase change exists, the parity operation can get mixed up with it. We can, if we wish, define parity to have the \( e^{i\alpha} \) in it, and no one can say that this is a wrong definition of the parity operator, because the physical effects will be the same. The important point is that it is only the total group of invariance operators that matters. To pick one of the operators out and argue that it is parity and the others are something else is foolishness. We have a group of invariance operations consisting of a parity operation, multiplication by \( e^{i\alpha} \), and the products of these, which corresponds to two definite physical properties of electrodynamics:
the phenomena of electrodynamics are invariant under inversion, and the number of particles minus the number of antiparticles is a constant. Which of the set of operators $\hat{P} \hat{U}$ one chooses to use as the parity operator for producing the state representing the spatial inversion of a given state makes absolutely no difference physically. If someone stubbornly insists that the operator $\hat{P} \hat{U}$ is really the parity operator, and that a man who uses $\hat{P}$ is actually employing the parity operator times the operator producing a phase shift of $-30^\circ$, nobody can contradict him on the basis of electrodynamics alone.

Now consider the frequently asked question: Isn't it reasonable to require that when you apply the parity operator twice, the net result should be the identity operator? When one asks why this should be so, the standard reply is that, physically, inverting the system twice produces no change in the system. And therefore, in quantum mechanics the mathematical effect of iterating the parity operator once should be the identity operator. This reasoning is false. The reason it is false is that in quantum mechanics, one cannot tell the difference physically between the identity operator and the operator $e^{i\alpha}$. You will sometimes encounter the argument in a highly mathematical form which goes like this. In quantum mechanics, the various geometric transformations on a physical system are represented by operators; for example, the actual operation of space inversion is represented by that parity operator. The operators therefore represent the operations in the sense that they obey the same multiplication tables. But this last allegation contains the falsehood; the operators need have the same multiplication table only up to a factor $e^{i\alpha}$, or even $\hat{U}$ if the number of particles is conserved, because such transformations make no difference, physically. Consequently it is not required that the square of the parity operator be the identity operator. All that is necessary is that the square of the parity operator have only diagonal matrix elements which are the same for those states which physically can communicate with one another. If there exist absolute rules that a certain set of states can never communicate with another set of states, then the square of the parity operator can be one constant with respect to the first set of states, and another constant with respect to the second set.

Such rules are called by Wigner, Wick, and Wightman "super-selection rules".¹

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¹ Wigner, Wick and Wightman, Phys. Rev. 88, 101 (1952)
A super-selection rule is one that is never violated. We don’t know really which are the super-selection rules, because we do not know today that any of our rules are absolutely absolute, but the ones which exist as far as we know are:

1. the conservation of leptons;
2. the conservation of baryons;
3. the conservation of charge.

These super-selection rules can be related to gauge transformations which leave the equations of physics invariant in the same way as we showed that in electrodynamics, the possibility of making a gauge change of the first kind implied the conservation of electrons.

Actually the group of invariance operations is larger than what we have been considering. In electrodynamics, we also have charge conjugation and gauge transformations of the second kind, while in mesodynamics the group contains elements associated with rotations and reflections in isotopic spin space. The whole group of invariance operations is the only thing that matters, and so people can adopt different definitions of the parity operator, which differ, for example, by a gauge transformation or a rotation in isotopic spin space. However, to the extent that one of these operators produces exact invariance, while another produces only approximate invariance, then it does matter which operator one chooses to call the parity operator. Because rotations in isospin space do not give invariant equations if one adjoins electrodynamics to mesodynamics, it would not be wise to define the parity operator so that it involved a rotation in isotopic spin space. Of course, when we consider the weak interactions too, we find that none of the things we have considered yet is really suitable as the spatial inversion operator. When we look at nature with all her physical phenomena, we realize that CP is really the spatial inversion operator, the parity operator, under which all of nature seems to be invariant. If there are degrees of invariance of the mathematical system under consideration, then the various parity operators are not equivalent, and one set of them may be much more useful than another.

When we wish to consider the parity operator as an observable in discussing the selection rules for a certain subset of natural phenomena, we will always choose a parity operator which is hermitian insofar as the part of quantum mechanical space under consideration is involved. Therefore, in such a discussion that block of the square of the parity operator for the subspace
involved is the identity operator for that subspace.

Particle-Antiparticle Conjugation

The equations of electrodynamics are invariant under the application of \( P \), but if we investigate the weak interactions, we find that some of the equations are not invariant under \( P \), no matter how much we play around with the definition of the parity operator, unless we include something which turns particles into antiparticles. This operation is, however, another one of the invariance operations of electrodynamics and mesodynamics, and is commonly called charge conjugation, although a more suitable name is particle-antiparticle conjugation.

The idea which suggests the existence of such an invariance operator is that perhaps the description of physical phenomena is such that it doesn't matter whether things are called particles or antiparticles, just as the existence of \( P \) is suggested by the idea that perhaps the description of phenomena is such that it doesn't matter whether we employ a left-handed coordinate system instead of a right-handed one.

Particle-antiparticle conjugation, despite rumors or myths to the contrary, and despite its name, is accomplished by a unitary operator. There is no complex conjugation involved. In our earlier discussion, which dealt with wave functions of a single particle, complex conjugation entered, but the discussion was in the old-fashioned sense, before the second quantization of the Dirac field had been carried out. When we use the language of second quantization, the Dirac field operators are carried into their hermitian conjugates by a unitary operator. This point is confusing, because the student hears about charge conjugation before he learns the formalism of second quantization.

Our program will be very similar to that used in the discussion of space reflection. First we will find a new set of field operators which leave the equations of electrodynamics invariant under the operation 
\[
e A_\mu \rightarrow e' A'_\mu = -e A_\mu.
\]
Then the new field operators are related to the old ones by a unitary transformation \( \zeta \), which we can find. We will be able to choose \( \zeta \) such that it is hermitian. From the invariance of the Lagrangian density under \( \zeta \), we get \( \zeta \zeta^\dagger = \zeta^\dagger \zeta \), and hence the eigenvalue of \( \zeta \) is conserved.

Ideally, one would hunt around until one found a suitable new set of field operators. But we can guess, using our background of experience, such
a set very easily.

\[ \psi^C(x,t) = F^{\alpha\beta} \psi^+(x,t) = C \psi(x,t) C^{-1} \]

\[ A^C(x,t) = -A(x,t) = C A(x,t) C^{-1} \]

where \( F \) is the four-by-four Dirac charge conjugation matrix, which may be taken as \( \gamma_2 \) in the standard representation, or simply 1 in the Majorana representation.

**PROBLEM 12:** Show that the transformation given above leaves the Lagrangian, and the commutation relations invariant, and hence that \( \psi^C \) and \( A^C \) satisfy the field equations, if and only if
\[ F F^+ = 1, \quad F^+ \gamma_{\mu} F = -\gamma_{\mu}^*, \quad \text{and} \quad F^+ \gamma_{\mu} \gamma_{\nu}^* F = \gamma_{\mu}^* \gamma_{\nu}^*. \]

We can find the unitary operator \( C \) quite readily by noticing that

\[ C a_{ks} C^{-1} = b_{ks}, \quad C b^+_{ks} C^{-1} = s^+_k, \quad C c_{kp} C^{-1} = -c_{kp}, \text{ etc.} \]

It follows that \( C \) operating on an electron state produces the corresponding positron state. It is evident that this \( C \) is also hermitian, so \( C^2 = 1 \). But again, we must remind ourselves that many choices for \( C \) are available, and most of them do not have the property \( C = C^+ \), which implied \( C^2 = 1 \).

Many examples of the use of the conservation of \( P \) to discuss selection rules are familiar, but it is worthwhile to give an example or two of the applications of charge conjugation invariance. Some features of positronium can be understood very nicely through charge conjugation arguments. A state of positronium may be represented by

\[ \sum_{s_1, s_2} \sqrt{\frac{d^3 x_1}{d^3 x_2}} \chi(x_1, s_1^1; x_2^1, s_2^1) N [\psi^+_s(x_1) \psi_s(x_2)] | \text{vac} \rangle, \]

where \( \chi \) is something like the wave-function of non-relativistic quantum mechanics. On applying \( C \) to this state, in the Majorana representation, we get:

\[ \sum_{s_1, s_2} \sqrt{\frac{d^3 x_1}{d^3 x_2}} \chi(x_1, s_1^1; x_2^2, s_2^2) N [\psi^+_s(x_1) \psi_s(x_2)] | \text{vac} \rangle \]

\[ = - \sum_{s_1, s_2} \sqrt{\frac{d^3 x_1}{d^3 x_2}} \chi(x_1, s_2^1; x_2^1, s_1^1) N [\psi^+_s(x_1) \psi_s(x_2)] | \text{vac} \rangle \]

where we have adopted the natural convention that the charge conjugation quantum number of the vacuum is +1. On comparing the two expressions it is evident that if the wave function \( \chi \) is symmetric under the exchange of the electron and positron, the state is an eigenstate with \( C = -1 \); while if the wave function is antisymmetric, \( C = +1 \) for the state. We may characterize states of positronium by their rotation properties, and by their charge conjugation quantum number as follows:
Now let us construct a typical $n$-photon state:

$$\sum_{p_1\cdots p_n} \int \cdots \int d^3y_1 \cdots d^3y_n \phi(y_1, p_1, y_2, p_2, \cdots y_n, p_n) N[A_{p_1}(y_1) A_{p_2}(y_2) \cdots A_{p_n}(y_n)]^{\text{vac}}$$

On applying $C$ to such a state, it is immediately clear that the state is an eigenstate with $C = (-)^n$.

Since $C$ is conserved in electrodynamics, the electron and positron in the states $3S_1$, $1P$, etc. upon annihilation make an odd number of photons. It is most likely that they produce three photons; the production of five photons is roughly $(137)^2 = 20,000$ times rarer, and similarly the production of more than five photons is vanishingly small. Likewise the states $1S_0$, $3P$, $\cdots$ decay into states with an even number of photons, the most likely of which is that of two gamma rays. This decay is about $137$ times faster than the three photon decay characteristic of $3S_1$, $1P$, $\cdots$. Of course, in the presence of an external field, these considerations do not hold any more, because the external field disturbs the invariance. You would have to change its sign, too, in order to preserve the invariance. So all these selection rules refer to an isolated positronium atom.

The decay of $3S_1$ into two photons also happens to be forbidden by a geometrical rule that no system of angular momentum $1$ can disintegrate into two photons.*

Suppose we consider the emission of light by positronium without annihilation. Charge conjugation then tells us that the emission of one photon takes place only in transitions between an odd and an even charge parity state. No electromagnetic transition between an odd and an odd or an even and an even state of positronium in the absence of an external field can occur, unless at least two photons are made.

From this example, we see that $C$ is very much like $P$. However, it is much less useful as a conserved quantum number, because the only eigenstates

* C. N. Yang, Phys. Rev. 77, 242 (1950)
of $\mathcal{C}$ that are interesting are for systems with zero charge. In fact, the system must contain equal numbers of particles and antiparticles for each type present. Other eigenstates of $\mathcal{C}$ can be constructed, but these are composed of linear combinations of states with different total charge, or lepton number, or baryon number, and thus are of no physical interest. So $\mathcal{C}$ is useful mostly in special problems like positronium and the annihilation of nucleon-anti-nucleon pairs.

However, invariance of a system under $\mathcal{C}$ and $\mathcal{P}$ has a second type of consequence besides telling us that a system can't go from one eigenvalue to another. Besides the selection rules, the invariance under charge conjugation implies a symmetry in the matrix elements. The symmetry in the matrix elements guarantees that a state and its image state, under $\mathcal{C}$ or $\mathcal{P}$, must have the same properties: the same energy, decay rate, etc. Invariance under $\mathcal{C}$ tells us that the $\mu^+$ and $\mu^-$ must have exactly the same mass, and opposite magnetic moments. Actually CP invariance is sufficient for most statements like this, so the fact that $\mathcal{C}$ and $\mathcal{P}$ are violated by the weak interactions doesn't change the conclusions. But either $\mathcal{C}$ or CP will tell you that a particle and its antiparticle have similar properties, and $\mathcal{P}$ tells you that a system and its mirror image have identical properties.

**Time Reversal**

Now let us go on to $\mathcal{T}$. $\mathcal{T}$ differs fundamentally from $\mathcal{C}$ and $\mathcal{P}$ in that the time-reversed field operators $\psi^\tau_{\alpha}(\mathbf{x},t)$ and $\lambda^\tau_{\alpha}(\mathbf{x},t)$ cannot be related to $\psi_{\alpha}(\mathbf{x},t)$ and $\lambda_{\alpha}(\mathbf{x},t)$ by a linear transformation. Many reasons for that can be given. Perhaps the best is to realize that all such transformations leave the Heisenberg equations of motion invariant, and thus if

$$\psi^\tau_{\alpha}(t') = E_{\alpha\beta} \psi_{\beta}(t)$$

where $t' = -t$, $i \frac{\partial}{\partial t} \psi^\tau_{\alpha}(t') \neq [\psi^\tau_{\alpha}(t'), \mathcal{H}]$

since $i \frac{\partial}{\partial t} \psi^\tau_{\alpha}(t') = -i E_{\alpha\beta} \frac{\partial}{\partial t} \psi_{\beta}(t) = -E_{\alpha\beta} [\psi_{\beta}(t), \mathcal{H}] = -[\psi^\tau_{\alpha}(t'), \mathcal{H}]$

There is a way out. If in addition to the linear transformation we complex conjugate the operators, then $\psi^\tau_{\alpha}(t') = E_{\alpha\beta} \psi^\dagger_{\beta}(t)$ does indeed satisfy the time-reversed Heisenberg equation of motion:

$$i \frac{\partial}{\partial t} \psi^\tau_{\alpha}(t') = -E_{\alpha\beta} i \frac{\partial}{\partial t} \psi^\dagger_{\beta}(t) = E_{\alpha\beta} [\psi^\dagger_{\beta}(t), \mathcal{H}^*] = [\psi^\tau_{\alpha}(t'), \mathcal{H}^*].$$
Such transformations are unlike anything we have ever dealt with before. Time reversal is much more complicated from the beginning than space reversal, even in non-relativistic quantum mechanics.

Another way to see that a new type of transformation is needed to discuss time reversal is to consider the commutation relations of the electromagnetic potential field operators:

$$\left[ \frac{\partial A_\mu(x,t)}{\partial t}, A_\nu(y,t) \right] = -i \delta_{\mu\nu} \delta(x - y).$$

Under the reversal of the sign of time, it is evident that, unless I am able to pull some sort of conjuring trick, a change of sign in the left-hand side of the commutation relations will be introduced. But I never can get anything different on the right-hand side by means of a linear transformation. Thus no linear transformation can be found which leaves the equations of electrodynamics, including the commutation rules, invariant under time reversal.

In order to relate the time-reversed quantities to those which are not time reversed, we will need, therefore, to employ a transformation which consists of complex conjugation followed by a linear transformation. Such an antilinear transformation can be obtained from a unitary operator, $T$:

$$A_\mu'(t) = T A_\mu(t) T^+; \quad \psi_\mu'(t) = T \psi_\mu(t) T^+. $$

but the transformation law for $T$ under a change of basis transformation generated by $U$ is not the usual $T' = U T U^+$, but rather $T' = U^* T U^+$. These features are most peculiar in quantum mechanics; none of our other transformations possess them. For example, under spatial inversion the matrix element of the spatially reversed operator between two reversed states was obviously the same as the matrix element of the original operator between the two original states, because of the unitarity of the transformation. In contrast, all matrix elements are transformed into their complex conjugates when the states and operators are transformed by time reversal. This, however, doesn't affect the physics, because the only things we measure are transition probabilities, which depend on the absolute squares of matrix elements, and the diagonal matrix elements of hermitian operators, which are real. All interference behavior, all probability laws, all the eigenvalues of energy, angular momentum, all the intensity formulae, etc. remain exactly the same as before.

In working with time-reversal mathematically, there appears to be a variety of technically acceptable procedures. Many people employ non-linear
operations on state vectors, in which effectively bras become kets and vice versa. This appears to me to be somewhat confusing, and unnecessary, at least for obtaining the results we shall be concerned with. We shall work instead with a unitary operator \( T \), in the same way as we worked with unitary operators \( P \) and \( C \) in the discussions of space reversal and particle-antiparticle conjugation. The results of this discussion of time reversal will be of quite a different character than those of the previous discussions of \( P \) and \( C \), but at least the mathematical formalism will be the same. There is only one thing with which one has to exercise some special care. That arises when we change the basis in the representation of the state vectors. For example, if we put \( | \psi \rangle = U | \alpha \rangle \), then we must choose \( T \) such that \( \Psi^\tau (t) = T^\dagger \Psi^\tau (t) T^+. \) Since

\[
\Psi^\tau (t) = E_{a}^{\alpha} \Psi^\ast (t^\dagger) = E_{a}^{\alpha} U^\ast \Psi^\ast (t^\dagger) U^+
\]

\[
= U^\ast T \Psi (t^\dagger) T^+ U^+ = U^\ast T U^+ \Psi (t^\dagger) U T^+ U^+
\]

it is clear that the transformation rule for \( T \) is \( T^\dagger = U^\ast T U^+ \).

Now let's try to find a new set of field operators which obey the equations of motion and the commutation relations with the sign of \( t \) reversed. Under time reversal, since \( A_j \) is generated by currents which reverse their direction, we expect that \( A_j \) reverses sign. \( A_0 \) is generated by the charge density which doesn't change sign, but \( A_\mu \) contains a factor of \( i \). Thus we would guess that:

\[
A_j^\tau (x, t) = -A_j^\ast (x, -t) = T A_j (x, t) T^+. \]

Now this looks a little peculiar if one says: "Isn't the combination \((A_j; A_0)\) a four-vector?" If it were, then under time reversal \( A_0 \) should change sign, but \( A_j \) should not. Under our parity transformation, \( A \) indeed was a four-vector, but under time reversal \( A \) behaves like a pseudovector. This is perfectly possible; the behavior under parity and time reversal need not be the same.

Now, what do we want to choose for \( \Psi^\tau (x, t) \)? We shall try

\[
\Psi^\tau (x, t^\dagger) = E_{a}^{\alpha} \Psi^\ast (x, t); \quad t = -t^\dagger,
\]

and seek to determine a Dirac matrix \( E \) such that

\[
(\not\tau + m) \Psi^\tau = i e \not\tau \Psi^\tau.
\]

To do this, we may multiply the equation of motion for \( \Psi^\ast \) on the left by \( E \), obtaining:
(E γμ E⁻¹ ωμ + m)ψτ = i e Aμ E γμ E⁻¹ ψτ

But since ωμ = ωμ, we see that the conditions on E are:

E γμ E⁻¹ = γμ.

If one writes

γ4γ5γμ E⁻¹ γμ γ4γ5 = γ4γ5γμ γ4γ5 γμ = - γ4γ5γμ γ4γ5 γμ,

one may notice that the conditions of the matrix (γμ γ5E) are the same as the conditions on the charge conjugation matrix F, since the invariance of the fermion commutation relation requires E⁻¹ = E⁺. Thus we may take

E = γ5γ4 F.

It is an easy exercise to verify that the remaining equation of motion remains invariant in form.

\[ \Box^2 A^μ = - \Box^2 A^μ = - \{ + i e N [ γμ γ5 ] \} \]
\[ = - i e N [ \psi^τ E γμ γ5 E⁻¹ ψτ ] \]
\[ + - i e N [ \psi^τ γμ ψτ ] . \]

Thus the equations of electrodynamics are invariant under time reversal.

We can construct an operator T which produces this transformation, in the representation whose basis kets are eigenstates of the uncoupled Hamiltonian, with specified momenta and polarizations. A suitable operator is:

\[ T A^μ = T A^μ = \pm A^μ_{ks}, \]  \[ T A^μ = \pm A^μ_{ks}, \]  \[ T A^μ = \pm A^μ_{kp}, \]

The consequences of time-reversal invariance are unlike those of the other invariance operations. Under parity and charge conjugation, we have

P H P⁺ = H  and  C H C⁺ = H, which imply P S P⁺ = S  and  C S C⁺ = S.

The time reversal operator has the property that in electrodynamics and other theories invariant under time reversal,

\[ T H(t) T⁺ = [H(-t)]^* \]

and therefore,

\[ T S T⁺ = P(\exp \{ -i \int [H(-t)]^* dt \} \]
\[ = P^{-1}(\exp \{ +i \int [H(t')]^* dt' \}) = S^{+k} = S^{-l} \].

So under time reversal, S → S⁻¹. Part of this result should have been obvious from the beginning. The inverse of the S-matrix leads you backwards.
in time, and thus it is exceedingly reasonable that when the S-matrix is transformed by the time reversal operator, you should get something like its inverse.

The fact that \( T S T^+ \neq S \) means that there are no selection rules due to the invariance under time reversal. However, this invariance does imply the relation

\[
\langle f^\tau | S | i^{\tau^*} \rangle = \langle f^\tau | T S T^+ | i^{\tau^*} \rangle = \langle f^\tau | T S T^+ | i^{\tau^*} \rangle,
\]

which tells you the phases of certain matrix elements. In the analysis of the synchrotron experiments here several years ago on the photoproduction of pions, this relation enabled one to deduce the phases of the photoproduction matrix elements from the known phase shifts for pion scattering. In this way the number of parameters to be determined from the analysis of the data could be reduced by a factor of two.

March 10, 1959

Consequences of CP Invariance for the Weak Interactions

We now have the machinery to substantiate several statements that were alleged to be equivalent to the assumed CP invariance of \( \beta \)-decay phenomena. In particular, the realness of the coupling constants in \( \beta \)-decay under the assumption of CP invariance was used to deduce from the absence of Fierz terms that the Fermi interaction is either \( V \) or \( S \) while the Gamow-Teller interaction is either \( A \) or \( T \).

In our investigation last time, we found an operator \( P \) such that

\[
P \, \psi(x) \, P^+ = \beta \, \psi(-x),
\]

and an operator \( C \) such that

\[
C \, \psi(x) \, C^+ = F \, \psi(x).
\]

Thus

\[
C \, P \, \psi(x) \, P^+ \, C^+ = \beta \, F \, \psi(-x),
\]

and

\[
C \, P \, \psi^+(x) \, P^+ \, C^+ = \psi(-x) \, F^+ \, \beta.
\]

It would be tedious to go through all the various couplings to verify that CP invariance holds if and only if the \( G \)'s are real. If one considers
an example, the manipulations involved in the general case become clear, and one can see just why the realness of the coupling constant is necessary and sufficient for invariance under CP. Let us consider therefore just one of the possible couplings:

\[ G^V \left( \bar{\psi}^p \gamma_\alpha \psi_n \right) \left( \bar{\psi}^e \gamma_\alpha \gamma_5 \psi_n \right) + \text{H.A.} \]

To find out what happens under CP, we have to remember that

\[ F^\beta F = -\beta^*, \quad F^+ \gamma_\alpha ^\beta F = \beta^* \gamma_\alpha, \quad F^+ \gamma_5 F = -\gamma_5^*. \]

Then under CP the nucleon part of the interaction becomes:

\[ \left( \bar{\psi}^p F^+ \gamma_\alpha ^\beta F \psi_n^+ \right) = \left( \bar{\psi}^p \beta^* \gamma_\alpha \psi_n^+ \right) = \left( \bar{\psi}^p \gamma_\alpha \psi_n \right)^+. \]

Similarly,

\[ C^P \left( \bar{\psi}^e \gamma_\alpha \gamma_5 \psi_n \right) F^{\dagger P}_n = \left( \bar{\psi}^e F^+ \gamma_\alpha ^\beta \gamma_5 F \psi_n^+ \right) \]

\[ = -\left( \bar{\psi}^e \gamma_\alpha \gamma_5 \psi_n \right)^*. \]

Since the net effect of CP on the interaction turns the spinor operator combinations into their Hermitian adjoints, it equivalently just complex conjugates the \( G^V \). The interaction is therefore invariant under CP if and only if \( G^V \) is real.

**PROBLEM 13:** Find the effect of CP on some other four-fermion coupling, e.g. the direct tensor coupling, and thereby show that the interaction is invariant under CP if and only if the coupling constant is real.

Precisely the same condition is necessary and sufficient in order that the system shall possess time reversal invariance. If the system is invariant under time reversal, then \( H(t) \) must go into \( [H(-t)]^* \) upon the application of the unitary operator \( T \). But,

\[ T \{ G^V \left[ \bar{\psi}^p(t) \gamma_\alpha \psi_n(t) \right] \left[ \bar{\psi}^e(t) \gamma_\alpha \gamma_5 \psi_n(t) \right] \} T^+ \]

\[ = G^V \left[ \bar{\psi}^{p*}(-t) E^+ \gamma_\alpha \psi^{*n*}(-t) \right] \left[ \bar{\psi}^{e*}(-t) E^+ \gamma_\alpha \gamma_5 \psi^{*n*}(-t) \right] \]

which, since \( E^+ \gamma_\alpha \psi = \gamma_\alpha^* \), is equal to

\[ G^V \left[ \bar{\psi}^{p*}(-t) \right] \left[ \bar{\psi}^{e*}(-t) \right] \]

\[ = G^V \left[ \bar{\psi}^{p*}(-t) \right] \left[ \bar{\psi}^{e*}(-t) \right] \]

Therefore, \( T H(t) T^+ = [H(-t)]^* \) if and only if \( G^V = G^V^* \). It should be quite evident from this exercise that any of the other \( \beta \)-decay couplings has a real coupling constant if the system is to be invariant under time reversal.
We have gone into these formal demonstrations of invariance, not because they are of exceedingly great logical importance, but rather because they are quite useful in theoretical work for corroborating the results obtained by physical arguments.

Suppose we consider that example of a coupling Hamiltonian density as a function of the coupling constant: \( J^f(G, \vec{x}, t) \). Then we have shown mathematically that \( CP J^f(G, \vec{x}, t) \frac{T^+}{T} = J^f(G^*, -\vec{x}, -t) \) and that \( T^f \frac{J^f(G, \vec{x}, t) T^+}{T} = \left[ J^f(G^*, -\vec{x}, -t) \right]^* \). From the combination of these two results, we see that \( TCP J^f(G, \vec{x}, t) \frac{T^+C^+T^+}{T} = \left[ J^f(G, -\vec{x}, -t) \right]^* \). Since, as we have discussed, the complex conjugation of everything does not affect the physics, it is apparent that the system described by such an interaction Hamiltonian is invariant under the product of the three operations, space reflection, particle-antiparticle conjugation, and time inversion, regardless of the phase of the coupling constant. This automatic invariance under TCP is a general property of any of our relativistically invariant local field theories, and is equivalent precisely to the statement that an antiparticle may be regarded as a particle moving backwards in space-time.

Consequences of Time Reversal Invariance in The Photoproduction of Pions

It has been mentioned that people were able to deduce the phases of certain matrix elements from time reversal considerations, and thereby greatly simplify the analysis of the experiments on the photoproduction of pions at low energies. The needed formal machinery is now at our disposal, and so let us examine how the phases can be obtained.

But first, the pion-nucleon theory under consideration should be shown to possess invariance under time reversal. The theory will be invariant provided a transformation rule for the pion field operators can be found so that \( \varphi^\lambda(t) \rightarrow [\varphi^\lambda(-t)]^* \). That part of the Lagrangian density which involves the pion fields is:

\[
-\frac{1}{2} N[\mu \phi^\lambda \phi^\lambda + \partial_\alpha \phi^\lambda \partial_\alpha \phi^\lambda] - ig N[\overline{\psi} \not\tau \psi] \not\tau \phi^\lambda + e N[\phi^\lambda \phi^\lambda] A^\lambda - e^2 N[A^\lambda A^\lambda] N[\phi^\lambda \phi^\lambda] + \frac{1}{2} \partial_\alpha \phi^\lambda \partial_\alpha \phi^\lambda.
\]

Our task is to define \( T \frac{\phi^\lambda(t) T^+}{T} \) in such a way that \( T \frac{\phi^\lambda(t) T^+}{T} = \left[ \phi^\lambda(-t) \right]^* \). Since \( \tau_1 = -\tau_1 \), \( \tau_2 = -\tau_2 \), and \( \tau_3 = \tau_3 \), it is easy to see that the problem is solved by choosing...
If instead of the real fields \( \phi_1 \) and \( \phi_2 \), the equations were written for the complex field \( \phi_c^+ \), all the equations would have the (-) sign.

It was shown in the last lecture that time reversal invariance leads to the following relation:

\[
\phi^* = \phi^* \quad \text{where} \quad \phi^* = \{\phi^*\}^*.
\]

To employ this, however, some further relation must be found which will essentially define the states |\( \phi^* \)\rangle. One way to find such a relation is to start with the equation which defines the essential properties of the operator \( \Theta \), namely

\[
\Theta \phi(t) \Theta^* = \phi(-t),
\]

where \( \Theta \) is any operator representing an observable, and \( \Theta \) is its classical behavior under time reversal. For example, \( \Theta = +1 \) if \( \Theta \) represents the energy, square of the angular momentum, charge, parity, etc.; whereas \( \Theta = -1 \) for the current, momentum, angular momentum, and so on. Suppose now one is dealing with an eigenstate \( |a\rangle \) of \( \Theta \), that is, \( \Theta |a\rangle = \Theta_a |a\rangle \). Then

\[
\Theta |a^*\rangle = \Theta |a^*\rangle = \Theta \Theta^* |a^*\rangle = \Theta \Theta^* \Theta^* |a^*\rangle = \Theta \Theta_a |a^*\rangle,
\]

since \( \Theta_a \) is real by virtue of the fact that \( \Theta \) represents an observable. From the equality it is clear that the state |\( a^* \)\rangle is an eigenstate of \( \Theta \) with the eigenvalue \( \Theta \Theta_a \).

We shall label the states under consideration by the quantum numbers \( J, M, a \), where \( a \) denotes all those remaining quantum numbers, such as the energy, number of particles type, parity, and so on, needed to specify the state. The quantities represented by \( a \) are taken so that \( \Theta_a = +1 \). Now because \( \Theta \) is unitary, and because an eigenstate is transformed into the complex conjugate of the corresponding eigenstate by \( \Theta \), the following must be true:

\[
\Theta |a, J, M\rangle = e^{-i\eta(a, J, M)} |a, J, -M\rangle^*.
\]

Furthermore, by playing with the raising and lowering operators for angular momentum, \( J_+ \) and \( J_- \), it is quite easy to show that \( \eta(a, J, M) = \xi(a, J) + M \xi \).

Finally, by changing the basis using the unitary operator \( \Upsilon \) defined so that

\[
|a, J, M\rangle = \Upsilon |a, J, M\rangle = e^{-\frac{i}{2}(\xi(a, J) - M)} |a, J, M\rangle,
\]

we can obtain new operators, \( \Theta', S', \Theta' \), etc., and employ the convenient identity
\[ T' | \alpha, J, M \rangle = e^{-i \pi (J+M)} | \alpha, J, -M \rangle \]. Having shown it is possible to eliminate most of the phase dependence by a suitable unitary transformation, we may assume that we are working in this representation and omit the primes.

Now consider just a block of the S-matrix dealing with a given value of J. Inserting the value for \( T | \alpha, J, M \rangle \) just obtained into the relation
\[ \langle f | S | i \rangle = \langle i^{\text{TK}} | S | f^{\text{TK}} \rangle \], there results:
\[ \langle \beta, J, M | S | \alpha, J, M \rangle = \langle \alpha, J, -M | S | \beta, J, -M \rangle. \]
But the system is invariant under rotations in ordinary space, so that the matrix element \( \langle \beta, J, M | S | \alpha, J, M \rangle \) is independent of M. Therefore, we can drop the label M, and just write:
\[ \langle \beta, J | S | \alpha, J \rangle = \langle \alpha, J | S | \beta, J \rangle. \]
This relation tells us that by making a suitable unitary transformation affecting only the phases of the basis states, we can work in a representation in which the S-matrix is symmetric. It is a trivial point that the S-matrix may be made to be symmetric; all we need to do is diagonalize it. However, the result that has been found is not trivial. It states that without essentially changing the formulation of the problem, i.e., retaining the basis states to be of the same nature, a representation of the states can be found in which the S-matrix is symmetric if the system is invariant under time reversal.*

The symmetry of the S-matrix was used successfully to determine the phases of the photoproduction amplitudes at low energies. At low energies, for a given J, M, parity, there are only three channels, which may be taken to be (1) \( \gamma p \), (2) \( [\pi N]_{I=\frac{3}{2}} \), and (3) \( [\pi N]_{I=\frac{1}{2}} \). Exploiting the symmetry, we can write the S-matrix block as:
\[
S = \begin{pmatrix}
1 & e^A & eB \\
e^A & e^{2i\delta_1/2} & 0 \\
e^B & 0 & e^{2i\delta_3/2}
\end{pmatrix} + \text{terms of order } e^2.
\]
The \( \delta \)'s are the phase shifts due to the pion-nucleon scattering if electromagnetism is "turned off". The unitarity condition, considered to order \( e \), then requires:
\[
e^{2i\delta_1/2} + e^A = 0, \quad e^{2i\delta_3/2} + eB = 0.
\]
Hence, \( A = \pm |A| i e^{i\delta_1/2} \), and \( B = \pm |B| i e^{i\delta_3/2} \). Therefore, instead of four

* F. Coester, Phys. Rev. 82, 619 (1953).
unknown parameters to be determined from the analysis of the data on photoproduction, there are only two.

It is evident that the success of such a treatment depends upon two factors: (1) there must exist both a strong and a weak interaction, so that higher powers of the weak coupling may be dropped; and (2) the eigenstates of the strong coupling must be known. For the analysis of photoproduction at around 1 BEV, such a treatment is not very useful, because the phases of the photoproduction matrix elements depend on phase shifts for complicated states containing various numbers of pions.
No 16: The Universal V-A Interaction - No Strangeness Change

We have seen, with a slight rewriting of the order of history, that the form of the \(\beta\)-decay interaction has been experimentally determined to be

\[
\mathcal{H}_\beta = \left\{\gamma_\alpha \left( G_V \gamma^\alpha - G_A \gamma^\alpha \right) n \right\} \left\{ \gamma_\alpha \frac{1 + \gamma_5}{\sqrt{2}} v \right\} + \text{H.A.}
\]

This form was distilled from the results of many experiments. First of all, the existence of allowed \(\beta\)-transitions indicated the necessity of couplings of both the Fermi and the Gamow-Teller types. From the rates, it was possible to conclude that the F and G-T couplings were approximately equal in strength. Furthermore, assuming CP or T invariance, one could tell from the absence of Fierz terms in the spectrum that the F coupling was of one kind, and similarly the G-T coupling was not a mixture of two types. Angular correlation experiments yielded the information that the F coupling is vector and the G-T is axial vector. The study of the strange particles then led to the investigation of whether the weak interactions were parity conserving or not. The \(\beta\)-decay interaction would be parity non-conserving if there were both odd and even couplings. We discussed the admixture of odd couplings in terms of a parameter \(c\), whose phase was intimately related to the invariance properties of the interaction. If \(c\) were real, the interaction would be CP and T invariant, whereas if \(c\) were imaginary, it would be invariant under TP and C. And if \(c\) were complex, the interaction would possess no particular symmetry, except TCP, of course. Landau's and Salam's conjecture that \(c\) was real and of magnitude unity, which meant a two-component neutrino, was confirmed by the experiments which detected the maximal effect in phenomena like the longitudinal polarization of \(\beta\)-rays. The sign of \(c\) happened to be +, which means that the neutrino is left-handed, and the antineutrino is right-handed.

Quantitatively, the rates of the decay of \(^3\text{He}\) and the neutron were used to calculate that:

\[
|G_V| = (1.01 \pm 0.01) \times 10^{-5} M_p^2
\]

\[
|G_A/G_V| = 1.19 \pm 0.04
\]

Let us look in detail at one more experiment which is of interest for two reasons. First of all, it is representative of that class of experiments which first demonstrated parity non-conservation in \(\beta\)-decay, and secondly, it
is the only experiment so far which is capable of measuring both the sign and magnitude of $G_A/G_V$. That type of experiment is one in which the nucleus is originally aligned, and the angular distribution of the $\beta$-rays is then examined. Due to the parity non-conserving nature of the $\beta$-decay interaction one gets a correlation between the direction of the $\beta$-rays and the spin of the nucleus. The first detection of the non-conservation of parity through the observation of such a term was carried out by the cryogenic group at the Bureau of Standards under the direction of Miss Wu at Columbia.

The first experiment was done on Co$^{60}$, and could have measured the interference between $G_V$ and $G_A$, but the nuclear matrix elements would come in and mess up the determination of $G_A/G_V$. The allowed Fermi matrix element is always known, up to coulomb effects, but the Gamow-Teller matrix element is usually not known reliably, since it involves the nuclear wave function. The only case in which we really know the Gamow-Teller matrix element is that of the neutron. Therefore, to measure both the sign and magnitude of $G_A/G_V$, an experiment has been performed on polarized neutrons at the Argonne, which measures the correlation between the $\beta$-ray direction and the direction of polarization of the neutron.

Let us work out the form of the correlation term in the neutron experiment. The neutron will be taken to be polarized in the direction of the unit vector $\vec{s}$. The four momentum of the resulting electron is $q$, while that of the antineutrino is $k$. By reference to page 238, one can check that the non-relativistic approximation to the R-matrix element is:

$$\sqrt{2m/2E} q \frac{G_v}{G_A} \left\{ p^+ u^{e^+}_q (1 - \alpha \cdot \vec{s} \cdot \vec{Z}^\dagger) \frac{1 + \gamma_k}{\sqrt{2}} \nu_k \right\}$$

where $x = -G_A/G_V$.

The quantity of interest is, as usual, $\Sigma |R|^2$, in which the sum is to be over the spin orientations of all the particles in the final state.

$$\Sigma |R|^2 = G_v^2 \Sigma \left\{ \nu^{e^+}_{-k} \frac{1 + \gamma_5}{\sqrt{2}} (1 - \alpha \cdot \vec{s} \cdot \vec{Z}^\dagger) \frac{2m}{2E} q \Sigma \frac{p^+ u^{e^+}_q}{q} \right\}$$

By making use of

$$n n^+ = \frac{1 + \vec{s} \cdot \vec{d}}{2}, \quad \Sigma \frac{p^+ u^{e^+}_q}{q} = 1, \quad \frac{2m}{2E} q \Sigma \frac{u^{e^+}_q u^{e+}_q}{q} = \frac{m - ig \beta^1}{2E q}$$

and

$$\Sigma \nu_k \nu^+_{-k} = -i \frac{k^1 \beta^1}{2k},$$

it is easy to reduce that expression to the following form:
By shuffling the matrices in the trace, remembering that the trace of an odd number of gamma matrices is zero, and defining \( k = \beta \gamma \), this can be reduced further to:

\[
G^2 \frac{\text{Trace }}{16 \ k E_q} \{(\gamma^1_5 - 1) \ \widetilde{\gamma}^1 \ (1 + \vec{\gamma} \cdot \vec{s})(1 - x \vec{\gamma} \cdot \vec{z}')(\vec{m} - i\vec{q}' \gamma^0)(1 - x \vec{\gamma} \cdot \vec{z}')(1 + \gamma^0_5)\}
\]

Employing the handy relation \( \vec{\sigma} \cdot \vec{A} \vec{\sigma} \cdot \vec{B} = \vec{A} \vec{B} + i \vec{\sigma} \cdot \vec{A} \vec{\times} \vec{B} \), and doing the trace over the nucleon spins, one is left with just a trace over the leptonic matrices, and we may drop the primes:

\[
G^2 \frac{\text{Trace }}{8 \ k E_q} \{(\gamma^1_5 - 1) \ \widetilde{\gamma}^1 \ (1 + \vec{\gamma} \cdot \vec{s})(1 - x \vec{\gamma} \cdot \vec{z}')(\vec{m} - i\vec{q}')(1 - x \vec{\gamma} \cdot \vec{z}')(1 + \gamma^0_5)\}
\]

First averaging over the neutrino direction, and then shuffling the Dirac matrices, one can obtain

\[
G^2 \frac{\text{Trace }}{4 \ k E_q} \{(\gamma^1_5 - 1) \ i k \beta \ [(1 + 3x^2) - 2x(1-x) \vec{\gamma} \cdot \vec{s} ] \ \vec{q} \}
\]

If one now makes use of the fact that \( \vec{\gamma} \cdot \vec{s} = -i \ \gamma^5_5 \beta \), one has only to evaluate

\[
\text{Trace } \{(\gamma^1_5 - 1) \ [(1+3x^2)i \beta + 2x(1-x) \gamma^5_5 \beta] \ \vec{q} \}
\]

which is

\[
\frac{\text{dE}_q}{(1+3x^2) + 8x(1-x) \vec{r} \cdot \vec{s} \gamma}.
\]

If again we define \( \vec{q} = \vec{E}_q \), we get at last:

\[
\frac{1}{4\pi} \int |\Sigma|^2 \ d\Omega_k = G^2 \frac{1 + 3x^2}{(1 + 3x^2)} \{1 - \frac{2x(x-1) \vec{r} \cdot \vec{s} \gamma}{1 + 3x^2} \}
\]

The result of this experiment was that the correlation term is very small. (The correlation has the extrema \( +1/3 \) at \( x = 1/3 \), and \(-1 \) at \( x = -1 \)). Again, however, the experiment initially gave wrong answers, even though the equipment worked perfectly. The reason was that, for security, they measured the recoil proton as well as the electron. In other words, to avoid background effects, they checked that all their counted electrons were accompanied by recoil protons. The trouble was that not all of the recoil protons were picked up in their experimental arrangement; only those with their momenta in a certain solid angle were detected. But their definition of a true event then meant that a recoil proton had to come off in a definite portion of the solid angle. They were prejudicing the results, therefore, because there is a multiple angular
correlation involved. First of all, there is the correlation between the spin of the neutron and the electron's momentum, and secondly, there is the correlation between the electron's direction and that of the antineutrino. Thus a correlation also exists between the neutron spin orientation and the proton's momentum, which is the negative of the sum of the momentum of the electron and the antineutrino's momentum. Therefore, by forcing the protons to go into a certain solid angle, they were exerting a bias as to the electron direction. In this way, the results were loused up, and instead of getting about an 11\% term, they got a 37\% term. When they improved the experimental arrangement so that recoil protons were picked up in essentially every direction, the result dropped down to around 11\%. This implied that \( x \) is either slightly larger than 1, or slightly less than 0. The root slightly larger than 1 is chosen because from the rates we know that the magnitude of \( x \) must be about 1.

By investigating the difference between the results from the two experimental arrangements, they could find the correlation between the neutron spin and the antineutrino direction. This told them the nature of the \( \beta \)-decay interaction, namely that it's vector and axial vector, which of course, we know now from other experiments. But that experiment all by itself really determines almost everything. Its main value now, however, is that it determines \( x \). The result is:

\[
x = -\frac{G_A}{G_V} = 1.25 \pm 0.04.
\]

This agrees well with the \( \beta \)-decay rates which gave \( |x| = 1.19 \pm 0.04 \).

So \( x \) is presumably very close to 1, but it is significantly different from unity. How are we to interpret that? If \( x \) were exactly equal to 1, we could write the interaction in this way:

\[
\sqrt{2} G_V \left[ \bar{p} \gamma^\alpha \frac{1 + \gamma_5}{\sqrt{2}} n \right] \left[ \bar{e} \gamma^\alpha \frac{1 + \gamma_5}{\sqrt{2}} \nu \right] + H.A.
\]

We would conclude that a certain chiral vector charge exchange interaction of quite short range exists between a nucleon and a lepton. The current for the interaction is completely parity non-conserving, and insofar as participation in the interaction is concerned, the electron, neutrino, proton, and the neutron, each participate only when its helicity is -1. This would be extremely simple, and one must ask why the observed \( \beta \)-decay interaction is not like this. The interpretation of the difference is fairly straightforward actually, and such a difference was expected - although not precisely of the kind found.

What we have been dealing with heretofore is presumably not the fundamental interaction; instead it is the result of the modification of the fundamental
interaction by the mesonic cloud. The nucleon has strong interactions and, as a result of these strong interactions, the β-decay interaction that we test in nuclear physics, whose parameters we have evaluated, is not the interaction of the electron and neutrino with the bare, or theoretical, proton and neutron.

From all our experience, the simplicity in an interaction occurs only when one discusses the bare particles. For example, we believe that the strong interactions, when stated in terms of the bare particles and the unrenormalized coupling constants, have a great deal of symmetry, which is disturbed in practice by the electromagnetic interactions not possessing the same symmetry. There is no reason why the situation should not be entirely similar in the case of the weak interactions. The bare particle weak couplings may be expected to show their own symmetries which are disturbed in practice by the strong interactions.

The factor 1.25 is thus viewed as arising from the relative renormalization of the vector and axial vector parts of a symmetrical bare interaction. Although this renormalization effect is compatible with our ideas on how these things should go, it is still a little mysterious as to why the factor is so close to unity. Many people had expected a factor of four or more, and it is not idiotic to suspect that the factor could have changed to 1.25 from -1 instead of from +1, although it is much more pleasing if it was +1.

Assuming that the bare vertex is given by

\[ G(\gamma_a \tau^+ + \gamma_a \gamma_5 \tau^+), \]

let us now take a look at what we might expect the complete vertex to be. This is exactly the same question that came up in the treatment of electrodynamics. It was found by the application of relativistic invariance and symmetry considerations that all the higher order corrections could only lead to a change of the original vertex \( e \gamma_a \) to a complete vertex of the form:

\[ e \gamma_a \frac{F(q^2)}{2m} \cdot \sigma_{\alpha\beta} q_\beta G(q^2). \]

Analogously, we expect that the weak bare vertex, when one puts in all the corrections due to the strong and electromagnetic interactions, will be changed to:

\[ G_V \gamma_a \tau^+ F_1(q^2) + A \sigma_{\alpha\beta} q_\beta \tau^+ F_2(q^2) \]
\[ - G_A \gamma_a \gamma_5 \tau^+ F_3(q^2) + B \gamma_5 q_\alpha \tau^+ F_4(q^2). \]

Now there is a point here that we shall give some attention. In the case of electrodynamics, at zero \( q^2 \) the coefficient of \( \gamma_a \) in the complete vertex was the same as the coefficient of \( \gamma_a \) in the original vertex. That's
because the current in electromagnetism is conserved, and thus the charge of the proton including mesonic effects is exactly the same as the charge of the proton not including mesonic effects. Earlier in the course, this situation was expressed by saying that in electrodynamics $Z_2 = Z_1$, or the corrections from the vertex to the charge are cancelled exactly by corrections to the propagator. In the case of the weak interactions we have no assurance that there is an analogous conserved current, and therefore neither $G_V$ nor $-G_A$ is expected to equal $G$

So far in our work we have put $F_1(q^2)$ and $F_3(q^2)$ equal to 1; we have ignored any effects of a shape of the vertex, and we have ignored the induced terms by using $A = B = 0$. This neglect is very reasonable, since the energies of the $\beta$-decays are of the order of at most a few $\text{MEV}$, whereas the shapes of the nucleon due to mesonic effects involve in the Fourier transform momenta of the order of the meson mass. At the very low momentum transfers characteristic of nuclear $\beta$-decays, all the corrections we have omitted are extremely tiny.

Upon reference to the derivation of the form for the complete vertex in electrodynamics (see pages 135-6), it will be noted that we made some use of the existence of so-called "mirror" diagrams. The "mirror" diagrams do not exist for the vertex which changes the charge, and so to eliminate some of the other possible forms for terms in the complete vertex a few additional assumptions have to be made. These are not very restrictive in nature, but unless they are made the complete vertex can have some more tiny corrections.
March 31, 1959

Last time the idea was set forth and discussed that the phenomenological \( \beta \)-decay interaction may be regarded as the result of the modification of a fundamental four-fermion point interaction

\[
\sqrt{2} G \left\{ \bar{p} \gamma_a \frac{1 + \gamma_5}{\sqrt{2}} n \right\} \left\{ \bar{e} \gamma_a \frac{1 + \gamma_5}{\sqrt{2}} e \right\} + H.A.
\]

due to the action of meson clouds around physical nucleons. The strong couplings do not affect the leptonic factor in such an interaction, but may be expected to change the nucleonic factor in the following way:

\[
G \gamma_a \rightarrow G_V \gamma_a F_1(q^2) + A \sigma_{q\rho} q_p F_2(q^2)
\]

\[
G \gamma_a \gamma_5 \rightarrow -G_A \gamma_a \gamma_5 F_3(q^2) + B \gamma_5 q_a F_4(q^2),
\]

where \( q \) is the four-momentum transfer from the nucleons to the leptons at the weak interaction vertex, and \( F_n(0) = 1 \). Altogether in \( \beta \)-decay, \( G_V \) and \( G_A \) are about the only things we can expect to see; \( A \) and \( B \) cannot be expected to be large enough to overcome the smallness of the momentum transfer factor \( q \) in \( \beta \)-decay, and the form factors \( F_n(q^2) \), if our knowledge of electromagnetic nucleon form factors is used as a guide, will not begin to deviate significantly from unity until \( q^2 \) becomes of the order of the square of a few hundred MEV, whereas in \( \beta \)-decay \( q^2 \approx (10 \text{ MEV})^2 \).

### Muon Decay

We have spoken about the coupling of the muon in our qualitative discussion of a possible universality among the weak couplings. The question of such a universality is now a precise quantitative query: Is the muon coupling identical to that of the electron? In this case for every interaction involving the electronic factor

\[
\left\{ \bar{e} \gamma_a \frac{1 + \gamma_5}{\sqrt{2}} e \right\},
\]

there is a corresponding interaction involving the muonic factor

\[
\left\{ \bar{\mu} \gamma_a \frac{1 + \gamma_5}{\sqrt{2}} \mu \right\}.
\]

The \( \nu' \) is to be massless, like the \( \nu \), but we need not commit ourselves further as to whether it is identical with \( \nu \) or not. The possibility that \( \nu' \) is \( \bar{\nu} \) can be excluded by experiment.
Now how would one find out the coupling of the muon with the nucleon? Nuclei do not undergo muon decay, because the energies involved are far less than the mass of a muon. However, much like electron capture in $\beta$-decay, the process $\mu^- + p \rightarrow n + \nu'$ does occur, and the inquiry about universality now takes the form of asking whether the $\mu$-capture process goes in exactly the same way as the electron capture process. To answer this we would need a lot of experiments on $\mu^-$ absorption in nuclei: an experiment to determine the ratio of Fermi couplings to Gamow-Teller couplings; a couple more to determine if the Fermi coupling is vector, and the G-T is axial vector; another to determine that there is 100% parity non-conservation, but that $T$ or $CP$ is conserved; and finally, one to determine whether it is $(1 + \gamma_5)$ or $(1 - \gamma_5)$, i.e. whether the muon is coupled in a left-handed fashion or a right-handed fashion.

None of these experiments has been done. All that is known from $\mu^-$ absorption is that if the interaction is of the same form, i.e. V-A, then $G_V^\mu$ and $G_A^\mu$ for the muon are the same as $G_V^e$ and $G_A^e$ for the electron to within 20% or so. Experimental information on the process $\mu^- + p \rightarrow \nu' + n$ is very scarce; mainly what there is, is just the rate of $\mu^-$ absorption in certain simple nuclei, which can be compared with a corresponding $\beta$-decay. For example, the decay rate of

$$B^{12} \rightarrow C^{12} + e + \bar{\nu} \quad \Delta J = 1, \text{ no}$$

is known, and if the capture rate for the process

$$\mu^- + C^{12} \rightarrow B^{12} + \nu', \quad \Delta J = 1, \text{ no}$$

going to the ground state of $B^{12}$, is measured, one can determine the ratio of the muon axial vector coupling constant to the electron axial vector coupling constant, since after a small correction due to the different momentum transfers, the nuclear matrix elements are the same. Within the various errors, which amount to about 20%, the ratio is unity.

However, there is one extremely sensitive test of the identity of the muon interaction with the electron interaction, which gives a perfect result. That is the measurement of the ratio of the rates of $\pi^+ \rightarrow \bar{e} + \nu$, and $\pi^+ \rightarrow \mu + \nu'$. If the couplings are identical, then for every diagram involving a mess of strong interactions in the $\pi$-$e$ decay matrix element symbolized by
there is a corresponding diagram involving the same structure of strong interactions for $\pi - \mu$ decay matrix element, which may be symbolized by

\[ \pi^+, q \leftrightarrow \mu^- \]

Now there are also some other diagrams yielding a contribution to the $R$-matrix element which are of first order in the weak coupling constant. However these other diagrams involve the exchange of photons between the final charged lepton and the charged baryons and mesons symbolized by the blob. The existence of such diagrams will be neglected in the following argument, because they involve the relatively small electromagnetic coupling constant.

In comparing the rate of $\pi - e$ decay to that of $\pi - \mu$ decay it does not matter at all what the absolute value of the meson-baryon factor in the $R$-matrix element may be, since it will cancel out assuming the weak coupling of $\mu\nu'$ is identical to that of $e\nu$. The only things that cause a difference in the rates are the difference in the leptonic factor in the $R$-matrix element and the variation in the density of final states. This simplicity was noticed about ten years ago by Finkelstein and Ruderman, who made predictions for the ratio of the rates for all the various $\beta$-decay interactions.

Let us calculate the rate for $\pi - e$ decay by symbolically summing all the diagrams of the kind indicated above. All we have to know about the blobs is their form, which, since the $\beta$-decay interaction is vector-axial vector, must be a combination of vector and axial vector, too. But there is only one parameter characterizing the meson-baryon blobs; that is $q$, the four momentum of the pion. From only one four vector it is impossible to form a pseudovector, and so all the blobs must be vector in form. Therefore, neglecting electromagnetic radiative corrections, we may write to first order in the weak couplings:

\[ \pi^+, q \leftrightarrow e^- \]
\[ R_{fi} = \sqrt{2m_{\pi}E_p} \{ \nu^\mu \gamma_a(1 + \gamma_5) \nu_e \} q_a F(q^2)/\sqrt{2E_q} \]

where \( q = p + k \), and \( q^2 = -m^2_{\pi} \).

To get an expression for the rate, we have to sum the absolute square of the R-matrix element over all the leptonic spin states, and then calculate the density of final states. The first step is to notice that \( q_a \gamma_a = \gamma = \gamma + \kappa \), and since \( -\gamma \nu_e = i \gamma_p \gamma_e \), we have

\[ \{ \bar{\nu}^\mu_{\kappa} \gamma_a(1 + \gamma_5) \nu_e \} q_a = -i \gamma \{ \bar{\nu}^\mu_{\kappa}(1 - \gamma_5) \nu_e \} \].

Therefore,

\[ \Sigma |R|^2 = m^2 \frac{|F(-m^2_{\mu})|^2}{2E_q} \text{Trace} \left\{ \frac{i(-m_{\pi} - i\gamma)}{2E_p} (1 + \gamma_5) \frac{1}{2k} (1 - \gamma_5) \right\}. \]

Using the relation \( k^2 = 0 \), one obtains:

\[ -p \cdot k = (k - q) \cdot k = k(1 - \nu_{\mu} \cdot \kappa). \]

We now will specialize to the case \( \nu_{\mu} = 0 \), i.e. pion at rest, to calculate the density of states. In the rest system of the pion, \( E = k + E_p = k + \sqrt{m^2 + k^2} \), and thus

\[ \frac{dE}{dk} = 1 + \frac{k}{E_p} = \frac{E}{E_p} = E_{\pi}, \]

Also, as a result of

\[ m_{\pi} = k + \sqrt{m^2 + k^2}, k = \frac{m_{\pi}}{2} (1 - m^2_{\pi}/m^2_{\mu}). \]

Therefore,

\[ \Gamma = 2\pi^2 |R|^2 \frac{m_{\pi}}{(2\pi)^3} \frac{dE}{dk} \frac{k^2}{m_{\pi}} \frac{F(-m^2_{\mu})^2}{m^2_{\mu}} m^2(1 - m^2_{\mu}/m^2_{\pi})^2. \]

If the weak coupling of \( \mu \nu \) is the same as the weak coupling of \( e \nu \), then we conclude that, up to radiative corrections,

\[ \frac{\Gamma_{\mu \nu}}{\Gamma_{e \nu}} = \frac{m^2_{\mu}(1 - m^2_{\mu}/m^2_{\pi})^2}{m^2_{e}(1 - m^2_{e}/m^2_{\pi})^2} = 1.3 \times 10^{-4}. \]

Recent experiments, among the earliest being the one by Tollestrup and others at CERN, do indeed find that the decay of the pion into an electron and neutrino occurs with the predicted branching ratio.

Despite the extreme paucity of experiments on muon absorption in nuclei, we may consider that this agreement in a branching ratio constitutes very convincing proof that the muon weak coupling is essentially identical to the
electron weak coupling. I say "essentially" because this branching ratio is
obviously independent of whether the muon coupling occurs through
\{\bar{\mu} \gamma_\alpha (1 + \gamma_5) \nu \}' or through \{\bar{\mu} \gamma_\alpha (1 - \gamma_5) \nu \}'}, i.e. whether the \nu' and \mu are
left-handed as are the \nu and e, or whether they are right-handed. The sign
of \gamma_5 may be inferred directly by examining the longitudinal polarization of
the muons in the decay of a pion. In this two-body decay, the muon comes out
in exactly the opposite direction from the neutrino. The \nu' will be completely
polarized, left-handed if \(1 + \gamma_5\) occurs in the coupling, and right-handed
if \(1 - \gamma_5\) occurs. But since the pion is spinless, the longitudinal polar-
ization of the muon will be the same as that of the \nu', and therefore such
an experiment determines directly the sign of the \gamma_5 term. Unfortunately
the experiment is quite difficult, and has not been done yet.

However, we may go on, using the hypothesis of a universal Fermi inter-
action as symbolized by the Feynman triangle, to obtain the sign of \gamma_5 indirectly.
For this purpose, we now shall look at the features of the decay of the muon.

The obvious thing to try in an attempt to explain \mu-decay, in the spirit of
universality, is the coupling:

\[ \frac{1}{\sqrt{2}} G \{\bar{\mu} \gamma_\alpha (1 \pm \gamma_5) \nu \}' \{\nu \gamma_\alpha (1 \pm \gamma_5) e\} + H.A. \]

What we wish to do is to measure the sign of the \gamma_5 term which is left open,
and then to check precisely that such an interaction really does explain all
the properties of \mu-decay, and finally to measure G. The measurement of G
is especially interesting, because G is expected to be the unrenormalized
coupling constant of \beta-decay. The muon, electron, \nu, and \nu' do not have
any strong couplings, and thus, apart from a tiny, calculable electromagnetic
effect, the G here should be the same as that characterizing the bare coupling
of e, \nu to n, p.

If the spectrum of electrons produced in \mu-decay is calculated for
the two cases \((1 + \gamma_5)\) and \((1 - \gamma_5)\), it is found that the two spectra are
completely different, and the experimental spectrum agrees exactly with the
\((1 + \gamma_5)\) one. Therefore, \nu' is left-handed, and thus \nu' may or may not be
identical with \nu. This will remain a permanently unanswered question;
there is no remotely practical way of telling. There are, of course, experi-
ments in principle which could distinguish between the case \nu' = \nu and the
case \nu' \neq \nu, but no such experiment is feasible. For example, if one could
accelerate an atomic pile to the speed of light, one could check to see
whether the very-energetic, meta-violet shifted pile antineutrinos produce \mu+
in the same way that e\(^+\) are produced in the Reines and Cowan experiment. No particularly practical way of doing this comes to mind, but in principle this does distinguish between the two cases. Or another possibility would be to carry out a Reines and Cowan experiment with the \(\bar{\nu}'\) from \(\pi^-\mu\) decay. But the intensity of \(\bar{\nu}'\) from an accelerating machine producing pions is so tiny compared to the intensity from a pile that the experiment is again impossible. So all we can say is that \(\nu'\) acts like \(\nu\) in that it is massless and left-handed, but we don't know if \(\nu'\) is \(\nu\) or not. It may be remarked now, that later on we shall encounter a possible test of this identity which is feasible, but unfortunately the interpretation is by no means unique.

Knowing the helicity of \(\nu'\) from a rough inspection of the electron spectrum in muon decay, we now have a complete theory, and we should be able to predict all the features of \(\mu\)-decay precisely. Every experimental confirmation of these predictions provides a check on the theory. In particular, we can now go back and compute the electron spectrum more exactly to see just how good the agreement really is. The next thing we shall look at is the rate of \(\mu\)-decay, which determines \(G\). Then the asymmetry of electrons from polarized muons may be computed. One hundred percent polarized muons are produced in pion decay, and we think we know the sign of the polarization, so we can make an exact prediction of magnitude, sign, and energy dependence of the asymmetry, which may then be compared with experiment. All of these predictions agree quite well with the experimental observations, and so the conclusion is that, to within the accuracy of the experiments, we have a perfect theory of \(\mu\)-decay. It will be interesting to see, when the accuracy of the experiment is improved, whether it remains perfect. Eventually highly refined studies of the spectrum might show some non-locality in the interaction. At present the point interaction is sufficient to explain everything that has been observed.

With this theory, let us now calculate some of the features of \(\mu\)-decay, starting with a \(\mu^+\) at rest with its spin in the \(\hat{s}\) direction. However, to perform the algebra in a more or less covariant fashion, I shall assign to the \(\mu^+\) a four momentum \(q\), and a spin in the \(s\hat{q}\) direction, \((s = \pm 1)\). The momentum of each decay particle is shown in the following diagram:

(See next page)
For such a decay the lowest order contribution to the R-matrix element is:

\[
\frac{G}{\sqrt{2}} \left\{ \gamma_v \gamma^e_v (1 + \gamma_5^e) \gamma_{-p} \right\} \left\{ \gamma_{-q} \gamma^e_{-q} (1 + \gamma_5^e) \gamma_{+k} \right\} \sqrt{m_\mu/m_p} \frac{E_p E_q}{p \cdot q}.
\]

As usual we desire to sum the absolute square of this expression over the spin states of the $e^+$, $\nu$, and $\bar{\nu}'$. Using the technique of traces, one can easily obtain:

\[
\sum |R|^2 = \frac{G^2}{2E_p E_q} \left( \frac{(-i\mathbf{y}) \gamma^e_\alpha (1 + \gamma_5^e)(-i\mathbf{y}) \gamma^e_\sigma (1 + \gamma_5^e)}{2} \right) \times \frac{1}{4} \text{Trace} \left\{ (1 + \gamma_5^e) \gamma^e_\sigma (1 - \frac{sL \cdot \hat{q}}{2} (\mathbf{\mu} - i\mathbf{d}) \gamma^e_\sigma (1 + \gamma_5^e) \right\}.
\]

Note first that the mass terms in the traces can be omitted since the trace of an odd number of $\gamma$-matrices is zero. Then it is apparent that only the second trace need be evaluated, for the first can be obtained from the second by doubling it, setting $s = 0$, and replacing $k^1$ by $p$, $q$ by $k$. By juggling some of the matrices, the second trace can be put in the form

\[
-\frac{1}{4} \text{Trace} \left\{ (1 + \gamma_5^e) (1 - \frac{sL \cdot \hat{q}}{2}) \mathbf{d} \right\} \gamma^e_\sigma (1 + \gamma_5^e) \right\}.
\]

which can be evaluated in four parts.

\[
-\frac{1}{4} \text{Trace} \left\{ \mathbf{d} \gamma^e_\sigma (1 + \gamma_5^e) \right\} = \delta_\sigma^\alpha \mathbf{q} \cdot k^1 - \mathbf{q}_\sigma \cdot k^1 \right\}.
\]

\[
-\frac{1}{4} \text{Trace} \left\{ \gamma_5^e \mathbf{d} \gamma^e_\sigma \right\} = -\epsilon_{\lambda\rho\sigma\sigma} \mathbf{q}_\lambda \cdot k^1 \right\}.
\]

The evaluation of the third and fourth parts is facilitated by the relations:

\[
\hat{z} \cdot \hat{q} = -i \gamma_5 \mathbf{\beta} \hat{d} \right\} \gamma^e_\sigma (1 + \gamma_5^e) \right\} = -i \mathbf{q} \cdot \mathbf{d} = \mathbf{q}_\mu \hat{d} \right\}.
\]

Employing these, we can show:

\[
-\mathbf{s} \frac{1}{4} \text{Trace} \left\{ (\mathbf{\beta} \cdot \mathbf{q}^e_q - \mathbf{q}_\mu \hat{d} \right\} \gamma^e_\sigma (1 + \gamma_5^e) \right\} = -i \mathbf{q} \cdot \left( \delta_\lambda^\sigma \mathbf{k}^1_\lambda + \delta_\lambda^\omega \mathbf{k}^1_\omega - \delta_\alpha^\sigma \mathbf{k}^1_\mu \right) + i \mathbf{q}_\mu \left( \mathbf{q}_\sigma \cdot \mathbf{k}^1_\sigma + \delta_\alpha^\sigma \mathbf{k}^1_\sigma - \delta_\alpha^\sigma \hat{d} \cdot \mathbf{k}^1 \right).
\]

Similarly,
Using the symmetry mentioned, the result for the first trace can be written immediately. It is

\[ 2 \{ \delta_{\alpha\sigma} p^k - p_\alpha k_0 - p_\sigma k_\alpha + \varepsilon_{\lambda\rho\sigma} p_\lambda k_\rho \}. \]

If one makes use of \( \varepsilon_{\lambda\rho\sigma} \varepsilon_{\omega\gamma\delta} = 2(\delta_{\lambda\omega} \delta_{\rho\gamma} - \delta_{\lambda\gamma} \delta_{\rho\omega}) \) and exploits fully the symmetry properties of the various parts of the traces in the indices \( \alpha \) and \( \sigma \), it is not very laborious to calculate the product of the two traces, which comes out to be:

\[ 8 \{ \langle \mathbf{p} \cdot \mathbf{k} + s [(-i\mathbf{q}_\perp)^2] \mathbf{q} \cdot \mathbf{k} - (-i\mathbf{q}_\perp)^2 \mathbf{q} \rangle \}. \]

Therefore,

\[ \Sigma |R|^2 = 4 G^2 \left\{ \frac{p^k}{\mathbf{E}^2} \left\{ \frac{q^k}{\mathbf{E}^2} s(\mathbf{q} \cdot \mathbf{k} - |\mathbf{v}|) \right\} \right\}. \]

Writing \( \mathbf{p} = \mathbf{p} / E \), and specializing to the case \( q = 0 \), spin in \( \mathbf{s} \) direction, one obtains simply:

\[ \Sigma |R|^2 = 4 G^2 \left( 1 - \mathbf{p} \cdot \mathbf{k} \right) \left( 1 - \mathbf{s} \cdot \mathbf{k} \right). \]

From this formula, we see that there are two correlations in the decay of the muon. The \( e^+ \) and \( \nu \) tend to come off in opposite directions, and the \( \nu \) tends to be emitted opposite to the direction of the spin of the \( \mu^+ \).
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To obtain the spectrum, rate, and angular correlation, it is only necessary to insert the density of states and to integrate over the unobserved parameters.

\[
d\Gamma = \frac{4G^2}{(2\pi)^5} (1 - \frac{\vec{p} \cdot \vec{k}'}{\mu - k' - E_p - k}) \delta(\mu - k' - E_p - k) \frac{d^3p}{d^2k'}
\]

The kinematics of the process yields the equations

\[
\vec{k} = -\frac{\vec{p} + \vec{k}'}{\mu - k' - E_p}, \quad k = \sqrt{k'^2 + p^2 + 2p \cdot k'}.
\]

It is convenient to work with the variables \( p, k', x = \frac{\vec{p} \cdot \vec{k'}}{\mu - k' - E_p}, y = \frac{\vec{p} \cdot \vec{s}}{\mu - k' - E_p} \), and two other angular variables, which may be quickly integrated out to give \((2\pi)^2\).

\[
d\Gamma = \frac{4G^2}{(2\pi)^3} (1 - \frac{x}{\mu - E_p} - \frac{1 - y}{\mu - E_p} - x - y) \delta(\mu - k' - E_p - k) \frac{dx dy dp dk'}{(2\pi)^3}
\]

For simplicity in the following integrations, we shall neglect the electron's mass \( m \). This is a very good approximation except at the low energy end of the electron spectrum, because \( m/\mu < 1/200 \).

If one writes:

\[
\delta(\mu - k' - p - \sqrt{k'^2 + p^2 + 2pk'x}) = \frac{\mu - k' - p}{pk'} \delta(x - x'),
\]

where \( x' = 1 + \mu^2/2pk' - \mu/p - \mu/k' \), the integration over the variable \( x \) is trivial. The condition that \( x' \) be within the range of integration, i.e. \(-1 \leq x' \leq 1\), is equivalent to the following conditions:

\[
p \leq \mu/2, \quad k' \leq \mu/2, \quad \text{and} \quad p + k' \geq \mu/2.
\]

\[
d\Gamma = \frac{G^2}{2\pi^3} \frac{\mu/2}{\mu/2} \int \frac{d^3p}{\mu/2} \left( \mu k' + \mu p - \mu^2/2 \right) \left[ (\mu - k' - p)(1 - y) - y \frac{\mu}{p} (k' - \mu/2) \right] dk'.
\]

The integration is simplified if we define \( P = p/\mu \), and change from the variable \( k' \) to \( w = (k'/\mu + P - 1/2) \). Then,

\[
d\Gamma = \frac{G^2}{2\pi^3} \frac{\mu/2}{\mu/2} \int \frac{P}{w} \left[ \left( \frac{1}{2} - w \right)(1 - y) - y \frac{\mu}{p} \right] dw
\]

\[
d\Gamma = \frac{G^2}{2\pi^3} \frac{\mu/2}{\mu/2} \int \frac{P}{w} (3 - \mu P) \left( 1 - y \frac{1 - \mu P}{3 - \mu P} \right) dP dy.
\]

One hundred percent polarized muons are obtained from pion decay. In practice, if these are positive muons, they retain their polarization to a
The depolarization factor varies from material to material, and results for no depolarization can be obtained by a fairly reliable extrapolation. On the other hand, the $\mu^-$ gets caught in matter and very complicated things ensue. Because the depolarization of negative muons is tremendous, the experiments are almost always done with $\mu^+$. If at any given value of the positron energy, the angular distribution of the positrons is measured, one gets $(1 + \alpha \cos \theta)$, with an asymmetry parameter $\alpha$, which is a function of the energy:

$$\alpha = \frac{1 + \lambda P - 1}{3 - 4\lambda P}$$

The asymmetry parameter varies from $-1/3$ at the minimum positron energy, where it gets very hard to measure, to $+1$ at the maximum positron energy. It changes sign at $P = 1/4$, i.e., at half the maximum energy. In an actual experimental situation, we would start with $\mu^+$, which decays into $\mu^+$ polarized opposite to their direction of motion. This means that the high energy positrons will come off backwards preferentially. This statement that something moves preferentially in a direction opposite to the direction of motion of some other thing is obviously invariant under CP. Therefore, the high energy electrons from the decay of $\mu^-$ also come off preferentially backwards, although the magnitude of the asymmetry is much smaller due to the large amount of depolarization.

To obtain the spectrum, we just integrate over $y$, which knocks out the asymmetry term, and gives us a factor of 2:
In the old days when there were many possible muon spectra because there were many possible theories, there used to be a parameter $\rho$ which characterized all the various spectra. For this particular spectrum, $\rho = 3/4$. This $\rho$ is no longer of use because we know the theory. If there are departures from the theory, they are presumably not in the direction of that family of spectra which people characterized by various $\rho$-values. So the $\rho$-value is an obsolete concept. All we can say is that the theory predicts this spectrum, with tiny electromagnetic corrections, which have been computed by Berman, and to the extent that this spectrum differs from experiment, something is wrong, but something is not the matter in the direction of a different $\rho$-value, because this would imply there should be a little admixture of tensor or something else in the interaction, which is ridiculous. If something is the matter, it's not that there is a little tensor coupling, but that there is some departure from a point interaction, or some mechanism underlying the weak interactions which we are just beginning to see. However, at the present time there is no discrepancy in this spectrum when the electromagnetic corrections are included.

The lifetime of the muon is obtained by performing the remaining integration, which gives easily

$$\Gamma = \frac{G^2 \mu^5}{192\pi^3} = \frac{1}{\tau}.$$  

Recently at Chicago the lifetime of the $\mu^+$ has been measured quite accurately to be $(2.261 \pm .007) \times 10^{-6}$ sec., and if the electromagnetic corrections are put in, which amount to a percent or so, the value of the bare weak coupling constant comes out to be:
If you refer to the value that was obtained from $\beta$-decay for $G_\nu$, you will find that it came out to be that same value. (However, we did not correct that determination for the effects of electromagnetism, so the agreement is a percent or so less perfect.) If we assume that the bare weak coupling constant for the nucleon is also $G$, then we see that the nuclear axial coupling constant has been modified by the meson cloud by a factor of about 1.2, whereas the vector constant has apparently not been changed by mesonic effects. That's queer! We did not expect them to come out so closely equal, and so we shall want to consider that further. It's either an accident, or it isn't, and if it isn't, one can speculate about what it might mean.

The only remaining quantity in muon decay which has been measured is the polarization of the resulting electrons. The experiments have very large errors, but do support the theory in that the positrons are right-handed, and the electrons show left-handed polarization.

Let us now pass on to a point of interpretation. It has been mentioned that the $\nu'$ and $\nu$ might be the very same particle, or might be different, and that there is no unambiguous feasible way of telling. From this point of view, let's see what people mean by saying that there is conservation of leptons. We have heard already of the conservation of baryons, which tells us that one cannot create or destroy a heavy, nucleon-like fermion - the number of baryons minus the number of antibaryons is a constant of nature. Can some similar statement be made about the leptons: $\mu$, $e$, $\nu$, and $\nu'$?

In our experience with $\beta$-decay, $\mu$-absorption, and $\mu$-decay, we notice that every time a $\mu$ is created, a $\nu'$ is destroyed, and that every time an electron is created, a $\nu$ is destroyed. Therefore, if the $\nu'$ is distinct from the $\nu$, there are evidently two types of leptons, and each type is conserved. That is, if $N_e$ equals the number of electrons minus the number of positrons, etc., from the structure of the couplings, the following holds:

$$N_\mu + N_{\nu'} = \text{constant},$$

$$N_e + N_\nu = \text{constant}.$$
These two rules are in accord with what is seen in nature. Furthermore, by adding the two relations, we have:

\[ N_{\text{leptons}} = N_e + N_\mu + N_\nu + N_{\nu'} = \text{constant}. \]

On the other hand, if \( \nu = \nu' \), the first two rules clearly do not hold, but we do have the relation:

\[ N_{\text{leptons}} = N_e + N_\nu + N_\mu = \text{constant}. \]

So in either case, we can define a family of particles called leptons, such that the number of particles in this family is conserved.

It is now a remarkable fact that those members of the lepton family which are called particles all seem to enter the weak interactions with the same "handedness". There is no known reason why this simplicity must occur.

It is just another one of those fundamental laws which are observed but not understood. From the existence of this one-to-one correlation between helicity and the "particles" of the lepton family, one is led to suspect a similar simplicity in the weak couplings of the baryons. That is, it is not unlikely in the bare weak couplings of the baryons, all the baryons enter with the same helicity and thus all the antibaryons enter with the opposite helicity. Assuming that the mesonic effects did not change the relative sign of the vector and axial vector couplings in \( \beta \)-decay, then under this conjecture the basic interaction is left-handed for all the baryons also. This conjecture cannot be verified yet because not enough is known about the strange particles.

**A Current-Current Interaction**

So far in our examination of the complete picture of the weak interactions, we have proposed that a simple interaction of the same form, when referred to "bare" particles, exists between any two of the following pairs of particles: \( np, ev, \) and \( \mu \nu' \). From this, one might be led to conjecture that all the pairs may interact with themselves in the same way that they interact with each other. This is equivalent to saying that if we construct two currents, made up from all the pairs and very closely related to each other, the fundamental weak interaction is just that of the two currents. There is no experimental evidence at the moment on the question of whether a pair interacts with itself, but it is not beyond experimental test.

The suggested theory for all the weak interactions would be:

\[ \mathcal{N}_{\text{weak}} = - J_+ a J_- a, \]

where
\[ J_{-a} = i \frac{1}{2} \left\{ \left[ -\bar{n} \gamma_a \frac{1}{\sqrt{2}} p \right] + \left[ -\bar{e} \gamma_a \frac{1}{\sqrt{2}} \nu \right] + \left[ -\bar{\mu} \gamma_a \frac{1}{\sqrt{2}} \nu' \right] + \ldots \right\} \]

and

\[ J_{+a} = i \frac{1}{2} \left\{ \left[ -\bar{p} \gamma_a \frac{1}{\sqrt{2}} n \right] + \left[ -\bar{\nu} \gamma_a \frac{1}{\sqrt{2}} e \right] + \left[ -\bar{\nu'} \gamma_a \frac{1}{\sqrt{2}} \mu \right] + \ldots \right\} \]

The \ldots represent whatever other stuff that might be needed to account for the strangeness changing weak interactions. The cross terms are just exactly those which explain \( \beta \)-decay, \( \mu \)-absorption, and \( \mu \)-decay. But the "squared" terms, or the self terms, represent entirely new physical interactions, which we have not discussed at all, and for which so far there are no clear-cut experimental tests. But in principle experiments can be designed which can test their existence, and these are presently just beyond the limits of experimental technique, but in the not-too-distant future I think they may become feasible.

The simplest one to discuss theoretically would be the interaction \((\bar{\nu} e)(\bar{e} \nu)\), which would give rise to scattering. The most observable process would be the production of energetic recoil electrons by antineutrinos from an atomic pile: \( \bar{\nu} + e \rightarrow e + \bar{\nu} \). The cross section for this first order weak process would be of the same order as that for the first order weak process \( \bar{\nu} + p \rightarrow n + e \), namely about \( 10^{-20} \) barns. However, it is much harder to do the scattering experiment than the Reines and Cowan experiment, because the coincidence technique which proved so valuable in the detection of \( \bar{\nu} \)-absorption is not available. The only thing one has to observe is that every once in a while an electron in matter will acquire an energy of the order of a few MEV, since the scattered \( \bar{\nu} \) does nothing. This is obviously much more difficult to detect than a coincidence between a neutron capture \( \gamma \)-ray and a positron annihilation quantum, but in principle it is not impossible.

Actually, Reines and Cowan were at one time planning to do such an experiment for a different reason, so it may not be so completely insane to think about doing it. Then it was thought that the neutrino might have a magnetic moment, which would give rise to scattering. A two-component neutrino, i.e. a neutrino such that \( \gamma_S \nu = \nu \), cannot have a magnetic moment, but it can still scatter off electrons by this weak interaction.

The fact that a neutrino which has only states of a definite helicity, say left-handed, cannot have a magnetic moment due to the weak interactions, is easy to demonstrate mathematically. Since \( \gamma_5 \bar{u}_k^\nu = u_k^\nu \),

\[ \bar{u}_k^\nu = -u_k^\nu \gamma_5, \quad \text{and} \]
The neutrino can have a charge distribution due to the weak interactions, in the same way that the neutron has a charge distribution due to the strong interactions. Because the neutrino has no charge, its charge form factor $F(q^2)$ has to be zero at $q^2 = 0$. For this reason, the electromagnetic interaction between a neutrino and an electron should be smaller than the proposed weak interaction, and so a measurement of scattering would definitely indicate a direct interaction.

There is a second possibly practical way to test for the self terms in the weak couplings, although it again would be very hard. That involves examining the consequences of the interaction $(\bar{p}n)(\bar{n}p)$ in nuclear physics. The interaction gives rise to a force between a neutron and a proton, which is of quite small range. The potential has a 6-function nature like a piece of the static nuclear potential which we calculated from meson theory, but when better account is taken of the relativistic effects, the potentials are spread out to at least the order of the nucleon Compton wavelength. In addition, the potential may be expected to be spread out to the order of a meson Compton wavelength since each piece of the interaction has a form factor due to the strong couplings.

This force has the property that it is extremely weak, which certainly doesn't help to detect it, but also the special property of being parity non-conserving. Thus if this self term is present, the nuclear forces are slightly parity non-conserving. The first question which must be posed is whether this feature is a unique characteristic of the self term, or whether it would come about in any case. Before Yukawa invented his meson theory of nuclear forces, people were talking about nuclear forces due to $\beta$-decay. And in fact, they supposed that such forces were responsible for the attraction between nucleons. Those forces arose from this process

and now that it is known that the $\beta$-decay interaction is parity non-conserving, such forces must also be parity non-conserving. Such forces must certainly exist, if it is true that at such weak strengths the concepts of field theory continue to apply. This was the earliest guess as to the origin of nuclear forces. It had the slight flaw that it was too small by a factor of $G^2$ or $10^{-14}$. 

\[
\bar{u}_p, \sigma_{\alpha p} u_\nu = \frac{1}{4} \bar{u}_p (1 - \gamma_5) \sigma_{\alpha p} (1 + \gamma_5) u_\nu = \frac{1}{4} \bar{u}_p, \sigma_{\alpha p} (1 - \gamma_5) u_\nu = 0.
\]
That didn't bother people, because divergent integrals were involved, and so they figured the divergences, if cut off at a sufficiently high point, would cancel the smallness of the coupling constant. Yukawa had a better theory of nuclear forces, but the original guess remains as a suspected contribution to the nuclear force, which is parity non-conserving. However, it is very tiny, being of order $G^2$, and so the parity non-conserving force due to the self term, which is of order $G$, can certainly be distinguished from the other one.

Thus the thing to do is to look for very small contaminations of parity non-conserving nuclear forces, which can be identified with the self term. An experiment has recently been done to detect a parity non-conserving nuclear force, but the results do not shed any light on the existence of the self term. What they claim to have done is such an accurate experiment that anything stronger in the way of a parity non-conserving force than that due to the self term can be excluded. They are just at the level where they might expect to see the self term, but that they haven't doesn't prove it isn't there. But anything stronger doesn't exist. In other words, no kind of a strong interaction fails to conserve parity, according to this result. Some people had conjectured, for example, that possibly the strong K-particle couplings were parity non-conserving. This experiment, if taken at face value, squelches that idea pretty thoroughly, because one would certainly expect the K-couplings to make some small contribution to nuclear forces in nuclear physics, even if it's only one part in $10^5$, or something. Adair, Haas, and Leipuner claim that there is no parity non-conserving part of the nuclear forces to about one part in $10^8$. The force due to a self term would be in this neighborhood so that it is not excluded. But it will be most interesting to see whether such an experiment can be pushed further, so that the question of the existence of a self term in the weak interactions can be answered.

Let us discuss what type of experiment was carried out and the nature of the force due to the self term in some detail. The basic idea is that a neutron is absorbed in a heavy nucleus, in this case Cd$^{113}$, and a capture resonant state of Cd$^{114}$ is made, which then decays to the ground state by the emission of a $\gamma$-ray. Very slow s-wave neutrons, polarized by magnetic scattering, are used, and their energy is adjusted so that the resonant capture state is a $1^+$ level. The ground state of Cd$^{114}$ is a $0^+$ level, and so if everything is perfectly reasonable and parity is conserved, the gamma ray is a $M_1$ photon. But if parity to a slight extent is not conserved, the nuclear states will have small admixtures
of the wrong parity, then there will also be a sort of a tiny, ghost-like El transition between the capture state and the ground state, which will interfere with the Ml and lead to a term in the angular distribution of the gamma rays of the form $\mathbf{s} \cdot \mathbf{k}$, (where $\mathbf{s}$ is the polarization direction of the neutron, and $\mathbf{k}$ is the momentum of the photon). Now they are, of course, not looking for an effect of order 1 part in $10^8$, but rather the argument is that in a heavy nucleus the effect will be tremendously enhanced.

Bearing such an experiment in mind, let us now look at what kind of an effect and what order we might expect theoretically if the self term in the weak interactions exists. The only part of the self term of interest is the parity non-conserving part, because the parity conserving part will just get lost in the tremendous jungle of nuclear forces due to the strong and electromagnetic interactions. We shall neglect in this discussion the various meson corrections which will change the interaction somewhat, giving it extra pieces and broadening the potential out to a distance of the order of a meson Compton wavelength. Our problem then is to derive from the self term an effective non-relativistic potential.

In this experiment, for the first time, the sign of the weak coupling constant becomes interesting. In most problems concerning the weak interactions, where the intensity of something or other is observed, the formulae involve $G^2$, and so the sign of $G$ doesn't matter. But in this problem, the interference term will involve the sign of $G$ relative to the sign of the regular nuclear forces, which is known. On the basis of our theory of the weak interactions so far, there is no way of telling what the sign of $G$ is. But further considerations, which are tied up with the idea that the weak interactions are mediated by a boson field, give a clear indication that the situation should be analogous to the case of electromagnetism, where in order that the photon field has a positive energy, there is one particular sign to the interaction, namely two like charges repel. Since the weak interactions are vector-axial one expects, if there is any kind of a field mediating the interactions, the same would be true, namely the two like pairs would repel. Thus the sign of $G$ might be expected to be positive, although we certainly cannot prove that it should be this way.

The parity non-conserving part of the R-matrix element due to the self term is:

$$\frac{G}{\sqrt{2}} \left\{ \left[ \bar{u}^p_{q^1} \gamma_\alpha \bar{u}^p_{k^1} \right] \left[ \bar{u}^n_{k^1} \gamma_\alpha \gamma_5 u^n_{q^1} \right] + \left[ \bar{u}^p_{q^1} \gamma_\alpha \gamma_5 u^n_{k^1} \right] \left[ \bar{u}^n_{k^1} \gamma_\alpha \bar{u}^p_{q^1} \right] \right\}.$$
In the standard representation, it is an easy task to pass to the non-relativistic limit by employing the approximation

\[ u_k = \begin{pmatrix} \chi \\ \sigma \cdot k \chi \\ 2M \end{pmatrix}. \]

Eliminating the lower components of the spinors in this way, one obtains the following expression for the matrix element

\[ \frac{G}{M \sqrt{2}} \{ \chi_{11}^+ \chi_{12}^+ \otimes \chi_{12}^- \chi_{11}^- \} , \]

where

\[ 2\theta = \sigma_2 \cdot (k' + q) - (\sigma_1 \cdot q^1 \sigma_{1j} + \sigma_{1j} \sigma_1 \cdot k) \sigma_{2j} + \sigma_1 \cdot (q' + k) \]

\[ - \sigma_{1j} (\sigma_2 \cdot k' \sigma_2 \cdot q') + i \sigma_1 \times \sigma_2 \cdot (q' - k). \]

If one judiciously employs the relation \( \sigma \cdot A \sigma \cdot B = A \cdot B + i \sigma \cdot (A \times B) \), and the conservation law \( q' + k' = q = k \), the above expression can be simplified to be:

\[ \theta = (\vec{\sigma}_1 - \vec{\sigma}_2) \cdot (\vec{q'} - \vec{q}) + i \vec{\sigma}_1 \times \vec{\sigma}_2 \cdot (\vec{q'} - \vec{k}). \]

Now if one prefers to talk about potentials, it is probably possible to invent some sort of potential which will give this matrix element in first Born approximation. It is pretty easy to do that for the second term, which clearly is given by the static potential

\[ \frac{G}{M \sqrt{2}} \{ \sigma_1 \times \vec{\sigma}_2 \cdot V x_1 - x_2 \delta^2_3(x_1 - x_2) \} \]

but the first term must come from a more complicated velocity-dependent potential. Since however it is unnecessary to work with a potential, we shall not bother to concoct one. All the inferences we wish to draw can be inferred from the nature of the matrix element.

The operator \( \Theta \), being parity non-conserving and antisymmetric in the spin matrices, has the property that it has matrix elements, in the two body np system, between singlet-even and triplet-odd states and between singlet-odd and triplet-even states only. Since \( \Theta \) is rotationally invariant, it conserves the total angular momentum. Thus, for example, it has matrix elements between \( 1S_0 \) and \( 3P_0, 3S_1 \) and \( 1P_1 \), etc. In a many-body nucleus, of course, the rules are more complicated.

From this operator \( \Theta \) we can deduce what is to be expected in the line of
an El contamination of a Ml photon. Since $\Theta$ involves the momenta of various particles there will be a direct or "catastrophic" term representing a point interaction between a photon, neutron, and a proton, which arises from the replacement of the proton momentum $q$ by $q + eA$, and is of the form:

$$ -\frac{i e G}{M \sqrt{2}} \mathbf{q}_1 \times \mathbf{q}_2 \cdot \mathbf{A}.$$ 

A diagram for this matrix element has the following form:

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but it should be remembered that the point nature of the diagram is only a result of the non-relativistic approximation. Relativistically, the point of contact of the neutron and proton is distinct from the point where the photon is emitted. This direct interaction term will have a non-vanishing matrix element between a $1^+$ capture level and a $0^+$ ground state level in general, which leads to the emission of an El photon.

April 7, 1959

We may estimate the magnitude of an interference term in a light nucleus using the catastrophic term. The interaction term responsible for the Ml deexcitation of the capture level is just $-\mathbf{\mu} \cdot \mathbf{B} = -\mu \mathbf{V} \cdot \mathbf{A}$, and the magnetic moment $\mu$ will be of order $e/\hbar$. Thus the magnitude of an asymmetry term exhibiting the parity non-conserving nature of the nuclear force due to a self term in the weak interactions, which is of the order of the ratio of the matrix element of the "catastrophic" term to the Ml matrix element, is

$$(G e \delta/M) \div (e p/M) = G \delta/p,$$

where $p$ is the magnitude of the momentum of the emitted photon. The $\delta$-function brings in the amplitude that a proton and a neutron are at the same spot. To estimate the effect of $\delta$, we have to estimate the probability of finding two nucleons close to each other. The most important thing is how close must they be. If the interaction really had the form of a $\delta$-function, we would have to worry about the fact that the nuclear forces are very repulsive at short distances, the so-called hard core, which prevents two nucleons from getting very close to each other. However, we may argue that the $\delta$-function is really quite spread out by the form factors, and that we don't have to worry too much about the effect of a hard core.
Probably the suppression effect is not too tremendous, and one can take the δ-function almost at its face value. The δ then brings in something like \(1/\text{(volume of a nucleon)}\) for the probability that a neutron and a proton are close to each other. Estimating that at \(m_n^3\), we have for the order of magnitude of an interference term:

\[
\frac{G_\delta}{p} \approx 10^{-5} \frac{m_n^3}{m^2} \approx 10^{-6}.
\]

Besides this direct term which gives an E1 transition between a \(1^+\) and a \(0^+\) state with a matrix element of order \(eg\), we also get a contribution of order \(eg\) from second order perturbation theory by going through \(1^-\) and \(0^-\) intermediate states. That is, there is a parity non-conserving portion of \(R_{\text{Pi}}\) coming from

\[
\sum_n \frac{H_{\text{in}}^H H_{\text{ni}}^H}{E_i + \Delta E - E_n},
\]

where one of the matrix elements is that of the self term in the weak interactions, and the other is the ordinary electric dipole matrix element. (Relativistically all the E1 matrix element to order \(eg\) comes in this two-stage form; only in the non-relativistic formalism does it appear that part of the E1 matrix element comes in the first order of perturbation theory from the "catastrophic" term.)

In a light nucleus where the density of levels is not too high, one would expect the two-stage contribution to be of the same order as the direct contribution, and thus the estimate of the interference term is unchanged. For a heavy nucleus, on the other hand, the density of levels is very high, and it is conceivable that the two-stage contribution might get much larger than the direct one. Now we shall not go into the arguments for the magnitude of the enhancement based on nuclear structure, but only state that Adair alleges that a factor of \(10^3\) in enhancement is not unreasonable in a heavy nucleus like Cd\(^{114}\). That would result in an interference term of order \(10^{-3}\), which is around the present sensitivity of their equipment. It would be very interesting if the experiment can be refined somewhat so that they could state definitely whether the interference term from a self-pair interaction does or does not exist.

The next thing to take up is the question of how, if a current-current interaction exists, we might "explain" it or "understand" it. The obvious mechanism for understanding it is the same as the electromagnetic one. In the electromagnetic case, the interaction is \(-j_a(x)D_F^{\alpha}(x-y)j_\beta(y)\), where \(D_F\) is the photon propagator. \(j_a(x)\) is the sum of a large number of terms, one for each of the charged fundamental particles. Such a structure bears a similarity to the proposed form for the weak interactions, where \(J_\alpha(x)\) consists of a number of
terms also. Moreover, the charge of every elementary particle is the same -- it is always $e$ -- and it appears that in the weak coupling, the constant may be always $G$. We do not know why these two things are so, but they could have a common explanation.

If one pursues this analogy, one might say that the reason why these similarities exist is that the weak interaction is really

$$-J_{+\alpha}(x) \Delta^F_{\alpha\beta}(x-y) J_{-\beta}(y),$$

where $\Delta^F$ is the propagator for some heavy particle, which is practically a $\delta$-function. If the particle, call it $\alpha\beta$ (symbol: $X$), has a sufficiently large mass, $M_X$, the momentum dependence of the propagator will not be seen. Conversely, the absence of momentum dependent terms in the matrix element which yields agreement with the electron spectrum in muon decay implies that $M_X^2$ must be large compared to $m_{\mu}^2$. If this requirement is met, there appears to be no objection to supposing that the weak interactions are mediated by a particle, i.e., are the results of a universal coupling to a certain field.

If we suppose that such a particle exists, we have to ask what its properties are. First of all, since it carries a vector interaction, it must be a vector particle (spin 1). It has already been remarked that its mass $M_X$ must be large. Also it must be charged, since the interaction is charge exchange in nature. Since it has a mass we must describe it with the equations for a vector meson, the Proca equations, and so whatever complications exist for a vector meson must be included. The bare propagator $\Delta^F_{\alpha\beta}(x-y)$ will be the Fourier transform of

$$\frac{\delta_{\alpha\beta} + p_{\alpha}\gamma(p^2 + M_X^2)}{p^2 + M_X^2 - i\epsilon},$$

and thus the interaction would be

$$M_X J_{-\alpha}(x) X_{\alpha}(x) + H.A.$$

(For a vector meson, the propagator is not just $\delta_{\alpha\beta}/(p^2 + M^2)$, which is what you might expect naively and which is what we used for photons, but is actually more complicated. You will remember that for photons we were forced at one point, when we were being fairly strict, to use $(\delta_{\alpha\beta} - p_{\alpha}p_{\beta}/p^2)/p^2$ instead of $\delta_{\alpha\beta}/p^2$. However, we argued that because the sources of the electromagnetic field are conserved, the $p_{\alpha}p_{\beta}$ part never played any role, and we could forget about it. In the case of the vector meson, the sources now being considered are not conserved, and thus we must employ the correct propagator.
\( \frac{p_\alpha p_\beta / M^2}{p^2 + M^2} \). It is really very easy to see physically why the correct propagator has the extra piece. For this purpose, imagine that we are in the rest frame of the vector meson, where only \( p_4 \) is non-zero. Then for a real vector meson, i.e. at the pole of the propagator, \( p = (0,0,0, \pm i M) \), and the residue of the propagator is \( \delta a_\alpha^R \), where \( R = 1 \) for \( \alpha = 1,2,3 \) and \( R = 0 \) for \( \alpha = 4 \). In other words, \( \delta a_\alpha^R \) exists only for the spatial components; for the 4th polarization it is cancelled by the extra piece. This is simply the mathematical expression of the physical fact that for a real vector meson there are only three polarizations, and if the meson is at rest these are all spatial. At rest you cannot have real temporally polarized vector mesons, because under rotations these would behave in a scalar fashion. This is the physical reason for the added complication in the propagator for a vector meson.

The dimensionless coupling strength would be \( a_w = \sqrt{2} G / 4 \pi \), and so if the uxl exists, there is a precise meaning to the "strength" of the weak interactions. If \( M_X \) were of the order of the nucleon mass, for example, then \( a_w \) would be of order \( 10^{-6} \). For the weak interactions to be characterized by a fairly strong dimensionless coupling, say as for electromagnetism, \( M_X \) would have to be about 100 nucleon masses.

As the mass of the uxl is varied, the dimensionless strength varies. We may impose a restriction on this strength by the fact that the uxl hasn't shown up experimentally. If \( M_X \sim 2m_n \), \( a_w \sim 10^{-7} \). The uxl may decay into anything it is coupled to, for example, pions (since it would be coupled to np), and the two pairs of leptons. Its lifetime would be of the order \( (1/a_w) 10^{-24} \) sec. \( \approx 10^{-17} \) sec. Not only would the uxl decay with such a lifetime, but the K particle would decay into the uxl with a comparable lifetime. The case \( M_X \sim 2m_n \) is thus impossible, because the K particle lives for \( 10^{-8} \) sec. The conclusion then is that \( M_X > M_K \), but it could be in the neighborhood of \( M_K \).

The uxl could be produced in almost any kind of high-energy collision, but the cross section would be very small -- on the order of \( 10^{-6} \) of geometrical. So it would be essentially impossible to detect it this way. Besides that, if produced, the uxl would decay practically immediately, therefore in the laboratory one would only observe a tiny cross section at high energies for the single production of an electron or muon. Another way to look for the uxl would be to produce it electromagnetically with its antiparticle.
This is also extremely difficult, since the pair-production cross section goes down tremendously fast with increasing mass. As a result of these considerations, it is clear that no experimental evidence exists today on whether the uxl exists or does not, once we impose the restriction that $M_X > M_K$.

If $M_X$ is not too large, perhaps the easiest way to test the existence of the uxl experimentally would be to measure the deviation in the electron spectrum in muon decay due to the momentum dependence of the propagator. If $M_X \sim M_K$, the change in the spectrum is an increase in the so-called $\rho$-value of .01, which is just about at the limit of experimental detectability now. Currently, the best measurement of the $\rho$-value involves an error of \( \pm .02 \). As $M_X$ gets heavier, the effect on the spectrum, of course, gets smaller and more difficult to observe. Thus for the direct detection of an uxl one must wait for further advances in experimental physics.

The best thing we can do now is to see whether there are any indirect theoretical tests for the existence of the uxl. If the uxl exists, can we find a situation that is affected and that can be checked out using known experimental information? Only one such thing has been suggested, and that is the effect of the intermediate field of the uxl on the decay of the muon. One makes the assumption that the uxl exists with $M_X \gg m$, and $\nu = \nu'$, and looks at the contribution to the rate for the decay $\mu \rightarrow e + \gamma$ from the following type of diagram in which there is a virtual neutrino loop.

\[ \begin{array}{c}
\mu \\
\downarrow
\end{array} \quad \begin{array}{c}
\epsilon \\
\uparrow
\end{array} \quad \begin{array}{c}
\text{XXX} \\
\end{array} \quad \begin{array}{c}
\mu
\end{array} \]

The photon can come off of any of the three charged particles. If there is no uxl a similar diagram can be drawn,

\[ \begin{array}{c}
\epsilon \\
\uparrow
\end{array} \quad \begin{array}{c}
\text{XXX} \\
\end{array} \quad \begin{array}{c}
\mu
\end{array} \]

but it is totally ambiguous whether this diagram should be included. It is highly divergent and seems to vanish by symmetry, but you can't be sure because it depends upon how you apply the symmetry arguments in a divergent integral. There is no way of saying definitely whether this diagram should be included, nor how it is to be evaluated, if the uxl does not exist. On the other hand, if the uxl exists, it is clear that the diagram definitely should be included. The integral is divergent, but only logarithmically in the
physically important part. Under the stated assumptions, the rate for the decay \( \mu \rightarrow e + \gamma \) may be calculated, and the result is:

\[
\frac{\Gamma_{\mu \rightarrow e+\gamma}}{\Gamma_{\mu \rightarrow e+\nu+\bar{\nu}}} = \frac{1}{1100} F(\lambda/M_X),
\]

where \( \lambda \) is the cutoff. For \( \lambda/M_X \gg 1 \), \( F(\lambda/M_X) \rightarrow (\log \lambda^2/M_X^2)^2 \), and for reasonable values of \( \lambda/M_X \), \( F \) is of the order of 1, so the branching ratio comes out to be about \( 10^{-3} \).

Experimentally, in 1954 Steinberger and Lokanathan set an upper limit on the branching ratio of \( 2 \cdot 10^{-5} \). Keeping all the assumptions, there is one way out. If \( \lambda/M_X < 1 \), \( F \rightarrow 0 \). But this is a rather stupid way out, for under that condition, you cannot believe field theory predictions concerning the behavior of the particle. There is still a meaning, of course, to the question of the existence of an uxl regardless of whether or not its mass is larger than the energy where field theory breaks down. But it is meaningless to discuss theoretically its behavior using field theory if \( \lambda/M_X \ll 1 \).

However, we may use the result that the effect of the loop diagram goes to zero for \( M_X \rightarrow \infty \) with a fixed cutoff to eliminate the loop diagram in the case of no uxl by defining the point interaction as the limit in which the mass of an intermediate boson gets very large for a fixed cutoff.

If we drop the assumption that \( \nu = \nu' \), then this loop diagram does not exist, and we have no contradiction with experiment, but also no test of the existence of the uxl. Thus it is perfectly possible that the experiment should be taken as evidence for a distinction between \( \nu \) and \( \nu' \), rather than as discouraging the existence of an uxl; we have no way of knowing which at the present.

Vector meson theory is very closely analogous to electromagnetism, which is a vector "meson" theory with a "massless meson", and just as electromagnetism is renormalizable, so neutral vector meson theory can be made renormalizable too. But charged vector meson theory including the coupling to electromagnetism is not renormalizable. If you believe very strongly in renormalization, then the uxl idea looks very bad. On the other hand, the four-fermion point interaction is violently unrenormalizable. If you are a fanatic for the idea that renormalizable theories are the only kind of physically useful theories, then you would have to say that the weak interactions have not been put on any kind of reasonable basis, with or without an uxl.

This is about all there is to say about the question of the universal interaction of two currents as being the explanation of the weak interactions. The status at this time is that it is an appealing possibility, but the evidence
on self-pair interactions is nil, as is the evidence on the existence of an intermediate boson.

The Conserved Vector Current Hypothesis

From now on, we shall suppose that the weak interaction is of the form $J_{\alpha} J_{-\alpha}$, but without an explanation, and we shall forget for the rest of the discussion any possible non-localities coming from a cutoff or coming from an ud. Nobody has seen any sign experimentally of such non-localities, and so it is an open question as to whether they are there or not. Our problem then is to try to determine the form of $J_{\alpha}$. We have certain terms which would have to be present, but there has to be at least one more term to account for strangeness-changing weak interactions, and there are probably many more terms.

For the neutron and proton, it seems extremely possible that the interaction in the bare form is the same as for the leptons, although we certainly cannot prove this. All we can say is that the experimental coefficients are very close to unity, namely they are 1.0 and 1.2, but whether that means that the theoretical bare coupling constants are both unity is just a matter of conjecture. These might even be 1 and -1, or $\sqrt{2}$ and $\frac{\sqrt{2}}{2}$, or something else. But anyway we shall assume provisionally that they are 1 and 1, and write for the form of the current:

$$J_{\alpha} = J_{\alpha}^\ell + J_{\alpha}^1 + J_{\alpha}^2$$

where $J_{\alpha}^\ell$ contains lepton pairs, $J_{\alpha}^1$ is strangeness conserving, and $J_{\alpha}^2$ is strangeness nonconserving. At this stage we write

$$J_{\alpha}^\ell = i 2^{1/4} g^{1/2} \left\{ \left[ \bar{e} \gamma_{\alpha} \frac{1 + \gamma_5}{\sqrt{2}} \nu \right] + \left[ \bar{\mu} \gamma_{\alpha} \frac{1 + \gamma_5}{\sqrt{2}} \nu' \right] + ... \right\}$$

leaving open the possibility that other leptons may exist, and

$$J_{\alpha}^1 = i 2^{1/4} g^{1/2} \left\{ \left[ \bar{\pi} \gamma_{\alpha} \frac{1 + \gamma_5}{\sqrt{2}} p \right] + ... \right\}$$

As far as nuclear physics is concerned, the only parts of the current that matter are those for which the strangeness does not change, and therefore we concern ourselves now with $J_{\alpha}^1$. So let's now return to the problem of nuclear $\beta$-decay and see what effects various forms for $J_{\alpha}^1$ would produce.

It was found that in nuclear $\beta$-decay practically the only things one could measure were the renormalized coupling constants $G_V$ and $-G_A$, almost anything else, involving some momentum transfer divided by a large energy like
the meson mass, constituting a very small correction. So the two important facts about $\beta$-decay are the effective coupling constants $G_V$ and $-G_A$. In the effective current only $(\bar{\nu}\bar{p})$ came into our considerations, but the term $(\bar{\nu}\bar{p})$ in the phenomenological current depends on all the terms in the bare current $J_\alpha^\perp$, because mesons and hyperons are in the cloud about a nucleon. We found, much to our surprise, that to within two percent, $G_V = G$. This fact is extremely remarkable, because one expects to get a renormalization factor which could be almost anything. Then we notice that $-G_A$ is also close to 1, 1.25 or something, but is significantly different from 1. What these two important facts mean is not clear. One suggestion that has been made is that the equality is very significant, whereas the closeness of $-G_A$ to $G$ is an accident. It may be, on the other hand, that neither of them is very significant -- that both of them are accidents. Or it could be that in some totally unexplicable way they are both understandable very easily -- it may be easily provable that they are both close to one. The only one of these that any theoretical substance has been given to is the conjecture that the equality $G_V = G$ means something, but that $-G_A \sim G$ is an accident. In other words, you can give an explanation, if you want to, which guarantees the equality, but nobody has ever found anything which explains the approximate equality.

If we seek an explanation of $G_V = G$, something which we do not have to do -- we may just regard it as an accident, we are led to a definite theory for $J_\alpha^\perp$. The theory is very simple, actually we can guess it immediately from our previous discussions. Recall that we compared the effects of meson couplings on the vector interaction in $\beta$-decay to the effects of strong couplings on the electromagnetic interaction. In the case of the electromagnetic interaction with the proton, we started with $e\gamma$ and stated that under the influence of the strong couplings, this would become effectively:

$$e\gamma F(q^2) - \mu e^{\perp} \sigma_{\alpha\beta} q_\beta G(q^2), \quad F(0) = G(0) = 1,$$

in which the $e$ is the same because the interaction was through a current that was divergenceless. This accounts for why a positron and a proton, starting out with the same bare charge, also has the same real charge, despite the fact that the proton has strong interactions, while the positron does not. Another way to describe this physically is to say that when the proton is dissociated, say, into a $\pi^+$ and a neutron, the charge is conserved. The statement that the charge is conserved means that the charge goes onto the $\pi^+$, so that the resulting system interacts with the electromagnetic field with
the same strength as the proton.

In $\beta$-decay, we had assumed no such thing, and thus we did not expect $G_V$ to equal $G$. The current $[\bar{n} \gamma_\alpha p]$ is by no means divergenceless, and the vector $\beta$-decay strength as given by that term is not conserved under the influence of the strong interactions. We can see this physically in the following way. A proton has a certain $\beta$-decay charge giving rise to the process $p \rightarrow n + \bar{\nu} + \nu$, but when it is dissociated into a neutron and a $\pi^+$, nothing can do that any more. So under the influence of the strong couplings, the effective $\beta$-decay charge of a proton will be changed from the bare value, i.e. $G_V$ will be entirely different from $G$. If we want to restore the electromagnetic situation, which is what we want to do in order to construct a theory of the equality of $G_V$ and $G$, all we have to do is to arrange it so as the $\beta$-decay charge is conserved. So we construct a theory in which the vector part of $J^\mu_\alpha$ is divergenceless mathematically, or physically we construct a theory in which the resulting system still has the same amplitude for $\beta$-decay when a proton is dissociated by the strong interactions.

Physically we have to postulate a new process by which the $\pi^+$ can undergo $\beta$-decay directly with the proper amplitude; then the system of a neutron and a $\pi^+$ will have the same $\beta$-decay charge as the proton. Mathematically, we postulate that the vector part of the current, $J^\mu_\alpha$, contains terms in addition to $[\bar{n} \gamma_\alpha p]$ such that its divergence is zero to some desired extent. We can construct such a quantity by looking for the current density of something that is conserved to the accuracy required. By writing $[\bar{n} \gamma_\alpha p]$ as $[\bar{N} \gamma_\alpha \tau \cdot N]$, it is obvious that the thing to use is the current density for the $(\tau \cdot \cdot)$-component of the total isotopic spin, which is conserved. That is, we write

$$2^{1/4} g^{-1/2} J^\mu_\alpha = \mathcal{J}^\mu_\alpha = i \{[\bar{n} \gamma_\alpha p] + \sqrt{2} \{n^0 \bar{\sigma}_\alpha \pi^+ + \bar{n}^0 \sigma_\alpha \pi^0 \} + \ldots\},$$

where the $\ldots$ represent the isotopic spin currents for the strange particles. The equality $G_V = G$ might be so approximate that the strange particle currents should not be included, and thus as closely as we want $G_V = G$, we insert as many of the terms in $\mathcal{J}^\mu_\alpha$ as needed. This expression for $J^\mu_\alpha$ will do physically what we said it had to do, viz. anything into which a proton can dissociate virtually due to the strong interactions will have the same $\beta$-decay charge.

For example, the $\pi^+$ resulting from the dissociation of a proton into $n + \pi^+$ can undergo $\beta$-decay, and thus the probability of $\beta$-decay is preserved.
We do not know whether the violations of the equality $G_V = G$ involve the strange particles or not. It could be that the strange particles do not play a great role in the structure of the nucleon, and that therefore their currents would not be absolutely needed in $J_{QA}^{V}$ to give the degree of equality which exists. However that seems unlikely -- the strange particles are coupled in rather strongly and you would think that they do play a significant role, and therefore the isotopic spin currents of all the particles should be included. The $K$ particle would then possess a direct coupling, as would the $\Xi$ and $\Omega$ particle, and they would all be of the same form: the $(-)$-component of the isotopic spin current. If all the strange particles were so included, the violations of the rule $G_V = G$ would be due to electromagnetism. So up to corrections of order $\alpha$, $G_V$ would equal $G$.

This is called the conserved vector current (CVC) theory. The idea was proposed originally by two Soviet physicists, Gershtein and Zeldovich, a couple of years ago at a time when the vector interaction wasn't very popular -- everybody "knew" at that time that the $\beta$-decay interaction was scalar and tensor. Later when the V-A theory was becoming more respectable, Feynman and Gell-Mann proposed the theory independently. It is a considerable comfort to know that there is some way of understanding the remarkable equality of $G_V$ and $G$. On the other hand, since $-G_A/G$ is also fairly close to unity, and nobody understands at all why this should be true, there still remains the possibility that the equality is an accident, too.

It is therefore of interest to examine experiments which would test whether the pion has its own $\beta$-decay charge, or whether it undergoes $\beta$-decay only by virtue of its strong couplings to fermion pairs. Several experiments along this line can be devised, some of which are more practical than others. The obvious one is to measure the rate of $\pi^+ \longrightarrow \pi^0 + \bar{e} + \nu$. On the ordinary theory, this process proceeds via virtual baryon loops

and, of course, the amplitude cannot be calculated. The crudest of estimates, however, will indicate that the branching ratio is very small -- of order $10^{-7}$ to $10^{-9}$. With the CVC theory, the rate is predictable, but comes out to give a branching ratio of order $10^{-8}$. If the rate was measured to be that predicted by the CVC theory, that would make two fantastic accidents, and then
you would probably have to conclude that the CVC theory were correct. On
the other hand, if the rate came out definitely different from the prediction,
then the CVC theory is wrong. So this experiment is a rather decisive test,
but unfortunately, it is almost experimentally impossible to make an accurate
measurement of a branching ratio at a level of 1 part in a hundred million.

The reason that the rate for the process \( \pi^+ \to \pi^0 + \bar{\nu} + \nu \) is pre-
dictable, assuming the CVC theory holds, is that it is a very low momentum
transfer process, and therefore it is essentially only the total isotopic spin
of the pion system that determines the matrix element. Recall that in an
allowed Fermi transition between two brother states, the matrix element was
known. That is because the relevant matrix element is that of the operators
\[
\int \mathcal{A} \langle \bar{x} | e^{-i\mathbf{k} \cdot \mathbf{x}} | \mathbf{L}^3 \rangle d^3 x,
\]
and when we make the allowed approximation, which consists of replacing the
exponential by \( l \) and using only the \( l \)th component, the operator becomes
\[
\int \mathcal{A} \langle \bar{x} | \mathbf{L}^3 \rangle d^3 x = i \mathbf{l}_{z},
\]
whose matrix elements are known exactly. The same thing holds for the pion
in the CVC theory; the relevant matrix element is again \( \sqrt{2} \). So if the usual
nuclear physics formula for the \( ft \) value is employed, one should get the
rate for the process \( \pi^+ \to \pi^0 + \bar{\nu} + \nu \).

This remarkable possibility of predicting the decay rate exists only
if the CVC theory holds, which reminds us that the first statement is also
true only if the CVC theory holds. That is, the statement about the matrix
element for the decay \( 0^{+} \to 1^{+} \) was really wrong. Previously we
computed the transition matrix element and said it was obviously \( \sqrt{2} \), because
it was just the sum over all particles of the isospin lowering operator for
each nucleon, which we tacitly assumed to be the lowering operator for the
total isotopic spin. But we forgot that these particles actually have mesons,
and things may be much more complicated. The total isotopic spin of a nucleus
includes the isotopic spins of the mesons too. So the famous theorem that in
an allowed Fermi transition the matrix element is exactly known is actually
false except in the conserved vector current theory.

The question then immediately arises as to why the applications of
that theorem were so successful in the first place if it is wrong unless the
vector current is conserved. The answer can be the following. You may
suppose that in nuclei the meson effects are not very important, except in
the individual nucleons. Every individual nucleon has its own meson cloud,
and these particles with their clouds interact through the nuclear forces, but they behave like individual particles, i.e. these clouds are not greatly modified by the presence of other nucleons. This is a possible picture of nuclei, and if that picture is substantially correct, then the theorem would be true for nuclei without the conserved vector current. Thus there are two ways to understand the success of the predictions of the rates of Fermi transitions between brother states: (1) The conserved vector current theory is true. That explains immediately why $G_V = G$, and it also explains why the matrix elements are so perfect. Furthermore it enables one to predict the matrix element for a complicated system like the pion. (2) If the conserved vector current hypothesis is wrong, then two hypotheses must be made to replace it. Accidentally $G_V = G$, and secondly that the nucleon clouds in nuclei are rather independent.

April 9, 1959

We have two problems: the form of the strangeness conserving current $J^l$, and the form of the strangeness changing current. The first of these is completely specified by the hypothesis that the vector part of $J^l$ is conserved as far as the strong interactions are concerned. Actually an addition to that hypothesis is needed to specify $J^l$ completely, because we have determined only the vector part. But the axial part may be specified very simply by the postulate that, just like all the lepton pieces, all the baryon pieces involve $(1 + \gamma_5)$. This then is one possible theory of the strangeness conserving current $J^1$:

$$J^1_{-a} = i \frac{1}{4} G_{1/2} \{ [\gamma_a (1 + \gamma_5) \tau_a] + [\pi \times \sigma_a]_+ \} + \{ [\gamma_a (1 + \gamma_5)] \tau_3] + \ldots \}.$$

At least for the vector part of this current, we began to look for experimental tests. The one test that we had was that $G_V = G$ for the O-O transition from O, and a second test is that this same value of $G_V$ also works for the O-O transitions from Al, Cl, and Ca. That the same value for $G_V$ works in these four cases would seem to be a confirmation on the CVC theory, but it is also possible that nuclei behave in a rather simple way, i.e. the meson clouds of the individual nucleons in a nucleus are not greatly modified by the other nucleons. Without the conserved vector current, the only departures of the matrix elements from $\sqrt{2}$ come from the modification of the individual meson clouds in the presence of other nucleons, and this might be small.
Therefore, the fact that the same value for $G_V$ works for four different nuclei is not such strong evidence for the conserved vector current. If it could be used to predict the rates of some different things like $\pi^+ \rightarrow \pi^0 + e^- + \nu$, or $\Sigma \rightarrow \Sigma^0 + e^- + \bar{\nu}$, which checked out experimentally, that would constitute some solid evidence for the CVC theory. But such experiments would be exceedingly hard.

We are thus led to the problem of finding a more feasible test. One has been suggested recently,* and the corresponding experiment is now being carried out in Kellogg Lab. To examine the proposal, let us recall that, due to the strong couplings, the isotopic vector part of the electromagnetic coupling to the nucleon $\gamma^*_z$ is modified, and becomes

$$\tau_z \{ \gamma^*_a F^V(q^2) - \frac{\mu^V}{2M} \sigma_{a\beta} q^\alpha G^V(q^2) \}.$$  

($\mu^V = \mu^p - \mu^n = 1.79 + 1.91 = 3.7$). In the various measurements of electron scattering at Stanford, the form factors have in principle been determined. Now the isotopic vector part of the electromagnetic coupling is precisely through the $z$-component of the isotopic spin current density, and, in the CVC theory, the vector part of the weak coupling is precisely through the $(\pm)$-component of the isotopic spin density. These are related by rotations in isotopic spin space, and therefore up to electromagnetic corrections, the modified vector weak vertex should be given by

$$\tau^- \{ \gamma^*_a F^V(q^2) - \frac{\mu^V}{2M} \sigma_{a\beta} q^\alpha G^V(q^2) \},$$

with the same form factors.

This theorem tells us the magnitude of the corrections to the vector part of the weak nucleon vertex, and now we can compute whether they are of importance in nuclear $\beta$-decay. Possibly some part of it can be detected and checked. The departure of the form factors from unity in the range of nuclear $\beta$-decay momentum transfers is certainly too small to be seen, and in that range it is of the order of an electromagnetic correction. But what might be seen is that induced "magnetic" coupling. In the CVC theory, for say $q = 15$ MEV, it constitutes a correction of order

$$\frac{3.7 q}{2M} \sim 3\%.$$  

It is hard to detect a 3 percent effect, but if one can find some aspect of the

---

process where the effect is multiplied by 3 or 4, there might be a chance of testing the theory.

A determination of the existence of this effect would provide a test of the CVC theory because, although in the old theory the bare coupling $G \gamma_\alpha \tau_-$ would be modified by mesonic effects to

$$\tau_- \{G_\gamma \gamma_\alpha F_1(q^2) + A \sigma_{\alpha\beta} q_\beta F_2(q^2)\},$$

the value of $A/G_\gamma$ would be expected to differ substantially from $-3.7/2M$. The principal difference between the old theory and the CVC theory has to do with whether the pion carries $\beta$-decay charge or not. In the old theory it was just the nucleon that carried the vector current in $\beta$-decay; the pions carried current only by virtue of their connection with nucleons. In the new theory the pions of themselves have $\beta$-decay charge, so that when the nucleon "dissociation" into a nucleon plus pion, the $\beta$-decay charge is carried by the pion. Now the situation is similar to that of electromagnetism. Suppose we abolish the pion interaction with the electromagnetic field. What then would the anomalous magnetic moments of the proton and neutron be? In other words, how much of $\mu^V = \mu^P - \mu^n$ comes from pion current, and how much comes from nucleon current? If we could answer that question, we would have some idea of how big $A/G_\gamma$ would be in the old theory. In the old theory, $A/G_\gamma \sim - (\text{nucleon current contribution to } \mu^V)/2M$. Well, it can easily be seen from a charge symmetry argument that the pion current contribution to $\mu^S = \mu^p + \mu^n = -0.12$ vanishes. So if we neglect the contributions from strange particles, it appears that the nucleon current contribution to $\mu^S$ almost vanishes. We can't be even that sure about the nucleon current contribution to $\mu^V$, but we can guess that it would not be a large fraction of $\mu^V$. This is only a guess, but it seems plausible physically that since $m_n/M \sim 1/7$ the pion contribution to the anomalous moments should be several times the nucleon contribution. A test of the CVC theory is, on the basis of these arguments, a test of whether $A/G_\gamma$ is approximately $-3.7/2M$ or zero.

The term of interest involves the $\sigma_{\alpha\beta}$ matrices, so we want to look for an effect which is entirely analogous to the magnetic effect of an anomalous moment in electromagnetism. In testing for such an effect in electromagnetism, we look at a magnetic dipole transition, i.e., one in which $|\Delta J| = 1$, and the parity doesn't change. Similarly to check $-A/G_\gamma$ in $\beta$-decay, we look at the equivalent of a $M1$ transition, preferably a pure one like that from a $1^+$ level to a $0^+$ level. Moreover, we look for the highest energy transition of
this nature available, and that turns out to involve an energy of about 15 MEV.

This situation as a whole is somewhat different from electromagnetism in that the matrix element from $1^+$ to $0^+$ is dominated by the axial vector piece and the magnetic piece is only a small correction. This axial vector piece is what people have measured through the rates, and thus we know what this is for the purpose of calculating the expected effect of a magnetic term. But we also have to know the magnitude of the magnetic matrix element. It would, of course, be possible to calculate it crudely by assuming some simple model of nuclei, but there is an easier and much more reliable way to get it. Consider a case where the $1^+$ level is an isotopic triplet, and the $0^+$ level is an isotopic singlet. The $I_z = 0$ member of this triplet will decay by the emission of a magnetic dipole photon through only the isotopic vector part of the electromagnetic coupling, since $I$ does change from 1 to 0. By measuring the lifetime of the $I_z = 0$ member of the triplet, we can get the value of the matrix element of the $z$-component in isospace of the vector current. In $\beta$-decay we need the $(\pm)$-components in isospace of the same vector current, and charge independence tells us that these are related simply by the factor $\sqrt{2}$. So the magnetic dipole part of the vector matrix element for these special $\beta$-decays can be determined from the corresponding $\gamma$-transactions. Thus we know experimentally both pieces of the transition matrix element for these $\beta$-transitions, and all we have to do now is to compute physical effects of the interference between the two pieces, and hope we find one that comes out to be several multiples of $3.7\hbar/2M$. It turns out that the most easily measured effect, which does in fact get multiplied by a sizable factor, is a term in the spectrum. The spectrum turns out to be that expected normally multiplied by the factor

$$\left\{1 \pm \frac{8}{3} \frac{\eta}{2M} (E - \frac{1}{2}A)\right\},$$

which, since it is antisymmetric about the midpoint, contributes essentially nothing to the total rate. This parameter $\eta$ can be determined experimentally as outlined above, and is similar to what you might calculate theoretically using a simple argument which neglects orbital magnetic moments, coming out to be near $(3.7+1)$.

There is one thing that is exceedingly interesting about this effect, which is that in the $\beta^-$ transition and the $\beta^+$ transition the sign of the effect is reversed. The reason for this is that under charge conjugation the vector piece of the matrix element changes sign while the axial vector does not, so that a vector-axial interference term will always change sign. Therefore if
one looks at both spectra, and the ratio of the spectral anomalies is examined, the effect can be doubled, and also certain systematic experimental errors will cancel out. Furthermore, a number of complications might possibly arise from other types of little effects, the so-called forbidden pieces of the transition matrix element. In particular, the axial vector contribution alone has a lot of little forbidden terms in it which give such spectrum anomalies; however, these will appear as axial-axial interference terms and will not change sign under charge conjugation. So in the ratio these will cancel out, and we need not worry about them.

The best set of transitions satisfying our requirements are those from $B^{12}, C^{12*},$ and $N^{12}$ to $C^{12}$. In this case the maximum effect is around 20 percent. The experiment is presently being done only at one laboratory, Kellogg, but since it is a difficult one, we would like to see several groups try it.

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Having discussed qualitatively the essential features of the experiment which will test the conserved vector current hypothesis, we can now profitably go on to calculate the effect.

The situation we are concerned with is as follows:

The three states with $J^M = 1^+$ form an isotopic triplet, and the ground state of $C^{12}$ is an isosinglet. Because of the spins and the parities, the vector matrix elements are purely magnetic dipole in nature, which is particularly
convenient for investigating the proposed term in the effective vector coupling.

The big term in the $\beta$-transitions is the allowed part of the axial vector matrix element, and the vector matrix element plays the role of one of the forbidden corrections. In addition to the vector, there are forbidden corrections due to the modification of the axial vector coupling by the strong interactions. These latter effects we cannot calculate, but would probably be of the same order as the effect being searched for. The nice thing is, though, that when the interference of the little corrections with the allowed axial vector piece is considered, the vector-axial interference term changes sign upon going from the $\beta^-$-decay to the $\beta^+$-decay, whereas the axial-axial interference term remains the same. Thus if we take the ratio of the correction factors for the transitions $B_{12} \rightarrow C_{12} + e + \bar{\nu}$ and $N_{12} \rightarrow C_{12} + \bar{e} + \nu$, the axial-axial corrections will cancel out and only the interference terms we are interested in will be present.

Let us calculate the features of the decay of $B_{12}$. Let the momentum of the electron be $p$, and that of the antineutrino be $k$. Taking the state of $B_{12}$ to have $J = m$, we define a quantity $M^V_{\alpha}$ as the sum of a vector $M^V_{\alpha}$ and an axial vector $M^A_{\alpha}$ which are matrix elements of the meson-baryon part of the weak interaction Hamiltonian. In the CVC theory, neglecting electromagnetic corrections, we can express $M^V_{\alpha}$ concisely as

$$M^V_{\alpha} = -i \langle C_{12} | g \gamma_{\alpha}^+ (\rightarrow \pi^+ - p - k) | B_{12}, m \rangle$$

where

$$\gamma_{\alpha}^+ (\rightarrow \pi^+ - p - k) = \int e^{-i (p+k) \cdot x} \gamma_{\alpha}^+ (x) d^3 x$$

is a Fourier transform of the $+$ component of the isospin current density. No such expression can be written for $M^A_{\alpha}$; to identify it we will just write

$$M^A_{\alpha} \equiv \langle C_{12} | \sigma \gamma_{\alpha} (\rightarrow \pi^+ - p - k) | B_{12}, m \rangle.$$  

With these definitions, the $R$-matrix element to lowest order is:

$$R_{f1} = \sqrt{m/E_p} \left[ \frac{1 + \gamma_5}{\sqrt{2}} \gamma_\nu v^{-1} \right].$$

As usual we wish to compute the square of this matrix element and then sum over the lepton spin states. If $\gamma_{\nu}$ is defined as $\beta \gamma_{\nu} \beta$, then

$$\sum_{\text{lepton spins}} \left| R_{f1} \right|^2 = \frac{1}{E_p k} \frac{1}{4} \frac{1}{4} \text{Trace} \left\{ (1 - \gamma_5) \gamma_{\nu}^* (m - i\gamma_5) \gamma_{\nu} (-1) \right\}$$

$$= \frac{1}{E_p k} \{ 2 \text{Re} \left( M^* p^* (M - k) (\gamma^* K + p) \right) - \gamma_{\alpha} \gamma_{\nu} \gamma_{\alpha}^* (p \cdot p) \gamma_{\nu} \}.$$  

The next step is to determine the nature of $M^A_{\alpha}$, i.e. its transformation
properties under rotations, so that one can determine the behavior of $M_\alpha$ as a function of the spin state, $m$, of $B_{12}$. Knowing that, one can then compute the average of $\Sigma |R|^2$ over the $m$ values. The transformation properties are independent of the complications due to the meson clouds, and so to exhibit them we may employ once again a simple model for nuclei. In this model,

$$M_\alpha^V = \langle c_{12} | \int G_V e^{-i(\vec{p}+\vec{k}) \cdot \vec{x}} \tau_+ \gamma_\alpha \psi(\vec{x}) \ d^3x | B_{12}^1, m \rangle.$$ 

To reduce this to a more familiar form, we can write things in terms of non-relativistic two-component spinor field operators: $\chi^+(\vec{x})$, and $\chi(\vec{x})$.

$$M_\alpha^V \sim G_V \sqrt{\frac{1}{2M}} \left\{ \chi^+(\vec{x}) \tau_+ \vec{\nabla} \chi(\vec{x}) - \vec{\nabla} \chi^+ \tau_+ \chi \right\}$$

$$+ \chi^+ \tau_+ \vec{\nabla} \frac{1}{\sigma} \chi + \frac{1}{\sigma} \chi^+ \tau_+ \chi \}$

$$\times e^{-i(\vec{p}+\vec{k}) \cdot \vec{x}} d^3x.$$

The first two of the four terms are simply proportional to the non-relativistic current for a spinless particle. The last two terms can be written, after integration by parts, as:

$$G_V \frac{\vec{p}+\vec{k}}{2M} \times \int \left\{ \chi^+(\vec{x}) \tau_+ \vec{\nabla} \chi(\vec{x}) \right\} e^{-i(\vec{p}+\vec{k}) \cdot \vec{x}} d^3x.$$

The effect of the meson currents in the CVC theory is to multiply this Dirac moment term by the factor $(1 + \mu^V) = 4.7$. The first forbidden part of $M_\alpha^V$ transforms like

$$-i G_V \frac{\vec{p}+\vec{k}}{2M} \times \int \left\{ \chi^+(\vec{x}) \tau_+ \vec{\nabla} \chi(\vec{x}) \right\} d^3x,$$

which carries negative parity and thus does not give a contribution in our case. So altogether, the vector part of $M_\alpha$ will have only spatial components, and these will have the form

$$G_V \frac{\vec{p}+\vec{k}}{2M} \times \langle c_{12} | 1_{\mu+} | B_{12}^1, m \rangle,$$

where $\mu_+$ transforms like an axial vector. Furthermore, in a simple nuclear model, the major part of $\mu_+$ is $(1 + \mu^V)'' \tau_+ \vec{\sigma}''$.

Now we have to look at the axial vector part of $M_\alpha$. The big piece is just the allowed term, which we may indicate as the matrix element of

$$-i G_A \int \left\{ \chi^+(\vec{x}) \tau_+ \vec{\nabla} \chi(\vec{x}) \right\} d^3x = -iG_A '' \int \tau_+ \vec{\sigma}''.$$

The first order forbidden terms in the spatial components arising from the expansion of the retardation factor carry the wrong parity and so are ineffective in our transitions. On the other hand, that expansion gives rise to a forbidden contribution to $M_\alpha^A$, which transforms like the matrix element of

$$- G_A \frac{\vec{p}+\vec{k}}{2M} \cdot \int \left\{ \chi^+(\vec{x}) \vec{\sigma} \cdot \vec{\tau} \tau_+ \chi(\vec{x}) - \chi^+(\vec{x}) \cdot \vec{\tau} \tau_+ \chi(\vec{x}) \right\} d^3x.$$
This piece is undoubtedly present, in addition to meson exchange current effects, but its effects will cancel out when the corrections to the spectrum in $\beta^-$ and $\beta^+$-decay are compared.

It should be realized that the terms in $M_\alpha$ are made up from a space vector $(p+k)$ and nuclear matrix elements, all of which transform like a spatial axial vector, in such a way that under spatial rotations and reflections $M_V$ is a vector $M_H$ is a scalar, $M_\alpha$ is an axial vector, and $M_\mu$ is a pseudoscalar. The axial vector nature of the nuclear matrix elements is fixed by the fact that the initial state has $J^m = 1^+$ and the final state is $0^+$. Since the nuclear matrix elements have the same transformation properties, they are simply proportional to one another, the factor of proportionality being independent of $m$. For example,

$$\langle c^{12} | \mu_{\alpha+2} | B^{12}, m \rangle = \eta \langle c^{12} | \mu_{\alpha+2} | B^{12}, 0 \rangle$$

$$\eta = \frac{\langle c^{12} | \mu_{\alpha+2} | B^{12}, 0 \rangle}{\langle c^{12} | \mu_{\alpha+2} | B^{12}, 0 \rangle}.$$ 

This result is guaranteed by the Wigner-Eckart theorem which states that if $T^{m}_{\lambda \ell}$ is an irreducible tensor operator of rank $\ell$, 

$$\langle a', J', M' | T^{m}_{\lambda \ell} | a, J, M \rangle = \langle J, M; \ell, m | J, \ell; J', M' \rangle \langle a', J' | T^{m}_{\lambda \ell} | a, J \rangle,$$

where the quantity $\langle a', J' | T^{m}_{\lambda \ell} | a, J \rangle$ is independent of the projection quantum numbers, and is called the "reduced" matrix element. In our case we are dealing with an axial vector operator, e.g. $\mu$, which is equivalent to an irreducible tensor of rank 1 in the following manner:

$$T^1_{\lambda \ell} = \sqrt{2} T^1_{\lambda 1}, \quad T^0_{\lambda \ell} = \sqrt{2} T^0_{\lambda 1}, \quad \mu_x - i \mu_y = \sqrt{2} T^{-1}_{\lambda 1}.$$

To proceed we need one further result. In determining the transition rates we will want to determine $\Sigma |R|^2$ in which we sum over the final spin states of $C^{12}$ and average over the initial spin states of $B^{12}$. In doing so, we need to evaluate expressions for the following form:

$$\Sigma \Sigma_{M' M} \langle a', J', M' | T^{m}_{\lambda \ell} | a, J, M \rangle \langle a', J', M' | T^{m}_{\lambda \ell} | a, J, M \rangle.$$ 

This can be accomplished by using the Wigner-Eckart theorem, and one of the sum rules for the Clebsch-Gordan coefficients. Since the Clebsch-Gordan coefficients are real, the above expression equals:
\[ \langle \alpha', J' | T^*_{\ell} | \alpha, J \rangle \times \]
\[ \sum M \sum M' \langle J, M; \ell', m' | J, \ell'; J', M' \rangle \langle J, M; \ell, m | J, \ell; J', M' \rangle \]
\[ = \delta_{m m'} \delta_{\ell \ell'} \frac{2 J + 1}{2J + 1} \langle \alpha', J' | T^*_{\ell} | \alpha, J \rangle \times \langle \alpha', J' | T^*_{\ell} | \alpha, J \rangle. \]

It is now fairly easy to calculate the spectrum. Defining a vector function of \( m \) by \( S(m) = \langle c^{12} | \tau_+ | b^{12} \rangle_{m} \), one can write \( M_{a} \) in the following way:
\[ M = -i G_A \gamma \cdot \vec{S} + G_V \gamma \cdot \frac{p+k}{2M} \times \vec{S}; \quad M_{a} = G_A \gamma \cdot (p+k) \cdot \vec{S}. \]

Time reversal invariance requires that both \( \eta \) and \( \xi \) be real.

In terms of this notation, the sum rule presented above reads:
\[ \sum m S_{a}^*(m) S_{a}(m) = \frac{1}{3} \delta_{ij} \left| \langle c^{12} | \tau_+ \sigma | b^{12} \rangle \right|^2 = \delta_{ij} | S_{a}(0) |^2. \]

Applying the sum rule, it follows that to terms of third order in the lepton momenta:
\[ \sum m (p \cdot k)^3 (M \cdot k) = G_A^2 | S_{a}(0) |^2 \{ \left[ E_k - E_{\vec{p}} - E_{\vec{p}} \right] \} \]
\[ \sum m (p \cdot k) (M \cdot M) = \frac{2}{3} G_A^2 | S_{a}(0) |^2 \{ \left[ E_k - E_{\vec{p}} - E_{\vec{p}} \right] \} \]
\[ \sum m \epsilon_{a} \beta \gamma \delta M_{a} p \gamma \delta = G_A G_V \frac{\eta}{2M} | S_{a}(0) |^2 \{ \left[ E_k^2 - m^2 \right] - E_k^2 + (k - \vec{p}) \cdot \vec{k} \}. \]

**Problem 14:** Verify these three relations.

Putting the pieces together, one obtains:
\[ \frac{1}{3} \sum |R|^2 = G_A^2 | S_{a}(0) |^2 \{ 1 - \frac{1}{3} \beta \cdot k - \frac{2}{3} \gamma \frac{m^2}{E_p} \}
+ \frac{8 \gamma V}{3 G_A^2} \frac{\eta}{2M} \left[ E_p - \frac{1}{2} \Delta - \frac{1}{2} \frac{m^2}{E_p} \right] - \frac{2 \gamma}{3 G_A} \frac{\eta}{2M} \left[ \Delta - \frac{m^2}{E_p} \right], \]
where \( \Delta = k + E_{\vec{p}}, \) and \( \beta = \vec{p}/E_p. \)

The inclusion of forbidden terms of first order in the lepton energy divided by \( M \) results in a modified angular correlation and a change in the spectrum. The second of these effects is by far the easier to look for, and is characterized by the shape factor:
\[ 1 + \frac{\epsilon G_V}{3 G_A^2} \frac{\eta}{2M} \left[ E_p - \frac{1}{2} \Delta - \frac{m^2/2E_p}{E_p} \right] - \frac{\gamma}{3 G_A} \frac{\eta}{2M} \left[ \Delta - \frac{m^2/2E_p}{E_p} \right]. \]

Neglecting the term \( m^2/2E_p \) which is quite small over most of the range of electron energy in a high energy transition, the vector term leads to a
correction to the shape which is linear in the electron's energy.

Now if we calculate the spectrum in the case \( N^{12} \rightarrow C^{12} + \bar{e} + \nu \), we find that a similar shape factor results, and, in fact, the only change is that the sign of the vector-axial interference term is reversed. If one checks back over this calculation, one can easily see that the only modification that counts is the interchange of \( \beta \) and \( \gamma \) inside the trace which just reverses the sign of the \( \epsilon_{\alpha \beta \gamma \delta} \) term. That the values of \( \eta \) and \( \zeta \) remain the same, as far as charge independence holds, is a consequence of the Wigner-Eckart theorem as applied to the isotopic spin variables. Thus in the ratio of the two shape factors one should observe principally the quantity:

\[
1 + \frac{16}{3} \frac{\eta}{2M} \frac{\mathcal{G}_V}{\mathcal{G}_A} \left[ E - \frac{1}{2} A \right].
\]

The remaining part of the calculation is to predict the value of \( \eta \) in the conserved vector current theory. In the simple nuclear model which was employed to elucidate the transformation properties of the matrix elements, \( \eta \) has a definite value if the orbital contribution to the magnetic moments is neglected, which is \( (1 + \mu^Y) = 4.7 \). However, we can do even better than this, for \( \eta \) is just the ratio of the \( M_1 \) vector matrix element to the allowed axial vector matrix element, the absolute values of which can be deduced from known lifetimes. The lifetime of \( B^{12} \) is determined to within a percent or so by the allowed axial vector matrix element, and thus:

\[
|S_{\gamma}(0)|^2 = 2 \frac{G^2_V}{G^2_A} \frac{1}{0} 1 e_0 \frac{G^2_A}{B_{12}}.
\]

The lifetime of the \( C^{12*} \) state is similarly determined by the isotopic vector part of the magnetic moment transition matrix element by the familiar formula:

\[
\Gamma_{\gamma} = \frac{4}{3} \alpha \omega^2_Y \frac{1}{2M} |\left< C^{12} | \mu_{\Lambda^0z} | C^{12*}, 0 \right>|^2,
\]

and by the Wigner-Eckart theorem:

\[
\left< C^{12} | \mu_{\Lambda^0z} | B^{12}, 0 \right> = \sqrt{2} \left< C^{12} | \mu_{\Lambda^0z} | C^{12*}, 0 \right>.
\]

Therefore:

\[
|\eta| = \frac{\left< C^{12} | \mu_{\Lambda^0z} | B^{12}, 0 \right>}{|S_{\gamma}(0)|} = \frac{G_A}{G_V} \left\{ \frac{3 \Gamma_{\gamma} M^2 (ft)_{B^{12}}}{\alpha \omega^3_Y (ft)_{C^{12}}} \right\}^{1/2}
\]

\[
|\eta| = 2 \frac{G_A}{G_V} (2.34 \pm .25) = 5.7 \pm 0.6.
\]

If the conserved vector current theory is not true, there should be a completely
different value of $\eta$. The magnetic dipole transition matrix element for the $\gamma$-ray transition in $^{12}C$ would then not be related to the magnetic vector matrix element in the $\beta$-transitions. Returning to the simple nuclear model, the magnetic vector matrix element would not get a contribution from $\mu^v$, and thus it seems quite plausible that $\eta$ would only be about $1/4$ of its value in the CVC theory, since on comparing our estimate of $\eta$ with the value obtained experimentally in the CVC theory it becomes apparent that the $\gamma$-ray transition matrix element in $^{12}C$ is due very largely to the anomalous moments.

If the conserved vector current theory holds, the beautiful agreement between our rough theoretical estimate of $\eta$ and the value calculated on the basis of charge independence has the important consequence that the sign of $\eta$ is virtually certain to be positive, as in the simple nuclear model. This prediction of the sign of the effect is the only thing that we have to use a model to obtain.

In the ratio of the spectrum anomalies for the $\beta^-$ and $\beta^+$-transitions the complete effect is a linear term with a slope of

$$\frac{16}{3} \times 5.7 \frac{1}{12} = 1.3 \text{ (MEV)}^{-1},$$

or a total of $1.3 \times 15 = 20\%$ over the entire range of energy.

The experiment is being carried out in Kellogg, and the first run failed to detect an effect. But there were some systematic errors which rendered it impossible to draw an airtight conclusion. In the next run they will attempt to minimize further any sources of error so that a definite conclusion can be drawn.

The experiment is slightly complicated by the fact that a similar spectrum correction comes from another source, which we have not considered at all. That is the Coulomb effect which violates charge independence and requires us to modify the electron wave function. The effect is small and can be treated by perturbation theory. If one includes the Coulomb field of the nucleus and the effect of a finite radius, $R$, for the nucleus, in the spectrum, a little correction of order $\pm Z a E R$ comes in. However, one can estimate this effect, at least to within $50\%$ or so, and in the ratio the effect comes out to be $- (0.2 \pm 0.1)\%$ (MEV)$^{-1}$, and so the analysis can be corrected for it and be good to at least $0.1\%$ per MEV. Such an uncertainty is comparable with the experimental errors and should not cause any serious trouble.

There is a further electromagnetic effect which is different in the

case of positron emission from that in electron emission, which also has to be corrected for. That is the inner bremsstrahlung, the emission of soft photons. The process can be represented by the following diagram:

A real photon comes off, and thus there is no quantum-mechanical interference between this process and the one we discussed. However, the photons are not counted in the spectrometer, and thus the measured electron spectrum will be the sum of those from two distinct processes. Now is this inner bremsstrahlung effect different from B\textsuperscript{12} to N\textsuperscript{12} or will it cancel in the ratio? If you look at it hastily, you realize that it is obvious that it will cancel out in the ratio, because as far as the photon is concerned, it makes no difference at all whether it is emitted by an electron or a positron. But if you think some more about it, you realize that there is a significant difference in the two cases. The rate for inner bremsstrahlung is strongly dependent upon the energy given to the photon, and as that energy goes to zero, that rate becomes very large. (Keeping only terms of order e in the matrix element, the rate goes to infinity as the energy goes to zero, but if one calculates to all orders in e, the rate would not become infinite.) Therefore, the number of electrons with a given energy resulting from the inner bremsstrahlung process depends sensitively upon how much energy there is available to the photon. The end point of the spectrum differs by 2.7 MEV from boron to nitrogen, and thus the correction for inner bremsstrahlung is not the same. This correction is, however, easily calculable, and is very small if one stays away from the end points of the spectra.

A final experimental correction has to be introduced due to the other levels in C\textsuperscript{12}. The β-transition can lead to several levels in C\textsuperscript{12} besides the ground state, and when you measure the electron spectrum, you also count electrons resulting from a transition, for example, to the 4.43 MEV state in C\textsuperscript{12} if the energy of the electron is more than 4.43 MEV less than the end point. Since as much of the spectrum must be measured as possible to detect our effect, accurate knowledge of branching ratios is necessary to subtract out the unwanted contribution; fortunately, the branching ratios are only of
the order of a percent.

If the experiment turns out to show that the conserved vector current hypothesis is correct, we shall be one step ahead theoretically in that direction. On the other hand, if the experiment shows definitively that the pion contribution to the $\beta$-decay current is absent, then the theoretical conclusion will also be quite interesting. The theoretical conclusion presumably would then be that mesons do not have anything to do directly with the weak interactions -- that the weak interactions are just associated with fermions, probably always involving $(1 + \gamma_5)$. But also one other thing would be indicated. If the weak interactions do not involve mesons directly, there must be an intrinsic difference between mesons and baryons. In other words, if the pion does not couple directly to the leptons, that implies that the pion exists as a distinct particle. It could then not be regarded, as some people are trying to do, as a bound state of a nucleon and an antinucleon. If the mesons are baryon compounds, the coupling to the isotopic spin current of the baryons automatically becomes a coupling to the total isotopic spin current. So if the CVC hypothesis is false, it seems exceedingly plausible, although not rigorously deducible, that the pion is not a bound state, but a distinct entity.

April 16, 1959

The question has been asked as to how we know the form of the effective vertex, say for the electromagnetic interaction of a nucleon or for the weak interaction producing $\beta$-decay, which results from the modification of the bare vertex by the strong couplings. An analogous problem was considered for the theory of quantum electrodynamics, which consisted of the deduction of the form for the complete vertex which results from the modification of the bare electrodynamic vertex, not by other types of interactions, but rather by higher order electrodynamic corrections, "radiative" corrections. In that problem, three principles were employed to deduce the form of the vertex. The first principle was that the vertex must be vector in nature, and there are only a few vectors which one can construct from the parameters present. From these possibilities a couple could then be eliminated in two ways. Between free particle spinors certain vectors are equivalent to linear combinations of the others, and can therefore be omitted. Finally, we noted a certain symmetry among all the diagrams -- the existence of mirror image diagrams -- which consequently had to be present in the sum of all the diagrams,
but which is contained in only some of the vectors. In fact, the term which
was eliminated by the symmetry argument may also be eliminated more easily by
noting that the electromagnetic current is conserved, which implies
\[ q_\alpha \left\langle p^1 \left| j_{\alpha} \right| p \right\rangle = 0. \]
The first two principles, and the requirement that the
matrix element of the current be divergenceless, may clearly be employed in
the same way to restrict the form of the effective electromagnetic vertex for
a strongly interacting particle to what we have assumed all along, i.e.
\[ \gamma^\alpha F(q^2) - \frac{\mu}{2M} \sigma_{\alpha \beta} q_\gamma G(q^2). \]

However, in the case of the weak interactions, how do we know that the
effective form of the vertex
\[ \tau^\alpha \left\{ G_V \gamma^\alpha F_1(q^2) + A \sigma_{\alpha \beta} q_\gamma F_2(q^2) - G_A \gamma^\alpha \gamma_5 F_3(q^2) + B \gamma_5 q_\beta F_4(q^2) \right\} \]
does not contain in addition the terms
\[ \tau^\alpha C \sigma_{\alpha \beta} r_\gamma F_5(q^2), \quad \text{and} \quad \tau^\alpha D \gamma_5 r_\alpha F_6(q^2), \]
where \( r = p_1 + p_2 \)?

Indeed we do not know that they are not present because we do not know the
structure of \( J_{-\alpha} \). (If the CVC hypothesis is correct, the C term is absent,
but nothing considered so far implies anything about the existence of the D
term). The absence of such extra terms will be a consequence of some symmetry
principle, which implies the existence of "mirror diagrams" or something similar.
The question of interest is therefore what symmetry principle of the weak inter-
actions is being assumed to eliminate those terms.

It is conjectured that if \( J_{-\alpha} \) behaves the same way under the operation
\( B = P C I^2_{\alpha}(\pi) = P G \) as does its first term, \( n \gamma_\alpha (1 + \gamma_5) p \), then the C and D
terms will be absent. Let's look at how this first term is transformed. For
this, one must recall that since \( I_{-\alpha}(\pi) = e^{-i\pi x} \),
\[ I_{-\alpha}(\pi) \ n(x) I_{-\alpha}^T(\pi) = p(x), \]
and
\[ I_{-\alpha}(\pi) \ p(x) I_{-\alpha}^T(\pi) = -n(x). \]

Also, use must be made of the relations:
\[ C \ n(x) C^+ = F \ n^T(x), \quad \text{and} \quad C \ p^+(x) C^+ = p^T(x) F^+, \]
where the symbol T means the transpose of the spinor components. Thirdly,
\[ P \ p(x) P^+ = \beta p(-x), \quad \text{and} \quad P \ n^+(x) P^+ = n^+(-x) \beta. \]

By applying these relations, remembering that the convention we use is that
spinor field operators for different particles anticommutate, and using the
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properties of the F matrix, viz.

\[ F^+ \gamma_\mu F = \gamma^* \gamma = \gamma^T \quad \text{and} \quad \gamma^{\mu}_\mu F = -\gamma^* = -\gamma^T, \]

it is possible to show:

\[
\begin{align*}
P & \equiv \begin{pmatrix} n^+(x) & \beta \gamma_\alpha (1 + \gamma_5) p(x) \end{pmatrix} \begin{pmatrix} n^+(x) & \beta \gamma_\alpha (1 + \gamma_5) p(x) \end{pmatrix}^T
\end{align*}
\]

\[
\begin{align*}
&= -P \left\{ \begin{pmatrix} n^+(x) & \beta \gamma_\alpha (1 + \gamma_5) p(x) \end{pmatrix} \begin{pmatrix} n^+(x) & \beta \gamma_\alpha (1 + \gamma_5) p(x) \end{pmatrix}^T \right\}
\end{align*}
\]

\[
\begin{align*}
&= +P \left\{ \begin{pmatrix} n^+(x) & \beta \gamma_\alpha (1 + \gamma_5) p(x) \end{pmatrix} \begin{pmatrix} n^+(x) & \beta \gamma_\alpha (1 + \gamma_5) p(x) \end{pmatrix}^T \right\}
\end{align*}
\]

The conjecture thus involves the assumption that \( \gamma^T \) transforms in the following way:

\[
\begin{align*}
B & \equiv \begin{pmatrix} n^+(x) & \beta \gamma_\alpha (1 + \gamma_5) p(x) \end{pmatrix} \begin{pmatrix} n^+(x) & \beta \gamma_\alpha (1 + \gamma_5) p(x) \end{pmatrix}^T
\end{align*}
\]

Now assuming that \( \gamma^T \) behaves this way, what restrictions are thereby imposed upon the other terms in the current? We shall not look at complicated combinations of l and \( \gamma_5 \), but rather shall restrict our inquiry to those combinations of fermions with \( \gamma_5 \). The possible terms are then:

\[
\begin{align*}
&= \begin{pmatrix} n^+(x) & \beta \gamma_\alpha (1 + \gamma_5) p(x) \end{pmatrix} \begin{pmatrix} n^+(x) & \beta \gamma_\alpha (1 + \gamma_5) p(x) \end{pmatrix}^T
\end{align*}
\]

The \( \gamma^T \) term is exactly analogous to the \( \gamma^T \) term, and automatically transforms in the same way under B; Thus no restriction is placed upon this term. But under the operation B, \( \gamma^T \rightarrow \pm \gamma^+ \), \( \gamma^0 \rightarrow \pm \gamma^0 \), and \( \Lambda \rightarrow \pm \Lambda \), so that if the terms \( \gamma^T \gamma^0 \) and \( \gamma \Lambda \) are present, the behavior of \( \gamma^T \gamma^0 \) required under B automatically determines both the sign and magnitude of the terms \( \gamma^T \gamma^0 \) and \( \gamma \Lambda \), respectively. What the signs are depends upon one's convention as to the phase of the \( \gamma \) states, etc., but we need only state that if the signs are chosen as when constructing an isotopic spin current, the result will transform under B as desired. The two possible \( \gamma^T \gamma^0 \) terms will automatically transform in the assumed manner under B, as will the two \( \gamma^T \gamma^0 \) couplings. As long as we restrict ourselves to these simple forms for the non-strangeness changing current \( J^T \), the only condition prescribed for the desired behavior under B is that the \( \Sigma^0 \) and \( \Lambda^+ \) terms are determined by the \( \Sigma^+ \) and \( \Lambda^+ \) terms, respectively. If these conditions are fulfilled, one can prove that the "mirror" image diagrams exist, and thus that the two extra terms in
the effective weak vertex are not present.

The strangeness changing parts of the current play no role in this discussion because we are concerned with a vertex where the strangeness does not change, and so any contribution from strangeness changing parts would have to be that coming from a second order effect in $G$, which is very tiny. However, for the investigation of the strangeness changing current it is extremely interesting to define operations analogous to $B$, under which, for example, $p \rightarrow \Lambda$ or $\Xi$.

No. 17: The Weak Interactions -- Strange Particle Decay

We turn now to the problem of what can be deduced from the strange particle decays about the strangeness changing part of the weak interactions, and how, by specifying a form for the interaction, we can understand many of the features of strange particle decay phenomena. Let's first write down some of the characteristics of the strange particles, so that we may become familiar with the facts that we would like to understand theoretically. These are now pretty well known for most of the strange particles, although some of them are much harder to observe than others.

We shall concern ourselves with the characteristics only of the decays of the strange particles; everything that is observed about their strong interactions will be assumed to be known, because there is not very much that has been learned. The conservation of isotopic spin and of strangeness in the strong interactions fairly well describes the extent of our theoretical knowledge, although there is the possibility that the couplings of the baryons to pions may be described correctly by a global symmetry scheme.

We shall study first the decay pattern of the $K^+$. The following list of modes will contain only the observed ones. There are many other decay modes which must exist from the point of view of field theory, but these occur too infrequently to have been seen. The rates will be given in multiples of $10^6$ sec$^{-1}$, and the errors are generally a few percent except where noted explicitly.

<table>
<thead>
<tr>
<th>MODE</th>
<th>RATE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^+ \rightarrow \mu^+ + \nu^+$</td>
<td>48.0</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^+ + \pi^0$</td>
<td>20.9</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^+ + \pi^- + \pi^+$</td>
<td>4.62</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^+ + \pi^0 + \pi^0$</td>
<td>1.38</td>
</tr>
</tbody>
</table>
The decay rate of the \(K^+\) is, of course, the sum of all the partial rates, 81.6, and hence the lifetime of the \(K^+\) is about \(10^{-8}\) sec.

There are a couple of features concerning leptonic modes which bear explicit mention. First, the mode \(K^+ \to \bar{\nu} + e + \nu\) has not been detected, and it should be clear why.

Our calculation on the branching ratio for \(\nu^+ \to \bar{\nu} + e + \nu\) can be carried over easily to this case to predict

\[
\frac{\Gamma_{K^+ \to \bar{\nu} + e + \nu}}{\Gamma_{K^+ \to \bar{\nu} + \mu + \nu'}} = \frac{m_e^2}{m_\mu^2} \frac{(1 - m_e^2/m_K^2)^2}{(1 - m_\mu^2/m_K^2)^2} \approx \frac{1}{4} \times 10^{-4},
\]

and thus \(\Gamma_{K^+ \to \bar{\nu} + e + \nu} \approx 10^{-3}\) in our units! Thus it is quite understandable why this mode hasn’t been seen. Experimentally, the upper limit for its rate is about \(1\). Secondly, you will notice that rate for the mode \(K^+ \to \pi^0 + \bar{\nu} + \nu\) is roughly the same as the rate of \(K^+ \to \pi^0 + \mu + \nu'\). This is quite reasonable for the \((\bar{\nu}e)\) is coupled by the weak interactions in the same way as the \((\bar{\nu}\nu')\), and so the matrix elements should be comparable in magnitude. (In the two body decay, the form of the matrix element in vector and axial vector coupling is so special that the matrix element reduces essentially to the mass of the electron or muon, which causes the great difference in the rates.)

A few of the other modes which must occur are decays into two pions plus a lepton pair, and any of the above modes can be accompanied by a \(\gamma\)-ray. The \(\gamma\)-ray may be considered to come from regular inner bremsstrahlung, or it can come out of the middle of one of the complicated diagrams. The amount of inner bremsstrahlung is predictable in the limit of very low photon energies just from the rate of the process without the \(\gamma\)-ray and the fine structure constant, and the spectrum is of the form \(\frac{d\phi}{\omega}\). But then there are corrections and other diagrams, which contribute to the photon spectrum higher terms proportional to \(1 \, d\omega, \, \omega \, d\omega, \, \omega^2 \, d\omega, \) etc. The coefficients of these higher terms are not predictable merely from the rate of the process without the \(\gamma\)-ray, but depend upon the detailed structure of the process.

Let us now turn our attention to the decays of the neutral \(K\) particle. The neutral \(K\), as we have remarked a number of times, behaves in a rather peculiar fashion. To start with, we look at the strong interactions
where we find that the $K$ is an isodoublet like the nucleon, and so there are two neutral $K$'s, the $K^0$, and its antiparticle, the $\bar{K}^0$. The $K^0$ has strangeness $+1$ and the $\bar{K}^0$ has strangeness $-1$, which results in totally different behavior for them in the strong interactions. But in the decay of the $K$'s we have to adopt a different point of view, because strangeness is not conserved by the weak interactions. CP is conserved, however, by the weak interactions, and so we define linear combinations of the states $|K^0\rangle$ and $|\bar{K}^0\rangle$ which are eigenstates of $CP = +1$ and $CP = -1$, and we call these states $|K^0_1\rangle$ and $|K^0_2\rangle$ respectively. Since for spinless particles, $CP = PC$, we can set

$$|K^0_1\rangle = 2^{-\frac{3}{2}} (|K^0\rangle + \frac{1}{\sqrt{2}} |\bar{K}^0\rangle),$$

and

$$|K^0_2\rangle = 2^{-\frac{3}{2}} (|K^0\rangle - \frac{1}{\sqrt{2}} |\bar{K}^0\rangle).$$

It is these particles, $K^0_1$ and $K^0_2$, which have different lifetimes, because the transformation properties of the initial states under $CP$ is a very important factor influencing the transition matrix elements. Therefore, we shall list the decay modes and partial rates for the $K^0_1$ and $K^0_2$.

<table>
<thead>
<tr>
<th>MODE</th>
<th>RATE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^0_1 \rightarrow \pi^+ + \pi^-$</td>
<td>7000</td>
</tr>
<tr>
<td>$K^0_1 \rightarrow \pi^0 + \pi^0$</td>
<td>$\sim 3500$</td>
</tr>
<tr>
<td>$K^0_2 \rightarrow \pi^0 + \pi^+ + \pi^-$</td>
<td>$\sim 3$</td>
</tr>
<tr>
<td>$K^0_2 \rightarrow \pi^+ + \mu^- + \nu$</td>
<td>$\sim 7$</td>
</tr>
<tr>
<td>$K^0_2 \rightarrow \pi^+ + \mu^- + \bar{\nu}$</td>
<td>$\sim 7$</td>
</tr>
<tr>
<td>$K^0_2 \rightarrow \pi^- + \bar{\mu} + \nu$</td>
<td>$\sim 7$</td>
</tr>
</tbody>
</table>

The number of events seen is fairly small, and consequently the errors are quite large, running on the order of 50 percent except for the mode $K^0_1 \rightarrow \pi^+ + \pi^-$ where the error is of order 5 percent. The total rate for the $K^0_1$ is about $10^{10}$ sec.$^{-1}$, and that of the $K^0_2$ is about $10^7$ sec.$^{-1}$.

There is no pure lepton decay for the neutral K particles, which is part of the statement that nobody has ever seen any kind of an emission of a lepton-antilepton pair where that pair was not charged. Nobody has ever seen an electron pair emitted presumably by the weak interactions, or a muon
pair, or even a neutrino pair. The detection of a neutrino pair mode would not be impossible, because when the data on the branching ratios were refined, its occurrence would result in the failure of the branching ratios to sum to unity. Another thing that has never been seen is the emission of combinations like $\bar{e}u$. On the other hand, leptons can be emitted when accompanied by a charged pion or a charged pion pair. For reasons we shall go into later on, no numbers have been given for the rates of such decay modes of the $K_1^0$. As with the decay of any particle, associated with the observed modes are a tremendous number of other modes which must also occur, although very infrequently; for example, a gamma ray can accompany the decay products of a mode.

The $K_2^0$ cannot decay into two pions, because by the generalized Pauli principle any two-pion state with zero total angular momentum is an eigenstate of $CP$ with $CP = +1$. The three-pion decay of the $K_1^0$ has not been observed, but the explanation of this requires some knowledge of the wave function of the final state of three pions. It is not difficult to see that three pions with zero total angular momentum have $P = -1$. Now if the three pions are all $\pi^0$'s, then clearly $C = +1$, and thus the decay mode $K_2^0 \rightarrow \pi^0 + \pi^0 + \pi^0$ is forbidden. Now the final state of a $\pi^+$, $\pi^0$, and $\pi^-$ can have $C = +1$ or $C = -1$, depending on whether the wave function is symmetric or antisymmetric respectively under the interchange of the $\pi^+$ and the $\pi^-$. For the simplest final configuration where all three pions are in s-states, $C = +1$, and so the $K^0$ cannot decay into such a configuration.

In order that the mode occur, therefore, the final state wave function must contain $p$-waves, and thus the centrifugal barrier will strongly depress the magnitude of the matrix element. For this reason, it is expected that the mode $K_1^0 \rightarrow \pi^+ + \pi^0 + \pi^-$ would have a rate of the order of $1/100$ of the rate of $K_2^0 \rightarrow \pi^+ + \pi^0 + \pi^-$, and consequently will be exceedingly difficult to detect, especially because the branching ratio would then be of the order $1$ part in $10^6$.

The large rates for the two-pion decay of the $K_2^0$ should be noted together with the fact that the two-pion decay of the $K_1^0$ is slower by a factor of $500$. We shall try to interpret this feature, but at present the interpretation is quite uncertain. It is also of considerable interest that the recent Berkeley results on the ratio of the rates of the modes $K_1^0 \rightarrow \pi^+ + \pi^-$ and $K_0^0 \rightarrow \pi^0 + \pi^0$ are consistent with the value $2:1$. 
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The only hyperon decay modes that are really well known are the following. The rates will be quoted in units of $10^9$ sec$^{-1}$, are comparable with the rates for the two pion decay modes of the $K_0^0$, and have errors of about 10 percent.

<table>
<thead>
<tr>
<th>MODE</th>
<th>RATE</th>
<th>$\alpha \bar{P}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda \rightarrow p + \pi^-$</td>
<td>2.4</td>
<td>$.7 \pm .1$</td>
</tr>
<tr>
<td>$\Lambda \rightarrow n + \pi^0$</td>
<td>1.2</td>
<td></td>
</tr>
<tr>
<td>$\Xi^+ \rightarrow p + \pi^0$</td>
<td>6.4</td>
<td>$.7 \pm .3$</td>
</tr>
<tr>
<td>$\Xi^+ \rightarrow n + \pi^+$</td>
<td>6.4</td>
<td>$.0 \pm .1$</td>
</tr>
<tr>
<td>$\Xi^- \rightarrow n + \pi^-$</td>
<td>6.4</td>
<td>$.0 \pm .1$</td>
</tr>
<tr>
<td>$\Xi^- \rightarrow \Lambda + \pi^-$</td>
<td>same order</td>
<td></td>
</tr>
</tbody>
</table>

The third column is a measure of the asymmetry in the decay mode. The hyperons, when they are made in the strong interactions, are often polarized. Since the strong interactions conserve parity, the polarization can be only perpendicular to the plane of the reaction, e.g. in the direction $\mathbf{n} = \frac{\mathbf{P}_\pi \times \mathbf{P}_\Lambda}{|\mathbf{P}_\pi \times \mathbf{P}_\Lambda|}$ for the production reaction: $\pi^- + p \rightarrow \Lambda + K^0$. The lack of parity conservation in the decay then implies that there can be an asymmetry term, $\mathbf{n} \cdot \mathbf{P}_\pi$, in the distribution of the pions resulting from the decay of the hyperon. For perfectly polarized hyperons, the distribution will be of the form: $1 + \alpha \cos \theta$. The interesting parameter, $\alpha$, cannot be measured directly because the hyperons will not be polarized to a known degree by the strong interaction. The actual distribution will then be of the form $1 + \alpha \bar{P} \cos \theta$, where $\bar{P}$ is the polarization of the hyperons averaged over all the production angles. If the product $\alpha \bar{P}$ for a given mode turns out to be large, then you can be sure that the asymmetry parameter $\alpha$ is also large, because $\bar{P}$ can never be greater than unity. On the other hand, if the product $\alpha \bar{P}$ comes out to be zero, then you really don't know anything about $\alpha$, because $\bar{P}$ could be zero. That's true unless you can determine $\bar{P}$ from an analysis of the hyperon production cross sections and angular distributions, or unless there are two modes of decay of the hyperon, in one of which $\alpha \bar{P}$ turns out to be non-zero,
and in the other $\alpha^P$ comes out zero.

The product $\alpha^P$ for the mode $\Lambda \rightarrow p + \pi^-$ is definitely quite large, which means that the polarization is almost complete and that the magnitude of the asymmetry parameter is also close to its maximum value, unity. The determination of $\alpha^P$ for the mode $\Lambda \rightarrow n + \pi^0$ will be a very important experiment. It is of course exceedingly difficult because in addition to the $\Lambda$, the two products are neutral, but it is not impossible because the neutrons can be detected by the recoil protons they produce, and the $\pi^0$'s by the materialization of the decay $\gamma$-rays. In particular it would be very nice just to know the sign of $\alpha^P$ for this mode, for if the $\Lambda\pi^0$s are produced in the same way as done in the experiments measuring $\alpha^P$ for the mode $\Lambda \rightarrow p + \pi^-$, the $\overline{\pi}$ would cancel out, and one would know the relative signs for the $\alpha$'s of the two modes, which would be very interesting. The recent Berkeley results on $\alpha^P$ for the modes of the $\Sigma^+$ are taken to indicate that the asymmetry parameter is rather large for $\Sigma^+ \rightarrow p + \pi^0$ and rather small for $\Sigma^+ \rightarrow n + \pi^+$, and in fact it is claimed that for the charged pion mode $|\alpha| < 0.03 \pm 0.11$. In the case of the $\Sigma^-$, nobody has ever succeeded in detecting an asymmetry, but once more it may be mentioned that this does not imply $\alpha$ is small because $\overline{\pi}$ could well be zero.

Essentially nothing is known about the decay of the $\Xi$. The mode $\Xi^- \rightarrow \Lambda + \pi^-$ has been observed, and the rate could be something like the other hyperon rates, within a factor of three or so. An event that is fairly certain to be an example of the mode $\Xi^0 \rightarrow \Lambda + \pi^0$ has also been observed. The decay of a $\Xi^-$ into a neutron, with a strangeness change of 2, has not been seen. There is more energy available for the $\Xi$ to go into a neutron than a $\Lambda$, and so if there is no rule or something to prevent the mode, you would expect to get more neutrons than lambdas. On the other hand, the identification of a neutron as coming from a $\Xi$ is not too likely, because so far the experimentalists identify an event as containing a $\Xi$ when they see a cascade. Thus they would probably have ascribed an event containing $\Xi \rightarrow N + \pi$ as being due to a new particle. In the opinion of the cosmic ray people here at Caltech, however, it is unlikely that the mode $\Xi \rightarrow N + \pi$ could be more frequent than the mode $\Xi \rightarrow \Lambda + \pi$, because otherwise they probably would have seen a couple of events and actually identified them as being due to a new particle. Therefore, the mode $\Xi \rightarrow N + \pi$ could well exist, but presumably is at least slightly forbidden.
For more information we have to wait, and what we can do now is to try
to construct theories, or rules, which will enable us to appreciate and under­
stand better this body of data now available. The chief advantage of this
theorizing is not that we are likely to get the right theory, but that it
adds tremendously to the interest in the facts if one has some theoretical
principles to argue about. Otherwise who would want to remember all these
facts if he didn't have some idea as to how they should go according to one
scheme or another? By discussing various theoretical possibilities, I think
we'll get a real insight into what the data mean, so that if a new piece of
information comes in, we will understand where it fits. That is really the
principal value of this sort of theorizing; a secondary value is that you
might actually learn the correct theory, but at the moment that appears not
too likely, except by accident insofar as if you list ten possibilities, it
might be one of the ten. Thus, let us try to discuss the strange particle
decays from a theoretical point of view, looking at theoretical principles
and seeing what kinds of experiments test these principles. Not all of
these principles will be right, but some of them may well be correct.

The first thing we discussed in connection with a theory of the weak
interactions was the pattern of a current-current interaction, which fits
what we know about the weak couplings of (ev), (μν'), and (np) to one another.
However, the real tests of the hypothesis concern the interaction of a term
like (ev) with its corresponding term (ve), and these have not been performed.
Nevertheless, for the moment, let us work with the current-current form of
the weak interactions. We now wish to split the many terms possible in the
current into five groups, by writing:

\[ J = J^e + J^l + J^2 + J^3 + J^4 \]

where \( J^e \) contains only leptons, \( J^l \) contains baryon and meson terms producing
no strangeness change, \( J^2 \) carries strangeness -1, \( J^3 \) carries strangeness +1,
and \( J^4 \) carries strangeness -2. As examples of these groups we may give a few
typical terms:

\[ J^e = (\overline{e}v), (\mu\nu'), \ldots \]
\[ J^l = (\overline{n}p), (\overline{\Xi}^0), (\Lambda\Sigma^+), \ldots \]
\[ J^2 = (\overline{\Lambda}p), (\overline{\Xi}n), (\Xi\Lambda), \ldots \]
\[ J^3 = (\overline{\Xi}^0), (\overline{\Xi}^0) \]
\[ J^4 = (\overline{\Xi}^0 p), (\Xi n) \]
We presume that the nucleon term in \( J_{-a} \) is \( i 2^{1/2} \alpha^2 \gamma_\mu (1 + \gamma_5) p \), although of course the bare coefficients are really unknown. The rest of \( J_{-} \) is completely specified by the hypothesis of a conserved vector current, together with the assumption that the axial current comes from the \((1 + \gamma_5)\) in every fermion term.

To explain the decays of the strange particles, at least one term in the current must exist which causes a change in the strangeness.

The first thing that should be noticed about the strangeness changing groups of terms is that with \( J^2 \) alone one can obtain a qualitative description of all the strange particle decays that have been observed so far. No terms of the groups \( J^3 \) and \( J^4 \) are needed to account for the modes known today. That we saw at the very beginning when we saw by augmenting the Puppi triangle by \( \gamma p \) vertex, everything could be understood qualitatively.

Now consider the decay mode \( K^+ \rightarrow \mu^- + \nu \). We know that \( J_+ \) is used to create the lepton pair, and so there must be at least one term of \( J_- \) carrying strangeness \(-1\). Therefore, the group \( J^2 \) is certainly present in the current. If you could insist on having only one of the three possible strangeness changing current groups, \( J^2 \) is the one that would work. And if such a pattern were correct, you would expect some other leptonic decays such as \( \Lambda \rightarrow p + e + \bar{\nu} \), which has been seen twice and \( \Xi^- \rightarrow n + e + \bar{\nu} \), which may have been seen once. The branching ratio for the leptonic lambda decay is uncertain, but so far is about 1 part in 1000. These being the only leptonic decays of hyperons observed, it is clear that \( J^2 \) is presently sufficient for everything.

The question of interest is thus: what decays, if observed, would absolutely require that terms of groups \( J^3 \) and \( J^4 \) be present? To get a handle on this problem, we have to work with leptonic decays, for in those we know what charge the term in the current producing the observed strangeness change carries. It is clear that the existence of the decay mode \( \Sigma^+ \rightarrow n + \bar{e} + \nu \) would be conclusive evidence that couplings of the type arising only from \( J^3 \) exist. Similarly, the detection of the mode \( \Xi^- \rightarrow n + e + \bar{\nu} \) would be direct proof of the existence of \( J^4 \). But since only three leptonic decays of hyperons have been reported, we have no knowledge yet as to whether \( J^3 \) and/or \( J^4 \) is present.

Summarizing the above considerations, we can say that the detection of any one of the following decay modes indicates without any doubt that in the weak interaction hamiltonian there is a coupling between the lepton current and a current of the indicated type:
The conclusion we reach from existing experiments is that a coupling with $J^2$ is definitely there, but about $J^3$ and $J^4$ we don't know. Probably neither $J^3$ nor $J^4$ is present, but we cannot be sure until people study leptonic decays more extensively, or until we have other evidence.

The next test has to do with the leptonic decay of the $K^0$. If $J^3$ is not present, then we have the remarkable result that the modes

$$K^0 \rightarrow \pi^- + \bar{\mu} + \nu$$
$$\bar{K}^0 \rightarrow \pi^+ + \mu + \bar{\nu}$$

are allowed, but not the other way around, i.e.

$$K^0 \rightarrow \pi^- + \mu + \bar{\nu}$$
$$\bar{K}^0 \rightarrow \pi^+ + \bar{\mu} + \nu$$

Since by CPT invariance, the rates for the two members of each pair are equal, this then means that the proportion of the signs of the leptonic charges coming from the decay of neutral K's at a given moment depends upon the ratio of the amount of strangeness +1 to the amount of strangeness -1 in the wave function of a neutral K. This ratio can be read off from the curve which gives $|\langle K^0 | K(t) \rangle|^2$ as a function of the time after the production of a $K^0$ whose subsequent behavior may be described by $|K(t)\rangle$. The nature of the curve depends critically on $\xi = \left| \frac{(m - m_c)}{\Gamma_{K^0} - \Gamma_{K^0}} \right|$ as the following three examples indicate.

<table>
<thead>
<tr>
<th>$J^2$</th>
<th>$J^3$</th>
<th>$J^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^+ \rightarrow \bar{\mu} + \nu$</td>
<td>$\Sigma^+ \rightarrow n + e + \nu$</td>
<td>$\bar{\Sigma} \rightarrow n + e + \bar{\nu}$</td>
</tr>
<tr>
<td>$\Sigma^- \rightarrow n + e + \bar{\nu}$</td>
<td>$K^+ \rightarrow \pi^+ + \pi^+ + e + \bar{\nu}$</td>
<td>$\bar{\Sigma}^0 \rightarrow p + e + \nu$</td>
</tr>
<tr>
<td>$\Lambda \rightarrow p + e + \bar{\nu}$</td>
<td>$K^- \rightarrow \pi^- + \pi^- + e + \nu$</td>
<td></td>
</tr>
</tbody>
</table>

![Graph showing the function $|\langle K^0 | K(t) \rangle|^2$ as a function of time.](image-url)
It is not known which type of curve correctly describes the behavior of a fresh $K^0$, because the magnitude of $\xi$ is unknown, but in any case the ratio of positive muons to all the muons is given by twice the curve divided by $(e^{-T_0 t} + e^{-T_0 t})$. The same statement also is true for the electrons coming from the decay of a fresh batch of $K^0$'s. These are further tests of whether $J_3^3$ is absent or not.

Let us go into this matter in a little more detail. If we have pure $K^0$'s or pure $K^0$'s, then they would decay into a 50-50 mixture of + and - leptons, on account of the CP invariance of the weak interactions. But if $J_3^3$ is not present, the rates of $K^0$ and $K^0$ for a given leptonic mode are equal, because then there is no interference between the $K^0$ and the $K^0$ components. We have then the interesting result that only about 1 in 1000 of the $K^0$'s would decay into leptons, since the leptonic decay rate would be about $10^7$ sec$^{-1}$ while the total decay rate is about $10^{10}$ sec$^{-1}$. If a fresh beam of $K^0$'s is looked at, a mixture of $K^0$'s and $K^0$'s will be seen. Now the point is that each of these will go into leptons at the same rate, namely $10^7$ sec$^{-1}$, and
the decay rate of the beam of fresh $K^0$'s will have about the same leptonic decay rate. There will, of course, be interference between the $K_1^0$ and $K_2^0$ components of the fresh beam, but such interference could not change very much the average magnitude of the leptonic decay rate of the beam. So the total probability of seeing a leptonic decay compared to the two pion decays would be about 1 part in 1000 in that region close to the production point of $K^0$. Further out, by the time the $K_1^0$ component of the beam has died out, that probability rises to about 2/3, but if you look very close, it is very small. Now if $J^3$ is present, $K^0$ can go into both $\bar{\mu}$ and $\mu$, and similarly for $\bar{K}^0$. The rates for leptonic decay of $K_2^0$ and $K_1^0$ are then not equal, and, in particular, the rate for $K_1^0$ might be considerably larger than $10^7$ sec$^{-1}$. It could be that the total probability of seeing a leptonic decay close to the production point of the $K^0$ is much greater than 1/1000; it might be 1 percent or 10 to 20 percent. Now, in fact, experimentally it is less than 1 percent, but it has not been concluded experimentally that it is as low as 0.1 percent, which is what it has to be if $J^3$ is absent. So a more accurate determination of that probability is another possible test of the existence of $J^3$.

So far, we have discussed tests of whether there are couplings between leptons and terms belonging to groups $J^3$ and $J^4$. Now let us go on to discuss the consequences of the existence of $J^3$ and/or $J^4$ in the non-leptonic strange particle decays. In so doing we will be making explicit use of the assumption that the cross terms in the scheme $J_+ J_-$ appear in the weak interaction hamiltonian. Up to now in talking about leptons, we have just been analyzing experiments, without any actual use of the current-current hypothesis. We have simply been determining what type of terms must be coupled to leptons. But from this point on, the assumptions shall be made that there is the contribution $J_+ J_-$ in the hamiltonian, and that all the currents which are coupled to leptons are coupled in this way to one another.

If $J^3$ and $J^4$ are absent, then the decay mode $E \rightarrow N + \pi$ is completely forbidden, because in first order of the weak coupling, the maximum strangeness change would be one unit. Notice that it is necessary for this conclusion that not only $J^4$ but also $J^3$ be absent, because the term $J_+^2 J_-^3$ carries strangeness +2, and would thus induce the decay. As mentioned earlier, such a mode has not been seen, but, in the first place, the experimental statement is not very strong. And, in the second place, the decay could be weakly forbidden; even though it might be allowed, some rule could exist
which might reduce the rate by a factor of ten or a hundred, and then it might not be observed experimentally for a very long time. None of our tests so far are really perfect in that one can never be sure that a group in the current does not exist because certain modes are not observed. On the other hand, if the special modes are observed, there is no doubt that the associated types of terms do occur in the Hamiltonian.

An excellent test of whether \( J^3 \) and/or \( J^4 \) exists or not has been devised which is not affected by the fact that some rule may exist, which may weakly forbid a process, as one presumably does since \( \Gamma_{K^+ \rightarrow n^+ + n^0} \) is only 1/500 of \( \Gamma_{K^0 \rightarrow 2\bar{n}} \). If \( J^3 \) and/or \( J^4 \) is present then there exists a non-vanishing matrix element of the weak interaction Hamiltonian between the states \( |K^0\rangle \) and \( |\bar{K}^0\rangle \) which is of first order in \( G \), but if both \( J^3 \) and \( J^4 \) are absent, the matrix element would be of order \( G^2 \). A transition matrix element between \( |K^0\rangle \) and \( |\bar{K}^0\rangle \) produces a mass difference between \( K^0 \) and \( \bar{K}^0 \) which is twice the transition matrix element. This is easy to see by simply diagonalizing the important block of the Hamiltonian containing the \( K^0 \) and \( \bar{K}^0 \) matrix elements, which is

\[
\begin{pmatrix}
K^0 & \bar{K}^0 \\
K^0 & A \\
\bar{K}^0 & B \\
\end{pmatrix}
\]

on account of the CP invariance of nature. The diagonalization yields

\[
\begin{pmatrix}
A+B & 0 \\
0 & A-B \\
\end{pmatrix}
\]

for the eigenstates \( |K^0_1\rangle \) and \( |K^0_2\rangle \), which implies a mass difference of 2\( B \).

The real test is therefore to investigate which of the three different curves shown before describes correctly phenomena involving a fresh beam of \( K^0 \)'s, and thereby to determine whether the mass difference between \( K^0_1 \) and \( K^0_2 \) is of order \( G \), in which case \( \xi \gg 1 \), or whether it is of order \( G^2 \), in which case \( \xi \sim 1 \). The difference between these two cases involves a factor of order \( 10^6 \) or \( 10^7 \), which is bound to make quite a difference notwithstanding any weak selection rule.

This test was proposed by Okun', and should be carried out relatively easily. The experimental procedure is to take a beam of fresh \( K^0 \)'s, made together with hyperons in some target, and record the distribution in time,
measured from the production point of the $K^0$, of those events where the $K^0$ component of the beam produces another hyperon. If $\xi \gg 1$ as it is if the mass difference is of order $G$, then immediately the beam becomes an oscillating 50-50 mixture of $K^0$'s and $\bar{K}^0$'s, but the oscillations would be of the order $10^{16}$ per second and thus unobservable. So the hyperon distribution would appear as that due to 50 percent of the beam being $K^0$'s. But if the mass difference is of the order of $G^2$, then it would take some time for the $K^0$ component of the beam to build up, and the secondary hyperons would not be made close to the production point of the $K^0$'s.

Whether terms from $J^3$ and $J^4$ must be included in the current is thus the first hurdle we must clear in setting up a correct theory of the weak interactions, but this cannot be done until we get more information on strange particles.
April 23, 1959

In the absence of good evidence on the question, let us proceed mainly on the assumption that the groups $J_3$ and $J_4$ are absent from the current. From time to time though, we will come back and say what would happen if such terms do exist, because they may very well be there. With $J_3$ and $J_4$ excluded, the number of possibilities of strangeness changing terms in the current is rather small. In fact, there are only six serious candidates:

$$J_2^-: (\bar{A}\rho), (\bar{\Sigma}^0\pi), (\bar{\Xi}^0\Lambda), (\bar{\Xi}^0\Xi^0), (\bar{\Xi}^0\Sigma^+) .$$

It is conceivable that meson terms of the form $(K\pi)$ might be in $J_2^-$, but the existence of such a direct coupling of $K$ to $\pi$ with a regulation $G$ would lead to a rate for the leptonic decay of the $K$ particle which would be $180$ times greater than what is observed. So it is a pretty safe bet that no such $(K\pi)$ term is present. For convenience in the subsequent discussion, let us once again exhibit the possibilities for the baryon-meson terms producing no strangeness change:

$$J_1^+: (\bar{\Sigma}^0\pi), (\bar{\Xi}^0\Lambda), (\bar{\Xi}^0\Xi^0), (\bar{\Xi}^0\Xi^0), (\bar{\Xi}^0\Xi^0), (\bar{\Xi}^0\Xi^0).$$

The conserved vector current hypothesis, together with the idea that every baryon term involves $l \gamma_5$, tells us to take the relative coefficients of the above baryon terms to be: $1, \sqrt{2}, -\sqrt{2}, \sqrt{2}, 1$ respectively, according to the sign convention we have used all along. But who knows that the hypothesis is correct? In the following, therefore, we shall not assume so restrictive a hypothesis about $J_1^+$.

The first thing to notice is that it costs very little in generality to require certain theoretical symmetry properties of $J_1$ and $J_2^-$, whose consequences we may explore and check against experiment. We have discussed one such symmetry before, namely the invariance of $J_1$ under $B$, whose consequences however, being the absence of certain small terms in the effective vertex, were not easily detectable. But the symmetries now being advanced give us a series of conditions which can be checked rather easily. The hypothesis is that under rotations in isotopic spin space $J_2^-$ transforms like a spinor, and $J_1^+$ transforms like a vector, i.e. $J_2^-$ carries isospin $1/2$, and $J_1^+$ carries isospin $1$. Such transformation properties hold provided only the two members of the following pairs of terms are present in the proportion shown:
These conditions impose very slight requirements on the couplings, and have very important consequences for experiment, which we can check.

It may be remarked at once that if there were some \( J^3 \) present in the strangeness changing current, it could not carry spin 1/2 exclusively, because all the terms of \( J^2 \) like \((\bar{\Sigma}^+ \Sigma^0)\) and \((\bar{\Sigma}^0 \Sigma^0)\) carry \( I^\prime = -3/2 \), and thus necessarily carry \( I = 3/2 \). So subsequently if we find some evidence that our predictions based on the isospinor nature of the strangeness changing current and the isovector nature of \( J^1 \) do hold, that will also be some evidence that terms of \( J^3 \) are absent. Similar considerations hold for \( J^4 \).

The consequences are of two types. (1) In leptonic decays of the strange particles, the meson-baryon part of the matrix element transforms like an object carrying isospin 1/2. And (2) in the non-leptonic decays which come from \( J^1 \) and \( J^2 \) taken together, the matrix element can be split into two pieces, one of which transforms like an object carrying isospin \( 1/2 \) and the other isospin \( 3/2 \). Agreement with the latter statement provides to some extent evidence on whether \( J^3 \) is present. In the leptonic decays, we first have to examine whether \( J^3 \) is present by observing all types of leptonic decays, and then we have to test the decay of the \( K^0 \) as to whether \( \Delta I = 1/2 \) or not.

The spinor nature of the baryon-meson matrix element has two consequences for the leptonic decay of the \( K \) particle:

\[
\Gamma(K^0 \rightarrow \pi^- + \bar{\nu} + \nu) = \Gamma(K^0 \rightarrow \pi^+ + e + \bar{\nu}^c) = \Gamma(K^+ \rightarrow \pi^0 + \bar{\nu} + \nu),
\]

and

\[
\Gamma(K^0 \rightarrow \pi^- + \bar{\mu} + \nu) = \Gamma(K^0 \rightarrow \pi^+ + \mu + \bar{\nu}^c) = \Gamma(k^+ \rightarrow \pi^0 + \bar{\mu} + \nu^c).
\]

The first equality in each line, as remarked in the last lecture, is guaranteed by CP invariance, and thus we need only demonstrate that:

\[
\langle \pi^- | J^2_- | K^0_2 \rangle = \langle \pi^0 | J^2_+ | K^+ \rangle.
\]

Now our hypothesis about \( J^2 \) is that it transforms like the irreducible tensor operator \( T^{\frac{1}{2}}_{\Lambda} \) in isospace, and so by the Wigner-Eckart theorem:

\[
\frac{\langle \pi^- | J^2_- | K^0_2 \rangle}{\langle \pi^0 | J^2_+ | K^+ \rangle} = 2^{-\frac{3}{2}} \frac{\langle \pi^- | J^2_- | K^0 \rangle}{\langle \pi^0 | J^2_+ | K^+ \rangle} = 2^{-\frac{3}{2}} \frac{\langle \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2} | \frac{1}{2}, -\frac{3}{2}, 1, -1 \rangle}{\langle \frac{1}{2}, -\frac{3}{2}, -\frac{1}{2} | \frac{1}{2}, -\frac{3}{2}, 1, 0 \rangle} = 1.
\]
It is quite obvious from the proof that in the equalities we could replace $K_2^0$ by $K_1^0$.

Now how well are these equalities satisfied? Our data on the leptonic decay rates of the $K^+$ can be combined to predict a total leptonic decay rate of the $K_1^0$ or the $K_2^0$ to be $13.4 \pm 1.4$ in units of $10^6$ sec.\(^{-1}\). The experimental values are still quite poor. A recent letter from Berkeley\(^*\) quotes results from an experiment at Columbia which indicate a total leptonic decay rate of the $K_2^0$ of about $12 \pm 5$, and, on the basis of their 8 events, report their value for the same total rate to be about $20 \pm 7$, assuming the same total leptonic decay rate for the $K_2^0$ and the $K_1^0$. There is agreement here, but clearly we really have to wait until the data are refined to be sure of it.

This is about all that can be said that is useful about the leptonic decays on the basis of $J^2$ carrying spin $1/2$. It would, of course, tell us the relative rates of $\Sigma^- \rightarrow n + e + \bar{\nu}$ and $\Sigma^0 \rightarrow p + e + \bar{\nu}$, but the latter decay mode will never be observed because $\Sigma^0 \rightarrow \pi^0 + \gamma$ so rapidly. Similar comments can also be made concerning the leptonic decay of the $\Xi$.

For the non-leptonic decays, what can we get out of the statement that part of the matrix element transforms like an object carrying $I = 1/2$, and the remaining part like an object carrying $I = 3/2$? In considering the two and three pion modes of the $K$ particle and the hyperon modes, we shall find that the hyperon decays shed essentially no light upon the validity of the hypothesis, whereas the data on $K$ decays do tell us something.

Let us look first at the modes: $K \rightarrow 3\pi$. It is known, from the very careful and extensive study of the $K^+$ spectrum carried out because of the parity question, that the three pion matrix element is essentially independent of the energies of the pions. Hence, the pions must on the whole all be in $s$-states with only small admixtures of other angular momentum states. Furthermore, in that region of space where the interaction occurs, the $s$-wave wave function is essentially a constant, which implies that the total wave function is completely symmetric under permutations of the pions, and consequently the isotopic spin wave function must also be completely symmetric, by the generalized Pauli principle. This, together with the assumption that the weak interaction hamiltonian carries $I = 1/2$ and $I = 3/2$ only, will determine the isotopic properties of the final state of three pions to a fairly high accuracy.

Consider the problem of determining the permutation properties of the isotopic spin states formed from the addition of three spins, each of magnitude 1. There are $3^3 = 27$ of these, which can be analyzed into irreducible sets, with definite transformation properties under rotations, i.e. with definite angular momentum. From the study of angular momentum in quantum mechanics, we know we will get a set of 7 states with $I = 3$, two sets of 5 states with $I = 2$, three sets of 3 states with $I = 1$, and one state with $I = 0$. But this doesn't tell us what the symmetry properties of these sets are under permutations of the particles. Each of these sets can be arranged to have definite symmetry characteristics under the exchanges of the three particles, or, in mathematical language, to correspond to a particular irreducible representation of the permutation group. Our problem is to find which of these sets are completely symmetric. To do that, it is easiest to rephrase things, and find the symmetry properties of the irreducible tensors formed from the product of three vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$.

The irreducible tensor with $I = 3$ is obviously totally symmetric under exchanges, because the only way you can make the component with $I_z = 3$ is to take the product of all three $I_z = 1$ components of the vectors, and the symmetry properties of each component of an irreducible tensor are all the same. To make the tensor with $I = 0$, one has to form a scalar (or pseudoscalar) from three vectors. There is only one such combination, which is $\vec{a} \cdot \vec{b} \times \vec{c}$, the triple scalar product, which is clearly totally antisymmetric under exchanges, and we are therefore not interested in it. Now let us make three vectors. If these are taken to be:

$$\vec{a}(\vec{b} \cdot \vec{c}) + \vec{b}(\vec{c} \cdot \vec{a}) + \vec{c}(\vec{a} \cdot \vec{b})$$

and

$$\vec{a} \times (\vec{b} \times \vec{c}), \quad \vec{b} \times (\vec{c} \times \vec{a})$$

it will be seen that of the three irreducible tensors with $I = 1$, one can be chosen to be completely symmetric, and the other two are partially antisymmetric. The latter two form an irreducible two-dimensional representation of the permutation group of three particles. We are not so familiar with the $I = 2$ combinations, which are traceless symmetric dyadics, as we are with vectors and scalars, but it is not too difficult to see that the $I = 2$ forms will be built mainly from things like $\vec{a}(\vec{b} \times \vec{c})$ and $\vec{b}(\vec{c} \times \vec{a})$, and will be partially antisymmetric, belonging to the two-dimensional representation of the permutation group.
It should be clear now that the only totally symmetric states of three particles are the seven belonging to the set $I = 3$, and three more which are members of a set with $I = 1$. Now the $K$ particle has $I = 1/2$, and if the non-leptonic part of the weak interaction carries either $I = 1/2$ or $I = 3/2$ only, the totally symmetric $I = 3$ states cannot be reached. Thus the isotopic properties of the final three pion state will be described, to the extent that the wave function is totally symmetric, by a single isotopic spin state with $I = 1$, and the appropriate $I_z$. Therefore, the distribution of charge is determined.

However, knowledge of the isotopic state does not determine the ratio of the three pion decay mode rates of the $K^+$ and neutral $K$'s, because the interaction does not carry a definite isospin, as it does in leptonic decays if the strangeness changing current transforms like an isotopic spinor. It is only the distribution of charge among the three pions that is determined for each particle by the isospin state.

The final state of three pions in the decay of the $K^+$ will have $I_z = +1$. Let the states of the pions be denoted by 1, 2, 3, which in general are all distinct, i.e. no two pions are in the same state. The easiest way to determine the charge distribution is just to construct a typical totally symmetric $I = 1$, $I_z = +1$ state by applying the operator

$$n^- (1) \left[ n^o (2) + n^o (3) \right] + n^- (2) \left[ n^o (3) + n^o (1) \right] + n^- (3) \left[ n^o (1) + n^o (2) \right]$$

to the vacuum state. If we denote a state by $|Q_1, Q_2, Q_3\rangle$ for brevity, since $n^- n^o = n^o n^- + n^o n^+ + n^+ n^-$, the symmetric $I = 1$ state is

$$|+00\rangle + |00+\rangle + |0+0\rangle + 2|++-\rangle + 2|+-+\rangle + 2|--+\rangle.$$

The charge distribution ratio is thus: $(2n^+ + n^-):(n^+ + 2n^o) = (2^2 + 2^2 + 2^2): (1^2 + 1^2 + 1^2) = 4:1$. Now because charged pions are 4.6 MEV heavier than neutral pions, the density of states for the two modes differs, and putting that in, one would expect $\Gamma_{K^+ \to n^+ n^- + n^o} / \Gamma_{K^+ \to n^+ + n^o + n^o}$ to be reduced to 3.08. Using the data presented earlier, the experimental value for this ratio is $4.62/1.38 = 3.3 \pm 0.6$, which does agree with the prediction.

In the three pion decay of the neutral $K$, the final state will have $I_z = 0$, and the corresponding operator generating the completely symmetric $I = 1$ state is:

$$n^o (1) \left[ n^o (2) + n^o (3) \right] + n^o (2) \left[ n^o (3) + n^o (1) \right] + n^o (3) \left[ n^o (1) + n^o (2) \right],$$
which makes:

\[ 3|000\rangle + |0+\rangle + |0-\rangle + |0+\rangle + |0-\rangle + |00\rangle + |00\rangle, \]

from which one gets the ratio: \( \frac{(3|0\rangle)}{(3^2): (1^2 + l^2 + l^2 + l^2 + l^2 + l^2)} = 3:2. \)

Taking into account the difference in the density of states, one would expect to be close to 2:1, assuming the symmetry of the final state wave function. Without using that symmetry under the permutation group, you can also derive inequalities for these charge ratios, but we may as well make use of it since it is pretty well established.

Next we want to examine the consequences of the hypothesis for the two pion decay modes of the K particle. This is a slightly different problem. The two pions are in a relative s-state, because the spin of the K particle is known to be zero, and therefore the space part of the final state wave function is symmetric under the exchange of the two pions. By the generalized Pauli principle, the isotopic wave function must then also be completely symmetric under the exchange of the two pions, and hence the isotopic wave function can be a linear combination of \( I = 2 \) and \( I = 0 \). The K particle has \( I = 1/2 \) and so the \( I = 1/2 \) part of the interaction can lead only to \( I = 0 \). This \( I = 1/2 \) piece is therefore of no consequence for the decay of the \( K^0 \), since the final state necessarily has \( I_z = 1 \), and hence is pure \( I = 2 \). The two pion decay of the \( K^+ \) depends only on the matrix element of the \( I = 3/2 \) part of the interaction, which leads exclusively to a final state with \( I = 2 \). Therefore, by using the rate for the two pion decay of the \( K^+ \), we can compute the magnitude of the \( I = 3/2 \) matrix element, and by using the rate of the decay of \( K^0 \) we can also compute the magnitude of the \( I = 1/2 \) matrix element. If we can also determine the relative phases of these matrix elements, we can then predict the charge distribution in the decays of \( K^0 \).

In the charge distribution resulting from the decay of \( K^0 \) the interference between the \( I=2 \) and the \( I=0 \) amplitudes does appear and thus we must relate the relative phase of the amplitudes to something else. (Clearly this interference does not manifest itself in the total rates because the \( I=2 \) and \( I=0 \) final states are orthogonal to each other.) Since the weak interactions are presumably invariant under time reversal along with all the other stronger interactions, and the weak interactions need only be considered in the lowest order of perturbation theory, one can relate the phases of the weak interaction matrix elements to the scattering phase shifts for two pions in the
I=2 and I=0 states with J=0, in the same way that the phases of the photo-production matrix elements at low energies were related to pion-nucleon scattering phase shifts. At the energy of the K particle, neglecting of course electromagnetism, there is only a single final channel for a given value of I, I, J and M for the two pion interaction, and so the weak interaction matrix elements must have the relative phase corresponding to

\[ \pm e^{i(\delta_2 - \delta_0)} \]  

(See page 269). \( \delta_2 \) and \( \delta_0 \) are the phase shifts for pion-pion scattering for J=0 at 495 MEV total energy for the I=2 and I=0 states, respectively.

April 28, 1959

(The lectures on April 28 and April 30 were delivered by Dr. Richard P. Feynman).

Let us define the following transition matrix elements:

\[ A = \langle 2\pi, I=0 \parallel R_{\lambda/2} \parallel K, I=1/2 \rangle, \]

and

\[ B = \langle 2\pi, I=2 \parallel R_{\lambda/2} \parallel K, I=1/2 \rangle, \]

where the subscript on the operator indicates its "isotopic spin", i.e. designates its transformation properties in isospace. Assuming time reversal invariance of the weak interactions, \( B/A = \pm |B/A| e^{i(\delta_2 - \delta_0)} \). Then we have

\[ \Gamma_{K \to \pi^+ \pi^0} = c |B|^2 \left\langle \frac{1}{2'} \left| \frac{1}{2'} \frac{1}{2} \right| \frac{1}{2'} \frac{1}{2} \right\rangle^2 = c \frac{3}{4} |B|^2, \]

where in the constant c, we have lumped the various factors such as the density of states, which are the same for all two pion modes of the decay of K particles, (to within a percent or so). Since the interactions are invariant under CP, and since the final state of 2 pions is an eigenstate of CP with eigenvalue +1,

\[ \langle 2\pi |B| K^0 \rangle = \langle 2\pi |R\pi| K^0 \rangle, \]

and hence

\[ \Gamma_{K^0 \to 2\pi} = c(2 - \frac{1}{\sqrt{2}})^2 \left\{ |A|^2 \left\langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, 0, 0 \right\rangle^2 + |B|^2 \left\langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0 \right\rangle^2 \right\}. \]

These two equations can be combined to yield:
Now because $|B/A|$ is so small, we can deduce fairly sharp limits on the charge distribution in the two pion decay of $K_1^0$, under our hypotheses about $J^1$ and $J^2$. If the phase shifts from pion-pion scattering were known for $s$-waves in the $I=2$ and $I=0$ states at about 495 MEV, the hypothesis could be checked to the extent that electromagnetic effects are negligible. Conversely, assuming the validity of the proposed isovector nature of $J^1$ and the isospinor nature of $J^2$, and the absence of $J^3$ and $J^4$, one can deduce the difference in the pion-pion phase shifts from an accurate measurement of the charge distribution. Let us see mathematically how this comes about. The rate for $K_1^0 \rightarrow \pi^0 + \pi^0$ is equal to the constant $c$ times the absolute square of the following matrix element:

$$2 \frac{3}{2} \left\{ a \langle \frac{1}{2}, \frac{1}{2}; 0,0 | \frac{1}{2}, \frac{1}{2}; 0,0 \rangle \langle 1,0; 1,0 | 1,1; 0,0 \rangle + b \langle \frac{1}{2}, \frac{1}{2}; 0,0 | \frac{1}{2}, \frac{3}{2}; 2,0 \rangle \langle 1,0; 1,0 | 1,1; 2,0 \rangle \right\}. $$

Putting in the values for the Clebsch-Gordan coefficients, one obtains

$$\Gamma_{K_1^0 \rightarrow \pi^0 + \pi^0} = c \frac{1}{3} | A + \sqrt{2} B |^2 = c \frac{|A|^2}{3} \left\{ 1 + 2 \sqrt{2} \text{ Re } B/A + 2 |B/A|^2 \right\}. $$

Therefore, the predicted branching ratio would be:

$$\frac{\Gamma_{K_1^0 \rightarrow \pi^0 + \pi^0}}{\Gamma_{K_1^0 \rightarrow \pi^0 + \pi^0}} \approx \frac{1}{3} \left\{ 1 + 2 \sqrt{2} \text{ Re } B/A \right\} = \frac{1}{3} \left\{ 1 \pm 0.15 \cos(62.5^\circ) \right\} < 0.38. $$

Experimentally, the latest data from Berkeley* indicates this ratio to be 0.32 ± 0.14, in agreement with the prediction.

If you now ask what are the consequences of the assumption that the non-leptonic weak interactions carry $I = 1/2$ or $I = 3/2$, you get so little of use that the subject is not worth discussing at the present time. The experiments are too limited at present, and there is such a wide range of possibilities available compatible with the assumption, that you don't get anywhere.

---

This brings us to the end of the line, as far as anybody can see fairly clearly. I would now like to summarize what our approach to this problem has been. One makes hypotheses as restrictive as possible, and then checks to see whether the experimental data available do or do not contradict their consequences. The first of these hypotheses was that the weak interactions could be described by a current-current coupling. Secondly we assumed that of the various strangeness changing terms possible in the current, only those belonging to the group $J^2$ are present. And thirdly, we postulated that perhaps $J^1$ transforms like an isovector, and $J^2$ like an isospinor. As of today, we find that all the consequences fit with the experiments, but we cannot thereby conclude that the hypotheses are correct. If we had found something which clearly was in conflict with reliable experimental results, then we could be sure that the proposed theory was wrong, and we could construct a better one by modifying the hypotheses. Unfortunately, from a certain point of view, there is no experimental information which forces us to abandon these hypotheses and so, much of the impetus for inventing other theories does not exist. The way to make advances in a subject like this is to find definitive tests for the hypotheses made, but because the treatment of the strong interactions, which play a great role here, is almost impossible, it becomes quite difficult to check any hypothesis unless it is based upon very weak symmetry conditions. The weak restrictions suggested so far seem to be right, but much important experimental information still remains to be obtained before we can be very sure even about these.

Under the stated hypotheses, the strangeness changing current is restricted to be of the form

$$J^2 = i \sqrt{2} \frac{1}{\hbar} \{ \alpha (\bar{\Lambda}p) + \beta \left[ \sqrt{2}(\bar{\Sigma}n) + (\bar{\Xi}p) \right] + \gamma \left[ \sqrt{2}(\pi^+\bar{k}^0) + (\bar{\pi}^0k^+) \right] + \delta \left[ \sqrt{2}(\Xi^0\bar{\Sigma}^+ - (\bar{\Xi}^0\bar{\Sigma}^0)) + \epsilon (\bar{\Xi}^0\Lambda) \right].$$

The direct coupling between pions and kaons is inserted just for completeness, for then we can assert that if only the particles we now know of are in some sense fundamental, the above is the most general current carrying $I = 1/2$, $S = -1$, and $Q = -1$. The absence of conflict between experiment and the predictions of this scheme indicates that it is likely, but not necessarily so, that the weak interactions may be accounted for by such a theory. And now we would like to proceed further, say by specifying the coefficients $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, but we have not been able to do so. It is of some interest to examine
some of the difficulties which beset us, and this I would like to do now.

The first difficulty is associated with the concept of universality. Neglecting the uncertainties in deducing the fundamental interactions from the effective ones, which are the result of modifications due to strong couplings, we know that the strengths of the couplings of \((\bar{e}v), (\bar{\mu}v')\) and \((\bar{\nu}_p)\) to one another are essentially equal. The concept of universality would then suggest that, in the current, all the terms have the same relative coefficient unity. But the presence of the \(\sqrt{2}\)'s from Clebsch-Gordan coefficients seems to automatically foul up this idea of universality. To fix that up, one can try a trick like defining a combination like \(Z^0 = (\Lambda + Z^0)/\sqrt{2}\), as was done in discussing a scheme like global symmetry (see page 211) and regarding the weak interactions as being universal when written in terms of \(Z^0, Y^0, n, p, \Sigma^+, \Sigma^-, Z^0, \bar{\Sigma}^\pm\), which would enable you to get rid of the \(\sqrt{2}\)'s. The point of this fiddling is that since we are not absolutely sure of the symmetries which may be present in the weak interactions because they may be almost obliterated by the importance of strong couplings, the universality of the weak couplings does not imply that \(a, \beta, \gamma, \delta, \varepsilon\) are either 1 or 0, but rather that they are of order 1 if they do not vanish.

That the direct \((K\mu)\) coupling is present, with full steam in the sense of universality, can be ruled out experimentally, because the rate for \(K^+ \rightarrow n^0 + \bar{e} + \bar{\nu}\) is about 180 times smaller than that calculated from such a coupling if \(\gamma = 1\). It is always, of course, possible that the observed low rate might be due to destructive interference between the direct term and those involving virtual baryon loops, but when we are groping in the dark trying to understand something, we prefer not to use such delicate things as interference effects to explain things when there are less complicated explanations. That doesn't mean that it is necessary that \(\gamma\) be small, because when you don't know what the truth is, no one of your arguments, no matter how ingenious or clever it may be, is necessarily correct, but only that it is the best bet that \(\gamma\) is small.

Next, let's consider the possibility that \(a = 1\). If we neglect again the effects of strong interactions, we can calculate the rate for the leptic decay of \(\Lambda\), and the branching ratios come out to be 1.6 percent for the mode \(\Lambda \rightarrow p + e + \bar{\nu}\), and 0.3 percent for \(\Lambda \rightarrow p + \mu + \bar{\nu}'\). Similarly if \(\beta = 1/\sqrt{2}\), one would expect the \(\Sigma^-\) to undergo leptic decay with the mode \(\Sigma^- \rightarrow n + e + \bar{\nu}\) having a branching ratio of 5.7 percent and \(\Sigma^- \rightarrow n + \mu + \bar{\nu}\), having a
branching ratio of 2.5 percent. Experimentally the situation is still a little indefinite quantitatively, but it seems certain that the leptonic decay of the \( \Lambda \) and \( \Sigma^- \) is slower than the rate predicted if \( \alpha = 1 \), and \( \beta = 1/\sqrt{2} \). Now if this is definitely true, we will still not know whether it is the result of a lack of universality, i.e., the coefficients in the strangeness changing current are not of the order of unity, but smaller by a factor of 3 or so, or whether simply the factor of 3 or so is due to the modification of the weak vertex by the strong couplings.

The slowness of these leptonic decays may suggest that we abandon the concept of universality and work with an empirical hypothesis that the strangeness changing weak interactions are inherently weaker than those conserving strangeness. On the other hand, the decay \( K^+ \rightarrow \bar{\mu} + \nu \) seems to require, as forcefully as anything can, realizing that reliable calculations are impossible, that the coefficients in \( J^2 \) be of order unity. Furthermore, the rates for the non-leptonic decays of the strange particles are again not incompatible with the strangeness changing terms being present in the full strength from the standpoint of universality.

There is another empirical rule suggested by the data, whose justification is hard to see if we adhere strictly to a \( J^-aJ^+a \) scheme for the weak interactions. The matrix elements of that part of the interaction carrying \( I = 3/2 \) are relatively much smaller than those of the part carrying \( I = 1/2 \). The best way to study such a rule is to ask what are the consequences of the rule that the weak interactions carry only \( I = 1/2 \), in the non-leptonic as well as the leptonic decays. There will be many of these, which we may compare against experimental results, and it would be a miracle if there were no serious conflict. The existence of this miracle is a possible fact, for in all cases where we can check, the predictions of \( I = 1/2 \) are in accord with the data. This situation is rather odd; it may be a coincidence or the result of some deeper symmetry.

We have already seen that a consequence of this rule is \( \Gamma_{K^0_1 \rightarrow n^0 + \pi^0} / \Gamma_{K^0_1} = 0.33 \), and experimentally, it is \( 0.32 \pm 0.04 \). The \( K^+ \) should not decay into \( \pi^+ + n^0 \), and experimentally the rate for this mode is 1/500 times the rate for \( K^0_1 \rightarrow 2\pi \), which is, relatively speaking, so close to zero that the possibility
suggests itself that the decay of the $K^+$ into two pions may be due to the fact that the electromagnetic corrections do not respect the integrity of isotopic spin selection rules. Many people have tried unsuccessfully to account for the $K^+$ mode in this way, getting too small values for the rate, but it is not absolutely certain that this can't be done.

Now we must look to see if there is a similar predominance of the $I = 1/2$ matrix element in other places. First, let us consider the decay of the $\Lambda$. If the interaction carries $I = 1/2$, then the nucleon and pion in the final state must have $I = 1/2$, $I_z = -1/2$, which means

$$\frac{\Gamma_{\Lambda \to p + \pi^-}}{\Gamma_{\Lambda \to p^+ \pi^-}} = 0.66.$$ 

Experimentally, the ratio is $0.63 \pm 0.03$. The purity of the isotopic character of the interaction has the further consequence that the asymmetry parameters must be exactly the same for the two modes. But that prediction cannot be checked until they measure $\alpha_\pi^\prime$ for the neutral decay mode.

The rule enables us to say a little about the non-leptonic decays of $\Sigma^+$ and $\Sigma^-$, viz.

$$\Sigma^+ \to p + n^0, \quad \Sigma^+ \to n + \pi^+$$

and

$$\Sigma^- \to n + p^-.$$ 

Let us define:

$$C = \langle \nu_{n}, I = \frac{1}{2} \mid \mid R_{\frac{1}{2}} \mid \mid \Sigma, I = \frac{3}{2} \rangle$$

and

$$D = \langle \nu_{n}, I = 3/2 \mid \mid R_{\frac{3}{2}} \mid \mid \Sigma, I = 1 \rangle.$$ 

Then, by the Wigner-Eckart theorem, we can express the matrix elements entering into the observed decay modes in terms of $C$ and $D$ as follows:

$$\langle \pi^0 \mid R_{\frac{1}{2}} \mid \Sigma^+ \rangle = -C \langle 1, 0 \mid \frac{1}{2} \mid 1, \frac{1}{2} \rangle \langle 1, \frac{1}{2}, -1 \mid \frac{1}{2} \rangle \langle 1, \frac{1}{2} \rangle$$

$$= \frac{1}{\sqrt{2}} (C - D)/3;$$

$$\langle \pi^+ \mid R_{\frac{1}{2}} \mid \Sigma^+ \rangle = 0 \langle 1, 1 \mid \frac{1}{2} \mid 1, \frac{1}{2} \rangle \langle 1, 1 \mid \frac{1}{2} \rangle$$

$$+ D \langle 1, 1 \mid \frac{1}{2} \mid 1, \frac{1}{2} \rangle \langle 1, 1 \mid \frac{1}{2} \rangle$$

$$= (2C + D)/3;$$

$$\langle \pi^- \mid R_{\frac{1}{2}} \mid \Sigma^- \rangle = D \langle 1, -1 \mid \frac{1}{2} \mid 1, \frac{1}{2} \rangle \langle 1, -1 \mid \frac{1}{2} \rangle$$

$$= D.$$
Therefore, the following relationship must hold:
\[ \sqrt{2} \langle p n^0 | R_{1/2} | \Sigma^+ \rangle = \langle n n^+ | R_{1/2} | \Sigma^+ \rangle - \langle n n^- | R_{1/2} | \Sigma^- \rangle. \]

Now first of all, these numbers are complex, but their relative phases are given by the scattering phase shifts for pion-nucleon scattering at the appropriate energy, which is about 189 MEV (assuming, of course, that T invariance holds). The maximum of these phase shifts is about $14^\circ$, and so to a pretty good approximation we may talk as though all the matrix elements are relatively real. But secondly, there are two final states for each mode, i.e. for $J = 1/2$, since parity is not conserved, we can get both s-wave and p-wave pions. The matrix element for each mode is thus to be regarded as a two-component vector, with components along an S and a P axis. The asymmetry parameter of the decay mode is determined by the orientation of the vector in the S-P plane, being given by
\[ \frac{2 \text{ Real } (S_P)}{|S|^2 + |P|^2}. \]

In the same notation, the decay rate is proportional to $|S|^2 + |P|^2$. Now the rates for all three modes are equal to within 10 percent or so, and the asymmetry parameter for the mode $\Sigma^+ \rightarrow n + n^+$ is essentially zero*, whereas that for the mode $\Sigma^+ \rightarrow p + n^0$ is almost certainly quite large. These pieces of information, assuming the $I = 1/2$ interaction, lead to the following plot of the three matrix elements:

\[ \langle n n^+ | R_{1/2} | \Sigma^+ \rangle \]
\[ \langle n n^- | R_{1/2} | \Sigma^- \rangle \]
\[ \langle p n^0 | R_{1/2} | \Sigma^+ \rangle \]

This orientation of the right triangle is not unique; it can be rotated by any multiple of $90^\circ$, and reflected in either the S or P axis. But no matter

which of those orientations one chooses, the $I = 1/2$ interaction rule requires
that the asymmetry parameter for the mode $\Sigma^- \rightarrow n + \pi^-$ be small, which is in
very nice agreement with the fact that $\omega$ for this mode is very small.

The restriction of the interaction to one carrying $I = 1/2$ only
enables us to relate the decay rate of the $K_2^0$ into three pions directly to
the decay rate of the $K^+$ into three pions, under the assumption that the
kinematics of the process strongly favors those final states in the $K^+$ decay
with $CP = -1$. It is then just another exercise in the techniques we have
used so often to show that $\Gamma_{K_2^0 \rightarrow 3\pi} = \Gamma_{K^+ \rightarrow 3\pi}$, neglecting the differences
in the density of states due to the various mass differences. From the
results of our analysis of the relation between the leptonic decays of $K_2^0$ and
$K^+$, we may predict therefore the total rate of decay of the $K_2^0$. Neglecting
the differences in the densities of states, because they are all small compared
to present experimental errors, one would predict:

$$\Gamma_{K_2^0} = (19 \pm 3) \times 10^6 \text{ sec}^{-1}.$$  

The average of the Berkeley and Columbia results* is $(16.3 \pm 3.5) \times 10^6 \text{ sec}^{-1}$, but we really have to wait for better data before any agreement is claimed
because the two labs differ by a factor of two in their estimates of the rate.

There is one more prediction. It is easy to see that the $I = 1/2$
interaction leads to

$$\Gamma_{\Sigma^- \rightarrow \Lambda + \pi^-} / \Gamma_{\Sigma^0 \rightarrow \Lambda + \pi^0} = 2.$$  

But there is nothing known at present about the $\Sigma^0$, except that it probably
exists!

---

I thought that it would be a good idea to describe very roughly some of the ideas which have been concocted to explain the pattern of the weak interactions, but which are all unsatisfactory. It is not my purpose to go into these schemes in great detail, and to show you exactly how each one doesn't work; the thought is just to present a handful of ideas, and maybe someday you can play around with one or two of them, and find out yourself where hitches develop. It may be objected that that is a waste of effort, but I disagree, because the next person who tries some idea may have one slight difference in emphasis, which somewhere along the line leads him to develop the idea in a new way. And so it is never certain that somebody will be able to show that difficulties arise, unless he has been supplied with all the details, which is then a waste of time. Therefore I shall only describe the general character of a few ideas, one or another of which might be fruitful, if somebody new works with it.

I) One of the ideas comes from the fact that the leptonic decays of the hyperons seem to be too slow by a factor of ten or so, and is that perhaps the overall relative coefficient of the strangeness changing current is smaller than that of the strangeness preserving current, being say one-fourth or something. Nevertheless, all of the non-leptonic decays of the hyperons seem to be pretty fast; they don't seem to need a "smaller than universal" coupling, although we can't calculate their rates.

Furthermore, such a scheme leaves untouched the great mystery of showing why the $|\Delta I| = 1/2$ rule works so well. It is impossible to set up an interaction for the strangeness changing decays carrying only $I = 1/2$ without using a neutral current in addition to a charged current.

II) Another possibility is that $H_{\text{weak}} = -J^\alpha_+ J^-_\alpha - K_\alpha K^\alpha$, where $K_\alpha$ is a neutral current, which can be set up so that it carries only $I = 1/2$. As far as $K_\alpha$ is concerned, in principle leptonic terms like $(\bar{e} \mu)$ or $(\bar{\nu} \nu)$ are possible, but these are ruled out by experiment. Therefore, the first "disease" of the neutral current idea is that no leptonic terms are present. It is to be emphasized that this last remark is most subjective. To Dr. Feynman the absence of neutral leptonic terms would be quite unexpected, and arbitrary, so that he feels that the neutral current idea has one strike against it.

Another, more serious, difficulty is that a rigorous $I = 1/2$ interaction makes it hard to see why $\Gamma_{K^+ \to 2\pi}$ is as big as $\frac{1}{500} \Gamma_{K_1^0 \to 2\pi}$. That is why it would
be so important if it were reasonably possible from some point of view to account for the $\Gamma \rightarrow 2\pi$ decay electromagnetically. If it is not possible, then the interaction cannot carry $I = 1/2$ exactly, and this would be a serious strike against the idea of introducing a neutral current.

III) Another attempt starts from the assumption that the relative coefficient of the strangeness changing interactions is smaller than unity, but that also in most non-leptic decays certain specific types of diagrams give abnormally large contributions to the matrix element. Furthermore, these abnormally large diagrams produce only $|\Delta I| = 1/2$. For a simple example of the suggestion, suppose that the non-leptic current is essentially $(\bar{\nu}p) + .2(\bar{\nu}p)$. Then consider the following type of diagram in the process $\Lambda \rightarrow N + \pi$:

\[ \begin{array}{ccc}
\bar{\nu} & \rightarrow & p \\
N & \rightarrow & n \\
\Lambda & \rightarrow & N + \pi
\end{array} \]

which is characterized by the fact that the p loop is "undisturbed". In other diagrams, such as

\[ \begin{array}{ccc}
\bar{\nu} & \rightarrow & p \\
N & \rightarrow & \Lambda \\
\Lambda & \rightarrow & N + \pi
\end{array} \]

the loop would be used as an essential part of the machinery. Now the hypothesis would be that the "undisturbed" loop diagrams are much larger than anything else that you get by disturbing them. For some reason unknown to man, the "short-circuit" loop is to give an unusually large contribution. Now you will observe that the "undisturbed" loop results only in $|\Delta I| = 1/2$. And that would be the reason why the $|\Delta I| = 1/2$ rule works so well.

We never can really get rid of these possibilities, until we find a correct theory. In this case, the hypothesis must be much more elaborate because of the existence of other strange particles, but presumably such a complete scheme could be cooked up.

IV) Perhaps a $|\Delta I| = 1/2$ rule is not the reason for the various simple ratios that are present, but that there is some other symmetry involved, which has many of the same consequences. And so there is a big game in which you take all the hyperons and assume that among them there exists a larger symmetry than
is indicated by their masses. The global coupling scheme is of this nature. On the basis of such approximate symmetries, one can invent new quantum numbers, and examine various selection rules, which will also be only approximate. For example, supposing the bare mass of the \( \Lambda \) and the \( \Sigma^0 \) to be the same, one might assign to the particles a quantum number \( M \), which would be identical to \( I \) except for the interchange of \( \Lambda \) and \( \Sigma^0 \). Then one could construct the weak interactions so that a \( |A^M| = 1/2 \) rule would follow.

V) Another idea comes from the concept that certain of the particles regarded as elementary may actually be compound. In terms of a very small number of elementary particles, the current would be of a pretty simple form, but when written in terms of all the particles known today, its form could be most complex. As an example, let us look at a model due to Okun', in which the \( \Lambda, p, \) and \( n \) are regarded as fundamental particles. The binding is to be due to a new force, which can be taken to be vector in character by analogy with electromagnetism, and which couples with the same sign to the three particles, and perhaps with the same strength if one cares to make the bare mass of the \( \Lambda \) different from that of the nucleon. One supposes that three bound states of the constitution

\[
p \bar{n}, \quad p \bar{p} - n \bar{n}, \quad n \bar{p}
\]

exist, and identifies these with the pions. The isotopic singlet state, \( p \bar{p} + n \bar{n} \) presumably is less bound, and disintegrates extremely fast. In the same way, the \( K^+ \) and \( K^0 \) would be thought of as \( \bar{p} \Lambda \) and \( n \Lambda \). The \( \Sigma \)'s and \( \Xi \)'s would be made of three elementary particles in an obvious fashion.

In such a system, the weak interaction current would be something simple like \( (\bar{n}p) + (\bar{A}p) \), which we know is sufficient to account qualitatively for all strange particle decays observed so far. But from this form of the interaction, one is again faced with the mystery associated with the \( |A| = 1/2 \) symmetries. This doesn't necessarily mean that such a scheme is wrong; it simply means that the simple ratios would be regarded more or less as accidental.

After playing with a number of such schemes, you find that qualitatively they all give essentially the same kind of predictions. Then the question arises of how are we ever going to be able to tell which are the three fundamental particles, assuming that all the baryons and mesons are made from three, when we cannot calculate anything. To avoid this dilemma, the following hypothesis is suggested: Nature is constructed in such a way that it shall be ultimately impossible to determine which are the fundamental particles and which
are the compounds. This is an interesting speculation, but so far I have only been able to deduce one consequence from it. The only place I used it was to discover the vector coupling for the pions in the weak interactions, by the following line of thought. If the pions can be thought of as being constructed from a nucleon and an antinucleon, then knowing the coupling of the nucleons, and knowing that vector coupling is conserved, like in electromagnetism, it is easy to deduce the rate for, say, the transition $\pi^+ \rightarrow \pi^0 + \bar{e} + \nu$. Then I say that I don't know what the pions are; if the pions are fundamental in the sense we usually think of, then there would have to be a direct pion-pion coupling in the weak interactions whose size I know. If nature did not have such a term in the current, I would be able to deduce that the pions are not compounds, which is contrary to the hypothesis. This speculation may or may not be true, but it is one that you may find of use to generate new ideas.

Continuing with these compounding schemes, the interesting question of the minimum number of elements needed to account for all the known particles might be asked. It is quite easy to see that only three are needed for the baryons and mesons, but then three more are needed to account for the leptons.

Finally, I might mention a modification of the Okun's scheme that showed some success in giving a simple derivation of a complicated expression for the weak interactions, which accounts for many of the regularities observed. This model uses three fundamental particles - $\lambda$, $P$, and $N$, which have the same characteristics (strangeness, isospin, etc.) as the real particles $\Lambda$, $p$, and $n$, respectively, and accounts for the particles as follows:

\[
\Sigma^- \leftrightarrow \lambda \bar{P} \lambda, \quad \Sigma^0 \leftrightarrow \lambda \bar{N} \lambda; \\
\Sigma^- \leftrightarrow \lambda \bar{P} \bar{N}, \quad \sqrt{2}\Sigma^0 \leftrightarrow \lambda(P \bar{P} - N \bar{N}), \quad \Sigma^+ \leftrightarrow \lambda P \bar{N}; \\
\sqrt{2}\Sigma^- \leftrightarrow \lambda(P \bar{P} + N \bar{N}); \\
n \leftrightarrow \lambda N \bar{\lambda}, \quad p \leftrightarrow \lambda P \bar{\lambda}.
\]

All the real baryons are thus compounds of three of our elementary chunks of matter. The pions are easily made from two chunks:

\[
\pi^- \leftrightarrow \bar{P}N, \quad \sqrt{2}\pi^0 \leftrightarrow (P \bar{P} - N \bar{N}), \quad \pi^+ \leftrightarrow P \bar{N}.
\]

Now if one assumes that the weak interactions are associated with $(\bar{N}\bar{P}) + (\bar{P}N)$, one gets a fairly complex expression with fixed coefficients for $J^1$ and $J^2$, which looks like it could be used together with the appropriate neutral current to account for many of the observed regularities in the strange particle decays. However, a hitch develops when one naturally tries to account for the K mesons as $K^0 \leftrightarrow N \bar{\lambda}, \quad K^+ \leftrightarrow P \bar{\lambda}$, for there is nothing to prevent the decay $K^0 \rightarrow \pi^- + \bar{e} + \nu$ from going far too rapidly.
Our discussion of the strangeness changing weak interactions has been concerned almost wholly with certain symmetries, and the associated selection rules. These selection rules are only approximate, and it proves convenient to classify them according to the extent of their possible validity. The strongest type of rules will be put in class A. The characteristic of class A is that to first order in the weak interactions, there is absolutely no violation of the selection rule. The most important example of such a rule would be that in the strangeness changing terms coupled to leptons, \( \Delta S/\Delta Q = +1 \).

The second major assumption which has been advanced belongs to a different class — class B — because its validity is violated by the electromagnetic interactions. That is, of course, the proposition that \( J^1 \) carries exclusively \( I = 1 \), and \( J^2 \) carries \( I = 1/2 \). The consequences of these hypotheses were examined and found to agree with nature, at least to the limited accuracy of present experiments.

Now these rules can be applied within the framework of the \( J_{\alpha} J_{\alpha} \) form for the weak interactions; they are just fairly weak restrictions on the terms in \( J_- \). This is as far as we can get in understanding the strange particle decays, if we do not introduce any new ideas. But this is not enough to explain the pattern of the weak interactions. For example, it is not enough to explain why the two pion decay rate of the \( K^+ \) is so much slower than that of the \( K^0 \), and it is altogether inadequate to explain the tremendous number of simple relationships that can be found among the strange particle decays. And so, people have tried various other ideas.

One of the ideas is a set of rules of class C, whose existence depends upon the possibility that the strong interactions might be split into two groups: the very strong (VS) interactions having a high symmetry, and the medium strong (MS) interactions, which break the symmetry and produce the observed splittings of the baryon isospin multiplets. The rules of class C would be valid as far as the VS couplings were concerned, but violated by the MS couplings. This is one way to increase the power of the theory, by obtaining more fairly weak selection rules.

A second way to get more symmetry is to introduce a neutral current coupling. This neutral current must not contain any leptonic terms, in order to agree with experiment, which is a new and somewhat ugly assumption.
I have developed a scheme, which will explain all the observed simplicities, but it is especially ugly because it makes use of both the division of the strong couplings into classes and a neutral current-current interaction, and furthermore because the form of the interaction, when it is written out, doesn't seem to have any aesthetically pleasing property. You can't say when you look at it: "Yes, that must be it; it's so simple!" Nevertheless it could be right, but there is certainly no compelling feature about it. We have the experience that in exploring laws at a sufficiently fundamental level, whenever we find the correct ones they are always also beautiful. They are very simple and possess much symmetry when viewed in certain ways. They are usually not what we expect, but they're quite simple, after we've found the correct mathematical framework. Nobody knows why this is so, and certainly nobody can guarantee that this will remain true as we probe deeper and deeper, but it can well be appreciated why we tend to use this subjective appraisal of possible theories so much. Basically, however, this is nothing but talk, because what you have to do with any theory is to check experimentally what it implies. That is really the only test. You cannot decide just by its aesthetic appeal whether it is correct or not. But you can try to guess in this way whether it might be right, and then this particular scheme doesn't look like the right one, because it introduces a number of peculiar assumptions, and then even does not have any especially apparent symmetry.

A third approach, which is not a part of the scheme I have concocted, is to attribute the observed regularities associated with $|\Delta I| = 1/2$ to certain dominant diagrams possessing this property. But this idea seems to be wrong. For example, consider the decay of the $\Lambda$, and in particular the following class of diagrams:

We can estimate the sum of such diagrams by employing the experimental rates for the leptonic decays of the $\Lambda$ and of the pion, and it is found that these diagrams account for say $1/5$ of the total rate. So obviously none of the diagrams dominates the matrix element, and correspondingly you can see that nothing else can really dominate it either.
At present it seems that all one can do is to say if the first two ideas are employed, the facts can be explained but not in a tremendously satisfactory way. What is really needed at the moment is to do a number of experiments quite well, so as to make sure what the facts are. As long as we try to interpret "facts" that are only approximately correct perhaps, we'll never get anywhere. Check these things first of all: Is the \( K^0_1 - K^0_2 \) mass difference large or small? Does the \( \Sigma^+ \) give leptons, or doesn't it? Are the leptonic decay rates of the \( K^+ \) and \( K^0_2 \) really related by a factor of 2, or is it some other factor? Are the rates for the two modes of the three pion decay of the \( K^+ \) really in the ratio 4:1, when the density of states is corrected for? When we can be sure about these facts, which check our assumptions of class A and class B, and get some information which bears on our ideas of class C, e.g. the asymmetry in the neutral mode of the \( \Lambda \), then I'm sure we will be able to discriminate between a correct theory and a false one, and we won't have to rely on just these aesthetic criteria.

As an exercise on this topic, it would be a good thing to look at the process \( K^+ \rightarrow \pi^+ + \pi^0 \), assuming that the weak interactions carry only \( I = 1/2 \), and compute an estimate for the rate assuming that electromagnetic corrections are responsible for the decay. That is, cook up a simple set of diagrams that would give the decay mode individually, but arrange things so that the sum cancels because of the symmetry involved. Then put in the various mass differences, and find out what order of magnitude the resulting matrix element comes out to be.
The strong coupling approximation was invented by Gregor Wentzel in 1940. It was invented because the properties of the experimental muon didn't agree with the properties of the theoretical $\pi$ meson. The muon exhibited very weak interactions with nuclear matter, but it was supposed to be the Yukawa meson, and something had to be done to account for this terrific discrepancy between theory and experiment. One idea was that maybe the coupling was very, very strong -- so strong that the interactions, scattering, absorption, and so on, were damped out. In order to test this idea, it was useful to attempt a study of meson scattering when the coupling constant was very large. This would have been a good project anyway, independent of its historical motivation, and Wentzel figured out how to do it for the charged pair theory in the static nucleon model. It has been applied to many other theories since then, but it has never been reduced to the same type of mechanical operations that the weak coupling approximation has been reduced to. The strong coupling method is always based on examining the Hamiltonian, guessing that certain approximations will introduce negligible errors in the limit of large coupling constants, and then noticing that this is in fact so after the problem has been solved.

The strong coupling treatment of a fully relativistic theory with pairs and recoil is still completely mysterious. Nobody has ever succeeded in setting up a strong coupling method to calculate the predictions of such a theory. But any of the static theories can be treated successfully in the limit of a large coupling constant. We shall sketch the method, which is applicable to any theory with minor modifications, for the most complicated one, the symmetrical pseudoscalar theory, not for pedagogical reasons but because that corresponds the closest to a correct theory. After the method has been presented, to compute a nucleon isobar spectrum analytically in an approximation without getting into too much labor, we shall pass over to the symmetrical scalar theory. Finally, the results of a slightly more involved approximate analytical treatment of the symmetrical pseudoscalar theory will be given.*

Let us employ the symbols $\alpha, \beta, \ldots$ for the isotopic indices, and $i, j, \ldots$ for those of ordinary space. Putting $\omega^2 = \mu^2 - \nabla^2$, the Hamiltonian is:

\[
H = \frac{1}{2} \int \left( \frac{\mu^2}{\lambda a} \sum_\lambda \phi_\lambda \omega^2 \phi_\lambda \right) d^3x + \frac{\varepsilon}{\mu} \tau_\alpha \int \phi_\alpha \sigma \cdot \nabla U(x) d^3x,
\]

where \(U(x)\) is the spherically symmetric source function, which is a spread out \(\delta\)-function centered at the origin. We shall not go through the transformation between unrenormalized and renormalized coupling constants. Instead the unrenormalized coupling constant will always be used. In the strong coupling limit, there is a finite proportion between them. You can, if you like, carry out a renormalization program and define a renormalized constant, but it differs only by a factor of three or something, and so no divergence is altered.

In the transformations to be carried out, certain integrals over the source function will come in, and we might as well define these all at this point:

\[
N = \frac{1}{3} \int ( VW)^2 d^3x;
\]

\[
X = \frac{1}{2} \Sigma U, \text{ which is a Yukawa function for the source } U;
\]

\[
K^{-1} \delta_{ij} = \int \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} d^3x;
\]

\[
\xi = K X, \quad \text{and} \quad M = \frac{1}{3} \int (\nabla \xi)^2 d^3x.
\]

We shall follow essentially the most recent paper on strong coupling, by Landovitz and Margolis,* and this peculiar notation is a hybrid of rationalizing the coupling constant and using their symbology, although slightly modified.

The \(f\) is supposed to be very large, and therefore, we are principally concerned with the coupling between the meson field and the nucleon. So we focus our attention on the coupling, and ask what part of \(\phi_\alpha\) is coupled to the nucleon. The field \(\phi_\alpha\) can be thought of as an assembly of oscillators, which conventionally are discussed by means of the introduction of normal coordinates. For the free field, \(f = 0\), the normal coordinates are the amplitudes for excitation of the field in modes specified by values of the wave number and frequency. But when the coupling constant is quite large, the normal modes are not similar to the uncoupled modes, and it is not useful to describe the system in terms of the free modes. Instead we may attack the problem by separating the field into those few modes which are coupled to the nucleon and the remainder of the modes. There will, of course, still be a coupling between

---
*Landovitz and Margolis, Annals of Physics 7, 52 (1959)
the nucleon-coupled modes and the others (otherwise no approximation would be needed) due to the free field part of the Hamiltonian, but this may be neglected as the nucleon coupling becomes more and more dominant.

The coupled modes are nine in number:

\[ \phi^o_{\alpha k} = \int \phi^o_{\alpha} \frac{\partial U}{\partial x_k} \, d^3x, \]

and their canonical momenta can be taken to be

\[ \pi^o_{\alpha k} = \int \pi^o_{\alpha} \frac{\partial \phi^o_{\alpha k}}{\partial x_k} \, d^3x. \]

Now if we define

\[ \phi^i_{\alpha k} = \phi^o_{\alpha k} + \phi^o_{\alpha} \frac{\partial \xi}{\partial x_k}, \]

\[ \pi^i_{\alpha k} = \pi^o_{\alpha k} + \pi^o_{\alpha} \frac{\partial U}{\partial x_k}, \]

then \( \int \phi^i_{\alpha} \, d^3x = 0 \), and \( \int \pi^i_{\alpha} \, d^3x = 0 \). It is easy to see that the \( \phi^o_{\alpha k} \) and \( \pi^o_{\alpha k} \) form a set of canonically conjugate variables, but \( \phi^i_{\alpha} \) and \( \pi^i_{\alpha} \) are not quite conjugate variables. The commutation relations are:

\[ [\pi^o_{\alpha j}, \phi^o_{\beta k}] = -i \delta_{\alpha \beta} \delta_{jk}, \]

\[ [\phi^o_{\alpha j}, \phi^o_{\beta j}] = [\phi^o_{\alpha j}, \pi^o_{\beta j}] = 0, \]

\[ [\pi^o_{\alpha j}, \pi^o_{\beta j}] = [\pi^o_{\alpha j}, \phi^o_{\beta j}] = 0, \]

\[ [\phi^i_j(x), \pi^i_{j'}(x')] = i \delta_{\alpha \beta} \{ \delta(x-x') - \nabla \xi \cdot \nabla \xi' \}. \]

In terms of these variables, the Hamiltonian is:

\[ H = \frac{1}{2} \int \left( \frac{1}{2} \pi^o_{\alpha k} \frac{\partial U}{\partial x_k} \pi^o_{\alpha j} + \frac{1}{2} N \pi^o_{\alpha j} \pi^o_{\alpha j} + \frac{1}{2} K(\phi^o_{\alpha j})^2 \right) \, d^3x + \frac{1}{2} \left( \omega^2 \pi^o_{\alpha j} \pi^o_{\alpha j} + \Gamma \phi^o_{\alpha j} \phi^o_{\alpha j} \right) \, d^3x. \]

Now this is still a mess, because the nucleon-coupled, or "local," degrees of freedom are coupled also to the truncated or "remnant" degrees of freedom. We would like to perform a transformation which will eliminate that coupling. It is impossible to do this exactly in any finite form, but it can be done in an approximation based on the largeness of \( f \). This is where we, for the first time, make the approximation that the coupling is very strong. The transformation will shift \( \pi^i_{\alpha} \) by an amount that depends on \( \pi^o_{\alpha} \). In that way we shall get rid of the second last term in the Hamiltonian. That transformation, though,
since it involves $\pi^0_\Lambda$, will also shift $\varphi^0_\Lambda$ by a certain amount which depends on $\varphi^1_\Lambda$, restoring the coupling, and so it appears we won’t accomplish anything. But, in strong coupling we can say that the behavior of $\varphi^0_\Lambda$, being the nucleon-coupled part of the field, will be essentially determined by terms like the second, third, and fifth in the above Hamiltonian, and the little shift in $\varphi^0_\Lambda$ by an amount depending on $\varphi^1_\Lambda$ can be neglected. If one solves the problem making this approximation and goes back later to check the neglected terms, it is found that their effects are of relative order $1/r^2$, and not important in strong coupling.

This, by the way, is the only important strong coupling approximation that must be made. The rest of the approximations that are employed afterwards will be made for convenience, but we could do without them. The rest of the problem could be solved on a machine.

The transformation is accomplished by means of the unitary operator:

$$S = \exp \left\{ -i \frac{1}{\hbar} \int \frac{\varphi^0_\Lambda}{\Lambda} \frac{\partial}{\partial x_k} d^3 x \right\}.$$ 

Such an operator changes only $\pi^1_\Lambda$ and $\varphi^0_\Lambda$:

$$S \pi^1_\Lambda S^+ = \pi^1_\Lambda - \pi^0_\Lambda \left( \frac{\partial U}{\partial x_j} - \frac{1}{M} \frac{\partial E}{\partial x_j} \right),$$

$$S \varphi^0_\Lambda S^+ = \varphi^0_\Lambda + \text{term in } \varphi^1_\Lambda \text{ which we neglect.}$$

The transformed Hamiltonian is:

$$S H S^+ \equiv \frac{1}{2} \int (\pi^1_\Lambda + \varphi^1_\Lambda)^2 d^3 x + \frac{1}{2M} (\pi^0_\Lambda)^2 + \frac{1}{2} \int K(\varphi^0_\Lambda)^2 + \frac{f_\tau}{\mu} \tau_j \sigma_j \varphi^0_\Lambda.$$ 

In this approximation, the local and remnant degrees of freedom are completely uncoupled, and the problem of nucleon isobars is reduced to the finite problem of nine oscillators coupled to the four nucleon degrees of freedom given by the spin and isospin states of the source. That finite problem could be solved on a machine exactly. Actually, what has been done is to make more approximations based on strong coupling and then solve it analytically. The first term in the Hamiltonian describes a slightly truncated meson field, which acts much like a free meson field except that the waves are slightly scattered off the absence of the nine degrees of freedom.

The first thing we are interested in, in strong coupling, is the structure of the nucleon and of the nucleon isobars. We may anticipate the results by saying that when the coupling is strong, the isobars, which are virtual scattering resonances for the weak coupling situation we have been
considering so often, get pushed down further and further. So we argue in this way: In very weak coupling, there are no isobars evident anywhere; all the isobars are at essentially infinite energy. As the coupling gets stronger and stronger, they move down gradually, making their presence known by scattering resonances. And when the coupling becomes very strong, they become bound isobars, i.e., the energies of these isobars, one after another, fall below the energy of a free pion, and\textit{ in practice} would decay by photon emission. In fact, the coupling is not quite strong enough to produce such bound excited states of a nucleon, but we wish to study that case anyway.

The structure of all the isobars in the limit of very strong coupling is just determined by the part of the approximate Hamiltonian involving $\Phi^0\lambda$ and $\pi^0\lambda$. Similarly in order to calculate an adiabatic nuclear force, one might perform a division of the field into two parts, using a source function for two nuclei. If we want to look at scattering, then we have to do two things. First of all, we look at scattering in the limit of infinitely strong coupling, where we may forget about the "local" degrees of freedom, and work with only the "remnant" degrees of freedom. We do our best to construct a plane wave from these, and we find that we get a plane wave plus a little bit, which is scattering. This scattering is of order 1 -- there is no $f$ entering into the formulae. However, if the coupling is not infinitely strong, then there is also some scattering arising from the "local" degrees of freedom in the following way. The isobars, which are bound for infinitely strong coupling, begin to rise as the coupling gets turned off and go up into the continuum, for instance where the $3/2, 3/2$ isobar actually is, and there those isobars behave like scattering resonances. Then the additional terms that come from the resonances must be included. That's hard to carry out in practice, but it is possible.

The most interesting thing is the strong coupling limit is to take the "local" part of the Hamiltonian and explore the isobar spectrum of the nucleon, which is the problem whose solution will now be sketched. Rather than solving the problem exactly on a computing machine, we shall sketch it using further approximations based on strong coupling. Furthermore, let us simplify the problem by using the symmetrical scalar theory. There are then only three local degrees of freedom instead of nine. The only changes are that there are no gradients and associated factors of $1/3$ in the various source integrals. After performing the unitary transformations, one ends up with:

$$S_H S^+ \equiv \frac{1}{2} \int \left( \pi^\alpha_\lambda \phi^\lambda (\varphi^{\alpha 0}) + \frac{1}{2} K(\varphi^{\alpha 0})^2 + \frac{1}{2M} (\pi^{\alpha 0})^2 + f \tau_\lambda \psi^0_\alpha \right).$$
In this theory, the truncated field has a piece of the s-wave left out, instead of three p-wave pieces as in the symmetrical pseudoscalar theory. There is no μ in the coupling, because there are no gradients. Here we have three local oscillators coupled to the two isospin degrees of freedom of the source.

The isobar problem is equivalent to a three-dimensional spherical oscillator with a spin degree of freedom, which is coupled linearly with the position vector \( \vec{x} \), in a rotationally invariant fashion. To solve it, we may work in a Schroedinger representation, with a set of states which are eigenstates of \( \sigma^2 \) and the spin of the source, i.e. in the language of the spherical oscillator, eigenstates of position and spin projection. Representing the states by a wave function, which is a spinor, we want to find the eigenvalues of the following Schroedinger equation:

\[
(-\frac{\nabla^2}{2M} + \frac{1}{2} K x^2 + f \vec{r} \cdot \vec{x}) \psi = E \psi.
\]

Now you will notice a rather unusual feature in this problem, which is that the eigensolutions do not have a "parity" because the coupling term is odd when you perform the transformation \( x \rightarrow -x \). The eigensolutions can be taken as eigensolutions of total angular momentum and one of its components, but to make spin \( I \), we will have to use orbital functions with both the values \( I - \frac{1}{2} \) and \( I + \frac{1}{2} \) of orbital angular momentum. Separating out the angular variables, we get a set of two coupled equations for two radial wave functions. That is, set

\[
r \psi = X_{I-\frac{1}{2}}(r) \begin{pmatrix} a Y_{I-\frac{1}{2}}^{I-\frac{1}{2}}(\theta, \phi) \\ b Y_{I-\frac{1}{2}}^{I+\frac{1}{2}}(\theta, \phi) \end{pmatrix} + X_{I+\frac{1}{2}}(r) \begin{pmatrix} c Y_{I+\frac{1}{2}}^{I-\frac{1}{2}}(\theta, \phi) \\ d Y_{I+\frac{1}{2}}^{I+\frac{1}{2}}(\theta, \phi) \end{pmatrix},
\]

where the \( a, b, c, d \) are ordinary Clebsch-Gordan coefficients, and the Schroedinger equation becomes:

\[
\begin{align*}
\left\{ -\frac{1}{2M} \frac{d^2}{dr^2} + \frac{1}{2} K r^2 + \frac{(I-\frac{1}{2})(I+\frac{1}{2})}{2M r^2} \right\} X_{I-\frac{1}{2}}(r) + f r X_{I+\frac{1}{2}}(r) &= E X_{I-\frac{1}{2}}(r); \\
\left\{ -\frac{1}{2M} \frac{d^2}{dr^2} + \frac{1}{2} K r^2 + \frac{(I+\frac{1}{2})(I+\frac{3}{2})}{2M r^2} \right\} X_{I+\frac{1}{2}}(r) + f r X_{I-\frac{1}{2}}(r) &= E X_{I+\frac{1}{2}}(r).
\end{align*}
\]

These are equations which could be solved exactly on a computer, but this is not too important, because we are doing strong coupling anyway, and so we may as well make a few more approximations based on the largeness of \( f \), in which case we can solve them analytically.
The method which can be employed in strong coupling is to treat the radial coordinate essentially according to classical mechanics, finding an effective potential energy, and then discussing the motion of a quantum mechanical particle in the well by WKB, or something. The first step is to ignore the kinetic energy term, the \( d^2 \) term, in which case the above equations take the form:

\[
(E - V_{I-\frac{1}{2}}) \chi_{I-\frac{1}{2}} = W \chi_{I+\frac{1}{2}}, \quad \text{and} \quad (E - V_{I+\frac{1}{2}}) \chi_{I+\frac{1}{2}} = W \chi_{I-\frac{1}{2}}.
\]

The equations imply:

\[
E = \frac{1}{2}(V_{I-\frac{1}{2}} + V_{I+\frac{1}{2}}) \pm \left\{ W^2 + \frac{1}{4}(V_{I-\frac{1}{2}} - V_{I+\frac{1}{2}})^2 \right\}^{\frac{1}{2}},
\]

which will now be used as an effective potential energy. We want to solve the equilibrium problem, and so we are interested in the lower of these two solutions. Since \( W \gg (V_{I-\frac{1}{2}} - V_{I+\frac{1}{2}}) \) in the region of values of \( r \) that is important, the effective potential is:

\[
E = \frac{1}{2} Kr^2 + \frac{(I+\frac{1}{2})^2}{2Mr^2} = nr,
\]

which looks something like:

\[\text{[Diagram of a potential energy curve]}\]

The equilibrium radius can be obtained to a very high accuracy by considering only the \( r \) and \( r^2 \) terms, and comes out

\[r_{\text{equil.}} = f/K,\]

and to high approximation, the bottom of the well is
Now we can put in crudely the quantum mechanical effects by saying that the lowest eigenvalue will be given by the above expression plus a zero-point kinetic energy which will be of order $\frac{1}{2} \sqrt{\frac{K}{M}}$. There will also be radially excited modes for a given value of $I$, but these will be separated by energies of order $\sqrt{\frac{K}{M}}$, which is quite high compared to the separations of the lowest isobars for various values of $I$ in strong coupling.

The above analysis results in one low-lying isobar for each value of the isotopic spin, and the spacing, of order $1/r^2$, involves the source function through the parameter $K^2/2M$. Since the isotopic spin takes on half-integral values only, the energy separations are proportional to $3, 5, 7, 9, \ldots$. If $K^2/2M$ is calculated for a source of zero extent, it comes out simply to be $\hbar \omega$, which is finite, and thus we can ignore the cutoff in this problem. In the pseudoscalar theory with pseudovector coupling, one cannot let the source collapse to a $\delta$-function, and get a finite isobar spacing as is possible here.

In the theory of actual interest, the pseudoscalar symmetric theory, the calculations are of a similar nature, but quite a bit more complicated because there one has three radial variables and six angles. The angular part of the problem turns out to be equivalent to the asymmetrical top in quantum mechanics. The calculations involved are discussed in the paper by Landovitz and Margolis; we shall only summarize the results, leaving the algebra to the diligent student. The physics is exactly the same as we have done here; it is just complicated by a few more dimensions. Again one treats the angular quantities quantum mechanically, and the radial quantities quasi-classically, getting an energy of an equilibrium position which varies with the angular quantum numbers, which is the energy of an isobar.

Before the results are quoted I would like to make a couple more comments. First, the quasi-classical treatment we gave can be said a little more simply. The term $\vec{r} \cdot \vec{x}$ is known to be dominant, and so in choosing our set of states, it is appropriate to take those which are eigenstates of $\vec{r} \cdot \vec{x}$ instead of $\vec{x}$ and, for example $\tau_z$. The lowest eigenstates will then be essentially eigenstates also of $\vec{r} \cdot \vec{x}$ to the extent that the kinetic energy is small compared to the coupling. You can thus describe the strong coupling situation by saying that the $\tau$ variable is "frozen in" so that $\vec{r} \cdot \vec{x} = -|\vec{x}|$. In a similar way in the symmetric pseudoscalar theory, both $\tau$ and $\sigma$ are
"frozen in" by the strong coupling.

The other point is that this problem, where we have reduced an infinite number of field oscillators coupled to a nucleon source to a finite number coupled to a source, is a special case, obviously, of a general treatment which can be tried for any coupling strength. We could try to approximate the complete problem by a problem in which only a finite number of degrees of freedom of the field are excited, and we can determine these degrees at will so as to minimize the energy. This is the method of Tomonoga, which is called intermediate coupling. It is so called, not because it is known to be valid for intermediate values of the coupling constant, but because it is evidently valid both in strong and weak coupling, and it may be used to generate a solution for any value of the coupling. Because the solution is exact at both extremes, there is some hope that it will also give one a useful idea of the situation in the intermediate region. The Tomonoga method is ideal for calculating an isobar spectrum, but can be generalized to discuss scattering and so on. Actually it appears as though it does give fairly respectable results in between. For example, if it is applied to the symmetrical pseudoscalar theory, as was done by Lee and Christian, results very similar to the Chew-Low results are obtained.

We now summarize the results of Landovitz and Margolis. There are two quantum numbers, \( I \) and \( J \), and it is mathematically convenient to define \( Y = I + J \). Assigning various integral values to \( Y \), one gets various bands of solutions. The lowest band has \( Y = 0 \), which means that \( I = J \), and has the spectrum:

\[
E = \text{constant} + \frac{3}{4} \mu (\mu a) \frac{I(I+1) - 3/4}{r^2/\mu} \frac{1}{4\pi}
\]

\( a \) is the source "radius" defined by

\[
1/a = \int U(x) |x - x'|^{-1} U(x') \, d^3x \, d^3x'.
\]

The first two of these are experimentally known; they are the nucleon, and the 3/2, 3/2 resonance. We are led to expect a 5/2, 5/2 resonance therefore at an energy of about 8/3 the energy of the 3/2, 3/2 resonance. Now of course the coupling is not too strong, and the calculation is not too good, but we might expect a 5/2, 5/2 resonance somewhere up there. However, this would not be easy to see because in most problems states of \( I = 5/2 \) are not reached.

These results were obtained by Pauli and Dancoff in 1943. But Pauli and Dancoff did not explore the details of the other bands. They had done
enough work getting this far, and besides they were way out on a limb. Nobody knew then that the mesons were pseudoscalar, and nobody knew that things were symmetrical; these were purely theoretical assumptions. Furthermore, the theory at that time agreed extremely poorly with experiments, because of the muon-pion mixup. So they did plenty for 1943 when they got to this point! But now that we know that this makes a certain amount of sense it is worthwhile to pursue it further, looking at the next band.

The second band was \( Y = 1 \) and thus \( I = J + 1, I = J, \) or \( I = J - 1. \)

If we define

\[
A = \frac{\mu \pi}{3} \int u^2 \, d^3 x,
\]

the spectrum is:

\[
\text{constant} + \mu \frac{1}{\mu a} \sqrt{2la^2} \left[ \frac{3}{8} \frac{I(I+1) + J(J+1) - 3/2}{f^2/\mu \pi} \right] \mu.
\]

The 2 in the square root comes from the factor \( Y(Y+1) \) in a more general formula.

Landovitz and Margolis speculate that the new resonances found in photoproduction and meson scattering may be interpreted as intermediate coupling remnants of the isobars of this second band. This may or may not be true. However, there is a little evidence that the second resonance in photoproduction, the one they would like to interpret as the lowest isobar of the second band with \( Y = 1, I = \frac{1}{2}, J = \frac{1}{2}, \) has negative parity, and thus could not be obtained at all from a theory with only p-wave mesons. So this speculation is actually on a very weak footing.
The many-body problem is indeed a very large topic. In these lectures we shall treat only one aspect of it which has had some attention these last few years, the problem of many identical fermions with two-body instantaneous interactions. This problem has actually been treated for a large number of different cases and situations, but our discussion shall be restricted to the simplest of these situations, in which the most progress has been made. So we will consider a highly degenerate system, that is to say, the temperature is down near zero. Of interest then are the ground state and the first sets of excited states, in which the number of excited particles is small compared with the number of unexcited particles, so that we can ignore essentially the interaction of the excited particles with one another and restrict ourselves to the interactions of the excited particles with the unexcited ones. This is speaking in a certain loose sense, which we will have to define more accurately, i.e. what we mean by excited and unexcited particles, but that is the general idea why it is simple to consider low temperatures.

The next simplification that is widely made is to consider a very large system, a macroscopic batch of the particles. This is, of course, especially useful for those problems where the properties of two batches of matter are the same as the properties of one batch, or, in other words, where the properties are intensive, and don't depend on the size of the sample.

Now what specific problems can be treated under these simplifying assumptions? One situation to which such a theory may be applied is an idealization of the electron gas in a metal. In a metal the conduction electrons see a periodic field generated by ion cores arranged in a lattice, and moreover they see, in general, a slight rattling of these ion cores. The motion of the lattice, of course, obeys the laws of quantum mechanics, and is quantized in the form of phonons. But we can idealize the system somewhat by smearing out uniformly the spatial distribution of the ion cores. The negative charge of the electrons is then still balanced, and so we have the problem of an electron gas in a uniform positive background instead of a periodic potential. However, the coupling of the electrons with a slight rattling motion of the ion cores, which can be described as a coupling between the electrons and the phonon field, can nevertheless be retained, if one likes. In such an idealization, one deals with instantaneous coulomb two-body interactions, and perhaps an electron-phonon coupling,
even though the ions, whose vibrations are the phonons, have been suppressed. It is possible, of course, to go further by restoring the periodic potential of the ions. The difference in the treatment for a periodic background rather than a uniform one amounts to using Bloch waves instead of plane waves; there one deals with Billouin zones rather than Fermi spheres, and so on. But we will certainly not get far into the problem.

Being a gaseous system, the volume of the system is completely arbitrary. By calculating the energy as a function of the volume, one can deduce the pressure, and the compressibility of the system.

As a second application, let us consider a liquid, e.g. He\(^3\). A liquid will not fill an arbitrary volume. In that case, when the volume assumed for the system exceeds the natural volume, we have nonsense. If we are putting the particles into a big box and assuming that the ground state of the system is such that the stuff uniformly fills the box, the situation is unphysical when the volume of the box is bigger than the natural volume of the liquid. However, we can still apply the theory in the region of volumes less than the critical volume. Outside that region, a solution will still exist, but it will not be the ground state, and thus not be anything like the behavior of the actual matter. From the complete solution as a function of volume, it is easy to find what the natural volume is, so that one can tell where the solution should describe the actual behavior of the system. It doesn't matter too much, therefore, whether we are dealing with a liquid or a gas; the method can be applied to either one.

A further example is "nuclear matter". Now this stuff does not exist anywhere. The reason it doesn't exist is due to the coulomb repulsion between protons, which disrupts the nucleus if Z is too large. If one abolishes the effects of electromagnetism, then there would be no reason for the number of protons and the number of neutrons in an arbitrarily large nucleus to become especially unequal, and thus, so far as we know, there would be no tendency towards fission. Arbitrarily large nuclei could then exist, which would resemble light nuclei, ignoring the surface effects. "Nuclear matter" would be sort of an extended light nucleus, whose properties continued intensively due to the saturation characteristics of nuclear forces. Now the calculations on "nuclear matter" are interesting in themselves, and they also shed some light on actual nuclei, because, to the extent that we can ignore surface effects and certain other effects of the finite number of particles in a nucleus, we can
predict nuclear properties somewhat by analyzing this fictitious macroscopic nucleus. Another thing is, that having done this, further improvements may be introduced into the theory, so that it can actually be applied to a finite nucleus. People have been working on that too for some years. But that's a complication which we shall not discuss. Here we shall treat only the simple basic ideas which arise in the problem of a very large system.

Now the method! Some progress has been made in the last few years in the study of such problems, but it is not of a very sensational nature. It consists, as the progress in field theory also consists, in the glorification of perturbation theory sufficiently so that it begins to give results looking like the behavior of the actual stuff. Again all we can say is that the agreement is semiquantitative, just as in field theory; there is no indication that one has obtained a completely legitimate approximation. What people have done is to conceive methods for glorifying perturbation theory in appropriate ways for these various problems, so that one gets a general idea of what is going on. The glorification required in each problem is slightly different. However, all of the treatments can be described within a general formal framework of perturbation theory with diagrams, just as in field theory, by saying that for each problem certain higher order diagrams have an exaggerated significance. If you sum these special diagrams and add them to the lowest order calculations, it is possible to get an inkling of what is happening, just as in meson field theory where one can get a description of resonance behavior by summing certain sets of diagrams besides the lowest orders. The resonance in the $3/2,3/2$ state is the crucial physical thing in low energy meson-nucleon scattering, and it is necessary to add up those diagrams which contributed importantly to that. Here the situation is similar: Ordinary perturbation theory cannot be used because it gives nonsense, but if it is glorified a little, a reasonable semi-quantitative description of the situation may be obtained. So we shall set up a formalism, which is almost exactly the same as the field theory formalism, with diagrams, and show which types of diagrams should be most important for certain processes, and exhibit a formal technique for summing them and what kinds of equations result for the description of various systems.

We shall set up a perturbation theory for describing almost anything: the energy of the ground state; the specific heat at low temperatures; the excitation energy of a given kind of level; the potential experienced by a particle moving through the stuff, which has a real and imaginary part; the compressibility; and, in general, anything that might be of interest about such
a bulk sample of matter. The theory is constructed along the same lines as is used in field theory, i.e. the expansion of $e^{-i\mathcal{H}t}$ in powers of the interaction Hamiltonian, which leads to P-brackets, propagators, diagrams, and so on. Some of the applications are actually approximations to the field theory problems, and so it should be clear that the diagrams are going to be much the same. There is, however, one difference which is physically quite important. Instead of the vacuum, the basic state is the ground state, and so the analogue of the free vacuum is the free ground state which is a Fermi sea of $N$ non-interacting fermions in a box of volume $V$. That means that most of our diagrams are similar to electron pair diagrams in electrodynamics, but instead of having antiparticles, we have holes in the Fermi sea. Since our problem is completely non-relativistic, the interactions will be simultaneous, and so roughly speaking all the diagrams will be composed of chunks like:

There are also a couple of other units which may appear in these diagrams which do not appear in field theory diagrams because there the basic state has no charge. Here the basic state has a lot of particles which interact with themselves and with excited particles and holes, giving structures like:
With the basic units, all the diagrams can be drawn, and if you sum all of them up for a certain process, you get a description of the propagation of something through the medium. For instance, the sum of all diagrams beginning with one excited particle will describe the propagation of that excited particle through the medium. If the arrows are reversed, on the other hand, the propagation of a hole through the medium will be described. You can also construct in this way two particle propagators, and so forth. It should be noted that one excited particle may end up as two particles plus a hole, which is an "inelastic" or "frictional" process. This cannot happen for a particle which is perched just on the edge of the Fermi sea, but it can happen for a hole which is down in or a particle which is excited above. That means, that unlike a field theory of stable fermions, the particle propagators have pieces corresponding to inelastic processes. That can't happen in electrodynamics, for example; a single electron doesn't turn into a lot of other stuff. It does to some particles in field theory, of course; a muon decays into several particles. These inelastic pieces mean that there is friction in the motion through the medium; the particle state is dissipated.

Let us now set up some of the terminology to be employed in the mathematical discussion. First of all, the free ground state is defined as that state with \( N \) particle occupying the \( N \) lowest free particle levels in a box of volume \( V \). All the other states of the basis will be labeled by listing those levels above the Fermi sphere which are occupied, and those levels inside the Fermi sphere which are vacant. For example, the state \( |m,n; \tilde{p}\rangle \) is the one with the two excited free particle levels \( m,n \) filled, and a hole in level \( p \). If the interactions are now turned on, i.e., if one acts with the interaction Hamiltonian to all orders in the perturbation theory on the free states, then, roughly speaking, each free state, for example, \( |m,n;\tilde{p},\tilde{q}\rangle \), will go into an exact eigenstate of the complete Hamiltonian, which we may label as \( |\Psi(m,n;\tilde{p},\tilde{q})\rangle \).

For the ground state, this is exactly true, but for all the other states this statement must be interpreted in a certain sense. All the other states are embedded in a tremendously degenerate set of states, so that in a sense it is undefined which state one gets to by turning on the interaction. We can, however, define the exact state, e.g. \( |\Psi(m,n;\tilde{p},\tilde{q})\rangle \), specifically by the requirement that in the distant past it looks like the complete ground state with two excited particles and two holes. All such incoming states are only metastable; in time, they develop in such a way that what started as an excited particle in the medium degenerates into a lot of other stuff. However, in many cases, the
particle can propagate a long way through the medium before the dissipation becomes very important, and then we can treat the excited particles almost as stable objects.

One of the restrictions on the types of problems to be investigated is that the temperature of the system is not to be high, so that the exact eigenstates of major interest correspond to free eigenstates in which the number of excited particles and holes is small. That doesn't mean in the sense of an expansion of an exact eigenstate in terms of our free basis that there are not many excited particles and holes in the eigenstate, but only that it is connected ancestrally to a state with very few excited particles and holes compared to the number of particles in the sea. For such exact eigenstates, the energy will be a linear function of the excitation, e.g. the energy of the state $|\Psi(m, n; p, q)\rangle$ will be given by:

$$E\{|\Psi(m, n; p, q)\rangle\} = E(m) + E(n) - E(p) - E(q).$$

Such a formula could never be written were it not for the fact that there are so few excited particles and holes compared to the $10^{23}$ or so of particles in the sea, because otherwise there would be important non-linear terms as well.

The point then is that, at low temperatures when one deals only with such eigenstates, everything of interest can be computed from the propagator of a single excited particle or hole. For example, the specific heat of the material at low temperatures is given simply by

$$C_V = \frac{n^2}{3} k^2 T g(E_f),$$

where $E_f$ is the value of $E(m)$ at the top of the Fermi sea, and $g(E)$ is the density of states per unit energy interval.* Another example: the potential experienced by a particle fired into the medium is just $E(p) - p^2/2m$. Therefore, the major goal is to calculate the exact propagator, $S_F(p, \omega)$, for a single excited particle or hole.

A few additional remarks should be noted. First of all, the propagator can be thought of as a function of a single variable, the frequency $\omega$, for all the values of a parameter $p$, the momentum of the excitation. Clearly the functional dependence of the propagator on the momentum can only be through the scalar $p^2$, since $p$ is the only vector in the problem. Secondly, we remarked that there was

dissipation except for \( p \) right on the edge of the Fermi sea, and therefore there is not a perfectly defined \( E(p) \) for \( p \neq p_F \). Instead the best that can be done is to define a complex energy with a small negative imaginary part: \( E(p) - \frac{1}{2} i \Gamma(p) \). Actually we shall see later that this is not exact either, but the complex energy approximation is quite useful. This complex energy has a ready physical interpretation: if \( p \) is imagined to be the momentum of an excited particle that is sent into the medium, \( E(p) - \frac{p^2}{2m} \) is the real part of the potential that it experiences, and \( -\frac{1}{2} \Gamma(p) \) is the imaginary part of the potential, which is a well-known approximation in nuclear physics. The complex energy can be deduced from the propagator: if one looks at the complete propagator, one sees that it doesn't exactly have a pole on the real \( \omega \) axis, but rather, with a slight analytic continuation, it can be said to have a pole just below the real axis at \( \omega = E(p) - \frac{1}{2} i \Gamma(p) \). Hence, roughly speaking, we can describe the excitation as a particle with an energy \( E(p) \), which includes all the interactions, and with a certain width for decay \( \Gamma(p) \).

The diagrams for the energy per particle of the system in the ground state are formed from vacuum closed loops, for example,

![Diagram]

but, just as in field theory, only the connected closed loops must be counted. The sum of all the closed loop diagrams is of the form \( e^{-iET} \), where \( T \) is the total length of time from \( -\infty \) to \( \infty \), and all we want is the linear term in the expansion of the exponential in order to get the ground state energy, which is composed of the sum of all the connected diagrams with no external lines. Any picture with two disjoint pieces represents the \( E^2 \) term in the expansion, and so on.

When something else is to be calculated, such as the propagator of a single particle, the disjoint vacuum closed loops are again to be ignored, for they just keep telling us the energy of the ground state, which, having been computed once, we do not have to calculate again.

The next point that should be noted is that it might seem crazy that everything of interest can be obtained just from the propagator \( S_F(p, \omega) \) of a
single particle. Surely the scattering of two particles in the medium, triple scattering, and so on, must be of some importance. The answer is that they are, but they are all contained in the one particle propagator. All the two particle scattering diagrams are included somewhere in the propagator, and similarly for all the three particle diagrams. As a result, the propagator of a single particle does contain sufficient information to tell us everything we would like to know.

Finally, it may be remarked that one can, if one likes, analyze everything in terms of a different propagator, namely that for the potential, which is analogous to vacuum polarization in field theory. The lowest order diagram is, of course,

and there are many higher order corrections which give a frequency dependence to the interaction; for example,

By adding up all such diagrams, one gets a propagator for the potential, which describes how the potential between two particles in the medium is modified by the existence of the medium. In a sense, it is the "polarization" of the medium that is being described.

In general, collective excitations of the matter will show up as "almost" poles in the propagator of the potential. In the case of the electron gas the most important collective mode is the plasmon. In the nuclear case, such "almost" poles exist corresponding to the nuclear analogues of the plasmon; the best example is the "giant resonance", which appears in the nuclear photoeffect. This collective mode does not exist as a single excited state, but rather shows itself in several excited states, which have very large photoeffect cross sections. What it represents physically is a two fluid motion of the protons versus the neutrons in the nucleus. Because the nuclear interactions are so strong, there is a tremendous amount of damping for this motion, which makes the
resonance quite broad, so that it spreads over several of the levels of a light nucleus.

Roughly, we have first described the perturbation expansion used in the many-body theory; (the mathematical rules will be given later). Secondly, we have said how the properties of the medium are extracted exactly, assuming the series to be summed. Thirdly, we now want to indicate what are the approximations that people have actually used for given problems, for instance, Brueckner's method for the nuclear matter problem or the method that has been used for the electron gas at high densities. Let us first look at the nuclear problem.

Brueckner revived some years ago the idea that one could glorify perturbation theory somehow and thereby give a semiquantitative description of nuclear matter. Nobody had believed this to be possible for the preceding two decades. It had been considered that nuclei were so tremendously complicated that it was completely impossible to apply anything resembling perturbation theory. The success of the shell model showed that there was some simplicity somewhere in nuclei, that in some sort of effective sense perhaps it was true that the nucleons were weakly coupled, and perhaps one could employ a perturbation theory if only it were jazzed up sufficiently. Then came the optical model. High energy experiments at Berkeley in which 50 to 100 MEV nucleons were shot into nuclei, were found to be describable by the optical model of Serber, in which such an excited particle was represented as passing through a potential with a real and an imaginary part. In other words, it was just as if, in the nuclear medium, such an excited particle had an energy and a width, i.e. it behaved like a modified unstable single particle. Fermi then suggested that maybe the particles down in the nucleus could be described by the same optical model of Serber, with slightly different values of the parameters because of the change in the momentum. If you go down into a momentum in the Fermi sea maybe one could still say that an optical potential will work. This became the physical basis of Brueckner's approximation, which we now state in this more fancy way that we have been describing. Then the idea was, can the optical potential really be calculated using something roughly like perturbation theory. The problem was to find which diagrams are most sensitive, what is the physics of the deviation from perturbation theory, or what is the effect that makes lowest order perturbation theory not applicable, and the program was to isolate that effect and try to correct for it by including all those higher order diagrams which essentially produce that effect.

Now, for that purpose one must go back to the high energy optical model
which originally suggested this treatment. There it was realized that the energy of a high energy particle as it passes through the nucleus, which is related to the real part of the "dielectric constant", could be expressed roughly in terms of the forward scattering amplitude. And indeed in our formalism here we are going to describe the energy in the medium by a propagator of a single particle in the medium, which is just like the forward scattering amplitude of an excited particle traversing the medium. Now, to what extent is perturbation theory bad for computing the forward scattering amplitude? The difference between the actual forward scattering amplitude and the Born approximation scattering amplitude amounts simply to the iteration of Born approximation many times; the two particles scatter off each other repeatedly, and that gives you the exact forward scattering amplitude instead of the lousy one that you get from considering them to scatter only once. The reason why the first Born approximation is so lousy is the shape of the nuclear potential. The nuclear potential has a deeply attractive well and then a repulsive core, and such a potential has Fourier transforms which in no way resemble the actual scattering amplitudes, until you get to such a high energy that they are of no interest anyway. In particular, the hard core is really the tough part. Imagine there were only the hard core, and imagine it were infinitely hard. The calculation of the exact scattering amplitudes is then a trivial problem, but if it is done by Born approximation complete nonsense results because the Fourier transforms of the potential do not exist. So the essential inapplicability of the Born approximation to the problem was guessed by Brueckner to lie simply in the fact that the scattering of two particles must be considered to all orders; whenever one has two particles scatter one must not allow them to scatter just once, but rather, one must insert the actual scattering amplitude. Apart from this, one can practically use perturbation theory.

Another way to state the Brueckner method is the following. In any such problem at very low densities one can examine what the dominant diagrams are, and it turns out that they are the ones involving binary collisions, something which is very reasonable at low densities. Thus, diagrams of the following structure
are the leading ones at low densities. These are also the ones that Brueckner needs to employ in order to juice up the first Born approximation to the scattering of two particles so that it becomes the actual scattering amplitude. One of the low order diagrams for the single particle propagator is:

but in such a diagram the scattering is considered only to the lowest order, and the point is that this is no good. One has to put into the calculation instead the sum of all such diagrams where the scattering is iterated, for example:

By adding up all such diagrams, we are making use of the exact scattering amplitude rather than the Born approximation scattering amplitude. This is the basis of the low density approximation for any many-body system, and, in particular, is the obvious suggested approximation for nuclear matter.

However, the actual Brueckner suggestion is slightly more complicated. In the scatterings of two particles by one another, the energy denominators, if the diagrams are calculated just as drawn, would involve only $p^2/2m$, the free particle energy. But the particles are all imbedded in the nuclear medium, and thus it would be very nice if the energy denominators could be corrected by including some of the potential energy that each particle experiences in the medium. This would be done by correcting the propagator, and Brueckner's suggestion involves the calculation of the complete propagator in a self-consistent manner. The scattering is calculated with an unknown propagator, and then is employed to determine that propagator. Actually such a program would be very complicated, and so Brueckner does not employ an arbitrary propagator but rather just shifts
the pole. In fact, he does not even use a pole at a complex energy, because he restricts the propagator to the form

$$\frac{1}{E(p) - i\epsilon - \omega}$$

where $E(p)$ is real. Using such a propagator with an unknown $E(p)$, the successive scattering of two particles is calculated in the medium with a nuclear two-body potential. This then is used to calculate the propagator, and thus determine $E(p)$ in a self-consistent procedure.

At low densities something very like this approximation is bound to be pretty good, because it can be proved that at low densities diagrams of this nature

![Diagram of propagator](image)

are the dominant ones. Furthermore, we have the physical argument that the chief reason for the inapplicability of perturbation theory in the nuclear problem is just the fact that in scattering the potential must be iterated many, many times in order to get a reasonable scattering amplitude. So it looks as if the Brueckner treatment should be semiquantitatively good. On the other hand, nobody has ever shown that it should be particularly very good, and estimates of the errors are still wildly conflicting. But there is no doubt that it does give a general idea of what nuclear matter behaves like.

In the case of the electron gas, the interesting application is at the opposite limit, in the region of high densities. There an entirely different set of diagrams are important, not the successive scattering of two particles by each other, but the successive iteration of these "sausages" in the propagator for

![Diagram of electron gas](image)

the interaction. So in the case of the electron gas for a high density expansion what was done was to sum the series of "sausage" diagrams.
So far, each of these two programs is of limited applicability to the actual regions of density encountered in nature, but they both represent improvements in our knowledge of the physics.

May 19, 1959

In the calculation of the propagator, two types of diagrams, which we have not encountered in the field theory, will play an important role. They do not occur in field theory because the basic state, the vacuum, has no charge, and so the diagrams of this character

are thrown out. However, these diagrams do exist and must be taken into account in a problem with matter present in the basic state. The way it is done mathematically is by a very simple trick of defining the value of \( S_F(x,t) \) for \( t = 0 \), which is left undefined in field theory. Normally, one specifies that

\[
S_F(x,t) = i \sum_n \psi_n(x,t) \psi_n^+(0,0) \quad \text{for} \quad t > 0, \quad \text{(the sum being over all those levels outside the sea), and} \\
S_F(x,t) = -i \sum_n \psi_n(x,t) \psi_n^+(0,0) \quad \text{for} \quad t < 0, \quad \text{(the sum being over the levels in the sea).}
\]

In order to take care of the diagrams with passive particles, all we have to do is to define \( S_F(x,0) \), because all such diagrams just involve a particle which propagates instantaneously. Hence by a suitable mathematical choice of \( S_F(x,0) \) one can formally include the effects of the passive particles. (It is not obvious by any means that this must be true; rather we are just stating a result.)

We shall not derive the rules for calculating, but only state the rather simple Feynman rules, which result from the standard analysis using Wick's theorem. They will be the same as those we know already except for the trick involving the passive particles, and except for the fact that it is convenient to employ a dimensionless notation.

The momentum will be measured in units of the Fermi momentum, \( p_F \), and the energy in units of twice the Fermi energy, \( p_F^2/m \). If the volume per particle
is $\frac{1}{3} \sqrt{2g/3\pi} = V/v$, and the number of spin states is $g$, then the Fermi momentum is given by: $p_F = 1/2\zeta$, with $\zeta = \frac{3}{2g/3\pi}$.

Furthermore, it is convenient to introduce the quantity:

$$r_s = r_0 \frac{e^2}{\ln} = r_0/(\text{Bohr radius}),$$

which will be employed as the expansion parameter instead of the fine structure constant. The Fourier transform of the potential as a function of the momentum transfer is denoted by $v(q)$, which for the electron gas problem is $e^2/q^2$ in ordinary units, or $\frac{1}{2g/2\pi} q^2$ in these peculiar units. Incidentally, for this problem $g = 2$ and thus $\zeta = 0.520 \approx \frac{1}{2}$. For the electron gas in metals $r_s$ ranges from 2 to 7. The high density calculations made with the theory are good perhaps up to $r_s = 1$ or 2, and so what is needed clearly is an intermediate approximation, and not just an expansion. But the expansion has helped a lot in providing a description of what is going on.

The mathematical rules for constructing the integrals corresponding to the diagrams are pretty obvious except for the trick of including the passive particles. For each potential line, one inserts a factor of $v(q)$. For each fermion line,

$$s_F(p,\omega) = \frac{1}{4\pi} \int \text{d}^3p \text{d}\omega,$$

one inserts $\frac{1}{4} s_F(p,\omega)$. The evaluation of the propagator is quite straightforward, and yields

$$s_F(p,\omega) = \frac{\eta(p-1)}{p^2/2 - i\epsilon - \omega} + \frac{\eta(1-p)}{p^2/2 + i\epsilon - \omega},$$

where $\eta(x) = 0$ for $x < 0$, and $\eta(x) = 1$ for $x > 0$. In the future we shall just write $s_F(p,\omega) = (p^2/2 \mp i\epsilon - \omega)^{-1}$, or perhaps even leave off the $i\epsilon$, assuming that the reader knows the correct pole prescription. Each order in the interaction carries a factor of $(-i)$ from the expansion of $e^{-i H \omega}$, and each closed loop brings in a factor of $(-1)$. Integrals are always of the form $(2\pi)^{-4} \int d^3p \text{d}\omega$.

The remaining rule formally takes care of the passive particles in the sea. The trick is to define

$$s_F(x,0) \equiv \text{Limit}_{t \to 0^-} S_F(x,t).$$

The most frequently encountered consequence of such a definition is:

$$\int \frac{d\omega}{2\pi} s_F(p,\omega) = -i \eta(1-p).$$
Now we can calculate! As was said, everything of practical interest can be obtained from the complete propagator $S_F^I(p,\omega)$, or if one desires to be fancier one can employ the propagator of the particle for some purposes and the propagator of the interaction for others. So let us restrict ourselves to computing just those, and in the course of doing that we shall have to make subsidiary calculations of many other things.

The complete propagator $S_F^I(p,\omega)$ can be indicated diagramatically as:

\[
\begin{align*}
\text{Diagram 1} & = \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \ldots
\end{align*}
\]

In general, we may write: $S_F^I(p,\omega) = \{p^2/2 - \omega + \Sigma(p,\omega)\}^{-1}$. $\Sigma(p,\omega)$ is then essentially the sum of all the proper self-energy diagrams, i.e. those which you cannot break up into two pieces connected by a single unadorned fermion line. To the lowest order, $S^I = S - S \Sigma S$, and thus

\[
\Sigma = -\frac{1}{i} \left\{ \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \ldots \right\}
\]

Let us compute first the lowest order direct term in $\Sigma$. Remembering that a closed loop is involved, one sees immediately that:
Indeed, it is clear that this contribution to $\Sigma$ is just the classical energy of interaction of the particle with the particles in the Fermi sea. For the case of the electron gas, this term is not finite because of the infinite range of the Coulomb potential, but it is exactly cancelled by the interaction of the particle with the assumed positive background. This cancellation of the direct repulsive energy is the only function of the background and thus we may simply neglect both of them henceforth.

If we deal with a scalar potential, which does not depend on $\sigma$ or $\tau$ matrices then the operation of taking the trace just brings in a factor of $g$.

Another contribution to $\Sigma$ comes from the first order exchange diagram.

In the conventional Hartree-Fock model, this term represents the exchange energy. In the problem of the electron gas, the exchange term is not cancelled by anything, and just gives a finite negative, stabilizing contribution to the energy per particle.

Now we can go on to the more complicated processes, for example
diagrams of the forms suggested by the diagram:

The complete propagator for the interaction may be written in the form

\[ v'(q, u) = \frac{v(q)}{1 + \Pi(q, u)} v(q) \]

and may be indicated diagrammatically as:

To lowest order, \( v'(q, u) = v(q) - v(q) \Pi(q, u) v(q) \), and \( \Pi(q, u) \) is computed by summing all the proper polarization diagrams. The lowest order term contributing to \( \Pi(q, u) \) will be called \( \Pi^{(o)} \), and is:

\[ \Pi^{(o)}(q, u) = \frac{i g}{(2\pi)^3} \int d^3r \frac{d\omega}{(2\pi)^1} \frac{1}{i^2} \{ \eta(r-1) \eta(1 - |r+q|) \} \{ u - \frac{1}{2} + i\varepsilon \} \]

If \( \Pi^{(o)}(q, u) \) is considered as a function of the complex variable \( u \) for a given value of \( q \), it evidently has a branch line on the real axis for

\[-(q + q^2/2) \leq u \leq (q + q^2/2)\]. The branch line runs over all the possible energies
for a particle-hole pair with momentum $q$. For a single pair, this restricts the energies of the excitation to a finite range. Now if one would compute higher order contributions to $\Pi(q, u)$, the excitations can involve many pairs, and the restriction on the energies of the excitations is much weaker. Indeed, as one goes higher and higher in the order of the interaction, the branch line extends further and further along the real axis. This means that the real excitations of the matter with a given total momentum that can be produced have higher and higher possible energies.

If now we consider some collective mode such as the plasmon, it will be represented by a pole in the propagator of the interaction, say in the following location:

\[
\begin{array}{c|c|c}
\text{plasmon} & -(q + \frac{q^2}{2}) & \text{plasmon} \\
\hline
u \text{-plane} & q + \frac{q^2}{2}
\end{array}
\]

Then in the lowest order of the interaction, the plasmon pole is completely separated from the branch line, which means that the plasmon mode is completely stable. But when one includes the higher order in the interaction it becomes embedded in the branch line, which means that a plasmon of momentum $\vec{q}$ has an energy equal to that of other excitations that can be made, and hence it will decay into them. The plasmon then is unstable. Effectively the singularity moves off the real axis; roughly speaking it becomes a pole somewhere in the complex plane, and twice the imaginary part of the pole location is the width for the decay.

$\Pi(q, u)$ describes the polarization of the medium, and its singularities, specifically the branch line, are reflected in $v(q, u)$. But the propagator may have more singularities than $\Pi(q, u)$. In particular, it may have a pole, and in fact, for the electron gas, $1 + \Pi(q, u)v(q)$ does vanish, which is mathematically equivalent to the existence of the plasmon collective mode. The pole is buried in a branch line of the propagator, and we must look into how to treat this situation in order to obtain the lifetime of the collective excitation.

The sum of all diagrams of the form

which includes the lowest order exchange diagram
gives a contribution to $\Sigma$ which is equal to:

$$\Sigma^{(\pi)} = (2\pi)^{-4} \int d^3r \, dw' \, v'(p - r, \omega - \omega') \frac{1}{i} S_F(r, \omega').$$

This contribution shifts the energy of the particle, and gives it a width for decay.

May 21, 1959

The formula obtained for $\Pi^{(0)}(\vec{q}, u)$ shows that in the lowest order the polarization function is analytic in the complex $u$-plane except for a limited branch line along the real axis, which is symmetric about the origin, and it defines the function along the cut as the limit as you approach the cut from the upper half plane for $\text{Re} \, u > 0$, and from the lower half plane for $\text{Re} \, u < 0$. This latter feature is typical to all orders in the interaction. On the other hand, the higher order contributions to $\Pi(\vec{q}, u)$ give it further singularities, extending the branch line indefinitely. The propagator of the interaction, $v'(\vec{q}, u)$ has the same singularities, except that in addition it may have some poles embedded in the cut. They are then no longer poles, but they give rise to what are effectively poles just off the real axis, which we will discuss mathematically.

But first let us look at the structure of the fermion propagator. The propagator is written in the form:

$$S^f(p, \omega) = \left( p^2/2 - \omega + i\Sigma(p, \omega) \right)^{-1}.$$

We have given formulae for $\Sigma^{(1d)}$, $\Sigma^{(1x)}$, and a general expression for $\Sigma^{(\pi)}$ involving the complete interaction propagator $v'(\vec{q}, u)$. The second order term in $\Sigma^{(\pi)}$ can be calculated from our expression for $\Pi^{(0)}(\vec{q}, u)$ and is given by the formula:

$$\Sigma^{(\pi2)} = - \int \frac{d^3q}{(2\pi)^3} \frac{d\omega}{4\pi} \frac{1}{i} \left[ \frac{v(\vec{q})^2}{\left(\vec{p} - \vec{q}\right)^2/2 + i\varepsilon - \omega} \right].$$

When the integral over $\omega$ is performed, we get two distinct terms which are the time-independent perturbation theory expressions of the form

$$\begin{align*}
\Sigma_{\text{mim}}^{H^I_{nm}} \frac{1}{(E_m - i\varepsilon)} H^I_{nm}\end{align*}$$

corresponding to the two time-ordered classes of second order diagrams:
Considering $\omega$ as a complex variable, it is evident that the singularities of $\Sigma(\pi^2)$ consist of a branch line along the real $\omega$-axis from $-\infty$ to $-\frac{1}{2}$ and from $+\frac{1}{2}$ to $+\infty$. In addition to this cut, the propagator $S_F(\tau,\omega)$ would have a pole at a point $\omega = w(p)$ where: $w = p^2/2 + \Sigma(p,w)$. However, this root may not occur on that Reimann sheet of the function $\Sigma(p,\omega)$ which is of physical interest. There will be a region of momenta where the root has a negative imaginary part and another region where it has a positive imaginary part. At the separation point $p_o$ of the two regions, evidently the imaginary part of $\Sigma(p_o,w_o)$ vanishes. Now the imaginary part is the absorptive part corresponding to real transitions, i.e. the real decay of the excitation, and hence can vanish only at a place where there is no density of states into which the particle can decay. That can be only at the edge of the Fermi sea. So $w_o$ must be the true energy of a particle at the top of the Fermi sea, and is the root of the equation:

$$\omega - \frac{1}{2} - \Sigma(1,\omega) = 0,$$

producing a genuine pole in the propagator.

For any other value of $p$, $w(p)$ will be complex. If the imaginary part of $w(p)$, or equivalently of $\Sigma(p,\omega)$, is small, we can find the root approximately by the following procedure. Define the energy of the excitation, $E(p)$, as the root of the following equation:

$$\omega - p^2/2 - \text{Re} \, \Sigma(p,\omega) = 0.$$ 

Then, to the first order in the imaginary part of $\Sigma(p,\omega)$, $w(p) = E(p) - \frac{1}{2} \, i \, \Gamma(p)$, for:

$$\omega - p^2/2 - \Sigma(p,\omega) = \omega - E(p) - \Sigma(p,\omega) + \text{Re} \, \Sigma(p,E(p))$$

and thus, 

$$\omega - p^2/2 - \Sigma(p,\omega) \approx [\omega - E(p)] \left[1 - \frac{\partial \Sigma(p,E(p))}{\partial E(p)}\right] - i \, \text{Im} \, \Sigma(p,E(p)) = 0$$

if:

$$\omega = E(p) + i \, \frac{\text{Im} \, \Sigma(p,E(p))}{1 - \frac{\partial^2 \Sigma(p,E(p))}{\partial E^2(p)}} = E(p) + i \, \frac{\text{Im} \, \Sigma(p,E)}{1 - \frac{\partial}{\partial E} \, \text{Re} \, \Sigma(p,E)}.$$
Therefore,

\[ \Gamma(p) = -2 \frac{\text{Im} \Sigma(p,E(p))}{1 - \frac{\partial}{\partial E} \text{Re} \Sigma(p,E)} \]

This zero of \( \omega - p^2/2 - \Sigma(p,\omega) \) gives rise only roughly to a pole in the propagator, for it occurs across the cut on a Riemann sheet of the function other than the one of physical interest. However, to the extent that the imaginary part of \( w(p) \) is small, the pole on another sheet will still have a dominating effect on the values of the function on the physical sheet. The imaginary part of \( \Sigma(p,E(p)) \) goes like \( (p - 1)^2 \), and so for momenta near the Fermi surface the description of the propagator in terms of this "almost" pole is a good approximation. In this region, if one defines

\[ \frac{1}{N(p)} = 1 - \frac{\partial \text{Re} \Sigma(p,E(p))}{\partial E(p)} \]

one can approximate the propagator as:

\[ S^l_F(p,\omega) \approx \frac{N(p)}{E(p) - \frac{1}{2} i \Gamma(p) - \omega} \]

\( N(p) \), the strength of the pole, has the simple significance, as we found in field theory during our discussion of the spectral representation of propagators, that it is the probability that the excitation is a single unadorned excited particle.

Exactly the same approximate analysis can be made for the propagator of the interaction, \( v'(q,u) \). It has a branch line which covers the real \( u \)-axis, and possibly some poles on Riemann sheets other than the physical one. But if the poles are close to the real axis, they dominate the behavior of the propagator in a certain region of the physical sheet, in which the propagator can be approximated by:

\[ v'(q,u) \approx \frac{M(q)}{[\Omega(q) - \frac{1}{2} i \gamma(q)]^2 - u^2} \]

\( \Omega(q) \) is the energy of the collective mode, and \( \gamma(q) \) is its decay rate.

Let us look in more detail at an example of this phenomenon, the plasmon mode of an electron gas. We shall work in this problem under the assumption that \( q \) is small, anticipating the well-known result that for small values of \( q \), the plasmon frequency is a constant. In the lowest approximation, for sufficiently small values of \( q \), the branch line does not extend out to this frequency, and the plasmon is completely stable. However, as \( q \) gets larger and larger, even in the lowest approximation, the branch line extends out past the plasmon energy, and then the plasmon becomes unstable, the pole being at a complex value of \( u \) which is off the physical sheet. So the plasmon is a useful concept only to the
extent that the zeroth order branch line has not engulfed its pole, and therefore
we are interested in it only for relatively small $q$. Hence, we can simplify the
algebra by working only to the lowest non-vanishing order in an expansion in
powers of $q$. Small values of $q$ correspond physically to long range in the inter-
action, and the plasmon mode is an extended excitation. Short range interactions,
associated with large values of $q$, do not in fact give large collective effects.
This shows up mathematically in the fact that for large $q$, $\gamma(q)$ becomes quite
huge, which means that the excitation dies out extremely rapidly.

DuBois has calculated $\Pi^{(0)}(q,u)$ in his thesis.\* The results of the inte-
gration are:

$$
\text{Re } \Pi^{(0)}(q,u) = \frac{1}{2\pi^2} \left\{ 1 - \frac{1}{2q^3} \left[ q^2 - (u - q^2/2)^2 \right] \log \frac{u - q^2/2 + q}{u - q^2/2 - q} \right. \\
+ \frac{1}{2q^3} \left[ q^2 - (u + q^2/2)^2 \right] \log \frac{u + q^2/2 + q}{u + q^2/2 - q} \},
$$

which can be expanded, assuming $u \neq 0$, as:

$$
\text{Re } \Pi^{(0)}(q,u) = -\frac{q^2}{3\pi^2} u^2 + O(q^4);
$$

and:

$$
\text{Im } \Pi^{(0)}(q,u) = 0, \text{ for } |u| > q + q^2/2 \\
\text{Im } \Pi^{(0)}(q,u) = |u|/nq + O(q^2), \text{ for } |u| < q + q^2/2.
$$

Inserting these expressions into the lowest order formula for the corrected propa-
gator,

$$
v'_{\Pi}(q,u) \approx \frac{v(q)}{1 + \Pi^{(0)}(q,u)} v(q) = \frac{\text{Im} \Pi^{(0)}(q,u)}{q^2 + \text{Im} \Pi^{(0)}(q,u)},
$$

we see that for $|u| > q + q^2/2$, the latter expression has two poles on the real
$u$-axis at the points $u = \pm \omega^{(0)}(q)$, where

$$
\omega^{(0)}(q) = \sqrt{\frac{\epsilon r_s}{3\pi}} + O(q^2).
$$

Again to the lowest order in the corrected propagator, one can construct the
dispersion relation between the momentum of the plasmon excitation and its energy
by carrying out the expansion to higher orders in $q^2$. The result turns out to be:

$$
[\omega^{(0)}(q)]^2 = \omega_p^2 \left[ 1 + \frac{12}{5} \frac{q^2}{\omega_p^2} + \left( \omega_p^2/2 - 6/35 \right) \frac{q^4}{\omega_p^4} + \ldots \right],
$$

where $\omega_p = \sqrt{\epsilon r_s/3\pi}$ or $\sqrt{Ne^2/mV}$ in ordinary units.

The important feature of this dispersion relation is that $g^{(0)}(q)$ tends to a non-zero constant as $q \to 0$. And so in lowest order, the pole is not embedded in the line of singularities for sufficiently small values of $q$, but is isolated. Therefore for very small $q$ in this order, the collective excitation is perfectly stable because there is no single pair excitation with the same energy and momentum for it to decay into. As $q$ gets larger and larger, the pole moves out but the branch line spreads faster, and so eventually they meet. As the cut engulfs the pole, it acquires an imaginary part and moves off the physical Riemann sheet, which represents the fact that for large enough $q$, even in the lowest order of the interaction, the plasmon is unstable. If we consider all orders of the interaction, then for any value of $q$, no matter how small, the pole would be engulfed in the branch line which extends over the entire real $u$-axis, and as a result, the pole has migrated off the physical Riemann sheet. But for smaller and smaller values of $q$, the imaginary part of the pole location becomes smaller and smaller, making the pole the dominant feature in the structure of the function even on the physical sheet.

In the case of real metals, the consideration of this idealized model of an electron gas gives reasonably well the frequency of the collective mode. But the lifetime of the plasmon is represented poorly by twice the imaginary part of the location of the pole, because the model omits some important things determining the lifetime, for example, the coupling with the phonons.

In the case of electron gas, the strength of the coupling and the density combine in a unique way to produce the expansion parameter $r_s$. In the high density calculations, consequently, an expansion in powers of $r_s$ is also an expansion in powers of the potential strength. In no other problem is that true; in the other problems the expansion in powers of the potential is not identical with an expansion in powers of a density parameter $r_0$.

The calculations in the high density approximation are asserted to be good up to $r_s \sim 1$, whereas for the case of actual metals $r_s$ ranges from 2 to 7. One might therefore inquire as to whether a low density approximation, which would be equivalent to a strong coupling calculation, might shed some light on the actual case. The strong coupling limit has been solved and is simple to describe. The problem is that of an electron gas with a uniform positive background at sufficiently low density that the potential energy is much larger than the kinetic energy. Let us, therefore, ignore the kinetic energy, in which case we no longer have a quantum mechanical problem because we have dropped the $p^2$, which is what
doesn't commute with the position variables. Hence, it simply reduces to the classical problem of finding a configuration of N particles in a box of volume V which minimizes the potential energy. The electrons, since they repel one another, will form a crystal with a spacing of order $r_o$. The leading term in the energy will just be given in terms of the classical Madelung sum for the crystal. Then if you wish to go a little further by putting in some corrections, you would include the zero-point kinetic energy of the electrons and the possibility of the rattling of the crystal lattice. So in the next approximation, we include the phonons, and in that way we get expansions in powers of $1/r_s$. The specific heat, the ground state energy, and so on, can be calculated in this way, and the computations are estimated to be good for $r_s$ larger than 15. Thus the physical region has been bracketed on the two sides by the low and high density expansions, but an intermediate coupling scheme is needed to describe the situation for those electron densities which are characteristic of real metals.

If you want to get results which can be compared with experiment in the case of the electron gas in a metal, you would have to make two major modifications of the model we have been using. The first thing would be to put in just the phonons, ignoring the lattice except for its vibrational modes. This introduces, besides the coulomb force between the electrons, a phononic force due to the exchange of one phonon. The phononic force, together with the coulomb force, is said to be responsible for many of the peculiar properties of the electrons in a metal, for example, superconductivity in those metals where the parameter of coupling to the phonons is strong enough for the phenomenon to occur. So they say! It may be, though, as is certainly true for many properties of a metal, that one has to go still further by considering, in addition to the rattling of the lattice, the periodic nature of the positive background which it produces. In this case the plane-wave wave functions of the electrons must be replaced by the wave functions of a periodic potential, which can be set into correspondence with plane waves by the method of Bloch. Instead of $e^{i\mathbf{p} \cdot \mathbf{x}}$, one has $e^{i\mathbf{p} \cdot \mathbf{u} (\mathbf{x})}$, where $\mathbf{u} (\mathbf{x})$ has the periodicity properties of the lattice. All our theory can be modified in this way, but it gets very complicated. New phenomena then will arise, of course. For example, a plasmon, which in the model we worked on dies just through the coupling to the ordinary electron excitation modes, can die here also through the coupling to the periodic potential due to the fact that momentum is not exactly conserved. Probably this fact and the coupling with the phonons are the dominating factors determining the lifetime of a plasmon. But there are also
many qualitative properties of a metal for which it is supposed to be sufficient to ignore the variation of the background potential from its average value.

All of the present research on superconductivity is devoted to the model of an electron gas with a uniform positive background, interacting by means of a coulomb force and a phononic force, the so-called Fröhlich force. The experimental evidence that the phononic force is crucial is that such a force, depending as it does on the rattling of the lattice, depends on the mass of the lattice particles, and therefore, for a given material, e.g. lead, would depend on what isotope is being studied. This isotope effect is very pronounced in the parameters of superconductivity; for example, the transition temperature is found to depend rather sensitively on the isotope mass. Therefore, it is considered that the crucial thing in superconductivity is the coupling of the electrons to the phonons. The periodic potential, on the other hand, does not depend in any way on the mass of the ion cores, and so it is considered that you can probably ignore it and still arrive at an understanding of the phenomenon. So they consider the rattling of the lattice, but no lattice! Using this mathematical model, it seems as though they have, more or less, accounted for superconductivity, qualitatively, but it is not absolutely clear. The theory is still mysterious. But a general idea of the theory was laid down by Bardeen, Cooper and Schrieffer*, and this general model is thought to contain the truth somewhere if only it is treated better than it is, since it is treated in not too good an approximation.

May 26, 1959

Let us turn our attention to the Brueckner approximation for the nuclear matter problem. The approximation is essentially a glorified perturbation method, which is appropriate for the case of low densities. Now, of course, such an approximation will not exactly be accurate for this problem, because nuclear matter, as it would exist in equilibrium if electromagnetism did not exist, does not have a large interparticle spacing compared to the radius of the force. However, the repulsive core and the saturation properties guarantee that the spacing does not become small compared to the range, so that one may get a general idea of the magnitudes of various quantities if one calculates in the low density limit, and especially if one improves a low density perturbation

In the first order, only these two diagrams exist:

\[ \begin{array}{c}
\text{and} \\
\end{array} \]

which give the interaction of the excited particle with all the particles in the sea directly, and by exchange, respectively. In second order, the significant diagrams are

\[ \begin{array}{c}
\text{and} \\
\end{array} \]

The first question to be faced is: Suppose one really has a material which is in equilibrium at low density; what diagrams would one have to include subsequently in order to "glorify" the perturbation theory? Then we shall see that the set of important diagrams obtained on such a basis is especially appropriate to the nuclear matter problem because of the nature of the forces.

For a low density approximation, in each order we would want to include those terms which involve the smallest powers of the density. Now the density is proportional to \( p_f^3 \), and so in every order of the potential one looks for those terms with as few powers as possible of \( p_f^3 \). Every term involves many integrals over the momentum variables, and these integrals extend either over the region outside the Fermi sea, for a line traveling forward in time, or over the region inside the Fermi sea, for a line going backward in time. Those integrals will depend largely on \( p_f^3 \) when they are over the sea, i.e., when they arise from lines traveling backwards. So we wish to analyze the terms of the expansion not in Feynman diagrams, but in the old-fashioned diagrams where the time ordering of the internal lines is distinguished, and we wish to select and retain those diagrams in which there are as few lines going backwards as possible, for a given order in the potential. For example, the third order diagram

\[ \begin{array}{c}
\end{array} \]

in the case of low density, will be less important than this third order diagram
because the former has two holes propagating, while the latter has only one.

As a result, diagrams of the following natures

will be the most important in each higher order.

It will be noted that the approximation procedure being sketched is an attempt to get the sum of a double expansion, in powers of the potential and in powers of the density. Certainly no arguments can be put forth which guarantee that the diagrams neglected are not more important than some of those which are retained. Nothing here is rigorous; all we can say is that this procedure generates fairly reasonable answers. Perhaps it does so by accident, perhaps not!

One can, of course, get a little check on the approximation by going back and putting in some of the neglected terms; if these do not appreciably alter the results, one acquires a little confidence that maybe one is on the right track.

In practice, the correction to the potential propagator due to the polarization of the medium as given by diagrams like

is of some importance in the nuclear problem. The corrected propagator includes the effects of collective excitations in the medium, in which category belongs the famous reststrahlung, the motion of the neutrons and protons opposite to each other in a nucleus, which is responsible for the giant resonance in the nuclear photoeffect. (If nuclei are bombarded with $\gamma$-rays, it is found that the total cross section has a broad resonance at around 15 to 20 MeV.) This huge resonance cannot be attributed to any particular nuclear level, in the sense that highly excited nuclear levels are distinct, but instead it arises from a coherent sum over a vast number of nuclear levels in the vicinity. A certain linear combina-
tion of these excited states can be formed which is an approximately stationary state of the nucleus, corresponding to a collective excitation of the material. The correction for such collective motion is a prominent example of what should come in if one includes such diagrams of higher order in the density.

Brueckner's original argument was along different lines than that of a low density approximation, but leads to much the same conclusion as to what higher order diagrams to include. All the low order diagrams are just the lowest order Born approximations for the scattering of two particles using the nuclear potential. But such a calculation is totally absurd, because of the nature of that potential. Therefore, anytime you have to compute something involving the scattering of two particles, you have to calculate that scattering to all orders in the potential. In that way you get the scattering amplitude, which is a reasonable thing even for hard core potentials, whereas the n-th approximation to the scattering amplitude for such a potential is nonsense.

The next thing that Brueckner suggested is a way of further improving the approximation, by more or less using the corrected propagator for a particle in the medium in all calculations, even in the calculation of the corrected propagator. Representing a corrected propagator by a double line, roughly speaking he proposed to calculate from the sum of all diagrams of the forms:

\[ \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array} \]

So the proposed procedure is a self-consistent one. However, such a calculation would be very complicated using an arbitrary function \( S^+(F, \omega) \), and therefore what he really proposed was a simplification of this in which the complete propagator is approximated by a function of the form
You should remember that in the vicinity of a quasi-pole of the complete propagator, it can indeed be well approximated by such a function. But in fact, Brueckner suggests that such a form be used everywhere, and furthermore, in his computations, he simplifies the arithmetic by setting \( N(p) = 1 \), and replacing \( \Gamma(p) \) by \( \pm \varepsilon \). These latter two computational simplifications are not necessarily such hot ideas, and can be corrected in future calculations with the theory. Thus, the Brueckner proposal involves correcting the propagator of a particle for the effects of the medium only in one respect, that of changing \( p^2/2m \) to a better energy \( E(p) \), which is determined by a self-consistent calculation.

To find the effective scattering amplitude for two particle scattering in the medium, which is the first step in the mathematical development, we must calculate the sum of all the scattering diagrams:

![Scattering Diagrams]

Now, how does this differ from the free particle scattering, or why cannot we just put in the experimental scattering amplitude instead of having to calculate it from a potential? There are two important differences. The most important is that because of the medium, the Pauli principle excludes a certain set of final states, namely those with \( p < p_f \). That reduces the amount of scattering tremendously, and makes a glorified perturbation theory much more applicable to the problem of nuclear matter. It would be particularly fortunate if the nuclear potential did not have a repulsive core, for then the effect of the exclusion principle would be such as to render a straight application of perturbation theory fruitful. (You may object that in making such statements about the importance of exclusion effects there is a discrepancy, because previously we argued along the lines of a low density expansion. There we claimed that terms of relative order \( p_f^3 \) were small, whereas now we are claiming they have real importance, so that logically our considerations appear to be quite sloppy. Nevertheless, they are both true, more or less -- it is roughly true that we can get away with a low density expansion, or something along the same philosophy,
and it is also more or less true that in the scattering, the amplitudes are considerably reduced by exclusion effects.) The second reason that the scattering must be calculated from the potential as opposed to just using the nucleon-nucleon scattering amplitudes from experiment is that we wish to put in a corrected energy for a particle in the medium. This means that the energy denominators occurring in the diagrams or in the resulting integral equation are different from those occurring in free particle scattering.

The major effects of a new energy function, \( E(p) \), are exhibited well by considering its power series expansion:

\[
E(p) = E(0) + \frac{p^2}{2m^*} + \ldots
\]

\(-E(0)\) gives the depth of the potential well, and effectively says that a particle of given energy has a much higher momentum inside the medium than outside it. Except for this effect occurring when particles enter or leave the medium, the \( E(0) \) term is not important because it could be eliminated by just a shift in the energy variable. The next term, however, is very important, and says that up to terms of order \( p^4 \), the particle of mass \( m \) behaves, in the medium, just like a particle of mass \( m^* \). \( m^* \) is called the "effective mass" and is given by the relation:

\[
m^* = \left[ \frac{\frac{dE(p)}{dp}}{E(0)} \right]_{p=0}.
\]

Numerically, the values obtained for \( m^* \) are about \( \frac{1}{2} m \), and the higher order terms in the expansion of \( E(p) \) have not turned out to be terribly important.

The scattering matrix, \( K(\vec{p}_3,\vec{p}_1;\vec{p}_4,\vec{p}_2;\omega_1,\omega_2) \), representing the sum of all the ladder diagrams like

\[
\text{is the solution of the following integral equation:}
K(\vec{p}_3,\vec{p}_1;\vec{p}_4,\vec{p}_2;\omega_1,\omega_2) = v(\vec{p}_3,\vec{p}_1) + \int \frac{d^3k}{(2\pi)^3} \frac{v(\vec{p}_3,\vec{p}_1-k) K(\vec{p}_1+k,\vec{p}_1;\vec{p}_2-k,\vec{p}_2;\omega_1,\omega_2)}{\omega_1 + \omega_2 - E(\vec{p}_1 + k) - E(\vec{p}_2 - k)}.
\]

Since we deal with a situation which is invariant under translations in space, the momentum is conserved in the scattering, i.e., \( \vec{p}_3 + \vec{p}_4 = \vec{p}_1 + \vec{p}_2 \). However, we cannot say, as we could in vacuo, that the K-matrix will be a function of
essentially only the relative momenta, because there is an absolute scale of momentum in the problem fixed by the Fermi sphere. Thus the determination of the scattering amplitudes in the medium is a more complicated problem than free particle scattering because of three things: (1) the Pauli principle restrictions on the final momenta; (2) the changed form of the energy-momentum relation, which at the very least involves an arbitrary well depth and an arbitrary mass; and (3) the additional variable which comes in because you cannot just transform simply to the center of momentum system of the two particles. But even with these complications, it is a fairly straightforward task to solve the integral equation on some machine.

Using this scattering matrix, we now wish to calculate the corrected propagator, or equivalently, $\Sigma(p, \omega)$. The way to proceed is fairly clear. For example, the contribution of the sum of the diagrams of the form

\[
\int \frac{d^3 q}{(2\pi)^3} \eta(1-q) \text{Trace} \{ K(p, q; q, q; \omega + E(q)) \}.
\]

The sum of the corresponding exchange diagrams is obtained similarly from the formula for $\Sigma(1x)$, (see page 380), and gives the following contribution to $\Sigma(p, \omega)$:

\[
-\int \eta(1-r) K(p, \omega + E(r)) \frac{d^3 r}{(2\pi)^3}.
\]

Upon employing these, one gets an approximation to $\Sigma(p, \omega)$. Then one has to go through this procedure until one finds a function $E(p)$ which gives more or less consistency, i.e. assuming the corrected propagator, one calculates $K$, and then $S'$, which is required to be sufficiently close to the one you start with. This is the Brueckner procedure for calculating the energy of a single particle in the medium.

It is possible by the methods we have developed to calculate also the ground state energy. For every approximation that is made in computing the corrected propagator, there is a corresponding approximation procedure for calculating the ground state energy. This type of calculation we have never done
before, but it proceeds along the same lines except for a slight numerical factor of $1/n$ in $n$-th order diagrams. The relevant diagrams are the same as those for $S_1(p,\omega)$ except that the external line closes upon itself. Some examples of the lowest orders are:

As usual, only the connected diagrams give a contribution to the ground state energy; the disconnected ones represent higher powers of $E$ in the expansion of $e^{-iET}$. The approximation for the single particle energy suggests an analogous set of diagrams to be summed to get an approximation for the average binding energy, and since the diagrams are almost the same, all we have to do is to find a relation between the average binding energy and $E(p)$.

The first thing that has to be worked out is a combinatorial problem, namely, for every diagram how many other diagrams are equivalent to it. In the expansion of $\exp\{-i/\hbar \ dt\}$ there is a factor of $1/n!$ in the $n$-th order, which in most problems is cancelled by a $n!$ coming from the $n!$ distinct ways of assigning a space-time label to each vertex. But in vacuum self-energy diagrams, and in some other problems, there may not necessarily be $n!$ distinct ways of labelling the diagram. For example, the second order diagram contributes a matrix element involving $1/2!$, but can be labeled in only one way, since the labelings

\[ x \quad \text{and} \quad y \]

both correspond to the same normal product in the Wick expansion.

Taking the combinatorial question into account results in a factor of $\frac{1}{2}$
for the following "direct" diagrams:

\[ \frac{1}{2} \text{Tr} \text{Tr}^\dagger \int \frac{d^3p \, d^3q}{(2\pi)^6} \eta(l-p)\eta(l-q) K(p,p;q,q;E(p) + E(q)) \]

to the energy of the ground state. There will be a similar contribution from the corresponding exchange diagrams.

In this approximation, the ground state energy per particle can be written in terms of \( E(p) \), and the Brueckner formula is simply:

\[ \frac{E_{\text{ground}}}{N} = \frac{1}{2} \int \frac{\eta(l-p) [E(p) + p^2/2m]}{\eta(l-p)} d^3p \]

This is all calculated for various values of the density \( N/V \). The Brueckner guess for the density of nuclear matter is that which minimizes this approximate value for the ground state energy.

Many people have criticized this procedure, arguing that the equilibrium properties should be obtained by a variational calculation, and this point has been investigated recently in the literature, notably by Brout. The result is that the procedure of using the value of the density which minimizes \( E_{\text{ground}} \) in order to predict the equilibrium properties of nuclear matter is justified in the practical sense.

May 28, 1959

The Brueckner calculations have been criticized to some extent because the separation energy of a nucleon does not come out equal to the average binding energy. Hugenholtz and Van Hove have drawn attention to a theorem which relates the energy of a particle at the top of the Fermi sea to the average energy per particle, \( \bar{E} \). The theorem is simply

\[ E(p_f) = \bar{E} - V \frac{\partial \bar{E}}{\partial V}, \]

or equivalently,

\[ E(p_f) = \bar{E} + \frac{1}{2} P_f \frac{\partial \bar{E}}{\partial P_f}. \]

If the material is in equilibrium, then \( \frac{\partial \bar{E}}{\partial V} = 0 \), and so \( \bar{E} = E(p_f) \).
Not all substances have an equilibrium density, excluding zero density and infinite density from consideration. But stuff like nuclear matter clings together at a definite equilibrium density. Now using a suitable potential, Brueckner and Gammel get results like \( E = -15.2 \text{ MEV} \), and \( E(p_f) = -27.5 \text{ MEV} \). Brueckner has written a paper* in which he discusses at great length the difference between these two quantities, and the various places where \( E \) is the proper thing to use and those places in which \( E(p_f) \) is to be employed. For example, he takes a particle entering nuclear matter from the outside at zero energy, and asks what function is one to use to determine the momentum inside. He concludes the momentum, \( p \), is not to be determined by \( E(p) = 0 \), but rather by

\[
E(p) + \left\{ E - E(p_f) \right\} = 0.
\]

Since the results for \( E \) and \( E(p_f) \) differ by quite a bit, he claims that it is a very important matter to know which function to use in a given situation.

The purpose of the paper is questionable if one really believes that the theorem with which we have started the discussion is true. From this point of view, the fact that the computed values of \( E \) and \( E(p_f) \) are different is just due to the approximations of the Brueckner approach. If you sum up some of the diagrams in order to get an approximate result for \( E(p_f) \), and proceed similarly for \( E \), it is not true that the results need to be equal, although it is true for the exact problem. It would seem, therefore, that the thing to do would be to set up the approximation in such a way that the two results agree, or to use the theorem, calculating only that one quantity for which the approximation is better. Thus the criticism appears to be just. If the approximation were really good, the results for \( E \) and \( E(p_f) \) would be the same.

If one calculates in perturbation theory for any potential, one can in fact verify to any order that the theorem is valid. And therefore, if you were to sum up all orders and find an equilibrium configuration for which \( \frac{\partial E}{\partial V} \) vanishes, the two numbers, \( E \) and \( E(p_f) \), would be equal.

The full form of the theorem may be checked for the electron gas problem. In our reduced units of energy and momentum for this problem the theorem reads:

\[
E(1) = \frac{5}{3} \bar{E} - \frac{1}{3} r_s \frac{dE}{dr_s}.
\]

In the zeroth order, \( E(1) = 1/2 \), and \( \bar{E} \) is the average of the kinetic energy of the particles in the sea, which is \( \frac{3}{5} \cdot \frac{1}{2} = 3/10 \). Since \( E(0) \) is independent of \( r_s \),

* Brueckner, Phys. Rev. 110, 597 (1958)
the theorem is seen to hold, in the zeroth order.

In the first order,

$$E^{(1)}(1) = -\frac{1}{(2\pi)^3} \int d^3q \quad \eta(1 - |p - q|) \frac{\ln \zeta_s}{q^2} = \frac{\zeta_s}{\pi} \int_0^2 dq \int_{q/2}^1 dx = -\frac{\zeta_s}{2\pi}. $$

This immediately gives us $E^{(1)}(1)$, with much less work than it would have been to calculate directly. $E^{(1)}(1) = -\frac{3}{2}\zeta_s/\pi$.

In the second order, one gets a divergence if he carries out the perturbation theory, which is well known and formerly bothered people. This divergence, of course, is really not present if higher order terms are included. In the diagram

there occur two factors of $1/q^2$, which gives a divergent integral. But other diagrams of the form

provide terms proportional to $(1/q^2)^n$, which give more highly divergent integrals. But the sum of all such terms gives a finite contribution, for clearly we are dealing just with the expansion in $1/q^2$ of the lowest order correction to the interaction propagator, which is:

$$\frac{\ln \zeta_s}{q^2} \frac{1}{1 + \Pi^{(0)}_{\ln \zeta_s} \frac{1}{q^2}}.$$

Hence the inclusion of the string of "sausages" cures the divergence. The divergent integrals are effectively cut off at a value of $q$ of order $\sqrt{r_s}$, and so what was previously a logarithmic divergence becomes simply the $\log r_s$. And in the higher order terms, in addition to powers of $r_s$ one may also get powers of $(\log r_s)$.

In the second order, one will obtain a formula for $E^{(2)}$ of the form:

$$r_s^2(A \log r_s + B).$$

By the theorem then, $E^{(2)}(1)$ must be

$$r_s^2 (A \log r_s + B - A/3)$$

which can be verified by direct calculation.
The theorem can also be used to calculate $\bar{E}$, which is a relatively complicated thing to compute since it involves an extra integral over the particles in the sea, from $E(1)$, which is a much easier thing since it is just the energy of a single particle perched on the top of the Fermi sea. However, it is necessary to know $E(1)$ as a function of $r_s$.

Now there is still another way to calculate $\bar{E}$ if one knows some relatively simple quantity as a function of $r_s$. DuBois gives the following formula for $\bar{E}$:

$$\bar{E} = \frac{3}{10} \frac{3}{2\pi^2 r_s^2} \int d^3 q \int_0^{r_s} dr_s \frac{r_s}{q^2} \frac{\Pi_{r_s}^{-1}}{1 + \frac{q^2}{4\pi^2} \Pi_{r_s}(q,u)}$$

Such a formula is derived very simply. The ground state energy is the sum of all the connected closed loop diagrams. We notice that all such diagrams can be obtained by closing the interaction propagator diagrams. But in addition we have to watch out for the fact that there is a factor of $\frac{1}{n}$ associated with the $n$-th order diagrams for the ground state energy. This factor is taken into account by performing the integral with respect to $r_s$, for if the integrand is imagined to be expanded as a power series in $r_s$, the coefficient of $(r_s)^{n-1}$ comes just from $n$-th order diagrams, and

$$\int_0^{r_s} (r_s)^{n-1} dr_s = \frac{r_s^n}{n}.$$

Both of the formulae for $\bar{E}$ involve knowing something as a function of $r_s$, but the latter uses $r_s$ in the character of a coupling constant, while in the Seitz theorem $r_s$ comes in as a density parameter.

There may now be many more useful formulae for calculating $\bar{E}$. For example, there should be an analogous formula which comes from closing all the particle propagators and summing the diagrams, putting in the $\frac{1}{n}$ by some simple trick. No way is presently known, however, of calculating $\bar{E}$ simply if something else is not known as a function of $r_s$. In the Brueckner approximation there is a very simple formula, namely

$$\bar{E} = \frac{1}{2} \left(\frac{p^2}{2m} + E(p)\right)_{\text{average}},$$

but this formula is only an approximation, and the discrepancy between the computed values of $\bar{E}$ and $E(p_f)$ shows that the formula is not true for many higher order diagrams that are important.

Now as to the future of this business, since the theory for a many body
system has been put on a fairly rational basis, people probably will be able to find rather intelligent intermediate coupling methods for those values of density which occur in nature. This rather straightforward approach to the many body problem is relatively new, especially in solid state physics. Almost all calculations on the many body problem, which date back further than a few years, were just very rough physical guesses at the situation, represented somehow mathematically. These rough guesses were pretty good, and they are certainly as good as anything we have today, except for the nuclear matter problem, which was not attacked because people felt for many years that there was no hope for any simple approach to the problem. In the case of metals, the new calculations have no advantage over the old ones except that they are rational, whereas the old ones involved a tremendous amount of art in the analysis of any given problem. The recent developments have been mainly in the direction of finding a rational language to discuss various methods of treating many body problems approximately. The major difficulty is, of course, that the rational language is itself based on some approximation like an expansion in powers of a coupling constant or of the density parameter, which is wrong for the physical values of the parameters concerned. And so we are not presently really much better off. But in the future we may be, if people, using the rational language, develop intermediate strength coupling methods which really work.

END OF THIRD TERM