CALIFORNIA INSTITUTE OF TECHNOLOGY
Division of the Humanities and Social Sciences
Pasadena, California 91109

# ON GAME SOLUTIONS
# AND REVEALED PREFERENCE THEORY

Charles R. Plott

Social Science Working Paper
Number 35
February 1974

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Charles R. Plott
California Institute of Technology

Several connections between the concepts underlying the theory of revealed preference and the concepts underlying solutions to cooperative games, have been established by Wilson [ 9 ]. In this paper we provide some new connections. Wilson established the relationship between the solution concept of Von Neumann and Morgenstern and the strongest forms of rational choice found at Richter [ 6 ] and Hansson [ 4 ]. Here, for the cases of finite sets of alternatives we provide connections with weaker "degrees" of rationality found at Plott [ 5 ], Richter [ 6 ], and Sen [ 7 ].

## NOTATION AND FORMULATION

We let E be a universal set of alternatives and let $\beta$ be a nonempty family of subsets of E. The family $\beta$ is a class of admissible sets of feasible alternatives. A choice function is a function which has $\beta$ as a domain and subsets of E as the domain. If C(v) is a choice function we assume that $\phi \neq C(v) \subset v$ for $v \in \beta$. The set C(v) is termed the choice set and v is termed the fcasible set.

We are interested in structures for which C(v) is a model of a process. The set v is the feasible set and C(v) are the "outcomes." If C(v) is not a single element of E, then the interpretation is that the process has multiple "equilibriums" for that feasible set. Since $C(\cdot)$ is a model of a process there must be, of course, other parameters (such as individual preferences). For purposes of this paper we assume they are constant and therefore not listed.

We will need the following definitions. For a given structure $\langle \beta, C(\cdot) \rangle$ we define a particular binary relation, V, on E. We say

$$xVy \iff [x = y] \text{ or } [(\exists v)_{v \in \beta} : x \in C(v) \,\&\, y \in v].$$

A choice function is rational in case there exists a binary relation, R, on E such that $(\forall v)_{v\in\beta} C(v) = \{x \in v : xRy \text{ for all } y \in v\}$.

Let Q be an arbitrary binary relation on E. We shall sometimes study choice structures, $\langle \beta, C(\cdot) \rangle$ , which have some of the following properties.

Congruence Property, CP(Q) : $(\forall x)_{x\in v}(\forall v)_{v\in\beta}[x \in C(v) \iff (\exists y)_{y\in C(v)} : xQy]$

Solution Property, SP(Q) : $(\forall x)_{x\in v}(\forall v)_{v\in\beta}[x \in C(v) \iff (\forall y)_{y\in C(v)} xQy]$

Core Property, KP(Q) : $(\forall x)_{x\in v}(\forall v)_{v\in\beta}[x \in C(v) \iff (\forall y)_{y\in v} xQy]$.

The first axiom has been motivated by the theory of revealed preference. The second and third axioms come from game theory. In order to see the first simply let Q be the binary relation V as defined above. The axiom, in this special case, becomes equivalent to Richters W - axiom which under certain conditions is equivalent to the weak axiom of revealed preference.

The second two axioms in this axiomatic form have not received much attention in the literature. Von Neumann and Morgenstern did axiomatize the solution concept [ 8 , pp. ] but since then, to my knowledge, Wilson [ 9 ] has provided the only application of the axiom. In order to understand these axioms, imagine that we are modeling a game. The universal set of alternatives is a set E, but the feasible set v may not be the universal set. The rules and structure of the game combine with individual preferences to form a dominance relation, D, over the elements of E. That is, for any pair if xDy we know the members of some coalition prefer x to y and have the "power to enforce" x over y. Now if we let xQy mean y "does not dominate" x (i.e. $\neg yDx$) we can see that if (and only if) the choice function satisfies SP(Q) the elements of C(v) form a Von Neumann-Morgenstern solution. That is, every feasible, not chosen element is dominated by some chosen element and if two elements are chosen together, then neither dominates the other. Similarly, if the choice function satisfies KP(Q), where Q means "does not dominate," then C(v), for any v, is the core of the underlying game.

## RESULTS: REVEALED PREFERENCE

We wish first to focus on the relationship between the three properties discussed and the propositions which underlie revealed preference theory (namely the V relation and the concept of rational choice). We show that when things are taken in the proper combination each of the properties is equivalent to a different "degree" of rational choice.

Theorem 1. Let E be finite and let $\beta$ be power set (set of all subsets) of E. The function $C(\cdot)$ satisfies CP(V) if and only if there exists a binary relation R on E such that

i) $(\forall v)_{v\in\beta} C(v) = \{x \in v : xRy \text{ for all } y \in v\}$



ii) R is total and reflexive

and

iii) R is transitive.

Proof. This theorem is a special case of theorems developed at ( 6 ) and ( 9 ).

Theorem 2. Let E be finite and let $\beta$ be the power set (set of all subsets) of E and let R be any total, reflexive transitive binary relation on E. Let $C(v) = \{x \in v : xRy \text{ for all } y \in v\}$. Then, in this case $C(v) \neq \phi$ for $v \in \beta$ and the choice structure $\langle \beta, C(\cdot)\rangle$ satisfies CP(V).

Proof. Again this is a special case of known results [ 6 ].

Definition: a binary relation R is said to be quasi-transitive in case $[xRy \ \& \ \neg yRx]$ and $[yRz \ \& \ \neg zRy]$ implies $[xRz \ \& \ \neg zRx]$, i.e., $xPy \ \& \ yPz \Longrightarrow xPz$.

Theorem 3. Let E be finite and let $\beta$ be the power set (set of all subsets) of E. If the function $C(\cdot)$ satisfies SP(V) then there exists a binary relation R on E such that

i) $(\forall v)_{v\in\beta} C(v) = \{x \in v : xRy \text{ for all } y \in v\}$

ii) R is total and reflexive

and

iii) R is quasi-transitive.

Proof. Suppose $\langle \beta, C(\cdot)\rangle$ satisfies SP(V) then $\{x \in v : xVy \text{ for } y \in v\} \subset C(v)$. In order to see this assume xVy for all $y \in v$ and note that this implies xVy for $y \in C(v)$ which, by SP(V), implies $x \in C(v)$. Now, suppose $x \in C(v)$, then, by definition $xVy$ for all $y \in v$ so $\{x \in v : xVy \text{ for } y \in v\} \supset C(v)$. From the two conclusions taken together we see that SP(V) implies

$(\forall v)_{v \in \beta} C(v) = \{x \in v : xVy \text{ for } y \in v\}$ which, by letting $V = R$ becomes (i).

Since $\{x, y\} \in \beta$, for all $x, y$, we obtain (ii). With $R$ as defined suppose $xPy$ and $yPz$. We know then, that $y \notin C(\{x, y, z\})$ since this would mean $yVx$ which, together with SP(V) means $y \in C(\{x, y\})$ and $yRx$ contrary to hypothesis. We know by a similar argument that $z \notin C(\{x, y, z\})$. Consequently, $x = C(\{x, y, z\})$ and $\neg zVx$, since, otherwise, by SP(V), we would have $z \in C(\{x, y, z\})$. But $\neg zVx$, by definition of $R$, means $xPz$, so $R$ is quasi-transitive as demanded by (iii). ∎

Theorem 4. Let $E$ be finite and let $R$ be any total, reflexive, quasi-transitive binary relation on $E$. Let $\beta$ be the power set of $E$ and define for $v \in \beta$, $C(v) = \{x \in v : xRy \text{ for all } y \in v\}$. In this case $\langle \beta, C(\cdot) \rangle$ satisfies SP(V).

Proof of 4. We know from [ 7 , theorem ] that $C(v) \neq \phi$ for all $v \in \beta$. We note now that $xRy$ if and only if $xVy$.

Suppose $xVy$. Then there is a $v$ such that $x \in C(v)$ and $y \in v$. But then by construction of $C(\cdot)$, it follows that $xRy$. Suppose $xRy$. From the construction of $C(\cdot)$ it follows that $x \in C(\{x, y\})$ so we have $xVy$. Chaining these two observations together with the definition of $C(\cdot)$ we get

(1) $(\forall x)_{x \in v} (\forall v)_{v \in \beta} \{x \in C(v) \Longleftrightarrow (\forall y)_{y \in v} xVy\}$ which implies

(2) $(\forall x)_{x \in v} (\forall v)_{v \in \beta} \{x \in C(v) \Longrightarrow (\forall y)_{y \in C(v)} xVy\}$.

Now suppose $x_0 Vy$ for all $y \in C(v)$ and that $x_0 \notin C(v)$. Since $x_0 \notin C(v)$ and since $C(\cdot)$ is by (1) all of the V maximal elements, there exists an $x_1$ such that $\neg x_0 V x_1$. Clearly $x_1 \notin C(v)$ for otherwise would yield $x_0 V x_1$ contrary to construction. Since $x_1 \notin C(v)$ there exists, by (1), an $x_2$ such that $\neg x_1 V x_2$. If $x_2 \in C(v)$ we would have $x_0 V x_2$ from our



initial hypothesis but this would violate the conclusion $\neg x_0 V x_2$ which follows from the quasi-transitivity of $R$ (and thus $V$). If $x_2 \notin C(v)$, as concluded, there exists, by virtue of (1), an $x_3$ such that $\neg x_2 V x_3$. By repeating the argument we obtain a sequence, $x_0, x_1, x_2, \cdots$, such that for each $i$ there exists an $i + 1$ such that $\neg x_i V x_{i+1}$ is the case and such that each element is distinct (If $x_i = x_k$ then we would have, contrary to hypothesis, $C(\{x_i, x_{i+1}, x_{i+2}, \ldots, x_{k-1}, x_k\}) = \phi$). This conclusion violates the assumption of finite $E$. We conclude then that $x_0 \in C(v)$ or, in general that

(3) $(\forall x)_{x \in v} (\forall v)_{v \in \beta} \{x \in C(v) \Longleftarrow (\forall y)_{y \in C(v)} xVy\}$.

We only need note that (2) and (3) together form a statement of SP(V) ∎

Theorem 5. Let $E$ be finite and let $\beta$ be power set (set of all subsets) of $E$. If $\langle \beta, C(\cdot) \rangle$ satisfies KP(V) then there exists a binary relation $R$ on $E$ such that

i) $(\forall v)_{v \in \beta} C(v) = \{x \in v : xRy \text{ for all } y \in v\}$

ii) $R$ is total and reflexive.

Proof. Suppose $\langle \beta, C(\cdot) \rangle$ satisfies KP(V). Then, by definition, $V$ satisfies (i) and since $\beta$ contains all two element sets $V$ satisfies (ii). Simply let $R = V$ and we are finished ∎

Theorem 6. Let $E$ be finite and let $\beta$ be the power set of $E$ and let $R$ be any binary relation such that $C(v) \neq \phi$ for $v \in \beta$ and $C(v) = \{x \in v : xRy \text{ for all } y \in v\}$. In this case $\langle \beta, C(\cdot) \rangle$ satisfies KP(V).

Proof. We note $xRy \Longleftrightarrow xVy$ and the statement of the definition of $C(\cdot)$ becomes KP(V) ∎

## RESULTS: REVEALED DOMINANCE

The above results establish the relationship between the axioms and a particular binary relation. This binary relation, V, is determined by the internal structure of the particular choice function under consideration. It remains to establish relationships between a choice function and the structure of some, arbitrary, underlying dominance relation.

Consider the following motivation. We have observed (or are observing) the operations of a process. We know little or nothing about the underlying institutions but we would like to begin an investigation of them. Only the choices $C(v,\vec{\alpha})$ where $\vec{\alpha}$ is a vector of parameters and v is the feasible set, are known.* Could this choice function have been generated by an underlying cooperative game? If so, what is the structure of the game?

These questions cannot be completely answered now, even though some headway has been made [Bloomfield ( 1 ), Bloomfield and Wilson ( 2 )]. For now we supply an answer to a subproblem. If the parameters $\vec{\alpha}$ are fixed and if $C(\cdot)$ is a Von Neumann-Morgenstern solution to some underlying dominance relation D, what is that dominance relation? Does the process act as if such an underlying dominance relation exists? We answer this question for the special case where E is finite and we have observed the value of C(v) for all possible states of v and most importantly where $C(\cdot)$ is rational.

Definition: Let D be a "dominance relation"; xDy means x "dominates" y in the sense of Von Neumann and Morgenstern. We define $xUy \iff \neg yDx$; xUy means x is undominated by y.

*The difficult operational problems posed by the possibility that $C(\cdot,\cdot)$ contains more than one element, will not be addressed.



Theorem 7: If $C(v) \neq \phi$ for all $v \subset E$, E is finite and there exists a total, reflexive binary relation, R, such that $(\forall v)_{v \subset E} C(v) = \{x \in v : xRy$ for all $y \in v\}$ then there exists an asymmetric binary relation D, over E, such that $(\forall v)_{v \subset E}[\{x, y \in C(v) \Longrightarrow [\neg xDy \,\&\, \neg yDx\}]$ and $\{y \in v \setminus C(v) \Longrightarrow (\exists x) x \in C(v) : xDy\}]$ if and only if R is quasi-transitive.

Proof. Suppose R is quasi-transitive. Then let V = U and by previous theorem 4, we know $\langle \beta, C(\cdot) \rangle$ satisfies SP(U). That is, we know

$$(\forall v)_{v \subset E}(\forall x)_{x \in v}[x \in C(v) \Longleftrightarrow (\forall y)_{y \in C(v)} \neg yDx]$$

which can be seen to be the desired property.

Now suppose $C(\cdot)$ is rational and that D is an asymmetric binary relation such that $\langle \beta, C(\cdot) \rangle$ satisfies SP(U). We need to show only that R is quasi-transitive. Assume $x = C(\{x, y\})$ and $y = C(\{y, z\})$. Since SP(U) is satisfied we know $[xDy \,\&\, \neg yDx]$ and $[yDz \,\&\, \neg zDy]$. We also know xPy and yPz which allows us to conclude, from the properties of R, that $x = C(\{x, y, z\})$. This observation yields $[xDz \,\&\, \neg zDx]$ to be the case, using SP(U) and the asymmetry of D. Using SP(U) again we see $x = C(\{x, z\})$ which implies xPz and allows us to conclude that R is quasi-transitive ∎

The fact that $\langle \beta, C(\cdot) \rangle$ was rational played no small role in the proof of the last theorem. This property of a choice function may be present but it does not follow from the nature of Von-Neumann-Morgenstern solutions alone. The following remark makes the point clearly.

Remark: There exists an asymmetric dominance relation, D, and a choice function $C(\cdot)$ such that

i) $C(v) \neq \phi$ for all subsets.

ii) $\langle \beta, C(\cdot) \rangle$ satisfies SP(U).

iii) There does not exist a binary relation, R, such that $(\forall v)_{v \in \beta} C(v) = \{x \in v : xRy \text{ for } y \in v\}$.

Proof. We need only consider an example. Let $E = \{x, y, z\}, \neg xDy, \neg yDx, yDz, \neg zDy, zDx, \neg xDz$; $C(\{x,y\}) = \{x,y\}$, $C(\{y,z\}) = y$, $C(\{x,z\}) = z$, $C(\{x,y,z\}) = \{x,y\}$. The reader can verify that $\langle \beta, C(\cdot)\rangle$ satisfies SP(U). To show $C(\cdot)$ satisfies (iii) simply note that $C(\{x,z\} = z$ implies $\neg xRz$ for any candidate, R, but this contradicts the fact that xRz for any candidate R, which is implied by the fact that $x \in C(\{x,y,z\})$ ∎

We can ask a question similar to the above only rather than demand that $C(\cdot)$ reflect a Von Neumann-Morgenstern solution(s) of an underlying dominance relation we can demand that it reflect the core.

Theorem 8. Let E be a finite set and let $\beta$ be the power set of E. Let $C(\cdot)$ be a choice function such that

i) $C(v) \neq \emptyset$ for $v \in \beta$.

ii) $(\forall v)_{v \in \beta} C(v) = \{x \in v : xRy \text{ for } y \in v\}$ for some total, reflexive R,

then there exists an asymmetric binary relation D such that

iii) $(\forall v)_{v \in \beta} C(v) = \{x \in v : \neg yDx \text{ for all } y \in v\}$. Furthermore,

let D be any asymmetric binary relation such that the choice function induced by (iii) satisfies (i). Then there exists an R satisfying (ii).

Proof. Let D be any asymmetric binary relation and let $C(\cdot)$ satisfy (i) and (iii). Now define $xRy \iff [\neg yDx \text{ or } x = y]$ and we get the statement (ii). Now take an arbitrary R which induces, by (ii) a choice function satisfying (i). Define $xUy \iff xRy$ and note that (ii) becomes (iii). The requisite relation D is simply $xDy \iff \neg yUx$ ∎



In passing we should note that the hypothesis (i) used in this theorem is not without considerable strength. In effect, it demands that the core, from any subset is nonempty. This requirement is clearly not met by all dominance relations. The theorem then, specifies that exactly those dominance relations whose negations form suborders [3 ]. have the property.

We can state one final theorem which deals with the case where the choice (or outcome) over any set is but a single element. This is a case where, from one point of view, all of these ideas become the same idea. We state, without proof, the following theorem and corollary.

Theorem 9. Let E be finite and let $\beta$ be the power set of E. Suppose for all $v \in \beta$, C(v) contains one and only one element. If $\langle \beta, C(\cdot)\rangle$ satisfies any one of KP(V), SP(V), CP(V) or the weak axiom of revealed preference then it satisfies them all.

Corollary. Let E be finite and let $\beta$ be the power set of E. Suppose for all $v \in \beta$, C(v) contains one and only one element. Let Q be any total, antisymmetric binary relation on E. If $C(\cdot)$ satisfies any of KP(Q), CP(Q) or SP(Q) then it satisfies them all, and, in addition, Q = V, Q is an order and $C(\cdot)$ satisfies the weak axiom of revealed preference.

## SUMMARY

From a very broad point of view we are pursuing the possibility of making inferences from the behavior of a process about the underlying institutions which define the process. Suppose we are observing the outcomes of a process and we are willing to assume that the process can properly be described as a cooperative game. Can we infer, from the process behavior, what the underlying dominance relation must be? Can we identify a "revealed" dominance relation? From the dominance relation we might then be able to infer something about the institutions we would expect to find upon closer examination.

Answers to this question, for the finite case, are partially provided by theorems 7, 8 and 9. Theorem 7 shows that a rational choice function can be viewed as a unique solution of a game if and only if it is quasi-transitive rational. Theorem 8 shows a choice function can be a core of a game if and only if it is rational. Theorem 9 and its corollary show that a choice function which chooses but a single element reflects some underlying game if and only if it is consistent with a strong ordering in the revealed preference sense.

The remaining theorems (1 thru 6) serve to connect game solution axioms with other axioms which have been evolving in the revealed preference literature. They show that KP(V), SP(V) and CP(V) are, under certain conditions, equivalent to rational choice, quasi-transitive rational choice and transitive rational choice, respectively. Under the same conditions each of these "degrees" of rational choice has been characterized in terms of both path independence type axioms and revealed preference axioms [4], [5], [6], [7]. These theorems, then, serve to establish equivalence between concepts of "solutions," concepts of "revealed preference," and concepts of "rational choice."



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