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RATIONALITY AND RELEVANCE IN SOCIAL CHOICE THEORY

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Presented at TIMS Meeting.

SOCIAL SCIENCE WORKING PAPER 5

August 1971

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The central argument of this paper is that concepts such as "social preference," "social rationality," "public interest," "social benefits" and "social welfare" are unnecessary for the development and application of welfare economics principles and the design and/or modification of political economic processes. The primary reasons for using these constructions as offered by Samuelson and Arrow are misleading if not simply wrong. The features of the concepts which make their use compelling, are also features of other approaches to problems. Furthermore, since the tools themselves automatically restrict analysis to a rather "uninteresting" family of political-economic processes, their use may even be detrimental to the development of a relevant body of theory.

The fundamental ideas presented here have been offered in more precise form elsewhere [9], [10]. But, the overriding issues are somewhat obscured there, by the detail. The broader, somewhat more philosophical implications discussed here, are, I suspect, of greater interest.

*Presented at the Seminar on Mathematical Theory of Collective Decisions, Harbor Town, South Carolina, August, 1971. The author wishes to thank Robert P. Parks for his comments on an early draft.

The arguments of the paper will proceed as follows. The first section is devoted to a discussion about the relationship between rational social choice, social preference, social benefits, welfare, etc. From the point of view of social choice theory they are all the same concept. Those readers who are impatient with "formalism" are urged to stay awake for the first section. The failure, in the literature, to make certain distinctions precise, has resulted in some rather wide-spread misunderstandings.

Having provided the reader with an understanding of the concepts the discussion moves to the two major justifications for the use of these constructions. The Samuelson argument is, roughly, that the employment of the social welfare function is simply a reflection of a requirement that social choices conform to some system of ethics. The criticism of this argument offered here is that systems of ethics, which are capable of being represented by such functions, are but a small, uninteresting subset of ethics. Scholarly preoccupation with this subset of ethical propositions has caused the scientific implications of other ethical propositions, to be completely neglected.

The second major argument in support of the use of social preference relations has been advanced by Arrow. His argument is, roughly, that the existence of a social preference relation is necessary in order for the outcome of social choice processes to be independent of the path (or sequence) of choice. The third section of the paper is used to demonstrate that such a claim is, in general, simply wrong.

The final section is a summary. It is argued there that Arrow's General Possibility Theorem is actually a criticism of the use of the above tools. Other approaches can both retain the advantages and circumvent the disadvantages.

SECTION ONE

The section proceeds as follows. First some notation and interpretations are provided for those unfamiliar with the social choice literature. The discussion then moves to the topic of "rational" social choice. Some examples of errors are given in order to clarify the precise meaning. The concept is connected, from the point of view of model building, to "social welfare," "social benefits," etc. I argue that, from this point of view the concepts are the same (they all imply rational social choice in any case, and also vice versa in the finite case). The discussion then jumps to a characterization of ethical propositions where it is argued that only a limited class of ethics would, if imposed, require choice to be rational.

Social Choice

In order to keep the discussion brief, the interpretations will be restricted to those of economic models. The following concepts will be used in notational form:

An alternative or social state will be denoted by the lower case letters $\{x, y, z, \ldots\}$. In the case of an exchange economy model an alternative would be a matrix, each row of which would indicate a given individual's consumption level of the various commodities. If the model is complicated by production, time, random elements, etc., the symbols and concepts are intended to hold there as well.

The set of conceivable alternatives will be designated as E. This is a universal set, such as a consumption set in (Debreu type) economic models. Sometimes such sets are referred to as a "commodity space."

An agenda is a subset of E and unless some particular subset is of interest, will be denoted as v. The agenda can be viewed as a production possibilities set or consumption possibilities set or activities possibilities set, depending upon the model. As distinguished from E, the agenda refers to that subset of E the existence of which would violate no laws of nature, resource endowments, etc.

The set of admissible agenda is simply a family of agenda. The symbol will be $\mathcal{U} = \{v_1, v_2, \ldots, v_m\}$, when no particular family needs to be isolated. An admissible agenda, then, is a set of subsets of E. It could be a family of consumption possibilities sets, production possibilities sets, etc., depending upon the interpretation of the social states and agenda. Social processes must operate under varying circumstances and one of the things which varies is the agenda. The admissible agenda is a set which designates the range over which the agenda might vary. Of course, which subsets of E, which agenda, are in the admissible agenda is intended to be dictated by the particular problem, or application, at hand. For example, in some cases $\mathcal{P}(E)$ would be a natural family.[1]

A society is simply a vector $(R_1, R_2, \ldots, R_m)$ where R_i is the preference relation[2] for individual i and where there are n members of society. A set, D, of societies will indicate the set of admissible societies.

A social choice function is a function with domain[3] $\mathcal{U} \otimes D$, and range $\mathcal{P}(E)$ such that

$$\forall (v, R_1, \ldots, R_n) \quad C(v, R_1, \ldots, R_n) \subset v .$$
$$(v, R_1, \ldots, R_n) \in \mathcal{U} \otimes D$$

[1] The symbol $\mathcal{P}(E)$ indicates the power set of E -- the set of all subsets of E. In the case of an infinite E the set of all subsets may be "too large" to be of interest. For example, choice over "open" sets or non convex sets, may be too much to demand. Again these technical complications will be overlooked here so that the general problem can be discussed.

[2] We assume individual preference relations are total, reflexive, transitive binary relations on E. More precisely, a binary relation R, on a set E is said to be

a) total in case $\underset{x \in E}{(\forall x)} \quad \underset{\substack{y \neq x \\ y \in E}}{(\forall y)} \quad [xRy \vee yRx]$

b) reflexive in case $\underset{x \in E}{(\forall x)} \quad xRx$

c) transitive in case $\underset{x \in E}{(\forall x)} \quad \underset{\substack{y \in E \\ y \neq x}}{(\forall y)} \quad \underset{\substack{z \in E \\ z \neq y \\ z \neq x}}{(\forall z)} \quad [xRy \wedge yRz \Rightarrow xRz]$

[3] The symbol $\otimes$ represents a cartesian product. Certainly, for some purposes one might want to restrict the analysis to subsets of $\mathcal{U} \otimes D$. This possibility will not be pursued here.

That is, for any particular agenda in the admissible set and any particular society in the admissible set, the "chosen" set is a subset of that particular agenda.

The idea of a social choice function will be used for two separate ideas in this paper. In some places the functional notation will be used to represent an ethic. In other places the functional notation will be used to represent a process.[4] The distinctions will be made clear below, but I wish to dwell for a moment, now, on the "process" interpretation.

By 'represent some process' I mean that the "chosen" elements are the equilibriums of the process. Each process has its own representation. For example, we could let the function $C^{C.E.}(v, R_1, \ldots, R_n)$ represent the competitive equilibrium allocations when v is the consumption possibilities set and $(R_1, \ldots, R_n)$ are the individuals' preferences. As v and/or individual tastes change, the function simply indicates the new equilibrium set. Different choice functions would indicate different processes. More importantly, however, we will say that two processes are the same in case they have the same choice function -- that is, two processes are the same in case they always have the same equilibriums. One purpose for making this distinction is that "imagined" or "abstract" processes can be postulated and then one can ask whether or not such processes can

[4] In the last section a distinction will be made between "implementable" processes and "non-implementable" processes. The difference is between choice functions which could be representations of actual processes and those which cannot.

actually be constructed.

Rational Social Choice

A choice function is rational in case it can be rationalized. If a binary relation exists such that the chosen elements can always be viewed as the maximal elements, according to this binary relation, then the choice function is rational. Notice first that the concept of rational refers to a sequence of choices -- to the behavior of the choice function over part of its domain rather than its behavior at a single point. Notice also that the "chooser" need not be "purposeful" or "cognitive." In fact, a given choice function, if rational, may be rationalized by several different binary relations simultaneously. Furthermore, a choice function may be rationalized by a binary relation which does not have the "usual" properties of "preference relations."[5]

In the case of social choice we proceed as follows. A social choice function is said to be rational in case, for given and fixed individual preferences, it chooses from the various agenda as if it used a binary relation, R, as a "criterion" and chose the "best" (the R-maximal elements[6]) accordingly. More formally, a social choice function is said to be rational over the domain $\mathcal{U} \otimes D$ in case for each $(\overline{R}_1, \ldots, \overline{R}_n) \in D$, there is a binary relation R such that

$$(\forall v)_{v \in \mathcal{U}} \quad C(v, \overline{R}_1, \ldots, \overline{R}_n) = \{x \varepsilon v : xRy \text{ for all } y \in v\}.$$

The social choice function is said to be total, reflexive, transitive rational in case, the social choice is rational and for each vector $(\overline{R}_1, \ldots, \overline{R}_n)$ the rationalization R is total, reflexive, transitive.

The following three examples serve to illustrate the technical meaning. In the example the notation "$h(\cdot)$" is used in place of $C(\cdot)$ because the latter has been given a rather precise definition. Notice, in these examples the individuals' preferences have been dropped as parameters. We are free to do this since they are held constant in the definition of rationality.

Example 1. The choice function $h(\{a,b\}) = \{a,b\}$, $h(\{a,c\}) = \{a,c\}$, $h(\{a,b,c\}) = \{b\}$, is not rational. Why? From the first two choices we get aRb & aRc as a property of any rationalization R. The "a" is R-maximal relative to $\{a,b,c\}$ for any rationalization which exists. So, if the choice is rational, that is, if a rationalization exists, the element "a" must be among those chosen from the set $\{a,b,c\}$. We see by construction $a \notin h(\{a,b,c\})$ so no rationalization exists and the choice

[5] The "indifference relation" may not be transitive or the "strong preference" relation may not be transitive, or it is not total, etc. In general one can view these properties as various "degrees" of rationality. Conditions on the choice function which guarantee various "degrees" of rationality have been developed [2], [5], [7], [9], [11].

From one point of view the theory of preference is simply an assertion that choice can be rationalized by some binary relation with a particular set of properties (total, reflexive and transitive). That is, the theory asserts that the "choosing agent," whatever it is, "chooses" as if it had a preference relation with the usual properties and in every situation it chooses those alternatives which are "most preferred."

[6] See the glossary.

is not rational.

Example 2. The choice function $h(\{a,b,c\}) = \{a\}$ is total, reflexive, transitive rational. We need consider only a few of the many rationalizations, say, aPb & aPc & bPc; or, aPb & aPc & cPb; or, aPb & aPc & bIc, where P and I are the usual "strong preference" and "indifference" relations.[7] We note, from this example, that rationalizations need not be unique.[8]

Example 3. The choice function $h(\{a,b\}) = \{a,b\}$, $h(\{a,c\}) = \{a\}$, $h(\{b,c\}) = \{b,c\}$, $h(\{a,b,c\}) = \{a,b\}$, is total, reflexive rational but not transitive rational. The rationalization is aIb, bIc, aPc. Notice that the binary relation I is not transitive and thus, neither is R. However, $h(\ v\ )$ is for each v, the set of R-maximal elements.

In order to explore further the technical meaning of rational choice we turn now to a rather important misunderstanding in the literature. The misunderstanding is over which of Arrow's conditions the method of "rank-order voting" violates. Arrow suggests, incorrectly, that this process violates his Independence of Irrelevant Alternatives Condition. Actually, it satisfies that condition but does violate the "rationality" conditions.

The following axiom is essentially a quote of Arrow's definition of

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[7] See the Glossary.

[8] In the case of this example we have a severely restricted admissible agenda -- $\mathcal{U} = \{\{a,b,c\}\}$. If $\mathcal{U}$ contains all two element sets then any reflexive rationalization will be unique.

Independence of Irrelevant Alternatives, with slight changes made to keep the notation uniform with this paper.[9] His discussion is then quoted.

Condition 3. Let $(R_1, \ldots, R_n)$ and $(R_1', \ldots, R_n')$ be two sets of individual orderings and let $C(S, R_1, \ldots, R_n)$ and $C(S, R_1', \ldots, R_n')$ be the corresponding social choices. If, for all individuals i, and all x and all y in a given agenda S, xR_iy if and only if $xR_i'y$, then $C(S, R_1, \ldots, R_n) = C(S, R_1', \ldots, R_n')$.

> The reasonableness of this condition can be seen by consideration of the possible results in a method of choice which does not satisfy Condition 3, the rank-order method of voting frequently used in clubs. With a finite number of candidates, let each individual rank all the candidates, i.e., designate his first-choice candidate, second-choice, etc., choices, the higher weight to the higher choice, and then let the candidate with the highest weighted sum of votes be elected. In particular, suppose that there are three voters and four candidates, x, y, z, and w. Let the weights for the first, second, third, and fourth choices be 4, 3, 2, and 1, respectively. Suppose that individuals 1 and 2 rank the candidates in the order x, y, z, and w, while individual 3 ranks them in the order z, w, x, and y. Under the given electoral system, x is chosen. Then, certainly, if y is deleted from the ranks of the

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[9] Arrow's exact statement is:

> Condition 3: Let $R_1, \ldots, R_n$ and $R_1', \ldots, R_n'$ be two sets of individual orderings and let $C(S)$ and $C'(S)$ be the corresponding social choice functions. If, for all individuals i and all x and y in a given environment S, xR_iy if and only if $xR_i'y$, then $C(S)$ and $C'(S)$ are the same (independent of irrelevant alternatives).
>
> [1, p. 27].

The reader may be interested in the comparison of this axiom with the one stated and used by Blau [3]. This reformulation of the Arrow framework is exceptionally interesting since it is the "popularized" version of Arrow's work which tends to obscure the interpretations explored here.

> candidates, the system applied to the remaining candidates should yield the same result, especially since, in this case, y is inferior to x, according to the tastes of every individual; but, if y is in fact deleted, the indicated electoral system would yield a tie between x and z.
>
> [1, p. 27]

Close examination of the axiom reveals that the claim in the quote is simply wrong. The axiom refers to the behavior of the social choice function in cases where S is fixed and the other parameters, individuals' preferences, change. On the other hand the example pertains to a situation where the set S changes (from {w, x, y, z} to {w, x, z}) and individual preferences are fixed. Suppose the agenda is {w, x, z} and the preferences are as initially given. The social choice is {x, z}. Now, suppose individuals 1 and 2 continue to rank w, x and z in the order x, z, w, and suppose that individual 3 ranks them in order z, w, x. The choice over {w, x, z}, using the rank order method, remains {x, z} regardless of how the individuals feel about y. The reader can continue to verify that the rank order method satisfies the axiom, contrary to Arrow's claim.[10]

[10] This example is not the only one in Arrow's writings which has served to detract scholars from a proper understanding of the axiom. Consider the following example which was called to my attention by R.P. Parks.

> Condition 3 is perhaps more doubtful. Suppose that there are just two commodities, bread and wine. A distribution, deemed equitable by all, is arranged, with the wine-lovers getting more wine and less bread than the abstainers. Suppose now that all the wine is destroyed. Are the wine-lovers entitled, because of that fact, to more than an equal share of bread? The answer is, of course, a value judgement. My own feeling is that tastes for unattainable alternatives should have nothing to do with the decision among the attainable ones; desires in conflict with reality are not entitled to consideration, so that Condition 3, reinterpreted in terms of tastes rather than values, is a valid value judgement, to me at least.
>
> [1, p. 73]

Arrow's argument, quoted above, does establish the fact that social choice derived from a rank order voting process is not total, reflexive, transitive rational. If it were then the choice function would satisfy the weak axiom of revealed preference, since we are talking about finite sets [2], [5], [11], and his example shows this axiom is clearly violated. As it turns out, other examples can be used to show that social choice by the rank order voting process is not rational at all -- to any degree.

This example has caused considerable confusion in the literature about the Independence of Irrelevant Alternatives axiom. Almost all criticisms which have been directed at this axiom are actually criticisms of the rationality axioms.[11] As a result the problems with rational choice have remained unrecognized while the possible interpretations of Independence of Irrelevant Alternatives have not been explored.

Actually the Independence of Irrelevant Alternatives axiom can be

Close examination shows, again, that this example, as stated, is unrelated to his independence of irrelevant alternatives condition. The problem is that in this example, as in the rank ordering example he is changing the feasible set. Had he simply required that to him the allocation of a fixed amount of bread should be independent of people's taste for unavailable wine, the example would be well taken.

[11] For example, the following "restatement" of the axiom is hopelessly entwined with the rationality postulates.

> 2. If the removal from or insertion into the set of possibilities of a certain possibility x results in no change in any individual order of the remaining possibilities, then it must cause no change in the collective order of those possibilities. This condition is named the 'independence of irrelevant alternatives.'
>
> [6, p. 422]

viewed as a defining characteristic of those choice functions which can be implemented as processes. The axiom simply says that equilibriums depend only upon preferences over the "feasible" set. This is a property of almost all behavioral models -- including the competitive model. A rather extensive discussion of this point can be found at [10].

Rational Social Choice: Interpretations

The idea of rational social choice is, from a technical point of view, rather pervasive in economic writings. Many tools, if used to forge an economic process or system, would necessarily cause rational choice to be a property of that system. The most important of these tools are social preference relations, social welfare functions and social benefit functions.

Many view the job of a "social engineer" to be one of desinging systems in a manner which assures that the process outcomes are always the "best" according to some "social preference relation." Without inquiring about the source of this "preference relation," which is usually at least a sub-order,[12] we can see immediately a consequence of the procedure. If the equilibriums of a process are viewed as a "choice" and if the process is designed so that the "outcomes" or "equilibriums" are maximal according to some social preference relation then that process chooses rationally when individuals' preferences are fixed and the consumption

[12] A binary relation is a sub-order in case it is total, reflexive and obeys the law $[\sim zRy \wedge \sim yRx \Rightarrow xRz]$. The binary relations xIy, yIz, xPz; and xPy, yPz, xIz are both sub-orders. Such binary relations have the property that every finite set has a "maximal" element. That is, from any v there is at least one $x \in v$ such that xRy for all y in v. [4]

possibility set varies. The rationalization is simply the initial "social preference relation."

We can see immediately that the same argument applies equally to the use of a social welfare function. Such a function is usually of the form $W(x)$ or $W(u^1(x), u^2(x), \ldots, u^n(x))$ where the $u(\cdot)$'s are numerical representations of individual preference relations. It is to be used as follows. The system is to be designed (perhaps including a public servant who always implements the "proper" outcome) so that the outcomes from any consumption possibilities set v, are those which maximize $W(\cdot)$ over v.

We see immediately that processes designed according to the maximization of social welfare are processes which "choose" rationally. In order to see this we simply define a binary relation R, for all x and y in E as xRy if and only if $W(u^1(x), \ldots, u^n(x)) \geq W(u^1(y), \ldots, u^n(y))$. We see then, for fixed individual preferences the outcomes from any v are the maximal elements of R in v. Hence the process is rational since R is the rationalization. One can then view $W(\cdot)$ as simply a numerical representation of R.

A procedure of altering processes in accord with certain types of cost-benefit methods will also induce rationality into social choice. Let $B(y,x)$ be the "net benefits" of state x over state y. If $B(y,x) > 0$ we say the "net benefits" of moving from y to x are positive and if $B(y,x) < 0$ we say they are negative. The design of systems should proceed by assuring that the outcome, from any agenda v are the allocations in v from which it is not "beneficial" to move. That is, the outcomes, over v, would be $y_o \in v$ such that $B(y_o, x) \leq 0$ for all $x \in v$. Now if $B(y,z) \equiv B(y,x) + B(x,z)$ there is a function $W(\cdot)$ such that

$B(y,x) = W(x) - W(y)$.[13] We need only note that the chosen alternatives from any set would be those which maximize $W(\cdot)$ -- this function is, in essence, a social welfare function. We can then, from $W(\cdot)$ deduce the implied preference relation following the procedure above. The process will choose rationally.[14]

We see then that if social choice reflects the use of a social preference relation, a social welfare function or a "benefits" function, then choice will be rational. A natural question is, why on earth would we want to use tools which place such a restriction on a model? Surely, "social preferences," "benefits" or "welfare" are not "experienced" in any meaningful sense. Few claim to "see" or "measure" social welfare. In fact very few claim that these objects are even amenable to such realizations. The two major arguments, one due to Samuelson and the other due to Arrow, are discussed below.

SECTION TWO

Throughout the development of economics, as well as the social sciences in general, there has been a tendency for scholars to describe events and activities by using terms which are, essentially, subjective. Political-economic events are often described in terms which are fully understood

[13] Where finite sets are involved, things get somewhat more complicated. See [13] for a discussion of addative representations of relations.

[14] There are ways of stating the "cost-benefit" principle which do not imply rational choice. For example $B(y,x)$ may be defined in a manner which depends upon v. The "costs" of producing x over y may depend upon whether or not it was possible to produce z. In this case the choice would not necessarily be rational.

only to the user. Equality, fairness, efficient, democratic, just, public interest, and social values are just a few. Occasionally such terms can be defined abstractly but their use in common parlance and in "scientific" writing is seldom accompanied by definition. The scientific development of a subject can be equated to the ability of "scientists" to communicate, in writing, in a manner which allows independent verification of events (cookbook procedures). This cannot be done when the terms are subjective.

This problem of subjective terms appears to be recognized at several points in the development of economics. It is probably the reason that "interpersonal comparisons of utility" is considered to be an "unscientific" procedure. The idea of another person's utility is not, even in principle, a sufficiently isolated "object of experience" to be used free of "researcher bias." On the other hand the idea of "preference" with its foundations in choice behavior is not complicated by such problems -- at the conceptual level at least.

This problem of subjective terms might also be the reason why <u>appraisals</u> or explications of events in terms of <u>appraisals</u> tend to be purged from the body of scientific procedures.[15] Such procedures allow discoveries to be known only to the discoverer, since, by definition, transmission of results cannot be achieved even at the conceptual level, in a manner which allows independent verification.

It is along these lines that the Bergson-Samuelson social welfare

[15] Notice that the word <u>appraisal</u> (perhaps <u>approval</u> would serve as well) is used rather than the term "value judgment" or the term "norm." The reason is that I find the latter terms somewhat hopelessly entangled with the concept of "preference." I would like to be free to discuss "value judgments" without appraising them. More importantly, however, I wish to make a clear distinction between "ethics" and "preferences" below.

function serves as an important bridge from the art of economics to the science. The tool allowed the economic practitioner to serve, for the first time, an engineering function. For the first time "ends" could be separated from "means." "Ends" could be stated in a manner which allowed systems to be designed in accord with those ends, by individuals other than the one(s) stating the "ends." If given the ends (the ends maximize a particular function), anyone could, independently, check to see if they were attained by a given process.

Now, the contribution outlined above should not be confused with problems pertaining to the source of the social welfare function. The economist is told to have this function maximized -- not "social welfare" without the latter being specified. The source of the function is an important but different issue. Unfortunately debate frequently finds the two problems confused so a recognition of the substantial contribution of the Bergson-Samuelson construction is lost.

The advantage of the tool (if not the reason for the advantage) is clearly stated by Samuelson.

> It is a legitimate exercise of economic analysis to examine the consequences of various value judgments, whether or not they are shared by the theorist, just as the study of comparative ethics is itself a science like any other branch of anthropology. If it is appropriate for the economist to analyze the way Robinson Crusoe directs production so as to maximize his (curious) preferences, the economist does not thereby commit himself to those tastes or inquire concerning the manner in which they were or ought to have been formed. No more does the astronomer, who enunciates the principle that the paths of planets are such as to minimize certain integrals, care whether or not these should be minimized; neither for all we know do the stars care.
>
> [14, p. 220]

He continues, however, along lines I believe to be misleading. This juncture is most important since it is along the intuitive lines, directed

by the following quotes, that welfare economics has proceeded.

> Without inquiring into its origins, we take as a starting point for our discussion a function of all the economic magnitudes of a system which is supposed to characterize some ethical belief -- that of a benevolent despot, or a complete egotist, or "all men of good will," a misanthrope, the state, race, or group mind, God, etc. Any possible opinion is admissible, including my own, although it is best in the first instance, in view of human frailty where one's own beliefs are involved, to omit the latter. We only require that the belief be such as to admit of an unequivocal answer as to whether one configuration of the economic system is "better" or "worse" than any other or "indifferent," and that these relationships are transitive; i.e., A better than B, B better than C, implies A better than C, etc. The function need only be ordinally defined, and it may or may not be convenient to work with (any) one cardinal index or indicator. There is no need to assume any particular curvature of the loci (in hyper-space) of indifference of this function.
>
> [14, p. 221]

The new welfare economics is characterized as

> . . . a systematic way of introducing from outside of economics various ethical norms (as embodied technically in what is called a social welfare function) -- and so ordering the exposition of the conditions for an optimum that we first state these which require only the weakest postulates, and which therefore hold for the widest possible set of cases, and only later introduce the narrower and more restrictive hypotheses.
>
> [15, p. 37]

The argument is repeated.

> Without norms, normative statements are impossible. At some point welfare economics must introduce ethical welfare functions from outside of economics. Which set of ends is relevant is decidedly not a scientific question of economics. This should dispel the notion that by a social welfare function is meant some one, unique, and privileged set of ends. Any prescribed set of ends is grist for the economist's unpretentious deductive mill, and often he can be expected to reveal that the prescribed ends are incomplete and inconsistent. The social welfare function is a concept as broad and empty as language itself -- and as necessary. Whether we call it W, or G, or describe it in words is, of course, immaterial.
>
> [15, pp. 37-38]

The key word in the above quote is the word "must" used in the second sentence. The implication of this statement and the other two quotes above is that the use of ethical postulates necessitates the use of a function. This implication is repeated in his discussion of the closely related rationality postulates in the Arrow model. "Give up Axiom 1 -- a well-behaved ordering with transitivity -- and the whole problem vanishes into thin air." [16, p. 49]

His arguments involve two separate propositions. 1) The design of economic systems along ethical lines presupposes an objectively stated set of ethics. 2) Ethical propositions in economic models can only be reflected in total, reflexive, transitive binary relations. I have no quarrel with the first, but I do disagree with the second. His assumption, as will be established below, places an arbitrary restriction on the class of ethical propositions with which economists can work.

Ethical Propositions

Ethical propositions are reflections of the primitive "should." You should not steal. You should not kill. You should treat others as you would like to be treated, etc. This idea of what you "should" do, may, or may not be consistent with some idea of "good," "better," or "best."

In short, an ethic[16] itself is in general represented by a choice function as opposed to a binary relation. From any set of alternatives it

[16] In [10] I make a distinction between "absolute" ethics and "relative" ethics. A relative ethic is a family of absolute ethics -- it is an ethic which requires actions to accord with at least one of a set of one or more absolute ethics. Several ethical propositions in economics are of this form.

(the ethic) indicates which alternatives should, or should not, be among the chosen elements. For example when faced with a set of alternatives, all alternatives involving the action "kill" should be in the set not chosen. Consider a version of the Pareto principle -- when faced with a set of opportunities, the social outcomes should not contain Pareto dominated alternatives. A little different version could state -- from any set of alternatives--all Pareto undominated alternatives should be among the outcomes.

Abstractly, let v be a set of alternatives, an agenda, let $S(v, R_1, \ldots, R_n) \subset v$ be the set which "should" be chosen from v, according to some given ethic when the elements of v are available and preferences are given as $R_1, \ldots, R_n$. Let $\overline{S}(v, R_1, \ldots, R_n)$ be the set which "should not" be chosen from v, according to some other ethic. We say that the functions $S(\ldots)$ and $\overline{S}(\ldots)$ are, respectively, positive and negative ethics (or "representations" of ethics). Notice that individual preferences are included as parameters since "ethics" are frequently involved with attitudes. Of course we could have included properties of "social states," other than how people feel about them, as parameters. Notice also that an ethic is of the same mathematical form as social choice functions. We assert that any ethic can be represented by a choice function and any choice function can be viewed as a representation of some ethic.[17]

[17] We can say the systems of ethics $S^1(v, R_1, \ldots, R_n)$, $S^2(v, R_1, \ldots, R_n)$, $\ldots$, $S^n(v, R_1, \ldots, R_n)$ and $\overline{S}(v, R_1, \ldots, R_n)$, $\ldots$, $\overline{S}^n(v, R_1, \ldots, R_n)$ imply the ethics

$$S(v, R_1, \ldots, R_n) \equiv \bigcup_{i=1}^{n} S^i(v, R_1, \ldots, R_n) \text{ and}$$

$$\overline{S}(v, R_1, \ldots, R_n) \equiv \bigcup_{i=1}^{n} \overline{S}^i(v, R_1, \ldots, R_n), \text{ respectively.}$$

So, a system of ethics is itself an ethic.

What is the relationship between ethics and the word "better" used by Samuelson? While ethics, or applications of the word "should," have in general a choice function representation, the idea of "better" is, inherently, only a binary relation. Formally we would say, where xGy means "x is at least as good as y," that x is "better" than y in case xGy and not yGx ; and, that x is "best," relative to some set v, in case x is G-maximal in v. We can say an ethic is consistent with some idea of "better" in case there exists a binary relation (an "at least as good as" relation) such that the chosen elements (parameters other than the feasible set v, being fixed) are "best" according to this relation. That is, an ethic is consistent with the word "better" or the phrase "at least as good as" in case the choice function representation of the ethic is rational. Furthermore, we would say that the idea of "at least as good as" employed is transitive in case the representation of the ethic is transitive rational.[18]

We can now make a very important observation. Not all ethics have rational representations. This means that, if we are to interpret a rationalization of a choice function (representing some idea of "should") as an "at least as good as" relation (that is, xGy means, when G is the rationalization, that x is at least as good as y) then not all ethics are consistent with the idea "better" or "worse" or "best" -- at least to the extent that "better" is a binary relation and "best" are the maximal elements. We need only observe that the set of "rational" choice functions (over some specified range) is but a small subset of the set of all choice

[18] We also suggest that a consistent use of the word "value," at least within political economic models, is as a numerical representation of an "at least as good as" relation.

functions.

The first assertion of this paper is now established. Use of the Bergson-Samuelson social welfare function places arbitrary restrictions on the set of admissible ethics. Why should analysis be restricted to ethics whose representations are rational (total, reflexive, transitive rational at that)? No answer has been supplied. In fact, many of the ethics actually stated in the economics literature are not rational [10].

The second major argument of this paper is that the family of ethics compatible with a Bergson-Samuelson social welfare function is, from one important point of view, very "uninteresting." This argument will be taken up in the final section with Arrow's General Possibilities Theorem.

We turn our attention now to the major advantages provided by the Bergson-Samuelson social welfare function. The advantages are derived from the fact that the function could be viewed as a representation of a system of ethics, in "objective" terms. The advantages remain if the function is a choice function as opposed to a real valued function or binary relation.[19] The welfare function has, then, no special advantages.

The design of systems can proceed as before -- only without the social welfare function. Let $S(v, R_1, \ldots, R_n)$ be a positive ethic and $\bar{S}(v, R_1, \ldots, R_n)$ a negative ethic. We say the process represented by the social choice function $C(v, R_1, \ldots, R_n)$, is compatible with $S(v, R_1, \ldots, R_n)$ in case, for all admissible $(v, R_1, \ldots, R_n)$, $S(v, R_1, \ldots, R_n) \subset C(v, R_1, \ldots R_n)$.

[19] While there is, in principle no problem of transmitting the concept there may be practical problems in cases of large numbers of alternatives. See the discussion in [10] on "goals."

We say $C(v, R_1, \ldots, R_n)$ is compatible with $\overline{S}(v, R_1, \ldots, R_n)$ in case, for all admissible $(v, R_1, \ldots, R_n)$, $C(v, R_1, \ldots, R_n) \cap \overline{S}(v, R_1, \ldots, R_n) = \emptyset$, where $\emptyset$ is the empty set. We say "social choice" is compatible with an ethic in case those elements which "should" be chosen are always chosen and those elements which "should not" be chosen are never chosen. Whether or not the chosen elements can be viewed as "best" is another, independent and perhaps irrelevant matter.

The Arrow Argument

At least three different, but distinct lines of argument can be found in Arrow's work, in favor of what has been called his "consistency conditions." These conditions simply assert that social choice must be rationalized by a total, reflexive, transitive binary relation. The rationalization is called the "social preference" relation. Precisely why one would make such an assumption weighs heavily on the interpretation of Arrow's major result.

The first type of justification given is essentially the same as that given by Samuelson. That is, the "social preference relation" is simply a reflexion of some system of ethics. The second justification is that the existential quantification or definition of something called a "social preference" is attempted.

These first two justifications are often confusingly intertwined in his arguments. Part of the problem is a failure to distinguish between value judgments, ethical propositions, preferences, tastes and values. The following quote seems to contain a bit of everything.

> . . . Given these basic value judgments as to the mode of aggregating individual desires, the economist should investigate those mechanisms for social choice which satisfy the value judgments and should check their consequences to see if still other value judgments might be violated. In particular he should ask the question whether or not the value judgments are consistent with each other, i.e., do there exist any mechanisms of social choice which will in fact satisfy the value judgments made? For example, in the voting paradox discussed above, if the method of majority choice is regarded as itself a value judgment, then we are forced to the conclusion that the value judgment in question, applied to the particular situation indicated, is self-contradictory.
>
> [1, pp. 4-5]

The first two sentences indicate that he clearly intends social choices to ultimately conform to some given set of ethics ("value judgments" to him). That is, one function of an applied welfare economist is to provide the design of an economic system along some externally stated ethical lines. The third sentence then declares that an ethical proposition which fails to dictate rational choice is "self-contradictory." From this we would conclude that the representation of an ethical proposition must be a transitive binary relation. This is the same formulation used by Samuelson and examined above.

The next quote is also of interest. Here, as a purpose of the research, he appears to seek a "reasonable" definition of a social preference relation. By virtue of being a "preference" it must have the properties (namely his consistency conditions) or preferences.

> In ideal dictatorship there is but one will involved in choice, in an ideal society ruled by convention there is the divine will or perhaps, by assumption, a common will of all individuals concerning social decisions, so in either case no conflict of individual wills is involved. The methods of voting and the market, on the other hand, are methods of amalgamating the tastes of many individuals in the making of social choices. The methods of dictatorship are, or can be rational in the sense that any individual can be rational in his choices. Can such consistency be attributed to collective modes of choice, where the wills of many people are involved?
>
> [1, p. 2]

This interpretation of the rationality postulate, along with the one above, I consider to be, for practical purposes, the same as the Bergson-Samuelson interpretation. The third justification Arrow gives is considerably different.

> It is against this background that the importance of the transitivity condition becomes clear. Those familiar with the integrability controversy in the field of consumer's demand theory will observe that the basic problem is the same: the independence of the final choice from the path to it. Transitivity will insure this independence; from any environment, there will be a chosen alternative, and, in the absence of a deadlock, no place for the historically given alternative to be chosen by default.
> That an intransitive social choice mechanism may as a matter of observed fact produce decisions that are clearly unsatisfactory has been brought out in different ways by Riker and by Dahl. Riker's emphasis is on the possibility that legislative rules may lead to choice of a proposal opposed by a majority, Dahl's rather on the possibility that the rules lead to a deadlock and therefore a socially undesired inaction. The notion of a "democratic paralysis," a failure to act due not to a desire for inaction but an inability to agree on the proper action, seems to me to deserve much further empirical, as well as theoretical, study.
> Collective rationality in the social choice mechanism is not then merely an illegitimate transfer from the individual to society, but an important attribute of a genuinely democratic system capable of full adaptation to varying environments.
>
> [1, p. 120, emphasis added]

The striking thing about this rather eloquent argument is that it is followed by neither elaboration nor clarification even though this type of justification is alluded to at several points in Arrow's writings. Several formalizations are possible. Only one will be followed here.

First, consider the idea of a rule for tabulating social choices over "small" sets. For example, majority rule is a rule for tabulating choices over two element sets. Now given such a rule, defined over a family of "small" sets,[20] social choice over "large sets" is accomplished by "repeated application" of the rule.

[20] The family, of course, could be viewed as a bases. In addition "status quo" concept may extend the "bases." See [10] for a method of introducing the concept into the analysis.

Consider majority rule for example. Let $M(\{x, y\}, R_1, \ldots, R_n)$ indicate the "majority winner" between x and y when preferences $(R_1, \ldots, R_n)$ are given. Suppose further that people vote in accord with their preference, that is, preferences are also decision rules,[21] the number of people is an "odd number" and indifference is not a property of preferences. Under these conditions there is always but a single element in the set $M(\{x, y\}, R_1, \ldots, R_n)$.

Suppose now, "society" is faced with a choice from a three element set. We wish to construct a function (or process), $C(\{x, y, z\}, R_1, \ldots, R_n)$, which indicates the "outcome" or "social choice" from the three element set. Since the "rule" can act only on two element sets it cannot be applied directly. We can, however, use the rule to define or derive an outcome for the three element set. That is we can use certain procedures to extend choice from two element sets to choice over larger sets.

This extension could be achieved in several different ways, by repeated application of the "rule." For example, we could define $C(\{x, y, z\}, R_1, \ldots, R_n)$ as follows:

$$C(\{x, y, z\}, R_1, \ldots, R_n) \equiv M(\{z, M(\{x, y\}, R_1, \ldots, R_n)\}, R_1, \ldots, R_n).$$

In words, choice over the set $\{x, y, z\}$ is defined by first taking the winner of a majority ballot between x and y and placing it against z. What about the four element set $\{w, x, y, z\}$? One way (there are others) would be to place the winner from the set $\{w, x\}$ against the winner of the set

[21] This assumption is not in general necessary. It could be achieved axiomatically by an application of Arrow's Independence of Irrelevant Alternatives axiom taken together with Pareto Optimality but, for purposes here, there is no reason to take that route.

$\{y, z\}$. The winner of winners would then be the "choice." That is, we would have

$$C(\{w, x, y, z\}, R_1, \ldots, R_n) \equiv M(\{M(\{w, x\}, R_1, \ldots, R_n), M(\{y, z\}, R_1, \ldots, R_n)\}, R_1, \ldots, R_n).$$

Notice that choice over the four element set is <u>derived</u> from choice over two element sets.

Those familiar with the voter's paradox may already have detected a problem. The outcome, or social choice, from $\{x, y, z\}$ and from $\{w, x, y, z\}$, in the case of the voter's paradox depends critically upon the "sequence" or, more broadly, "path," of choice over the two element sets. Had the winner from the set $\{x, z\}$ been put against y, the social choice from $\{x, y, z\}$ would have been different. Or, in the case of the four element set $\{w, x, y, z\}$ the choice would have been different had winners from $\{w, y\}$ and $\{x, z\}$ been pitted together.[22]

I take it that this dependence upon the "path" of choice is what Arrow finds objectionable. Of course, since the motivation is not precisely stated it is impossible to specify his argument exactly. At a broad level

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[22] In the case of majority rule we might want to define the social choice as the set of elements which could win through <u>some</u> sequence or path. That is, we may construct a process for which voters voted according to their preferences using some format of majority rule contests. However, any format may arise. There may be no "laws" governing this, so the "outcomes" or "equilibriums" -- before the fact -- would be the set of elements which could win by <u>some</u> format. In the three element case, for particular configurations of preferences, the choice would be all three elements.

The surprising thing about this process is that it violates Pareto Optimality. Consider the four element set $\{w, x, y, z\}$ and let the preferences of five individuals be: w, y, x, z for the first two; y, x, z, w for number three; x, z, w, y for number 4; and x, y, z, w for number five. Now suppose the path of voting is that the winner of $\{x, y\}$ is put against w. The winner of the second vote is then put against z. The outcome, along this path, is z. But a check of the preference relations reveals that x is unanimously preferred over z. Thus, z is chosen even though Pareto-dominated. The process does not satisfy the Pareto principle.

he demands that social choice be free from the arbitrary parameter "path."[23] He argues that the rationality postulates or "consistency conditions" are necessitated by this demand.

No such necessity is implied at all. Consider the example $C(\alpha) = \alpha$ for $\alpha \in [\{x\}, \{y\}, \{z\}, \{x,y\}, \{x,z\}, \{y,z\}]$ and $C(\{x, y, z\}) = \{x, z\}$. Without specifying the "rule" in mechanized terms or the "sets" over which it operates, we can see that this choice function satisfies Independence of Path. This claim can be seen at an intuitive level from the following observations.

$$C(\{C(\{x\}) \cup C(\{y,z\})\}) = C(\{\{x\} \cup \{y,z\}\}) = C(\{x,y,z\}) = \{x,z\}$$
$$C(\{C(\{y\}) \cup C(\{x,z\})\}) = C(\{\{y\} \cup \{x,z\}\}) = C(\{x,y,z\}) = \{x,z\}$$
$$C(\{C(\{z\}) \cup C(\{x,y\})\}) = C(\{\{z\} \cup \{x,y\}\}) = C(\{x,y,z\}) = \{x,z\}$$
$$C(\{C(\{x,y\}) \cup C(\{x,z\})\}) = C(\{\{x,y\} \cup \{x,z\}\}) = C(\{x,y,z\}) = \{x,z\}$$

etc., etc.[24]

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[23] We can pursue my interpretation of this formally -- from [9] --

Assume E is a finite set.

Let $V = \{v_1, \ldots, v_m\}$ be a family of subsets of E. For all $S \subset E$ define $V_S = \{V: \bigcup_{v \in V} v = S\}$. Assume that $v \subset E$ & $v \neq \emptyset \Rightarrow C(v) \neq \emptyset$

(I.P.) Independence of Path:

$$\underset{S \subset E}{(\forall S)} \; \{V \in V_S \;\&\; V' \in V_S \Rightarrow C(\bigcup_{v \in V} C(v)) = C(\bigcup_{v' \in V'} C(v'))\}$$

The axiom says we can choose over S directly, or arbitrarily segment S, choose over the parts, then choose over the choices without changing the ultimate result. Notice that the property holds for all S so one can further refine the segments without worry.

[24] A formal proof of the assertion is given in [9].

We can now make a very important observation. Although the example is a choice function which is Independent of the Path it is not a rational choice function.[25] That is, an assumption of rational choice is not necessitated by a demand that choice be independent of the path.

We have now established the fourth assertion of this paper. The argument provided by Arrow in support of the rational choice assumption is simply wrong. The importance of this observation should not be minimized. If Independence of the Path is demanded in place of rational choice, Arrow's major result, The General Possibilities Theorem, does not hold. There are decision processes which satisfy Independence of the Path and all of Arrow's conditions (other than rationality).[26]

Several interesting questions as well as interesting results have been omitted from this section. When the set of alternatives is expanded to an infinite set several complications arise. Also there are connections between Independent of Path behavior and "preferences," see [9]. But the major observation is established. If Independent of Path behavior is all that is desired then the rational choice assumptions -- even the weakest forms -- can be dropped.

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[25] We can show this as follows. Recall a choice function is rational if there exists a rationalization -- a binary relation for which the chosen elements are always maximal according to this binary relation. Suppose such a relation, R , exists. From the two element choices we conclude $yR\alpha$ for $\alpha \in \{x, z\}$ and thus y is maximal in $\{x, y, z\}$. But, since $y \notin C(\{x, y, z\})$, we contradict the assertion that R is a rationalization.

[26] Not all of Arrow's conditions are stated in terms of choice functions so his conditions must be reworded accordingly for this assertion to be technically true. When the alterations are made a process which chooses the Pareto Optimals serves as a counter example to a proposed impossibility result.

SECTION THREE

The first sentence of this paper asserted, in spirit at least, that concepts such as "social preference," "social rationality," "public interest," "social benefits" and "social welfare" should be dropped from the bag of tools. They are, as has been argued here, the same concept when looked at as a formal property of models.

The concepts do not span the range of interpretations which have been attributed to them. In the case of Arrow, we have shown that the set of rational social choice functions is but a subset of those which are characterized by the "independence of path" property. We have also argued that use of a Bergson-Samuelson social welfare function places an arbitrary restriction on the set of ethics which can be reflected in social choices.

We shall now argue that the class of rational social choice functions, while from either the Arrow or Samuelson point of view, is an "uninteresting" class. I say "uninteresting" because the intersection of this class of social choice functions with the class of social choice functions which could, conceivably, be implemented as efficient social decision processes, yields a set of choice functions which are representations of dictatorship processes. To repeat, any social choice function which 1) is transitive rational, 2) satisfies Pareto optimality, and 3) is capable of being implemented as a process is a dictatorial choice function.[27]

In order to establish this assertion we need only recall the discussion, above, concerning the independence of irrelevant alternatives

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[27] Assume there are at least three people, at least three alternatives and that the domain of the choice function is "large."

axiom and then appeal directly to Arrow's General Possibility Theorem. It was indicated above and is argued at somewhat greater length in [10] that the independence of irrelevant alternatives axiom can serve as the defining characteristic of a group decision process.[28] That is, any choice function satisfying the defining characteristic could, in principle, be the equilibriums of some process stated as a function the parameters. Furthermore, any choice function which does not satisfy the axiom is incapable of being an accurate representation of any process.

Arrow's result can be stated as follows. If you are given a set of ethics which a) are transitive rational and b) include pareto optimality then all processes[29] for which the equilibriums are always what they "should" be, according to the given system of ethics, are dictatorship processes.

The title of this paper includes the word "relevance." How can social choice theory be relevant in view of such a negative result? Easy! Simply drop the rationality assumptions. Very little would be lost and much would be gained.

The major advantage of the social preference concept is that it allows, at least in principle, applied economists to perform an engineering function. It allows the "objective" to be "objectively" stated. It allows communication among researchers as to the success of their efforts in

[28] The axiom states that choice over any arbitrary set S depends only upon individuals' preference over S and nothing outside of S . The assertion is that this is a characteristic of all socio-economic processes. A change in preferences for things infeasible does not change equilibrium over the feasible set.

[29] We assume the domain is sufficiently large.

relation to an attempted objective. This should not be confused with the problem of choosing an objective.

This advantage need not be lost. Ethics can still be captured in a formal way (in principle at least), by choice functions as opposed to binary relations or numerical functions. Such choice functions, naturally, may be difficult to express in words -- but so are binary relations and numerical functions. At the conceptual level, at least, such difficulties do not exist. More importantly, however, the more versatile method of representation, suggested here, allows research to move systematically into areas which have been previously unexplored in a formal way. Furthermore, since discussions of processes frequently take place in ethical terminology the possibility exists that models built with ethics as the objects of debate (variables) as opposed to allocations, will have better explanatory power.

Removal of the rationality conditions allows the possibility of discovering ways to build choice from the application of rules. If we replace the social rationality with Independence of the Path, Arrow's impossibility result is left behind. Many social choice functions satisfy his conditions with rationality replaced by the independence property. More importantly the power of axiomatic method can be used to discover the rules. Once one specifies how the system "should" behave the axiomatic method can be applied to find the proper equilibriums. The problem then becomes one of properly designing institutions.

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Glossary

$\emptyset$ is the empty set

ε means - is an element of

$\Rightarrow$ means - implies

[...] means - ... is a true statement

$A \subset B$ means $[x \varepsilon A \Rightarrow x \varepsilon B]$

$\sim$ means - not

$\vee$ means - or

$\wedge$ means - and (it does not mean and/or)

$\forall$ means - for all

$\exists$ means - there exists

$\{\theta: \ldots\}$ is read - the set of θ for which the statement ... is true

$A \cup B = \{x: x \varepsilon A \vee x \varepsilon B\}$

$A \cap B = \{x: x \varepsilon A \wedge x \varepsilon B\}$

$\bigcup_{v \varepsilon V} v$ means - the union of all sets v which are elements of V.

$(\forall_{\psi} \theta)$ [...] is read - for all θ such that the statement ψ is true the statement [...] is also true.

$f(x) \equiv g(x)$ means - $(\forall x)$ $[f(x) = g(x)]$

$A \otimes B = \{(x,y): x \varepsilon A \wedge y \varepsilon B\}$

R is a binary relation on A means - $R \subset A \otimes A$

xRy means - $(x,y) \varepsilon R$

xPy means - $[xRy \wedge \sim yRx]$ for some specific R

xIy means - $[xRy \wedge yRx]$ for some specific R

A binary relation R over a set A is

total in case $(\forall_{x \varepsilon A} x)$ $(\forall_{\substack{y \varepsilon A \\ y \neq x}} y)$ $[xRy \vee yRx]$

reflexive in case $(\forall_{x \varepsilon A} x)$ $[xRx]$

transitive in case $(\forall_{x \varepsilon A} x)$ $(\forall_{\substack{y \varepsilon A \\ y \neq x}} y)$ $(\forall_{\substack{z \varepsilon A \\ z \neq x \\ z \neq y}} z)$ $[xRy \wedge yRz \Rightarrow xRz]$

An element x is said to be R-maximal over a set A in case

$$x \varepsilon A \wedge [(\forall_{y \varepsilon A} y) \quad xRy].$$

A choice function C(v) with domain $\mathcal{U}$ is said to be <u>rational</u> in case there exists a binary relation R such that

$$(\forall_{v \varepsilon \mathcal{U}} v) \quad [x \varepsilon C(v) \Rightarrow x \text{ is R-maximal over } v]$$

The binary relation R is said to <u>rationalize</u> C(v) or be a <u>rationalization</u>.

A rational choice function C(v) is said to be

<u>reflexive rational</u> in case there exists a reflexive rationalization

<u>total rational</u> in case there exists a total rationalization

<u>transitive rational</u> in case there exists a transitive rationalization

<u>total, reflexive, transitive rational</u> in case there exists a total, reflexive, transitive rationalization.