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THE PROBABILITY OF A CYCLICAL MAJORITY

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CONSIDER A COMMITTEE or society attempting to order the alternatives (X_1, X_2, X_3) by use of majority rule. Each individual is assumed to have a strong ordering (called a profile) on the alternatives. "Indifference" is not a property of the profiles. The committee is said to "prefer" X_i to X_j, denoted X_iCX_j if X_i is preferred to X_j on a majority of the individual profiles. It is well known that if certain individual profiles are chosen, the resulting "social ordering" may be cyclical, i.e., X_iCX_j, X_jCX_k, X_kCX_i. Such a result is called a "cycle."

Two aspects of this problem have been of interest. The first is that of placing conditions on individual profiles necessary and sufficient for the resulting social ordering to contain a cycle (see [**1, 2, 6, 8, 9, 11, 12, 13**]). The second is that of obtaining the probability that certain types of cycles occur—given that individuals are allowed to choose at random among all possible profiles. This probability depends upon the number of people (always assumed to be odd) and the number of alternatives.

There are three different probabilities of interest. We let $n = 2m + 1$ be the odd number of individuals (m is a positive integer) and we let $r \geqslant 3$ be the number of alternatives. The probabilities of interest are:

(i) $Q(m, r)$: the probability that one issue is preferred by a majority to *all other* issues;

(ii) $P(m, r)$: the probability that the social ordering is completely transitive (contains no cycle);

(iii) $Z(m, r)$: the probability that one issue is preferred by a majority to *all other issues and* the complete social ordering contains a cycle.

Very little is known about these functions. Duncan Black [**2**] found that $P(1, 3) = .9444, \ldots$, by complete enumeration. David Klahr [7] found $Q(1, 4) = .8888, \ldots$, by enumeration. Monte Carlo techniques were used [**3, 7**] to estimate $Q(m, r)$ for small values of the variables. All of these probabilities are for the case where choices over the profiles are equally likely.

Our analysis will proceed as follows. In Section 1, we will derive the special case for $P(m, 3) = Q(m, 3)$ and the choices are completely random. This is done in order to acquaint the reader with the notation used in the following sections. In Section 2, we derive $Q(m, r)$. In Section 3, we derive $P(m, r)$. In the final section, we present some numerical values for $Q(m, r)$ and $P(m, r)$.

Before continuing, we can deal with $Z(m, r)$ directly. We simply observe that if the social ordering is completely transitive (contains no cycle), then one issue

[1] This paper was delivered at the meeting of the Econometric Society, Chicago, 1966. The material in Section 2 has been treated independently in two papers published since the writing of this paper [**5, 10**]. The authors wish to thank Otto Davis, Morton Kamien, and David Klahr for their comments and suggestions.

is preferred by a majority to all others. It follows directly that

$$(1) \qquad Z(m, r) = Q(m, r) - P(m, r).$$

1. $P(m, 3) = Q(m, 3)$ FOR THE EQUALLY LIKELY CASE

Since the notation of the following sections becomes rather cumbersome, we shall first derive the function for the special case of three alternatives and equally likely choices over the possible profiles. Observe $P(m, 3) = Q(m, 3)$ since in the case of three alternatives a cycle occurs if and only if no alternative is preferred by a majority to all others.

Let $Q(X_i)$ be the probability that X_i is preferred by a majority to the other two. Then

$$(2) \qquad Q(m, 3) = \sum_{i=1}^{3} Q(X_i).$$

Since, by assumption, the choices over profiles are equally likely,

$$(3) \qquad Q(X_1) = Q(X_2) = Q(X_3) = \frac{Q(m, 3)}{3}.$$

Consequently, in order to compute $Q(m, 3)$ we need only find $Q(X_1)$ and multiply by 3.

The $n = 2m + 1$ individuals choose at random from the elements of $S_r = S_3 = \{\sigma_1, \ldots, \sigma_6\}$. The set S_r contains all possible orderings (profiles) σ, of the r alternatives. It thus contains $r!$ elements as enumerated.

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6
X_1	X_1	X_2	X_3	X_2	X_3
X_2	X_3	X_3	X_2	X_1	X_1
X_3	X_2	X_1	X_1	X_3	X_2

Let U_i, where $0 \leqslant U_i \leqslant 2m + 1$, be the number of voters who select profile $\sigma_i \in S_3$, $1 \leqslant i \leqslant 3!$. U_i will, of course, always be an integer.

We know

$$(4) \qquad U_1 + U_2 + \ldots + U_6 = 2m + 1, \qquad 0 \leqslant U_i \leqslant 2m + 1,$$

or

$$(5) \qquad U_1 = 2m + 1 - \sum_{i=2}^{3!} U_i.$$

We write $\sigma_i(X_j) = k$ in case, on profile i, there are $k - 1$ alternatives preferred to X_j. That is, X_j is ranked in the kth place on profile i. If $\sigma_i(X_j) < \sigma_i(X_l)$, then X_j is ranked higher on profile i than is X_l. Thus the committee "prefers" X_i to X_j in case a majority of the voters choose profiles σ_l such that $\sigma_l(X_i) < \sigma_l(X_j)$. This is written $X_i C X_j$.

Define:

$$A_{1j} = \{\sigma_i \in S_3 | \sigma_i(X_1) < \sigma_i(X_j)\}, \qquad j = 2, 3,$$

$$A'_{1j} = \{\sigma_i \in S_3 | \sigma_i \notin A_{1j}\},$$

$$B_{1j} = \{i | \sigma_i \in A_{1j}\};$$

$$B'_{1j} = \{i | \sigma_i \in A'_{1j}\}.$$

Where the profiles are indexed as enumerated above

$$A_{12} = \{\sigma_1, \sigma_2, \sigma_6\}; \qquad A'_{12} = \{\sigma_3, \sigma_4, \sigma_5\};$$

$$B_{12} = \{1, 2, 6\}; \qquad B'_{12} = \{3, 4, 5\};$$

$$A_{13} = \{\sigma_1, \sigma_2, \sigma_5\}; \qquad A'_{13} = \{\sigma_3, \sigma_4, \sigma_6\};$$

$$B_{13} = \{1, 2, 5\}; \qquad B'_{13} = \{3, 4, 6\}.$$

If a majority of the voters prefer X_1 to both X_2 and X_3, we have

(6.1) $$U_1 + U_2 + U_6 > U_3 + U_4 + U_5,$$

(6.2) $$U_1 + U_2 + U_5 > U_3 + U_4 + U_6,$$

or

(7) $$\sum_{i \in B_{1j}} U_i > \sum_{i \in B'_{1j}} U_i, \qquad j = 2, 3.$$

Observe that (5) and (7) provide necessary and sufficient conditions for X_1 to be preferred to X_2 and X_3 by a majority.

Now substituting (5) into (6), we get

(8.1) $$2m + 1 > 2U_3 + 2U_4 + 2U_5,$$

(8.2) $$2m + 1 > 2U_3 + 2U_4 + 2U_6,$$

which simplifies to

(9.1) $$m \geqslant U_3 + U_4 + U_5,$$

(9.2) $$m \geqslant U_3 + U_4 + U_6,$$

or

(10) $$m \geqslant \sum_{i \in B'_{1j}} U_i, \qquad j = 2, 3.$$

System (10) simply stipulates that less than half of the individuals choose profiles on which either X_2 is preferred to X_1 or X_3 is preferred to X_1. Again, (5) and (10) are necessary and sufficient for $X_1 C X_i$, $i = 2, 3$. Any solution to the system (5) and (10) will be a distribution of the voters among the possible profiles such that $X_1 C X_i$, $i = 2, 3$. Further if voters choose profiles such that $X_1 C X_i$, $i = 2, 3$, then that distribution of voters will be a solution to the system (5) and (10).

If voters choose among the possible profiles in S_3, such that U_i of the voters choose profile σ_i, the probability that a particular $U_1^*, \ldots U_{3!}^*$ occurs is given by the multinomial formula as

$$(11) \qquad P(U_1, \ldots U_6) = \frac{(2m+1)!}{U_1! \ldots U_6!} \theta_1^{U_1} \ldots \theta_6^{U_6}$$

where θ_i is the probability that an individual chooses σ_i. Since, by assumption,[2] $\theta_1 =, \ldots, = \theta_6 = 1/r! = 1/6$, we can simplify (11) to

$$(12) \qquad P(U_1, \ldots U_6) = \frac{(2m+1)!}{(6)^{2m+1}} \prod_{i=1}^{6} \frac{1}{U_i!}.$$

We can now find $Q(X_1)$ by attaching to each solution to (5) and (10) the number dictated by (12) and summing all such numbers over all solutions to (5) and (10).

By substituting (5) into (12), using (3), and summing, we obtain

$$(13) \qquad 3Q(X_1) = Q(m, 3) = \frac{3(2m+1)!}{(6)^{2m+1}} \sum_{U_2=0}^{f(2)} \cdots \sum_{U_6=0}^{f(6)} \prod_{j=2}^{6} \frac{1}{U_j!(2m+1-\Sigma_j U_j)!},$$

$$f(k) = \begin{cases} 2m+1-\Sigma_{\substack{i<k \\ i \neq 1}} U_i & \text{if} \quad k \notin B'_{1l}, \qquad l = 2, 3, \\ \min \begin{cases} 2m+1-\Sigma_{\substack{1<k \\ i \neq 1}} U_i, \\ m - \Sigma_{\substack{i<k \\ i \in B'_{1l}}} U_i, & \text{all } l \text{ with } k \in B'_{1l}, l \in \{2, 3\}, \end{cases} \end{cases}$$

which is the desired expression. In terms of the profiles as indexed, this is

$$(14) \qquad Q(m, 3) = \frac{3(2m+1)!}{(6)^{2m+1}}$$

$$\times \sum_{U_2=0}^{2m+1} \sum_{U_3=0}^{\min\begin{Bmatrix}2m+1-U_2 \\ m\end{Bmatrix}} \sum_{U_4=0}^{\min\begin{Bmatrix}2m+1-U_2-U_3 \\ m-U_3\end{Bmatrix}} \sum_{U_5=0}^{\min\begin{Bmatrix}2m+1-U_2-U_3-U_4 \\ m-U_3-U_4\end{Bmatrix}} \sum_{U_6=0}^{\min\begin{Bmatrix}2m+1-U_2-U_3-U_4-U_5 \\ m-U_3-U_4\end{Bmatrix}} [U_2!$$

$$\times U_3! U_4! U_5! U_6! (2m+1-\Sigma_{i=2}^{6} U_i)!]^{-1}.$$

2. $Q(m, r)$: PROBABILITY OF PREFERENCE FOR ONE ISSUE

The derivation of $Q(m, r)$ is a straight forward generalization of the analysis contained in Section 1. We start by calculating the probability $Q(X_s)$ that X_s is preferred by a majority to all other alternatives. Observe that

$$(15) \qquad Q(m, r) = \sum_{s=1}^{r} Q(X_s).$$

Again individuals choose from the elements of $S_r = \{\sigma_1, \ldots, \sigma_{r!}\}$ with θ_i being the probability that any particular individual chooses profile σ_i.

[2] This assumption will be dropped in Sections 2 and 3.

Let us define:

$$A_{sj} = \{\sigma_i \in S_r | \sigma_i(X_s) < \sigma_i(X_j)\}; \qquad A'_{sj} = \{\sigma_i \in S_r | \sigma_i \notin A_{sj}\},$$
$$j = 1, 2, \ldots, s-1, s+1, \ldots r!.$$
$$B_{sj} = \{i | \sigma_i \in A_{sj}\}; \qquad B'_{sj} = \{i | \sigma_i \in A'_{sj}\}.$$

U_i is the number of voters that choose σ_i. The profile σ_s has X_s ranked higher than all other alternatives. That is, index the profiles such that σ_1 has X_1 as the most preferred, σ_2 has X_2 as the most preferred, etc.

We know

$$2m + 1 = \sum_{i=1}^{r!} U_i \tag{16}$$

or

$$U_s = 2m + 1 - \sum_{\substack{i=1 \\ i \neq s}}^{r!} U_i. \tag{17}$$

Further, $X_s C X_j$ for $j = 1, \ldots, s-1, s+1, \ldots, r$ only if

$$\sum_{i \in B_{sj}} U_i > \sum_{i \in B'_{sj}} U_i, \qquad j = 1, \ldots, s-1, s+1, \ldots, r. \tag{18}$$

Expressions (17) and (18) are necessary and sufficient for $X_s C X_j, j = 1, \ldots, s-1, s+1, \ldots, r$.

Observe that $\sigma_s \in A_{sj}$ for $j \neq s$, so $s \in B_{sj}$ for $j \neq s$. Thus substitution of (17) into (18) yields

$$2m + 1 > 2 \sum_{i \in B'_{sj}} U_i, \qquad j = 1, \ldots, s-1, s+1, \ldots, r, \tag{19}$$

which holds if and only if

$$m \geqslant \sum_{i \in B'_{sj}} U_i, \qquad j = 1, \ldots, s-1, s+1, \ldots, r. \tag{20}$$

Thus, systems (17) and (20) are necessary and sufficient for $X_s C X_j, j = 1, \ldots, s-1, s+1, \ldots, r$.

The multinomial formula can now be used in the same way as it was used in the previous section. If θ_i is the probability that any particular voter chooses $\sigma_i \in S_r$, the probability that a particular $\{U_1^*, \ldots, U_{r!}^*\}$ occurs is

$$P(U_1, \ldots, U_{r!}) = (2m+1)! \prod_{i=1}^{r!} \frac{\theta_i^{U_i}}{(U_i)!}. \tag{21}$$

Now to each solution to (17) and (20) we assign the number dictated by (21) and sum all such numbers over all solutions to (17) and (20).

The formula, after substituting (17) into (21) and summing, is

$$Q(s) = (2m+1)! \sum_{U_1=0}^{f(1)} \cdots \sum_{U_{s-1}=0}^{f(s-1)} \sum_{U_{s+1}=0}^{f(s+1)} \cdots \sum_{U_{r!}=0}^{f(r!)} \left[\prod_{\substack{i=1 \\ i \neq s}}^{r!} \frac{\theta^{U_i}}{(U_i)!} \right] \times \frac{\theta_s^{2m+1-\Sigma_{i \neq s} U_i}}{(2m+1-\Sigma_{i \neq s} U_i)!}; \tag{22}$$

$$f(k) = \begin{cases} 2m + 1 - \sum_{\substack{i<k \\ i \neq s}} U_i & \text{if } k \notin B'_{sl} \text{ for any } l \ (l = 1, \ldots, s-1, \ s+1, \ldots, r). \\ \min \begin{cases} 2m + 1 - \sum_{\substack{i<k \\ i \neq s}} U_i, \\ m - \sum_{\substack{i<k \\ i \in B'_{sl}}} U_i, \text{ all } l \text{ such that } k \in B'_{sl} \ (l \in \{1, \ldots, s-1, \ s+1, \ldots, r\}). \end{cases} \end{cases}$$

Substitution of (22) into (15) yields the desired expression.[3]

[3] For the special case where there are only three individuals and where $\theta_i = 1/r!$, $i = 1, \ldots, r!$, a considerably simplified formulation of $Q(1, r)$ can be deduced. Start by deriving the probability that some particular issue (X_i) is preferred by a majority to all others. Let 1 choose a profile $\sigma \in S_r$ on which $\sigma(X_i) = j + 1$. By assumption the probability that j takes any particular value in the interval $0 \leqslant j \leqslant r - 1$ is $1/r$. Given that 1 has chosen σ, let 2 choose a profile $\sigma' \in S_r$ such that $\sigma'(X_i) = k + 1$ and such that no issue is preferred to X_i on *both* σ and σ'. Otherwise, some issue would be preferred by a majority to X_i. The probability 2 makes such a choice is

$$\begin{cases} \dfrac{1}{r}, & \text{if } k = 0, \\ \dfrac{1}{r} \prod_{l=1}^{k} \dfrac{r - (j + l)}{r - l}, & \text{if } 1 \leqslant k \leqslant r - (j + 1), \\ 0, & \text{if } k > r - (j + 1). \end{cases}$$

Given that 1 and 2 have selected such profiles, let 3 choose $\sigma'' \in S_r$ such that $\sigma''(X_i) = q + 1$ *and* such that no issue which is preferred to X_i on σ'' is preferred to X_i on either σ' or σ. The probability with which this occurs is

$$\begin{cases} \dfrac{1}{r}, & \text{if } q = 0, \\ \dfrac{1}{r} \prod_{s=1}^{q} \dfrac{r - (j + k + s)}{r - s}, & \text{if } 1 \leqslant q \leqslant r - (j + k + 1), \\ 0, & \text{if } q > r - (j + k + 1). \end{cases}$$

Now the probability that X_i is preferred by a majority to all others is

$$Q(X_i) = \frac{1}{r^3} \sum_{j=0}^{r-1} \sum_{k=0}^{r-(j+1)} \sum_{q=0}^{r-(j+k+1)} F(j, k) \cdot G(j, k, q)$$

where

$$F(j, k) = \begin{cases} \prod_{l=1}^{k} \dfrac{r - j - l}{r - l}, & \text{if } k \geqslant 1, \\ 1, & \text{if } k = 0, \end{cases}$$

$$G(j, k, q) = \begin{cases} \prod_{s=1}^{q} \dfrac{r - j - k - s}{r - s}, & \text{if } q \geqslant 1, \\ 1, & \text{if } q = 1. \end{cases}$$

Simplification, translation of indices, and the observation that for the equiprobable case $Q(X_i) = (1/r)Q(1, r)$ yields

$$Q(1, r) = \frac{1}{r^2} \sum_{j=1}^{r} \sum_{k=1}^{r-j+1} \sum_{q=1}^{r-j-k+2} \frac{(r - k)!(r - j)!(r - q)!}{(r - 1)!(r - 1)!(r - j - k - q + 2)!}$$

which is the desired expression.

3. $P(m, r)$: PROBABILITY OF COMPLETE TRANSITIVITY

In this section a function indicating the probability that majority rule results in a completely transitive social ordering will be derived.[4] We begin by computing the probability, $P(s)$, that the profile σ_s corresponds to the social ordering. There are $r!$ transitive social orderings so

$$P(m, r) = \sum_{s=1}^{r!} P(s). \tag{23}$$

We establish the following definitions:

$f_i(\sigma_s) = \{j | \sigma_s(X_j) = i\}$. The expression $f_i(\sigma_s)$ is the index of the alternative in the ith place on σ_s.

$$A_{ij} = \{\sigma \in S_r | \sigma(X_i) < \sigma(X_j)\}; \qquad A'_{ij} = \{\sigma \in S_r | \sigma \notin A_{ij}\}.$$

$$B_{ij} = \{k | \sigma_k \in A_{ij}\}; \qquad B'_{ij} = \{k | \sigma_k \in A'_{ij}\}.$$

$$S_r = \{\sigma_1, \ldots, \sigma_{r!}\}.$$

U_i is the number of voters choosing profile σ_i. Thus, U_i is a nonnegative integer.

If the voters choose profiles such that under majority rule the social ordering corresponds to the profile σ_s, the following system of $r(r-1)/2$ inequalities must be satisfied:

$$\sum_{k \in B_{f_i(\sigma_s) f_j(\sigma_s)}} U_k > \sum_{k \in B'_{f_i(\sigma_s) f_j(\sigma_s)}} U_k, \qquad 1 \leqslant i \leqslant r-1; i+1 \leqslant j \leqslant r; \quad i, j \text{ integers}. \tag{24}$$

System (24) simply stipulates that if, for example,

$$\sigma_s = \begin{Bmatrix} X_3 \\ X_1 \\ X_2 \\ \vdots \end{Bmatrix},$$

the voters are distributed among the profiles such that $X_3 C X_i$, $i = 1, 2, 4, \ldots, r$, and $X_1 C X_i$, $i = 2, 4 \ldots r$, and $X_2 C X_i$, $i = 4, 5, \ldots r$, etc. Again we know

$$2m + 1 = \sum_{i=1}^{r!} U_i \tag{25}$$

or

$$U_s = 2m + 1 - \sum_{\substack{i=1 \\ i \neq s}}^{r!} U_i. \tag{26}$$

Observe that σ_s is the unique σ in

$$\bigcap_{\substack{1 \leqslant i \leqslant r-1 \\ i+1 \leqslant j \leqslant r}} A_{f_i(\sigma_s) f_j(\sigma_s)}.$$

[4] The solution to this problem as reported by the authors in [4] is wrong. We are indebted to Morton Kamien who found an error in an early draft of this paper.

Substitution of (26) into (24) with simplifications gives

$$m \geqslant \sum_{k \in B'_{f_i(\sigma_s) f_j(\sigma_s)}} U_k, \qquad 1 \leqslant i \leqslant r-1; i+1 \leqslant j \leqslant r. \tag{27}$$

Together (27) and (25) provide conditions necessary and sufficient that the social ordering correspond to the profile σ_s.

The multinomial formula (21) can be used as before. We assign to each solution to (27) and (25) the number dictated by (21). We then sum these numbers over all solutions. Substituting (26) into (21) we obtain the following expression for the sum:

$$P(\sigma_s) = (2m+1)! \sum_{U_1=0}^{g(1)} \cdots \sum_{U_{s-1}=0}^{g(s-1)} \sum_{U_{s+1}=0}^{g(s+1)} \cdots \sum_{U_{r!}=0}^{g(r!)} \left[\prod_{\substack{i=1 \\ i \neq s}}^{r!} \frac{\theta_i^{U_i}}{(U_i)!} \right] \times \frac{\theta_s^{2m+1-\Sigma_{i \neq s} U_i}}{(2m+1-\Sigma_{i \neq s} U_i)!}; \tag{28}$$

$$g(k) = \min \begin{cases} m - \displaystyle\sum_{\substack{1 \leqslant j \leqslant k-1 \\ j \in B'_{f_p(\sigma_s) f_l(\sigma_s)}}} U_j, & \text{all } B'_{f_p(\sigma_s) f_l(\sigma_s)} \text{ containing } k; \\ & 1 \leqslant p \leqslant r-1; p+1 \leqslant l \leqslant r; \\ 2m+1 - \displaystyle\sum_{\substack{j=1 \\ j \neq s}}^{k-1} U_j. & \end{cases}$$

Substitution of (28) into (23) gives the desired functions.

4. COMPUTATIONS FOR THE EQUALLY LIKELY CASES

Table I presents several values for $Q(m, r)$ when $\theta_i = (1/r!)$.[5] As is obvious from the formula, the computation involves some fantastically larger numbers. Nevertheless, the formula is computable (given sufficient time) but there is certainly room for simplification.

Several interesting problems remain. G. T. Guilbaud[6] has stated, without derivation, that

$$\lim_{m \to \infty} [1 - Q(m, 3)] = .0877.$$

We have not verified this result. Further, the asymptotic behavior of $Q(1, r)$ can be investigated. We *conjecture* that $\lim_{r \to \infty} Q(1, r) = 0$. The results on Table I, however, show that the function decreases very slowly.

Finally, there are many symmetrical aspects of the problem. Perhaps a proper characterization of these can yield a statement of the formulas which would allow easier calculations and investigations into the asymptotic behavior of the functions.

[5] This function was programmed by Mrs. L. Vilms. Computer time was provided by the Herman C. Krannert Graduate School of Industrial Administration, Purdue University.

[6] G. T. Guilbaud, "Les Théories de l'intérèst général et la probléme logique de l'agrégation," *Economic Applique*, Vol. 5, 1952, p. 519. This result has been verified by Garman and Kamien [**5**].

TABLE I

EVALUATION OF $Q(m, r)$

No. of people $n = 2m + 1$	Number of alternatives r 3	4	5	6	7	8	9	10	11	12	13	14	15
3	.94444	.8888	.8399	.7977	.7612	.7293	.7011	.6760	.6536	.6333	.6148	.5980	.5825
5	.93055	.8611											
7	.92490												
9	.92202												
11	.92019												
13	.91893												
15	.91802												
17	.91733												
19	.91679												
21	.91635												
23	.91599												
25	.91568												
27	.91543												
29	.91521												
31	.91501												

Table II presents several values of $P(m, r)$. Of course, $P(m, 3) = Q(m, 3)$. The new numbers are for $P(1, 4)$ and $P(1, 5)$ which are, as would be expected, lower than the corresponding values of $Q(m, r)$. Computation of $P(m, r)$ is somewhat easier than the computation of $Q(m, r)$. Even though the number of inequalities to check is larger, the number of solutions, which causes the problem, is considerably smaller.

TABLE II

EVALUATION OF $P(m, r)$

Number of people $n = 2m + 1$	Number of alternatives 3	4
3	.94444	.8298
5	.93055	.7896
7	.92490	
9	.92202	
11	.92019	
13	.91893	
15	.91802	
17	.91733	
19	.91679	
21	.91635	
23	.91599	
25	.91568	
27	.91543	
29	.91521	
31	.91501	

Nevertheless, the numbers involved in $P(m, r)$ are large. Computation of $P(2, 4)$ took five minutes of the IBM 7094. Our programmer estimates that the computation of $P(2, 5)$ would take over 100 *hours* on the same machine and the evaluation of $Q(2, 5)$ would take almost three times as long. If further evaluations of these functions are desired, it would certainly seem that simplifications are in order.

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