Nonlinear dispersion in a coupled-resonator optical waveguide

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The propagation of an optical pulse in a coupled-resonator optical waveguide may be calculated nonperturbatively to all orders of dispersion, in the conventional tight-binding approximation, even though the dispersion relationship is nonlinear. Working in this framework, we discuss limits of the physical parameters and approximations to the exact formulation that highlight the conditions under which pulse distortion can be minimized. The results are fundamental to the design of coupled-resonator optical waveguides and are also relevant to other applications of the tight-binding method.

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A coupled resonator optical waveguide (CROW) can be made in a wide range of materials, including photonic crystals, polystyrene microspheres, and optical fibers.[1] The basic geometry is composed of a linear array of structural elements, each capable of supporting resonant modes of the electromagnetic field, e.g., defect cavities in a photonic crystal, as shown in Fig. 1. Although the individual resonances may depend on Bragg reflection, waveguiding in such a composite structure is fundamentally a consequence not of total internal reflection or Bragg reflection from a periodic structure but instead of the overlap between the individual resonator modes. The analysis of the cw waveguide modes in a CROW was introduced by Yariv and co-workers[2,3]; recent experiments have confirmed the accuracy of the analytical framework.[4–6]

As photonic crystal waveguides, CROWs are especially attractive because the well-researched catalog of the properties of defect states[7] contributes directly to the analysis of pulse propagation,[8] and the roles of the critical parameters on which the waveguiding characteristics depend are readily apparent. In view of the wide applicability of these waveguides in microscale all-optical information processing devices,[8,9] we discuss the effects of higher-order dispersion and ways to minimize the distortion in terms of the structural parameters of the waveguide.

We assume that the structural elements comprising the periodic waveguide, e.g., defect modes in a photonic crystal or photonic wells in the description of superstructure gratings in fibers, are identical and lie along the z axis (with unit vector $\mathbf{e}_z$) separated by a distance R. The waveguide mode $\boldsymbol{\phi}_k(\mathbf{r})$ at a particular propagation constant k is written as a linear combination of the elemental modes $\boldsymbol{\psi}_l(\mathbf{r})$:

$$\boldsymbol{\phi}_k(\mathbf{r}) = \sum_n \exp(-inkR) \sum_l \boldsymbol{\psi}_l(\mathbf{r} - nR\mathbf{e}_z)\,, \tag{1}$$

where the summation over n runs over the structural elements and the summation over l refers to the bound states in each individual element. The dispersion relationship around a central propagation constant k_0 is

$$\begin{aligned}\omega_{k_0+K} &= \Omega(1 - \Delta\alpha/2) + \Omega\kappa\cos(KR) \\ &\equiv \omega_0 + \Delta\omega\cos(KR)\,,\end{aligned} \tag{2}$$

where Ω is the eigenfrequency of the (identical) individual resonators and both $\Delta\alpha$ and κ are overlap integrals involving the individual resonator modes and the spatial variation of the dielectric constant. We limit the terms in the integrals to nearest-neighbor coupling, as applicable to waveguides formed by coupling high-Q resonators. The parameter $\Delta\omega$ is given in terms of the spatial variation of the dielectric constant by

$$\begin{aligned}\Delta\omega = \Omega \int & \mathrm{d}^3\mathbf{r}[\epsilon_{\mathrm{res}}(\mathbf{r} - R\mathbf{e}_z) \\ & - \epsilon_{\mathrm{wg}}(\mathbf{r} - R\mathbf{e}_z)]\boldsymbol{\psi}_l(\mathbf{r}) \cdot \boldsymbol{\psi}_l(\mathbf{r} - R\mathbf{e}_z)\,,\end{aligned} \tag{3}$$

where ϵ_{res} is the dielectric constant of the individual resonators and ϵ_{wg} is the dielectric constant of the waveguide.[1] We assume that the integral does not change appreciably in value for different l; i.e., no degeneracy in the waveguide mode (eigen-) frequency; further, we consider narrow-band pulses (of sufficiently broad temporal extent) to restrict the range of K within the first Brillouin zone, $|K|R < \pi$.

The field describing a pulse $\mathbf{E}(\mathbf{r}, t)$ is written as a superposition of waveguide modes $\boldsymbol{\phi}_k(\mathbf{r})$ within the Brillouin zone, with the corresponding time-evolution propagators:

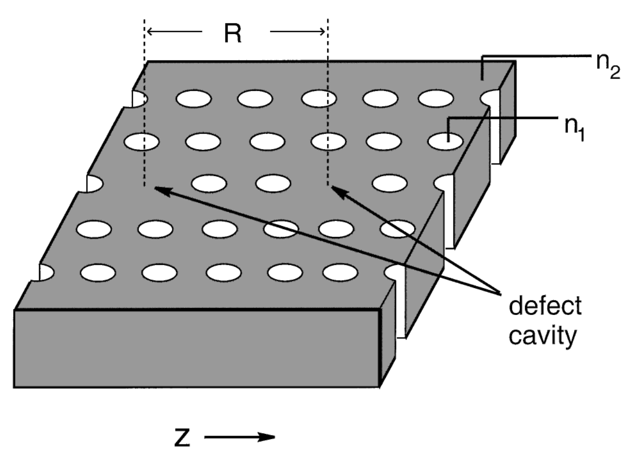

Fig. 1. Schematic of an infinitely long CROW with periodicity R along the z direction, consisting of defect cavities embedded in a two-dimensional photonic crystal.

$$\mathbf{E}(\mathbf{r}, t) = \exp(i\omega_0 t)\int_{-\pi/R}^{\pi/R} \frac{\mathrm{d}K}{2\pi} \times \exp[i\Delta\omega t \cos(KR)] c_{k_0+K}\boldsymbol{\phi}_{k_0+K}(\mathbf{r}) . \quad (4)$$

The boundary conditions specify a pulse shape at the $z = 0$ cross section of the waveguide and centered at the optical frequency ω_0:

$$\mathbf{E}(\mathbf{r}_\perp, z = 0, t) = \exp(i\omega_0 t) E(z = 0, t)\mathbf{G}(\mathbf{r}_\perp) , \quad (5)$$

where $\mathbf{G}(\mathbf{r}_\perp)$ describes the transverse mode profile and $E(z = 0, t)$ is the temporal envelope. The coefficients c_{k_0+K} in Eq. (4) are derived from the equality of Eq. (4) evaluated at $z = 0$ and Eq. (5):

$$E(z = 0, t)\mathbf{G}(\mathbf{r}_\perp) = \int_{-\pi/R}^{\pi/R} \frac{\mathrm{d}K}{2\pi} \times \exp[i\Delta\omega t \cos(KR)] c_{k_0+K}\boldsymbol{\phi}_{k_0+K}(\mathbf{r}_\perp, z = 0) . \quad (6)$$

The vectorial dependence of $\mathbf{G}(\mathbf{r}_\perp)$ is obtained directly from that of $\boldsymbol{\phi}_{k_0+K}(\mathbf{r}_\perp, z = 0)$. We use the notation $G(\mathbf{r}_\perp) = |\mathbf{G}(\mathbf{r}_\perp)|$.

Equation (6) is readily inverted in the limit of a linear dispersion relationship in place of Eq. (2) by use of the properties of the Fourier transform. But in the general case, we are faced with a more difficult problem: a Fredholm integral equation involving a nonlinear function of K in the exponent of the kernel. It is easily verified that using the Taylor series expansion of Eq. (2) does not lead to a tractable solution of Eq. (6) and that the equation is not invertible.

A closed-form solution of Eq. (6) can be found by expansion of the left-hand side of Eq. (6) in a Neumann series of Bessel functions.[10] In obtaining the coefficients of this expansion, certain contour integrals involving the Neumann polynomials are required, but there also exists, under most applicable conditions, a representation in terms of real integrals, which are easier to evaluate numerically. In defining the coefficients

$$\beta_n = \begin{cases} \dfrac{1}{2R} c_{k_0} & n = 0 \\ \dfrac{n}{i^n} \displaystyle\int_0^\infty \frac{\mathrm{d}t'}{t'} [E(0, t') - E(0, 0)] J_n(\Delta\omega t') & n \geq 1 \end{cases} , \quad (7)$$

there is one overall degree of freedom in c_{k_0}, representing a scale factor that is accounted for by Parseval's relationship.[10] The propagation of a pulse is given by

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4} \exp(i\omega_0 t) \sum_{m=0}^{\infty} b_m J_m(\Delta\omega t) \sum_{n=0}^{\infty} \beta_n G(\mathbf{r}_\perp) \times \left\{ \exp[-i(\pm m \pm n) + k_0 R] \sum_l \boldsymbol{\psi}_l[\mathbf{r} - (\pm m \pm n)_+ R\mathbf{e}_z] \right\} , \quad (8)$$

where the coefficients b_m are defined as

$$b_m = \begin{cases} 1 & m = 0 \\ 2i^m & m \geq 1 \end{cases} . \quad (9)$$

We have used the symbol $()_+$ in Eq. (8) as a compact notation for the sum over both choices of sign of $\pm$ that yield a nonnegative number for the expression inside the brackets.

In the following, we focus on the longitudinal waveguiding characteristics and set $G(\mathbf{r}_\perp) = 1$ for simplicity. We also drop the vectorial dependence of $\boldsymbol{\psi}(\mathbf{r})$ and of $\mathbf{E}(\mathbf{r}, t)$, since the two are the only vectorial functions in Eq. (8) and are related in a straightforward manner. The resulting field depends on only one spatial and one temporal coordinate, reflecting the intrinsic physical geometry of the waveguide.

We ask what temporal waveform would be measured by an observer sitting at one of the resonators, i.e., $E(z, t)$ as a function of t with the spatial coordinate z set to the location of one of the resonators (so that z/R is an integer). In the practically important limit of high-Q resonators, $\psi_l(z)$ is narrowly peaked around $z = 0$, $|\psi_l(z = \pm R)| \ll |\psi(z = 0)|$, and we can simplify the double summations over m and n in Eq. (8) by representing this condition as a Kronecker delta function. After some algebra, we may write an expression for the envelope $E(z, t)$ defined by the usual relationship, $|\mathbf{E}(\mathbf{r}, t)| \equiv \exp[i(\omega_0 t - k_0 z)] E(z, t)$, as the sum of two terms:

$$E(z, t) = \frac{1}{4} \sum_{n=-\infty}^{z/R} \tilde{b}_n J_m(\Delta\omega t) \beta_{z/R-n} \sum_l \psi_l(z = 0) \times \frac{1}{4} \sum_{n=z/R}^{\infty} b_n J_n(\Delta\omega t) \beta_{n-z/R} \sum_l \psi_l(z = 0) , \quad (10)$$

where

$$\tilde{b}_n = \begin{cases} 2b_0 & n = 0 \\ b_n & n \neq 0 \end{cases} . \quad (11)$$

Note that Eq. (11) equates the magnitudes of $\tilde{b}$ as a consequence of the original definition, Eq. (9).

As functions of t, the first term on the right-hand side of Eq. (10) represents a backward-propagating pulse and the second term gives the forward-propagating pulse. One way to see this is by using the fact that, for small t, $J_n(t) \sim t^n$, so as we increase n, $J_n(t)$ rises from zero at larger t. As z increases (and consequently so does the integer, z/R), a larger value of n is required for maintaining the same argument of β in the second term of Eq. (10). Through its corresponding Bessel function, this term will contribute significantly at larger t than a term involving a smaller n. Physically, this term describes a point on the envelope reaching greater values of z at later t, i.e., a forward-propagating pulse. A similar argument shows that the first term in Eq. (10), with the modified coefficients $\tilde{b}_n$, describes a backward-propagating pulse.

A pulse envelope as a function of t is therefore described by a contiguous set of Bessel functions:

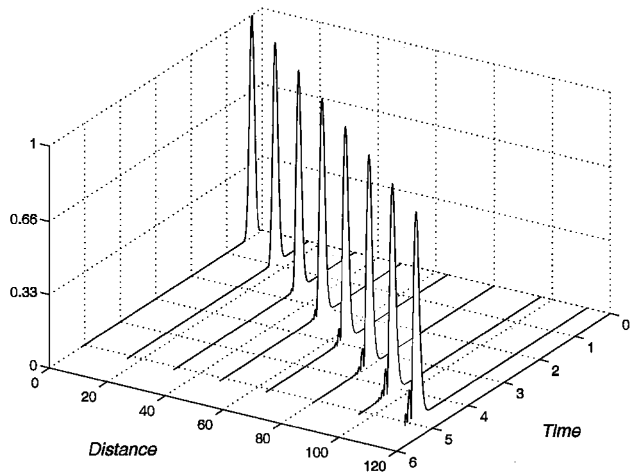

Fig. 2. Temporal evolution of a Gaussian envelope at specific distances inside a CROW, showing the effects of dispersive propagation. At greater depths, the peak of the envelope arrives at a later time, and ripples in the trailing edge indicate higher-order distortion.

The field at z is written as a superposition of an appropriately translated set of these functions $\{J_{z/R}(t), J_{z/R+1}(t), \ldots\}$, multiplied by the coefficients $\{\beta_0, \beta_1, \ldots\}$. Distortion accumulates with distance as a consequence of the changing interrelations between neighboring Bessel functions, e.g., the difference between the set $\{J_0(t), J_1(t), \ldots, J_p(t)\}$ and the set $\{J_5(t), J_6(t), \ldots, J_{5+p}(t)\}$ for a given $\{\beta_0, \beta_1, \ldots\}$. Note that the Bessel functions are replaced with sinusoids in the limit of a linear dispersion relationship in place of Eq. (2): These basis functions maintain the same relationships between neighbors irrespective of the origin of the set, which is why distortionless propagation may be achieved in this limit.[8]

Figure 2 shows the temporal profiles of an input Gaussian envelope as would be detected along such a waveguide. The crest of the envelope travels with a group velocity $\Delta z/\Delta t \lesssim \Delta\omega R$. The dispersion relationship is nonlinear, which precludes an exact group velocity that is valid to all propagation distances, but the error in assuming an equality is less than 0.5% for much of the regime shown in Fig. 2. Higher-order dispersion develops an oscillatory structure at the trailing edge of the pulse (see, for example, Agrawal, Fig. 3.7 of Ref. 11).

One additional simplification is illuminating and may also considerably speed up numerical computations. The normalizations in the identity[12]

$$n\int_0^\infty \frac{\mathrm{d}x}{x} J_n(bx) = 1 \qquad (12)$$

may be used in the definition of β_n in Eq. (7). For slowly varying envelopes $E(z = 0, t')$, the asymptotic limit of the Bessel function is a cosine that, when multiplied by a slowly varying function and integrated over several periods, averages out to zero. We replace $n/t' J_n(\Delta\omega t')$ with $\delta(t' - n/\Delta\omega)$ so that

$$\beta_n = i^{-n} E(z = 0, n/\Delta\omega), \quad n \geq 1; \qquad (13)$$

i.e., different values of β represent temporal samples of the input pulse envelope along the $\Delta\omega t$ axis. The infinite summations in Eq. (10) may thereby be restricted to a finite number, based on the temporal extent of the pulse, without significant loss of accuracy.

This analytical formulation allows waveguides to be designed to achieve the desired propagation characteristics. In a waveguide of a given length, the interresonator spacing, which determines the parameters in Eq. (2), may be chosen to limit the distortion and achieve a certain (effective) group velocity of propagation. Waveguides constructed in electro-optically tunable material will allow the waveguide modes to be altered in real time, through $\psi(\mathbf{r})$ in Eq. (1), and consequently the eigenmode overlap integrals that appear in Eq. (2). Similar effects may be achieved in microelectromechanical systems waveguides constructed out of a patterned membrane with piezoelectric actuators that cause a mechanical deformation; in this case, the physical geometry of the resonators is altered rather than the refractive-index difference between the resonators and their surroundings. This alteration leads to the development of microscale-tunable all-optical delay lines and signal-processing devices, such as interleavers and multiplexers. Our analysis yields a closed-form result for arbitrary input pulse shapes, and to all orders of dispersion, even though the dispersion relationship is nonlinear.

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