in the case of an external force F the coefficient $c_1 = MF$, where M is called the mobility). The causes of the second type are called *forces of diffusion* and produce an effect proportional to the negative gradient of the density $-c_2 \partial D/\partial y$. The accumulation of the elements in any point is, therefore, determined by the (negative) divergence of the total effect. This leads to the equation[3]

$$\frac{\partial D}{\partial t} = -\frac{\partial}{\partial y}(c_1 D) + \frac{\partial}{\partial y}\left(c_2 \frac{\partial D}{\partial y}\right). \tag{14}$$

Comparing this with the form (8) we find

$$b = c_2, \quad a - db/dy = c_1. \tag{15}$$

The expression for c_1 is in agreement with the conclusions arrived at above. It shows that the form (14) of the equation of diffusion is a good guide to the physicist as it brings in evidence the quantities significant from his point of view. On the other hand, Kolmogoroff's equations (3) and (4) are more general since they include the case of transition probabilities which are inhomogeneous in time.

[1] A. Kolmogoroff, *Mathematische Annalen*, **104**, 415 (1931).

[2] The substitution of the general integral of equation (13) for D_0 in (9) and (10) does not lead to the equation (12). Kolmogoroff claims the same results by imposing upon D_0 the condition $\lim D_0 = 0$ for $y = \pm\infty$. That this argument is unconvincing, appears from the fact that it would exclude the uniform density. It is inconvenient to restrict oneself to finite systems.

[3] E.g., P. S. Epstein, *Gerlands Beitr. Geophysik*, **35**, 154 (1932).

ON THE STATISTICAL THEORY OF TURBULENCE

BY TH. VON KÁRMÁN

GUGGENHEIM AERONAUTICS LABORATORY, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated January 2, 1937

G. I. Taylor[1] gave an important impetus to the statistical theory of turbulence by introducing the concept of "isotropic" turbulence, defined by the feature that the mean squares and mean products of the velocity components and of their derivatives are invariant with respect to rotation and reflection of the coördinate axes. Taylor found that under the assumption of the isotropy, the squares and double products of the first derivatives of the velocity components can be expressed by one single correlation function $R(y)$. This function is defined by the ratio $\frac{\overline{u_1 u_2}}{\overline{u^2}}$ where u_1 and

u_2 are the simultaneous values of the x-components of the velocity fluctuation at two points at a distance y perpendicular to the x-axis. Taylor defines a characteristic length λ by writing

$$R(y) = 1 - \frac{y^2}{\lambda^2} + \text{higher terms in } y. \tag{1}$$

Obviously $\frac{1}{\lambda^2} = -\frac{1}{2}\left(\frac{d^2R}{dy^2}\right)_{y=0}$ and Taylor shows that

$$\overline{\left(\frac{\partial u}{\partial y}\right)^2} = 2\frac{\overline{u^2}}{\lambda^2}. \tag{2}$$

Taylor shows furthermore that the rate of decrease of the kinetic energy of the unit volume by dissipation is equal to $15\mu\frac{\overline{u^2}}{\lambda^2}$ (μ = viscosity coefficient). He applies this result to the calculation of the decrease of the mean energy of the turbulent fluctuations downstream from a certain turbulence producing device such as a grid or a mesh. Denoting the velocity of the main stream by U, he obtains the equation

$$U\frac{d}{dx}\overline{u^2} = -10\nu\frac{\overline{u^2}}{\lambda^2} \tag{3}$$

where $\overline{u^2}$ and λ^2 are both unknown functions of the distance x from the grid; ν is the coefficient of kinematic viscosity. Taylor introduces a somewhat arbitrary hypothesis connecting $\overline{u^2}$ and λ^2 with the linear dimension of the turbulence producing device, and tries to show that the reciprocal $\frac{1}{\sqrt{\overline{u^2}}}$ of the root mean square of the velocity fluctuation is always proportional to x.

In this paper a second equation is established connecting $\overline{u^2}$ and λ^2, and it is shown that the decay of the turbulence is fully determined without further assumption by the correlation function $R(y)$. Taylor's result appears as a special case corresponding to a certain type of the correlation function.

We write the hydrodynamic equations in the form:

$$\frac{\partial\omega}{\partial t} + q\nabla\omega - \omega.\nabla q = \nu\nabla^2\omega. \tag{4}$$

In this equation q is the velocity vector and ω the vorticity vector. Multiplying the three-component equations by ξ, η, ζ (where ξ, η, ζ are the components of ω), respectively, and adding, the equation

$$\frac{1}{2}\frac{D}{Dt}\omega^2 - \omega(\omega.\nabla)q = \nu\omega.\nabla^2\omega \tag{5}$$

is readily obtained.

We consider a fluid without mean velocity enclosed in a very large vessel and assume uniform, isotropic turbulence over the whole space considered. It is assumed that the vessel is so large that the influence of the walls can be neglected. Then taking mean values over time and the space considered, $\frac{D\overline{\omega^2}}{Dt}$ will be replaced by $\frac{d}{dt}\overline{\omega^2}$ and $\omega.\nabla^2\omega$ by $\overline{(\nabla \times \omega)^2}$. The second term needs special consideration. Writing out the expression in coördinates

$$\omega.(\omega.\nabla)q = \xi^2\frac{\partial u}{\partial x} + \eta^2\frac{\partial v}{\partial y} + \zeta^2\frac{\partial w}{\partial z} + \xi\eta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \eta\zeta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \zeta\xi\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right). \tag{6}$$

It will be seen that the expression is built up as the sum of the products of the components of the tensor $(\omega\,\omega)$ and of the components of the deformation tensor. Now it can be shown that because of the isotropic feature of the correlations between the velocity components and their derivatives, expression (6) vanishes in mean value. The proof will be given elsewhere. As a matter of fact, any value of the expression (6) different from zero would mean that the vortex filaments had a permanent tendency to be stretched or compressed in the direction of the vorticity axis. With the expression (6) vanishing, equation (5) leads to the simple result:

$$\frac{3}{2}\frac{\partial\overline{\xi^2}}{\partial t} = -3\nu\overline{\left(\frac{\partial\xi}{\partial y} - \frac{\partial\eta}{\partial x}\right)^2} \tag{7}$$

and two similar equations obtained from (7) by cyclic permutation. Defining a length λ_ω by the relation

$$\overline{\left(\frac{\partial\xi}{\partial y}\right)^2} = 2\frac{\overline{\xi^2}}{\lambda_\omega^2}$$

which is analogous to (2), and computing the relations between squares and products of the derivatives of ξ, η, ζ, we obtain

$$\frac{d}{dt}\overline{\xi^2} = -10\nu\frac{\overline{\xi^2}}{\lambda_\omega^2}. \tag{8}$$

Because of the isotropic feature of the system considered, $\overline{\xi^2}$ can be expressed by $\overline{u^2}$ and λ^2; namely $\overline{\xi^2} = 5\,\frac{\overline{u^2}}{\lambda^2}$. Hence, equation (8) together with

Taylor's equation (1) determines $\overline{u^2}$ as a function of the time, provided a relation between λ and λ_ω is known.

We substitute in equation (3) $\frac{d}{dt}$ for $U\frac{d}{dx}$ and write

$$\frac{d}{dt}\overline{u^2} = -10\nu\frac{\overline{u^2}}{\lambda^2}. \tag{9}$$

Then we multiply (8) by $\overline{u^2}$ and (9) by $\overline{\xi^2}$ and subtract. Taking into account that $\lambda^2 = 5\frac{\overline{u^2}}{\overline{\overline{\xi^2}}}$, we readily obtain

$$\frac{d}{dt}\lambda^2 = 10\nu\left(\frac{\lambda^2}{\lambda_\omega^2} - 1\right). \tag{10}$$

Let us assume for the time being a constant value of the ratio $\frac{\lambda^2}{\lambda_\omega^2}$. The physical significance of this assumption is discussed later. Then it follows that λ^2 is increasing at a constant rate with the time. Denoting $\left(\frac{\lambda^2}{\lambda_\omega^2} - 1\right)$ by β

$$\lambda^2 = \lambda_0^2 + 10\beta\nu t \tag{11}$$

where λ_0^2 is the value of λ^2 at $t = 0$. Then from (9) we obtain

$$\overline{u^2} = \frac{\overline{u_0^2}}{\left(1 + \frac{10\beta\nu}{\lambda_0^2}t\right)^{1/\beta}} \tag{12}$$

denoting by $\overline{u_0^2}$ the initial value of $\overline{u^2}$ at $t = 0$.

Applying equation (12) to Taylor's problem, i.e., replacing t by x/U, we obtain

$$\frac{1}{\sqrt{\overline{u^2}}} = \frac{1}{\sqrt{\overline{u_0^2}}}\left(1 + \frac{10\beta\nu x}{\lambda_0^2 U}\right)^{1/2\beta}. \tag{13}$$

The symbols $\sqrt{\overline{u^2}}$, $\sqrt{\overline{u_0^2}}$ denote root mean squares. Taylor's result is

$$\frac{1}{\sqrt{\overline{u^2}}} = \frac{1}{\sqrt{\overline{u_0^2}}} + \text{const. } \frac{x}{U}. \tag{14}$$

Our equation (13) gives this result as the first term of a series development. Obviously

$$\frac{1}{\sqrt{\overline{u^2}}} = \frac{1}{\sqrt{\overline{u_0^2}}} + \frac{5\beta_1^\nu}{\lambda_0^2\sqrt{\overline{u_0^2}}}\frac{x}{U} + \text{higher terms in } x. \tag{15}$$

If the linear law is correct for arbitrary large values of x, β must be equal to 1/2. Now β can be expressed by the coefficients of the development of the correlation function $R(y)$. We write

$$R(y) = 1 - \frac{y^2}{\lambda^2} + by^4 + \ldots$$

First it can be shown that $\overline{\left(\frac{\partial^2 u}{\partial y^2}\right)^2} = \overline{u^2}\left(\frac{d^4R}{dy^4}\right)_{y=0} = 24b\overline{u^2}$. Then going through an analysis of the correlation between the various products of the second derivatives of u, v, w, similar to that performed by Taylor for the first derivatives,[2] we obtain

$$\frac{\lambda^2}{\lambda_\omega^2} = \frac{28}{45}\frac{\overline{u^2}\,\overline{\left(\frac{\partial^2 u}{\partial y^2}\right)^2}}{\left[\overline{\left(\frac{\partial u}{y\partial}\right)^2}\right]^2} = \frac{28}{45}\frac{\left(\frac{d^4R}{dy^4}\right)_{y=0}}{\left(\frac{d^2R}{dy^2}\right)^2_{y=0}}.$$

In order to estimate possible values of $\frac{\lambda^2}{\lambda_\omega^2}$, the error function was first assumed as correlation function, substituting $R(y) = e^{-\frac{y^2}{\lambda^2}}$. This leads to $\frac{\lambda^2}{\lambda_\omega^2} = 28/15$ and $\beta = 13/15 = 0.87$. Taylor gives an example of the correlation function (see loc. cit., p. 45). Estimating b from the diagram $\frac{\lambda^2}{\lambda_\omega^2} = 1.63$ and $\beta = 0.63$. Plotting $\frac{1}{\sqrt{\overline{u^2}}}$ as a function of x with values of $\beta > 1/2$, we obtain curves similar to those observed by H. L. Dryden. Unfortunately, no exact comparison between our theory and Dr. Dryden's experimental results is possible because apparently the decay of the inherent turbulence of the wind tunnel and the decay of the turbulence produced by the screens interfere with each other, and this interference modifies the law of decay, especially in the region of small intensity of turbulence in which the deviations from the linear law are significant.

Taylor found that the constant factor in the second term at the right side of (14) is proportional to $1/M$, where M is the linear dimension of the mesh which produces the turbulence. Obviously, we obtain the same result by assuming $\frac{\lambda_0^2\sqrt{\overline{u_0^2}}}{v}$ proportional to M. Taylor made the arbitrary assumption that $\frac{\lambda^2\sqrt{\overline{u^2}}}{\nu}$ remains constant and proportional to M along the whole stream. In this new theory, it is sufficient to connect the initial values λ_0^2 and $\overline{u^2}$ with the characteristic length of the turbulence producing device, the values of λ^2 and $\overline{u^2}$ downstream are determined by differential equations. It is felt that this represents a progress from the theoretical point of view.

Considering the problem from a general point of view, it seems to the author that in an isotropic field the mean value of the turbulent fluctuations and the correlation function $R(y)$ being given at a certain instant $t = 0$, both $\overline{u^2}$ and R are determined for all positive values of t. In other words we must be able to obtain a partial differential equation for $R(y)$ with t and y as independent variables. As an illustration of such a general theory, let us consider the case which can be described as "one-dimensional turbulence." Let us assume that the fluid has only one degree of freedom corresponding to a motion in the x-direction. We can imagine, for instance, that sheets of fluid extending parallel to the x-z plane are oscillating irregularly in the x-direction. The velocity is a function of y and t; however, it will be assumed that the oscillations of the fluid sheets are so fast and the "wave length" of the oscillations so small that the mean value of $\overline{u^2}$ can be considered as a slowly varying function of y and t. We consider the particular case in which $\overline{u^2}$ is independent of y (homogeneous turbulence) and the correlation between the instantaneous velocities u_1 and u_2 of two sheets at $y = y_1$ and $y = y_2$ is a function of the time t and the relative distance $\eta = y_2 - y_1$. Let us now calculate the derivative of the product with respect to the time. Obviously

$$\frac{\partial}{\partial t}(\overline{u_1 u_2}) = \overline{\frac{\partial u_1}{\partial t}u_2} + \overline{u_1\frac{\partial u_2}{\partial t}}. \tag{16}$$

Now the hydrodynamic equations applied to the motion considered lead to

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} \tag{17}$$

and consequently

$$\frac{\partial}{\partial t}(\overline{u_1 u_2}) = \nu\left(\overline{\frac{\partial^2 u_1}{\partial y^2}u_2} + \overline{\frac{\partial^2 u_2}{\partial y_2^2}u_1}\right). \tag{18}$$

If the correlation function is denoted by $R(\eta, t)$

$$\overline{u_1 u_2} = \overline{u^2}R$$

and

$$\overline{\frac{\partial^2 u_1}{\partial y_1^2}u_2} = \overline{u_1\frac{\partial^2 u_2}{\partial y_2^2}} = \overline{u^2}\,\frac{\partial^2 R}{\partial \eta^2}.$$

Hence equation (18) can be written in the form

$$\frac{\partial}{\partial t}(R\overline{u^2}) = 2\nu\overline{u^2}\frac{\partial^2 R}{\partial \eta^2}. \tag{19}$$

Applying this equation when $\eta = 0$ and taking into account that $R(0) = 1$ and $\left(\frac{\partial R}{\partial t}\right)_{\eta=0} = 0$, we obtain

$$\frac{d}{dt}(\overline{u^2}) = 2\nu\overline{u^2}\left(\frac{\partial^2 R}{\partial \eta^2}\right)_{\eta=0} \tag{20}$$

and eliminating $\overline{u^2}$ from (19) and (20)

$$\frac{\partial R}{\partial t} = 2\nu\left\{\frac{\partial^2 R}{\partial \eta^2} - R\left(\frac{\partial^2 R}{\partial \eta^2}\right)_{\eta=0}\right\}. \tag{21}$$

The partial differential equation (21) is the fundamental equation of the problem considered. If R is given for $t = 0$ as a function of η, equation (21) determines the whole further development of the correlation and the mean value of the velocity fluctuation for all time.

The most interesting case is that in which the shape of the correlation function does not change with time. If the shape of R remains similar, $R(\eta, t)$ must be a function of the one dimensionless variable $\frac{\eta}{\sqrt{\nu t}}$. One of these solutions is, for example, $e^{-\frac{\eta^2}{8\nu t}}$. However, it is evident that in order to obtain such solutions, the initial shape itself must satisfy a certain total differential equation. In general the shape of the correlation curve will vary with the time. The practical consequence of this result is the conclusion that simple rules like Taylor's linear law for the decay of turbulence represent only approximations or special cases.

Using for $R(\eta)$ the development

$$R(\eta) = 1 - \frac{\eta^2}{\lambda^2} + b\eta^4 \tag{22}$$

$$\left(\frac{\partial^2 R}{\partial \eta^2}\right)_{\eta=0} = -\frac{2}{\lambda^2}$$

and equation (20) reads

$$\frac{d}{dt}\overline{u^2} = -4\nu\frac{\overline{u^2}}{\lambda^2} \tag{23}$$

which is analogous to our former equation (8) for the three-dimensional case. Substituting (22) in (21) and comparing the terms in η^2, we easily obtain

$$\frac{d\lambda^2}{dt} = 4\nu(6b\lambda^4 - 1) \tag{24}$$

which corresponds $\left(\text{putting } 6b\lambda^2 = \frac{1}{\lambda_\omega^2}\right)$ to (10). From (23) and (24) it follows that

$$\frac{d}{dt}\left(\frac{\overline{u^2}}{\lambda^2}\right) = -\ 4\nu\frac{\overline{u^2}}{\lambda^2\lambda_\omega^2} \tag{25}$$

and introducing the vorticity fluctuation $\omega = -\frac{\partial u}{\partial y}$, $\overline{\omega^2} = 2\,\frac{\overline{u^2}}{\lambda^2}$.

$$\frac{\overline{\omega^2}}{dt} = -4\nu\frac{\overline{\omega^2}}{\lambda_\omega^2}. \tag{26}$$

Hence, the equation (23) for the dissipation of energy and the equation (26) for the dissipation of vorticity appear as applications of the general partial differential equation (21), to small values of η, i.e., to the immediate neighborhood of the point considered. In the special case that the correlation function preserves its shape throughout the whole motion, i.e., R is a function of $\frac{\eta^2}{\nu t}$ only, the ratio $\frac{\lambda}{\lambda_\omega}$ remains constant; this was the assumption made earlier in this paper. Then the whole process of the decay of turbulence is determined by the initial values of $\overline{u^2}$, λ and λ_ω, and we obtain equations analogous to (11) and (12).

The equation corresponding to (21) for three-dimensional turbulence is given elsewhere.

The author is indebted to C. B. Millikan and L. Howarth for discussions on the subject and for their assistance in calculations.

[1] *Proc. Roy. Soc. A*, **151**, 421 (1935).

[2] As a matter of fact, the analysis was carried out by means of a more general and more direct method than that used by Taylor. The new method which uses the conception of the "correlation tensor" will be published elsewhere.