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Discrete self similarity in filled type I strong explosions

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We present new solutions to the strong explosion problem in a non power law density profile. The unperturbed self similar solutions developed by Sedov, Taylor, and Von Neumann describe strong Newtonian shocks propagating into a cold gas with a density profile falling off as $r^{-\omega}$, where $\omega \leq \frac{7-\gamma}{\gamma+1}$ (filled type I solutions), and γ is the adiabatic index of the gas. The perturbations we consider are spherically symmetric and log periodic with respect to the radius. While the unperturbed solutions are continuously self similar, the log periodicity of the density perturbations leads to a discrete self similarity of the perturbations, i.e., the solution repeats itself up to a scaling at discrete time intervals. We discuss these solutions and verify them against numerical integrations of the time dependent hydrodynamic equations. This is an extension of a previous investigation on type II solutions and helps clarifying boundary conditions for perturbations to type I self similar solutions. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4835375]

I. INTRODUCTION

Expanding shock waves are naturally produced by diverse astrophysical phenomena, such as supernovae, gamma ray bursts, and stellar winds. So far, analytical self similar solutions have been found for several simple cases, of which we take special interest in the case of strong spherical shocks propagating into a density profile that decays as a power of the radius

$$\rho_a(r) = kr^{-\omega}. \tag{1}$$

The first solutions of this kind to be found, now commonly known as the Sedov-Taylor-Von-Neumann solutions,[17] for the case $\omega < 3$ describe decelerating shocks. The solutions are based on the conservation of energy inside the shocked region, and they are called type I solutions. If $\omega < \frac{7-\gamma}{\gamma+1}$, where γ is the adiabatic index of the ambient gas, then the explosion is filled, i.e., the pressure is greater than zero anywhere inside the shocked region. If $\frac{7-\gamma}{\gamma+1} < \omega < 3$, then the explosion is hollow, i.e., the pressure (and the density) vanishes at a finite radius.[19] If $\omega = \frac{7-\gamma}{\gamma+1}$, then the hydrodynamic equations admit a relatively simple solution known as the Primakoff solution.[16] If $\omega > 3$ the energy diverges at the center, so energy conservation no longer applies and a different condition must be used.[19] In this paper we will focus on filled type I explosions ($\omega \leq \frac{7-\gamma}{\gamma+1}$).

The solutions discussed above, while useful, falls short when describing shocks propagating into density profiles that deviate from a simple power law decay. This might occur in a variety of astrophysical scenarios. One example could be the propagation of an outward shock wave in a stratified core collapse supernova progenitor.[7] Another example might be the interaction of a supernova shock wave with a circumstellar bubble.[3] Such bubbles form around progenitors that emit strong stellar wind that pushes the circumstellar wind away, so when the shock emerges from the progenitor, it first interacts with a low density medium inside the bubble, and later with the higher density medium outside. One example that we will dwell on is the variation of the luminosity due to the interaction of a supernova shock wave with a heterogeneous interstellar material.

From the reasons mentioned above, one could understand the need to generalize as much as possible the external density profile for which we can obtain analytic solutions, and this is what we attempt here. This paper takes after a similar endeavor for type II solutions.[12]

The idea of applying perturbation theory to the strong explosion problem is not new, but so far it focused on stability analysis. Throughout the years it has stirred up many controversies, most of which regarding inner boundary conditions. The stability of type I explosions was first studied by Bernstein and Book,[1] but their analysis was later refuted by Gaffet.[5,6] Consequently, a new perturbation theory was proposed by Ryu and Vishniac.[13,14,18] However, Kushnir and Waxman pointed out a possible error with the analysis of Ryu and Vishniac, and proposed yet another boundary condition to the perturbation theory.[8] Numerical simulations[10] and experiments with high power lasers[4] are in general agreement with the results of Ryu and Vishniac. The bone of contention in these controversies is the inner boundary conditions, i.e., the value of the hydrodynamic variables at the center. This paper will attempt to shed light on the question of the correct boundary conditions.

The plan in this paper is as follows: In Sec. II we review the unperturbed solutions and the boundary conditions at the front and at the center. In Sec. III we develop the perturbation equations and boundary conditions. We then discuss the solutions to these equations and compare them to numerical results obtained from a full hydrodynamic simulation, and finally we conclude in Sec. IV.

II. THE UNPERTURBED SOLUTIONS

We proceed to give a quick review of the unperturbed solutions under considerations.[16] The physical scenario is the deposition of a large amount of energy from a point source at the center of a spherically symmetric distribution of cold gas. It may be noted that spherical symmetry was chosen for its relevance to most astrophysical scenarios, but planar and cylindrical geometries may readily be treated as well. The gas density follows a power law behavior (Eq. (1)).

A. The hydrodynamic equations

We begin with the Euler equation for an ideal fluid with adiabatic index γ in spherical symmetry

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho u\right) = 0, \tag{2}$$

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\rho c^2}{\gamma}\right) = 0, \tag{3}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)\ln\left(\frac{c^2}{\gamma \rho^{\gamma-1}}\right) = 0. \tag{4}$$

These equations feature the density ρ, velocity u, and speed of sound c as the dependent variables. They are usually expressed in terms of the pressure p rather than the speed of sound, and they are related by

$$c^2 = \gamma\frac{p}{\rho}. \tag{5}$$

We define dimensionless variables

$$r = R\left(t\right)\xi, \tag{6}$$

$$u\left(r,t\right) = \dot{R}\xi U\left(\xi\right), \tag{7}$$

$$c\left(r,t\right) = \dot{R}\xi C\left(\xi\right), \tag{8}$$

$$\rho\left(r,t\right) = kR^{-\omega}G\left(\xi\right), \tag{9}$$

$$p\left(r,t\right) = kR^{-\omega}\dot{R}^2 P\left(\xi\right), \tag{10}$$

where $R(t)$ is the shock radius. It is assumed that the shock radius has power law dependence on time

$$R\left(t\right) = A\left(t-t_0\right)^{\alpha}. \tag{11}$$

B. Boundary conditions

The boundary conditions at the front are determined by the Rankine Hugoniot shock conditions:[9]

$$U(\xi = 1) = \frac{2}{\gamma + 1}, \tag{12}$$

$$C(\xi = 1) = \frac{\sqrt{2\gamma(\gamma - 1)}}{\gamma + 1}, \tag{13}$$

$$G(\xi = 1) = \frac{\gamma + 1}{\gamma - 1}, \tag{14}$$

$$P(\xi = 1) = \frac{2}{\gamma + 1}. \tag{15}$$

The power law index α is determined by the boundary conditions at the center. In principle, the center of an explosion can either be a source or a sink of energy. If the energy injection is power law of the time, than it is possible to obtain self similar solutions.[14] It was shown that energy injection always creates a hollow explosion,[14] as if the extra energy was the work exerted by an expanding spherical piston. The condition that the energy is conserved is therefore equivalent to the condition that the velocity vanishes at the center.

The total energy contained in the explosion is given by

$$E = 4\pi \int_0^R \left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1}\right) r^2 dr \propto k R^{3-\omega} \dot{R}^2$$

and the right-hand side is independent of time only if

$$\alpha = \frac{2}{5 - \omega}. \tag{16}$$

C. Thin shell model

As $\gamma \to 1$, the compression (i.e., ratio between the shocked and unshocked matter) increases, and matter is concentrated into a thinner shell, while the interior contains gas with a finite pressure and negligible density.[13] The density in the shell diverges, but the surface mass density remains finite

$$\sigma = \frac{\rho_a(R) R}{3 - \omega}. \tag{17}$$

The density in the interior (behind the shell) vanishes. The pressure inside the shell is obtained from Rankine Hugoniot equations

$$p_f = \rho_a(R) \dot{R}^2, \tag{18}$$

but the pressure in the interior is

$$p_i = \frac{1}{2}\rho_a(R) \dot{R}^2, \tag{19}$$

this expression can be obtained from the implicit solution for the dimensionless pressure as a function of the dimensionless velocity.[9] The material velocity at the front is equal to the velocity of the shock

$$u_f = \dot{R}. \tag{20}$$

Since the density vanishes at the center, one might confuse it with a hollow explosion. However, in hollow explosions the pressure vanishes at a finite radius, while in this case the pressure remains finite throughout.

We now turn to the energy balance of such explosion. Energy can be distributed as either thermal or kinetic, and can be either inside the shell or behind it. The kinetic energy behind the shell is negligible because there is no mass there, and the thermal energy of the shell is negligible because its volume is very small. As $\gamma \to 1$, the kinetic energy of the shell remains finite, but the thermal energy behind the shell diverges, because it is proportional to $(\gamma - 1)^{-1}$. Hence most of the energy is concentrated behind the shell as thermal energy. We can also use this approximation to find the relation between the energy and the trajectory of the shock front

$$E = \frac{4\pi}{3} R^3 \frac{p_i}{\gamma - 1} = \frac{4\pi}{6} R^3 \dot{R}^2 \frac{\rho_a(R)}{\gamma - 1}. \tag{21}$$

Substituting Eq. (11) yields

$$A = \left[\left(\frac{5-\omega}{2} \right)^2 \frac{6(\gamma - 1)}{4\pi} \frac{E}{K} \right]^{1/(5-\omega)}. \tag{22}$$

We will later use this model to obtain analytic results for perturbations in a gas with $\gamma \to 1$. A relevant question in this context is whether outside perturbations manage to cross the thin, dense shell and affect the inner region. On the one hand, the width of the shell goes to zero, but on the other hand, so does the speed of sound. From mass conservation and the Rankine Hugoniot relations, the width of the shell is

$$\frac{\Delta R}{R} = \frac{\gamma - 1}{(\gamma + 1)(3 - \omega)} \tag{23}$$

while the speed of sound at the shock front goes as

$$c_f = \frac{\sqrt{2\gamma(\gamma - 1)}}{\gamma + 1} \dot{R} \tag{24}$$

so the time it takes for information to cross the shell scales as $\sqrt{\gamma - 1}$, and is therefore much smaller than the time it takes the explosion to double its size when $\gamma \to 1$.

D. Primakoff solution

As was mentioned earlier, when $\omega = \frac{7-\gamma}{\gamma+1}$ the hydrodynamic equations admit a simple analytic solution

$$U = \frac{2}{\gamma + 1}, \tag{25}$$

$$C = \frac{\sqrt{2\gamma(\gamma - 1)}}{\gamma + 1}, \tag{26}$$

$$G = \frac{\gamma + 1}{\gamma - 1} \xi, \tag{27}$$

$$P = \frac{2}{\gamma + 1} \xi^3, \tag{28}$$

we will later see that for this solution it is possible to obtain analytic solutions for the perturbation equations.

III. DISCRETE SELF SIMILAR PERTURBATIONS

A. The perturbation equations

We now come to the case of a perturbed density profile. For the perturbation equation to be tractable we aim at a self similar solution by carefully choosing a perturbation whose characteristic wavelength scales like the radius. Namely, we take the perturbed density profile to be

$$\rho_a(r) + \delta\rho_a(r) = k r^{-\omega} \left(1 + \varepsilon \left(\frac{r}{r_0} \right)^q \right), \tag{29}$$

where r_0 has dimensions of length and bears only on the phase of the perturbation, q is the growth rate of the perturbation, and ε is a small, real, and dimensionless amplitude. We take the real part of any hydrodynamic complex quantity to be the physically significant element.

We define perturbed flow variables

$$u(r,t) + \delta u(r,t) = \dot{R}\xi \left[U(\xi) + f(t)\,\delta U(\xi)\right], \tag{30}$$

$$\rho(r,t) + \delta\rho(r,t) = kR^{-\omega}\left[G(\xi) + f(t)\,\delta G(\xi)\right], \tag{31}$$

$$p(r,t) + \delta p(r,t) = kR^{-\omega}\dot{R}^2\left[P(\xi) + f(t)\,\delta P(\xi)\right], \tag{32}$$

$$R(t) + \delta R(t) = R(t)\left[1 + f(t)\right]. \tag{33}$$

To allow separation of variables, the function $f(t)$ must satisfy

$$f(t) = \frac{\varepsilon}{d}\left(\frac{R}{r_0}\right)^q \Rightarrow \frac{\dot{f}R}{f\dot{R}} = q. \tag{34}$$

Where the parameter d represents the amplification of each mode, and is determined by boundary conditions, as explained in Subsection III B. If q is imaginary, the real part of $f(t)$ is periodic, the solution is discretely self similar, i.e., it repeats itself up to a scaling factor in intervals of $\frac{\Delta R}{R} = \exp\left(\frac{2\pi}{|q|}\right) - 1$. While the unperturbed solution and the perturbations in their complex form are both self similar, the physical solution which is the real part of their sum is not.

Plugging the perturbed hydrodynamic variables into the hydrodynamic equations yields dimensionless ordinary differential equations (ODEs) for the perturbed variables.[12]

B. Boundary conditions for the perturbations

The boundary conditions for the perturbed variables at the blast front are derived in a similar way to Refs. 2 and 13 and are identical to those appearing in Ref. 12

$$\delta G(\xi = 1) = \frac{\gamma+1}{\gamma-1}(d-\omega) - G'(1), \tag{35}$$

$$\delta U(\xi = 1) = \frac{2}{\gamma+1}q - U'(1), \tag{36}$$

$$\delta P(\xi = 1) = \frac{2}{\gamma+1}\left[2(q+1) - \omega + d\right] - P'(1). \tag{37}$$

In analogy to the unperturbed solution, where the parameter α is determined by the inner boundary conditions or total conservation of energy, the parameter d is determined by the same considerations. Integration of the self similar ODEs from the front to center with the wrong value of d would yield non zero velocity at the center, so the energy flux does not vanish, and the total energy is not conserved. We recall that the energy flux is given by $u\left(\frac{\gamma}{\gamma-1}p + \frac{1}{2}\rho u^2\right)$, but since the unperturbed density and velocity vanish at the center in filled type I explosions, the first order contribution to the flux would be $\frac{\gamma}{\gamma-1}p \cdot \delta u$. Hence, it is sufficient to require that δu would vanish at the center. Near the center, the derivatives of the self similar variables reduce to

$$\frac{d}{d\xi}\left(\frac{\delta U}{U}\right) = \frac{-q(\delta P/P) - 3(\delta U/U)}{\xi} + O\left(\xi^0\right), \tag{38}$$

$$\frac{d}{d\xi}\left(\frac{\delta P}{P}\right) = 0 + O\left(\xi^0\right). \tag{39}$$

Hence for generic values, the pressure perturbation would be constant, and the velocity perturbation would diverge as ξ^{-3}. Recalling that the power radiated from the center is $r^2 p \delta u \propto \xi^3 \delta U$, we see that choosing the wrong boundary condition would mean energy transfer through the center (periodic, if q is imaginary). The condition for preventing the divergence of the velocity perturbation is

$$\frac{\delta U\,(\xi = 0)}{U\,(\xi = 0)} = -\frac{q}{3}\frac{\delta P\,(\xi = 0)}{P\,(\xi = 0)}. \tag{40}$$

In case of Primakoff explosions, the pressure also vanishes at the center, so they require a different treatment (the energy also does not change, but the conditions at the center are different). A more detailed discussion of perturbations to Primakoff explosions is given in Sec. III F.

We note that condition (40) is different from both Refs. 13 and 8. The reason is that they treated angular perturbations, where the total energy of every perturbation always averages out to zero after summing over all angles, so energy considerations do not apply. The method of Ryu and Vishniac, $\delta P(\xi = 0) = 0$, keeps the tangential velocity from diverging, so it is irrelevant for radial perturbations. Thus, we can understand why there should be two separate conditions for radial and angular perturbations. We also note that in similar problem, e.g., perturbations to type II explosions, the same inner boundary conditions are used both for radial[12] and angular perturbations.[15]

C. The discrete self similar solution

While self similarity simplifies the problem by reducing the partial differential equations (PDEs) to ODEs, the resulting ODEs, in general, do not admit analytic solutions. Therefore, for each specific set of parameters γ, ω, and q, the functions δG, δU, δP, and the parameter d are found numerically. Since the ODEs are linear, there exists a matrix that relates the vector of the values of the flow variables at the center to the same vector at the front

$$\begin{pmatrix} \delta G\,(1) \\ \delta P\,(1) \\ \delta U\,(1) \end{pmatrix} = \mathbf{M} \begin{pmatrix} \delta G\,(0) \\ \delta P\,(0) \\ \delta U\,(0) \end{pmatrix}. \tag{41}$$

It is possible to find this matrix numerically, since it is independent of d. Thus Eq. (41) and the boundary conditions constitute 4 linear equations for 4 variables (d, $\delta G(0)$, $\delta P(0)$, and $\delta U(0)$). Solving these equations yields the value of d.

A comparison between the solutions discussed above and a hydrodynamic simulation is presented in Figure 1. All curves seem to agree. The numerical calculations were carried out using the hydrocode PLUTO.[11] We have also verified that better accuracy can be achieved by increasing the resolution. However, infinite resolution will not reduce the error to zero, because of differences between the initial conditions in the simulation and those assumed in the mathematical formulation. One difference is the size of the initial hot spot. In the mathematical problem the hot spot is point like, while in the simulation it always has a finite size. Another difference is the ambient pressure, which is assumed to be zero in the mathematical problem, while in the simulation it is also finite in the simulation.

Figure 1 shows that the wavelength of the density fluctuations is shorter than those of the pressure and velocity. This happens because the density is affected by both traveling sound waves and entropy waves, while the pressure and velocity are affected solely by sound waves. From this argument it follows that the characteristic wavelengths are given by $\frac{2\pi}{|q|}\left(1 - \xi U \pm \sqrt{\gamma \frac{P}{G}}\right)$ for the pressure and velocity, together with $\frac{2\pi}{|q|}\left(1 - \xi U\right)$ for density perturbations.

Finally, Figures 2 and 3 show d as a function of $Im(q)$, relating the fractional perturbation in the shock position to the fractional perturbation in the external density, for $\omega = 0$ and $\gamma = \frac{5}{3}$. The oscillations are due to the diffraction of the incident wave from the blast front, with wave reflected from the center. This property is qualitatively different from the behavior of the same curves plotted for type II explosions.[12] In type II explosions, sound waves mostly travel from the front to sonic point, and not the other way around, and that is why the $d(Im(q))$ curves for type II explosions are monotonous.

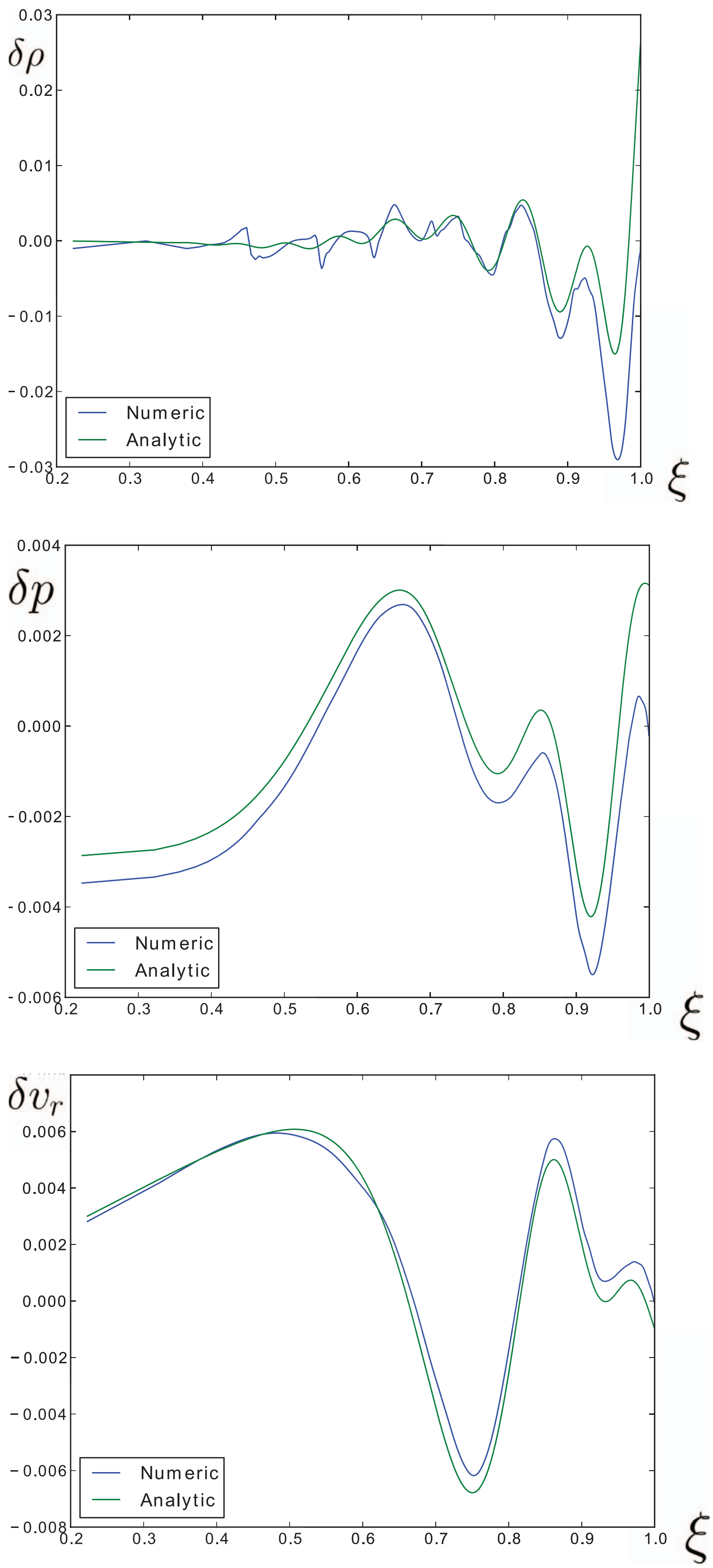

FIG. 1. Comparison of the analytic and numeric profiles of the perturbed hydrodynamic variables: density (top), pressure (middle), and velocity (bottom). The explosion parameters are $\gamma = \frac{5}{3}$, $\omega = 0$, $q = 20i$, and $\varepsilon = 0.01$.

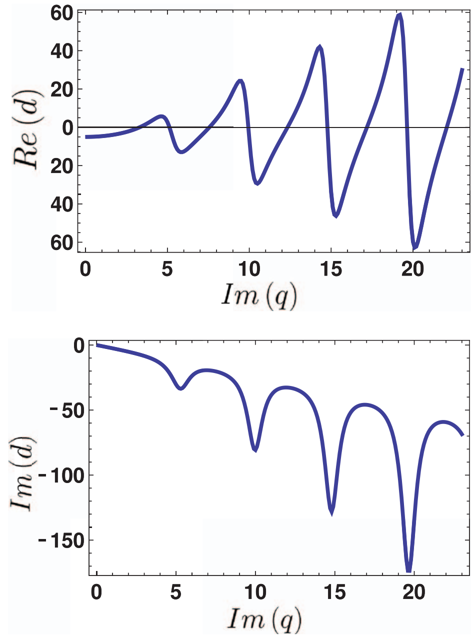

FIG. 2. The real (top) and imaginary (bottom) part of d as a function of $Im(q)$, for an explosion with $\gamma = \frac{5}{3}$ and $\omega = 0$.

D. Long wavelength limit

Perturbations with $q = 0$ correspond to perturbations in the coefficient K of the ambient density (Eq. (1)). From units considerations we know that $E \propto KA^{5-\omega}$, so if the energy is conserved $A \propto K^{\frac{1}{\omega-5}}$ and

$$d\,(q=0) = \omega - 5. \tag{42}$$

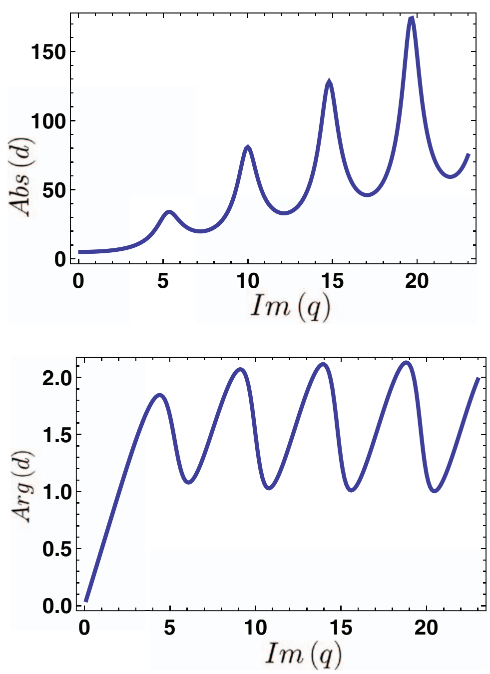

FIG. 3. Absolute value (top) and phase (bottom) of the parameter d as a function of $Im(q)$ for an explosion with $\gamma = 5/3$ and $\omega = 0$.

E. Thin shell model

In the thin shell model ($\gamma \to 1$) the total energy is given by $\frac{4\pi}{3} R^3 p_i = \frac{4\pi}{6} R^3 \rho_a (R) \dot{R}^2$. From the conditions that the energy remains constant $\delta \left(R^3 \rho_a \dot{R}^2\right) = 0$, we obtain the relation

$$d = \omega - 5 - 2q. \tag{43}$$

In the limit $q \to 0$ Eq. (43) reduces to (42).

F. Primakoff solution

In the case of the Primakoff explosion, the perturbation equations can be solved analytically. With the substitution

$$\mathbf{Y} = \left(\frac{\delta G}{G}, \frac{\delta P}{P}, \frac{\delta U}{U}\right)^T, \tag{44}$$

the system of ODEs can be reduced to the form

$$\frac{d\mathbf{Y}}{d \ln \xi} = \mathbf{M} \cdot \mathbf{Y}, \tag{45}$$

$$\mathbf{M} = \begin{pmatrix} \frac{6(\gamma-1)+q(\gamma+1)^2}{\gamma^2-1} & -\frac{2(-3+q+3\gamma+q\gamma)}{\gamma^2-1} & -\frac{2(7+q-\gamma+q\gamma)}{\gamma^2-1} \\ \frac{6\gamma}{\gamma+1} & -\frac{q+6\gamma+q\gamma}{\gamma+1} & -\frac{2\left(-3+(q+5)\gamma+q\gamma^2\right)}{\gamma^2-1} \\ \frac{3(\gamma-1)}{\gamma+1} & -\frac{-3+q+3\gamma+q\gamma}{\gamma+1} & -\frac{11+q+3\gamma+q\gamma}{\gamma+1} \end{pmatrix}. \tag{46}$$

The solution is

$$\mathbf{Y}(\xi) = \exp\left(\mathbf{M} \ln \xi\right) \mathbf{Y}(1). \tag{47}$$

Every term in $\mathbf{Y}(\xi)$ is the sum of 3 power laws in ξ, where each power is an eigenvalue of $\mathbf{M}$.

It is possible to perform the total energy integral explicitly for this case. The parameter d is chosen such that the total energy remains the same. Another way to find d by calculating the energy flux at the center and requiring that it be equal to zero. Both ways are mathematically equivalent, but the latter is computationally easier. We were not able to obtain an explicit expression for the parameter d, but for numerical values of γ, ω, and q the parameter d can be readily computed. The parameter d as a function of $Im(q)$ for $\gamma = \frac{5}{3}$ ($\omega = 2$) is given in Figure 4. We remark that that these curves are monotonous, whereas we saw earlier that for smaller ω the graphs are oscillating. The reason is that there is no reflection from the center in the case of Primakoff explosions, because the speed of sound vanishes there. Therefore, the short wavelength limit discussed in Ref. 12 also applies to the Primakoff solution, so

$$\lim_{q\to\infty} \frac{d}{q} = -\sqrt{2 + \frac{2\gamma}{\gamma - 1}}. \tag{48}$$

The derivation of this result is based on the assumption that there are no waves emanating from the center, so the outward going Riemann invariant does not change. The same argument cannot be applied to general filled type I explosions, because of the reflection from the center.

IV. DISCUSSION

We have laid out a method for solving the strong explosion problem in density profiles that deviate from a pure power law radial dependence. The key lies in choosing radially log periodic perturbations which do not introduce a new scale into the problem. This leads to self similar perturbation in the hydrodynamic quantities behind the shock, which can be found by solving a set of ordinary differential equations. It is possible to obtain self similar equations for the perturbations when the density perturbation is given in Eq. (29), but if q is imaginary, then the solution is only

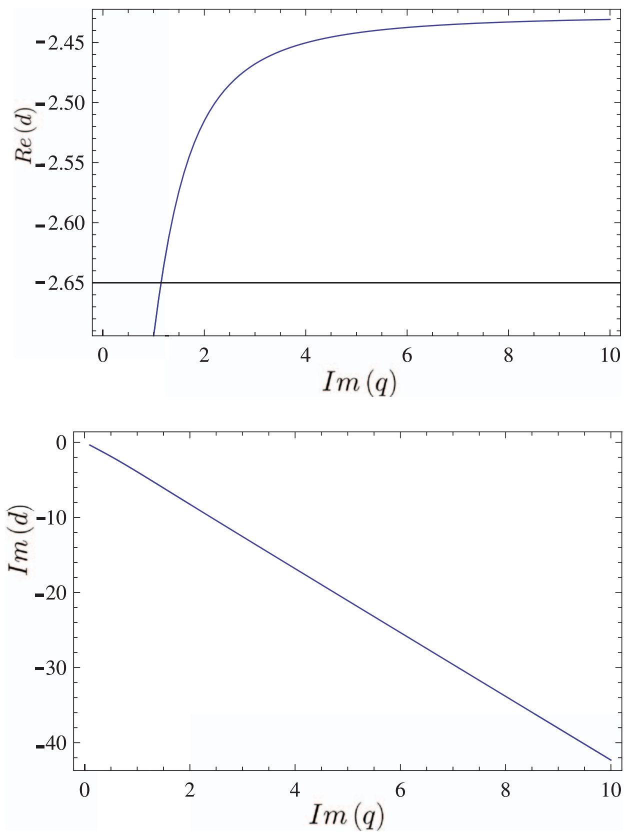

FIG. 4. The real (top) and imaginary (bottom) parts of d, as a function of $Im(q)$, for $\gamma = \frac{5}{3}$ and $\omega = 2$ (the Primakoff solution).

discretely self similar because of the periodic nature of the perturbations. We find that the coefficient d that relates the amplitude of the perturbations in the shock position with the amplitude of the density perturbations has a $O(1)$ real part and an $O(Im(q))$ imaginary part, so at the short wavelength limit, $Im(q) \gg 1$, $|d|$ increases. From the boundary conditions at the shock front (Eqs. (35)–(37)) we see that the absolute value of the dimensionless variables increases with q. The dimensional perturbed variables are proportional to the dimensionless variables divided by d, so at high values of q their amplitudes tend to a plateau.

The linearized perturbation treatment naturally ensures that the perturbations will be linear in ε. This simplifies the solution of the problem but limits the validity of the method to small perturbations. The perturbation theory developed above fails when ε becomes too large. The deviation from linear theory is of order ε^2. It is possible to obtain a more quantitative assessment of the difference by considering the long wavelength limit.

Since these perturbations are linear, it is possible to represent arbitrary small deviations of a density profile from a power law by a sum of different mode, as was done for type II solutions.[12]

The crux of the problem discussed is choosing the correct inner boundary conditions. The boundary conditions used here are different from both that of Ryu and Vishniac, and that of Kushnir and Waxman. However, they discussed angular perturbations, while we discuss radial perturbations only, and we claim that the inner boundary conditions for radial perturbations must be different from those of angular perturbations. The reason is that radial inner boundary conditions are based on energy conservation, which is irrelevant in angular perturbation as all modes conserve energy.

We conclude with an example of an astrophysical relation: the relation of a supernova remnant bolometric luminosity to density modulation in the interstellar medium. Let us consider a supernova remnant shockwave that propagates into the interstellar medium with a density ρ_a distributed in the form of Eq. (29). If the emitted flux would be some small fraction of the hydrodynamic energy flux ρv^3, the variation of the luminosity would be

$$\delta \ln L = \frac{\delta L}{L} = \delta \ln \left(\rho v^3 R^2\right) = \frac{\delta \rho}{\rho} - 3\frac{\delta v}{v} + 2\frac{\delta R}{R}. \tag{49}$$

We give explicit results for the case $\omega = 0$, $\gamma = 5/3$ and use the approximation for a thin shockwave $d = \omega - 5 - 2q$. From Eqs. (35)–(37) we get

$$\frac{\delta L}{L} = \frac{12 + 5q}{5 + 2q}\frac{\delta \rho_a}{\rho_a}. \tag{50}$$

This equation relates variations in the surrounding density to observed flux. In the limit $q \to 0$, where the wavelength of the perturbation is long, the relative variations in the luminosity are 2.4 times larger than the relative density variations, and both are in phase.

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