1928. (The transitions $4d^9 5p$ to $4d^9 5s$ of Cd III have since been published by McLennan, McLay and Crawford, *Trans. Roy. Soc. Can.*, **22**, III, 1928.) They did not determine the $^1S_0(4d^{10})$ level. The independent identifications made in the two laboratories are in complete agreement. Paper by the authors will appear in the May number of the *Physical Review*.

[5] A. G. Shenstone, *Physic Rev.*, **29**, 380, 1927; Laporte and Lang, *Physic. Rev.*, **30**, 378, 1927.

[6] Gibbs and White, *Physic. Rev.*, **31**, 309, 1928.

ON THE ENERGY AND ENTROPY OF EINSTEIN'S CLOSED UNIVERSE

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1. *Introduction.*—In a preceding article,[1] two principles have been proposed, expressed in a form valid for all sets of coördinates which seem suitable to serve as the analogues in general relativity for the first and second laws of thermodynamics. It is the purpose of the present article to apply these principles to the Einstein closed universe, regarded as filled with a perfect fluid, so as to obtain expressions which may be taken as representing the energy and entropy of this universe. In a following article we shall then use these expressions to investigate the equilibrium between radiation and matter in such a universe, a problem which has recently been attacked in a very stimulating manner by Lenz.[2]

In carrying out our computations, we shall not regard the pressure in the universe as necessarily negligible compared with the energy density, as has hitherto always been done in treatments of the Einstein universe. Our present necessity for abandoning this simplifying assumption arises from the fact that pressure and energy density are necessarily of the same order of magnitude for the case of radiation, and this rules out the simplification in a treatment of the equilibrium between matter and radiation. Owing to this fact that we do not neglect the pressure, the expression that we obtain for the energy of the universe will differ from that previously obtained by Einstein[3] with the use of the simplifying assumption. Our expression for energy also differs from that tacitly taken by Lenz for his treatment of the equilibrium between radiation and matter, but our expression for the entropy turns out to be the same as that which he assumed.

2. *Metrical Properties of the Einstein Universe.*—Before we can apply the laws of thermodynamics to the Einstein universe, we must consider

its metrical properties. These are contained in the form assumed for the line element

$$ds^2 = -R^2 d\chi^2 - R^2 \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) + dt^2 \tag{1}$$

giving us

$$g_{11} = -R^2, \quad g_{22} = -R^2 \sin^2 \chi, \quad g_{33} = -R^2 \sin^2 \chi \sin^2 \theta, \quad g_{44} = 1 \tag{2}$$
$$g^{\mu\nu} = 1/g_{\mu\nu} \; (\mu = \nu), \qquad g^{\mu\nu} = g_{\mu\nu} = 0 \quad (\mu \neq \nu)$$

and

$$\sqrt{-g} = R^3 \sin^2 \chi \sin \theta. \tag{3}$$

Furthermore, these values of the fundamental tensor $g_{\mu\nu}$ lead, as is well known,[4] to the following components of the contracted Riemann-Christoffel tensor

$$G_{11} = -2, \quad G_{22} = -2 \sin^2 \chi, \quad G_{33} = -2 \sin^2 \chi \sin^2 \theta, \quad G_{44} = 0 \tag{4}$$

and for the invariant spur of this tensor

$$G = 6/R^2. \tag{5}$$

3. *Relation between Metrical Properties and Energy Tensor.*—The above metrical properties will be related to the energy tensor $T_{\mu\nu}$ for matter and electricity by the fundamental equation[5]

$$-8\pi T_{\mu\nu} = G_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} G + \lambda g_{\mu\nu} \tag{6}$$

where the factor 8π simplifies the coördination of units and λ is a constant small enough so that the last term can be neglected except in the consideration of very large systems such as the whole universe. The justification for this equation is given by the consideration that the divergence of the left-hand side vanishes because of the physical facts concerning the behavior of energy and momentum, and the divergence of the right-hand side vanishes identically. Hence, if we desire a geometrical interpretation of energy and momentum the identification of the two sides of (6) appears natural.

To obtain an expression for the energy tensor $T_{\mu\nu}$ we shall assume the substance filling the universe to be a uniformly distributed perfect fluid, since we shall later wish to consider it as a mixture of perfect gas and radiation. For such a fluid it is well known that the energy tensor assumes the form[6]

$$T^{\mu\nu} = (\rho_{00} + p) \frac{dx_\mu}{ds} \frac{dx_\nu}{ds} - g^{\mu\nu} p \tag{7}$$

where ρ_{00} is the *proper macroscopic* density of the fluid, p its *proper* pressure and $d\chi_\mu/ds$ refers to its *macroscopic* motion. Since the fluid is taken as macroscopically stationary in the coördinates chosen, we have

$$\frac{d\chi}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0, \qquad \frac{dt}{ds} = 1 \tag{8}$$

and the energy tensor reduces to the terms in its leading diagonal

$$T^{11} = -g^{11}p, \quad T^{22} = -g^{22}p, \quad T^{33} = -g^{33}p, \quad T^{44} = \rho_{00} \tag{9}$$

or lowering suffixes

$$T_1^1 = -p, \quad T_2^2 = -p, \quad T_3^3 = -p, \quad T_4^4 = \rho_{00} \tag{10}$$

and lowering suffixes again

$$T^{11} = -g_{11}p \qquad T_{22} = -g_{22}p \qquad T_{33} = -g_{33}p \qquad T_{44} = \rho_{00}. \tag{11}$$

By substituting equations (2), (4), (5) and (11) into (6) we now easily obtain the desired relations between the material properties of the universe as given in terms of ρ_{00} and p and the metrical properties as given in terms of R and λ,

$$8\pi p = -\frac{1}{R^2} + \lambda \tag{12}$$

$$8\pi\rho_{00} = \frac{3}{R^2} - \lambda \tag{13}$$

$$\rho_{00} + p = \frac{1}{4\pi R^2}. \tag{14}$$

4. *Application of the First Law.*—We are now ready to apply the first law of thermodynamics as given in the previous article (l. c.) by the equation

$$\int\int\int\int\left(\frac{\partial \mathfrak{T}_\mu^\nu}{\partial x_\nu} - \frac{1}{2}\mathfrak{T}^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x_\mu}\right)dx_1dx_2dx_3dx_4 = 0 \tag{15}$$

where the integration is to be taken over the total spatial volume of the universe and over any desired time interval. Equation (15) is not a tensor equation since the integration is taken over a finite region, nevertheless as pointed out in the previous article the equation is true in all sets of coördinates since the integrand is itself equal to zero in all sets of coördinates at all points in space-time.

To use equation (15) for our present purposes it is desirable to transform the integrand into the form of an ordinary divergence by re-expressing the second term in the integrand in terms of Einstein's pseudo-tensor of gravitational energy. We obtain

$$\int\int\int\int\frac{\partial(\mathfrak{T}_\mu^\nu + t_\mu^\nu)}{\partial x_\nu}dx_1dx_2dx_3dx_4 = 0 \tag{16}$$

where the pseudo-tensor $\mathfrak{t}_\mu^\nu$ is known[7] to have the following value in terms of the fundamental tensor and its derivatives,

$$\mathfrak{t}_\mu^\nu = \frac{1}{16\pi}\left(g_\mu^\nu \mathfrak{L} - \mathfrak{g}_\mu^{\alpha\beta}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_\nu^{\alpha\beta}}\right) + \frac{1}{8\pi} g_\mu^\nu \lambda \sqrt{-g} \qquad (17)$$

with

$$\left.\begin{aligned} \mathfrak{L} &= g^{\rho\sigma}\sqrt{-g}\,[\{\rho\alpha,\beta\}\{\sigma\beta,\alpha\} - \{\rho\sigma,\alpha\}\{\alpha\beta,\beta\}] \\ \mathfrak{g}_\mu^{\alpha\beta} &= \frac{\partial}{\partial x_\mu}(g^{\alpha\beta}\sqrt{-g}) \\ \frac{\partial \mathfrak{L}}{\partial \mathfrak{g}_\nu^{\alpha\beta}} &= g_\alpha^\nu\{\beta\rho,\rho\} - \{\alpha\beta,\nu\}. \end{aligned}\right\} \qquad (18)$$

and

The calculation of the pseudo-tensor with the help of these equations is very lengthy. Nevertheless, evaluating the Christoffel three index symbols in terms of the fundamental tensor as given by equations (2), and substituting in (18) and (17), we finally obtain in agreement with Einstein (l. c.)

$$\left.\begin{aligned} &8\pi\mathfrak{t}_1^1 = R\cos^2\chi\sin\theta + \lambda\sqrt{-g}, \quad 8\pi\mathfrak{t}_2^1 = R\sin\chi\cos\chi\cos\theta \\ &8\pi\mathfrak{t}_2^2 = 8\pi\mathfrak{t}_3^3 = 8\pi\mathfrak{t}_4^4 = -R\cos^2 x\sin\theta + \lambda\sqrt{-g} \end{aligned}\right\} \qquad (19)$$

the values with all other combinations of the indices being zero.

We are now ready to apply equation (16). We shall be interested for thermodynamic purposes in the case $\mu = 4$ which gives information as to energy relationships, cases $\mu = 1$, 2 and 3 giving information as to momenta. Referring to equations (10) and (19), we note that $(\mathfrak{T}_4^\nu + \mathfrak{t}_4^\nu)$ and $\partial(\mathfrak{T}_4^\nu + \mathfrak{t}_4^\nu)/\partial\chi_\nu$ are equal to zero for all values of ν except $\nu = 4$ and our principle as given by equation (16) reduces to

$$\int_t^{t'}\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\frac{\partial}{\partial t}\left(\rho_{00}\sqrt{-g} - \frac{R}{8\pi}\cos^2\chi\sin\theta + \frac{\lambda}{8\pi}\sqrt{-g}\right)d\chi d\theta d\phi dt = 0. \qquad (20)$$

And substituting the values of λ and $\sqrt{-g}$ given by equations (12) and (3) this can be rewritten in the form

$$\int_t^{t'}\frac{d}{dt}\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\left\{(\rho_{00}+p)\sqrt{-g} - \frac{R}{8\pi}\cos^2\chi\sin\theta + \frac{R}{8\pi}\sin^2\chi\sin\theta\right\}d\chi d\theta d\phi dt = 0. \qquad (21)$$

On integration with respect to χ we find, however, that the last two terms contribute nothing to the expression, and we obtain the simple result

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} (\rho_{00} + p)\sqrt{-g}\, d\chi d\theta d\phi = \text{const.} \tag{22}$$

where the constant is independent of the time.

Furthermore, in our coördinates in accordance with equations (8), t is the *proper* time so that we can write

$$d\chi = \sqrt{-g}\, d\chi d\theta d\phi dt = dV dt$$

and

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \sqrt{-g}\, d\chi d\theta d\phi = 2\pi^2 R^3 \tag{23}$$

where dV is the element of *proper spatial* volume occupied by the material in the universe. Hence, equation (22) can also be written in the form

$$(\rho_{00} + p)V = \text{const.} \tag{24}$$

where V is the total spatial volume of the universe.

We may regard this quantity $(\rho_{00} + p)V$, whose value doesn't change with the time, as an expression for the energy of the universe. It differs from the quantity $\rho_0 V$ obtained by Einstein (l. c.), since we have taken the pressure into account, and also differs from the value assumed by Lenz, who used an expression which in our nomenclature would be $\rho_{00}V$.

5. *Application of the Second Law.*—We must now consider the application of the second law of thermodynamics as given in the previous article (l. c.) by the equation

$$\iiiint \frac{\partial \mathfrak{S}^\mu}{\partial x_\mu} dx_1 dx_2 dx_3 dx_4 \geqq 0. \tag{25}$$

For the entropy vector we have the equation of definition given in the previous article

$$S^\mu = \phi_0 \frac{dx_\mu}{ds} \tag{26}$$

where ϕ_0 is the proper density of entropy and dx_μ/ds refers to the macroscopic motion of the matter in the universe. Referring to equations (8), we see that the entropy vector reduces to a very simple form and equation (25) becomes

$$\int_t^{t'}\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \frac{\partial}{\partial t}(\phi_0\sqrt{-g})d\chi d\theta d\phi dt \geqq 0 \tag{27}$$

which in accordance with equation (23) can be rewritten in the form

$$(\phi_0 V)_{t'} - (\phi_0 V)_t \geqq 0. \tag{28}$$

Hence $\phi_0 V$ is a quantity which can only increase with the time and may be regarded as the entropy of the universe. This quantity proves to be equivalent to the expression that Lenz took for the entropy of the Einstein universe, but, in general, the expression for the entropy of a system in a gravitational field does not reduce to so simple a form.

[1] Tolman, R. C., these PROCEEDINGS, 14, No. 3, 268–272 (1928).

[2] Lenz, W., *Physik. Z.*, 27, 642–645 (1926).

[3] Einstein, A., *Berl. Ber.*, 448–459 (1918).

[4] See Eddington, A. S., *The Mathematical Theory of Relativity*, Cambridge, 1923, equations (69.21) and (69.22).

[5] See Eddington, l. c., equation (54.71).

[6] See Eddington, l. c., equations (54.81) and (54.82).

[7] See Eddington, equations (59.4), (58.1), (58.45) and (58.52). The additional term in λ can easily be shown necessary where the full form of equation (6) is used. Compare Einstein, l. c.

ON THE EQUILIBRIUM BETWEEN RADIATION AND MATTER IN EINSTEIN'S CLOSED UNIVERSE

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1. *Introduction.*—In two preceding articles,[1] the principles of thermodynamics have been expressed in a form suitable for applications in general relativity, and then applied to Einstein's closed universe, employing a set of coördinates in which the line element assumes the form

$$ds^2 = -R^2 d\chi^2 - R^2 \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2) + dt^2. \qquad (1)$$

Using these coördinates, and assuming the universe filled with a perfect fluid whose pressure is not necessarily negligible, the general relativity analogue of the first law of thermodynamics was found to lead to the equation

$$(\rho_{00} + p)V = \text{const.} \qquad (2)$$

where ρ_{00} is the *proper macroscopic* density and p the *proper* pressure of the fluid. And the analogue of the second law of thermodynamics was found to lead to the expression

$$(\phi_0 V)_{t'} - (\phi_0 V)_t \geqq 0 \qquad (3)$$

where ϕ_0 is the *proper* density of entropy of the fluid, and $\phi_0 V$ is found, as indicated, to be a quantity which can only increase with the time.