A NOTION OF EQUILIBRIUM AND ITS POSSIBILITY UNDER MAJORITY RULE

By CHARLES R. PLOTT*

Consider a committee that is faced with the task of deciding on how to change the magnitude of several variables. It is assumed that agreement is defined by majority rule. The decision-making body could be a board of directors attempting to decide on the magnitude of several investment projects, or a group of individuals deciding upon the allocation of a budget among several public goods. The setting makes no difference as long as the variables could conceivably be changed by any amount. If a change in the variables is proposed and the change does not receive a majority vote, then the "existing state" of the variables remains. If no possible change in the variables could receive a majority vote, then the "existing state" of the variables is an "equilibrium."[1]

The purpose of this paper is to make clear such a notion of equilibrium and to investigate the possibility of its existence. Section I sets forth the general setting, definitions, and assumptions. Section II pertains to situations where there is no constraint on the possible magnitude of the variables. Section III pertains to situations where there is one constraint (such as a fixed amount that the committee may spend). Section IV contains some general observations, possible applications, and questions brought forth by the procedure. An appendix contains an outline to the proofs of the propositions in the text.

Before continuing, it may be best to indicate some of the things *not* considered except by way of observation in the final section, if at all. Only "local" equilibriums are considered. Of course, "global" equilibriums must be special cases of these. There is no real theory of the path to equilibrium or even, for that matter, a convincing assurance that an equilibrium, if it exists, will be attained. Strategic considerations are ignored as are all second order conditions. These omissions are especially important in light of the results.

I. *The Basic Model*

Assume there are m individuals $(1, 2, \cdots, i, \cdots, m)$ who are attempting to decide on the magnitude of n variables $(x_1, x_2, \cdots, x_j,$

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[1] This notion of equilibrium was first examined by Duncan Black and R. A. Newing. Special cases of some of the theorems are implied by the diagrams in [1, pp. 19–28] and [2, pp. 137–39].

$\cdots, x_n)$. Each individual is assumed to have a differentiable utility function $U^i = U^i(x_1, x_2, \cdots, x_n)$ defined on the n variables.[2]

Consider now a specific, small change in the variables $(dx_1^*, dx_2^*, \cdots, dx_n^*)$ from some "existing situation" (say $\bar{X}$ in Euclidean n-space, E_n). This change in the variables shall be called a "motion."[3] Individual i would "vote for" the motion over the existing situation $\bar{X}$ if

$$\frac{\partial U^i}{\partial x_1} dx_1^* + \frac{\partial U^i}{\partial x_2} dx_2^* + \cdots + \frac{\partial U^i}{\partial x_n} dx_n^* > 0.$$

That is, he favors the motion if it would increase his utility. Adopting the notation to be used, we can say that he "votes for" the motion b_k if

$$\nabla U^i b_k > 0$$

where

$$\nabla U^i = \left(\frac{\partial U^i}{\partial x_1}, \frac{\partial U^i}{\partial x_2}, \cdots, \frac{\partial U^i}{\partial x_n}\right)$$

is the gradient vector of individual i's utility function, b_k is some *particular* "motion" $(dx_1^k, dx_2^k, \cdots, dx_n^k)$ from the infinite set of "small moves away from $\bar{X}$.[4] Further, if $\nabla U^i b_k < 0$, we say he is "indifferent." It shall be assumed that indifferent individuals behave in the same specified manner, i.e., always "vote for," always "vote against," or "never vote" as the case may be.

Graphically the situation is represented by Figure 1. The curve I is an indifference curve or level surface of the utility function. Suppose $\bar{X}$ represents the "existing state." This individual would "vote for" a proposed move from $\bar{X}$ such as b_f, vote "against" a motion such as b_a, and be indifferent toward a proposal such as b_i.

Consider now a group of individuals $(1, 2, \cdots, m)$ and the associated gradient vectors $\nabla U^i = (a_{i1}, a_{i2}, \cdots, a_{in})$. Ignoring problems of strategic behavior, all individuals would "vote for" any proposal, b, which satisfies[5]

$$Ab > 0 \tag{1}$$

[2] The variables are such that they behave as "collective goods." See [5].

[3] We shall assume the "motion" vector is normalized, i.e., where $(dx_1^*, \cdots, dx_n^*) = b^*$, then $|b^*| = 1$. For notational purposes, some particular motion, say j, will be denoted as b_j and the components will be denoted as $(b_{j1}, b_{j2}, \cdots, b_{jn})$.

[4] He votes for any motion for which the directional derivative, in that direction, is positive. ab, where a and b are vectors, denotes an inner product.

[5] The problem of finding such "acceptable proposals" is simply a linear programming problem. See [4].

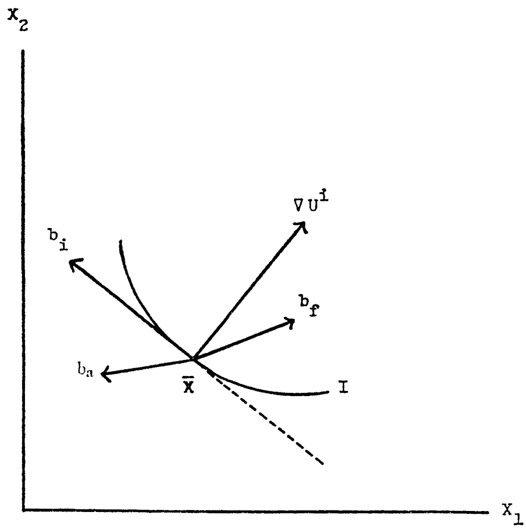

FIGURE 1

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

It is well known that this system of inequalities has a solution, b, if and only if the system

(2) $\quad yA = 0 \qquad y = (y_1, \cdots, y_m)$ and $y_i \geq 0$ not all $= 0$

has no semipositive solution y (see appendix).

Suppose the committee was operating under unanimity. Suppose further that an individual will vote "against" a motion that does not increase his utility (indifferent people vote "no"). Then any point in the space where (2) is satisfied would be an "equilibrium point." There would not exist a "motion" that could receive the required vote because, for any of the infinite possible motions, at least one person would vote "no."[6]

Suppose the committee was operating under majority rule. The committee would be at an equilibrium if and only if there does not exist a "motion" that could receive a favorable vote from a majority. This notion of equilibrium seems to accord well with the usual meanings of

[6] If it is supposed that individuals would "permit" a motion to which they are indifferent, an equilibrium point must be one where (2) has a strictly positive solution y. Every "motion" would either decrease the utility of at least one individual or leave everyone indifferent.

equilibriums. Certainly if the magnitude of the variables was such that some change in them could receive the required vote there is no a priori reason to suppose that the variables would not be so changed.

II. *Equilibrium: No Constraint*

The conditions for majority rule equilibrium when there exists no constraint on permissible motions will be given. Such a constraint is added in the following section. It will be assumed that there are m (an odd number) decision makers. Simple modifications can be made to account for situations where the number of people is even.

A. Existence of an unconstrained equilibrium requires that indifrerent individuals do not vote "yes."

If individuals who are indifferent are assumed to abstain from voting, such activities cannot be considered as "yes" votes. Otherwise, an equilibrium cannot exist. To prove this point, let M be an $(m+1)/2\times n$ matrix, the rows of which come from the rows of A. Since indifferent people vote "yes," equilibrium must be a situation where $Mb<0$[7] for all possible M that can be formed from the rows of A and all possible b. Pick an M and b that satisfy this equilibrium condition (say M_i and b_i). Then, where $b_k=-b$, $M_ib_k>0$. The motion b_k would receive a majority contrary to the assumption of equilibrium. Indifferent people cannot vote "yes."

The remaining equilibrium conditions must be satisfied by the gradient vectors. The proofs are in the appendix.

B. Any equilibrium must be a point of maximum utility for at least one individual.

If the point is a maximum for one and only one individual,[8] the gradients of the remaining individuals must satisfy C.

C. The remaining (even number of) rows of A can be divided into pairs for which there exists a strictly positive solution to

$$\begin{bmatrix} y_i \\ y_j \end{bmatrix}\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jn} \end{bmatrix} = 0 \qquad y_i, y_j > 0. \tag{3}$$

The last condition means that all individuals for which the point is not a maximum can be divided into pairs whose interests are diametrically opposed. The situation is shown diagrammatically by Figure 2. Points 1, 2, 3, 4, and 5 are the points of maximum utility for individuals 1, 2, 3, 4, and 5 respectively. The lines connecting the points are those

[7] The notation $Mb>0$ means that the inner product of b with each row of M is strictly positive.

[8] The conditions can be modified to account for the point being a maximum for more than one individual.

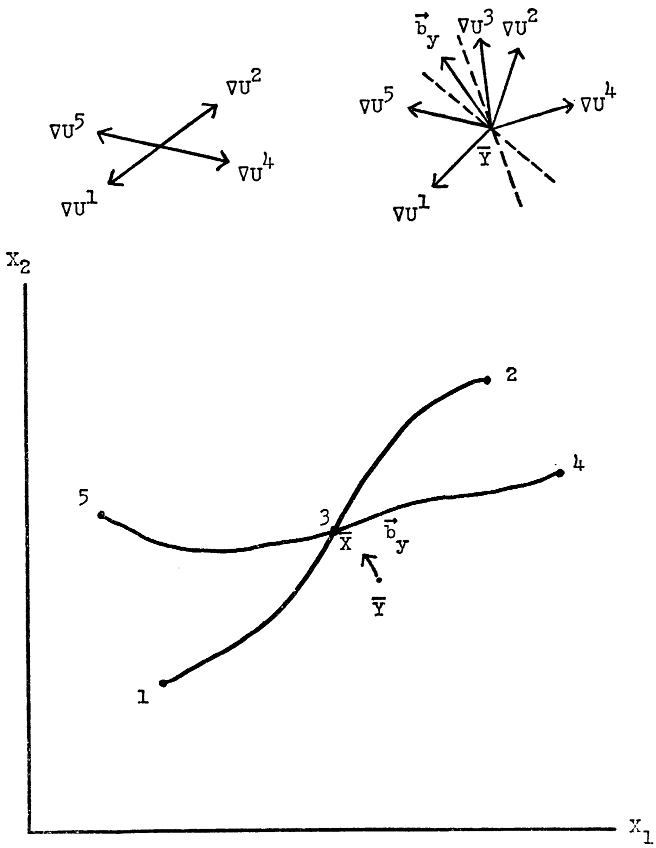

FIGURE 2

points for which (3) is satisfied—the "contract curves" as traditionally termed in economics. The only point which is an equilibrium point is $\overline{X}$, the point corresponding to the maximum of individual 3. At any other point there exists a motion which would receive a majority vote. For example, consider $\overline{Y}$. The motion b_y would receive the "yes" votes of individuals 2, 3, and 5.

The above conditions, as qualified, are both necessary and sufficient for a point to be an equilibrium.

Notice that any majority rule equilibrium is obviously Pareto Optimal (almost by definition). But, certainly not all Pareto Optimals are majority rule equilibriums. The condition for Pareto Optimality is simply condition (2), or the condition described in footnote number 6, depending upon how you choose to define Pareto Optimality.

The most important point is that there is certainly nothing inherent

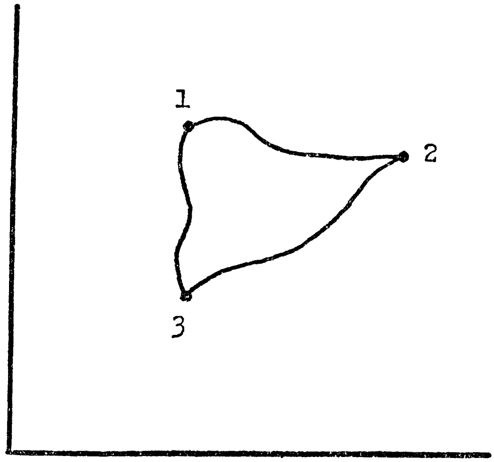

FIGURE 3

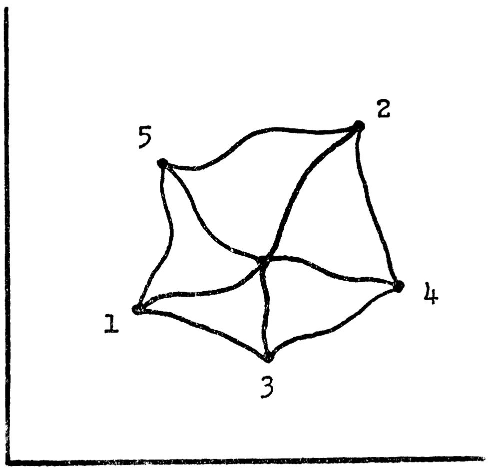

FIGURE 4

in utility theory which would assure the existence of an equilibrium. In fact, it would only be an accident (and a highly improbable one) if an equilibrium exists at all. For example, in Figures 3 and 4 there exists no equilibrium. And, it will be shown below that the addition of a constraint does little to relieve the situation if there are more than two variables to be determined.

III. *Equilibrium: One Constraint*

Suppose the committee was operating under a single constraint such as a budget constraint. Each of the variables has an associated price and the committee has only a fixed amount (I) to spend. The constraint is of the form

$$\sum_{i=1}^{n} P_i x_i \leq I. \tag{4}$$

If the committee is at a point such that

$$\sum_{i=1}^{n} P_i x_i < I$$

the constraint does not alter the range of possible motions and an equilibrium must satisfy the conditions A, B, and C above. Therefore, only points such that

$$\sum_{i=1}^{n} P_i x_i = I$$

are of interest.

The constraint can be treated as an individual who has veto power. Where the price vector, or the gradient of the constraint is denoted as P, the only admissible (or feasible) motions are those in a set β where

$$\beta = \{b \in E_n \mid Pb \leq 0\}.$$

The problem of finding majority rule equilibrium conditions is simply one of finding conditions on the gradients of the individuals such that

$$Mb \leq 0$$

for all M and all $b \in \beta$ where M again ranges over all $(m+1)/2 \times n$ matrices that can be formed from the rows of A, the matrix of all gradient vectors.

Again, just as in the unconstrained case, it is assumed that all indifferent individuals behave in the same manner. This assumption gives rise to the following behavioral condition.

D. If a constrained equilibrium exists, people cannot "vote for" a motion to which they are indifferent.[9]

Again the remaining conditions are on the gradient vectors of the individuals.

E. For a point to be an equilibrium, the gradient of at least one individual must satisfy

$$\begin{bmatrix} y_1 \\ y_P \end{bmatrix} \begin{bmatrix} a_{i1} & a_{i2} & & a_{in} \\ -P_1 & -P_2 & \cdots & -P_n \end{bmatrix} = 0 \qquad y_i > 0 \qquad y_P \geq 0. \tag{5}$$

[9] The proof is analogous to the proof of A above. The only difference is that rather than *any* b, one must be chosen from the set β' where $\beta' = \{b \in E_n \mid Pb = 0\}$.

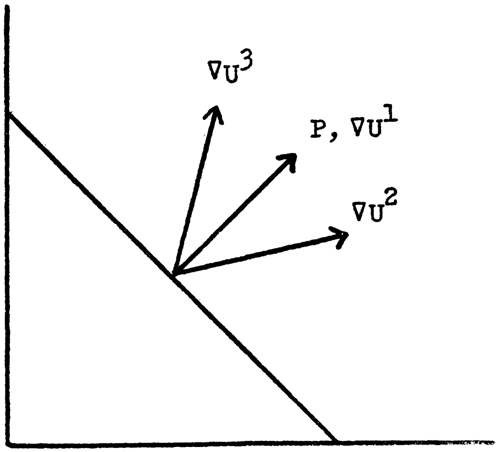

FIGURE 5

This says that the point is either a constrained maximum or a maximum for at least one individual. We shall assume for the following conditions that this is true for one and only one individual. Again, the conditions can easily be modified to account for other cases.

E. For a point to be an equilibrium point, those gradients (even in number) for which D does not hold can be divided into pairs such that there exists a solution to

$$\text{(6)} \qquad \begin{bmatrix} y_i \\ y_j \\ y_p \end{bmatrix} \begin{bmatrix} a_{i1} & a_{i2} \cdots & a_{in} \\ a_{j1} & a_{j2} \cdots & a_{jn} \\ -P_1 & -P_2 \cdots & -P_n \end{bmatrix} = 0 \qquad \begin{matrix} y_i, y_j > 0 \\ y_P \geq 0. \end{matrix}$$

In words this means that the individuals can be divided into pairs such that they are either diametrically opposed or at least opposed on every motion in the set β' defined in footnote 9. On Figure 5, the vectors P and ∇U^1 are parallel and lie in the same direction thus satisfying (5). P can be found as a positive combination of ∇U^2 and ∇U^3 thus satisfying (6).

It is certainly no trick to find a situation where there is no equilibrium at all. For example, there is no equilibrium on Figure 6, where points 1, 2, and 3 are points of constrained maximum for individuals 1, 2, and 3 respectively. Again there is nothing in utility theory that would guarantee the existence of an equilibrium. And again, if an equilibrium exists, it would be purely accidental. This result leads, at this stage of analysis, to rather pessimistic conclusions about the allocation of public goods. Samuelson [5] has demonstrated that the equilibrium attained by a market mechanism for public goods will, in general, fail to be an optimum. The analysis here implies that a majority rule political process

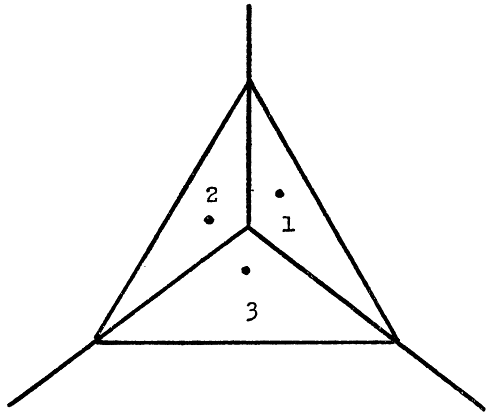

FIGURE 6

will fail to reach an equilibrium at all. Thus, in the case of public goods, society can count upon neither the market nor a majority rule political process to be a desirable allocative device.

IV. *Observations and Conjectures*

The results thus far seem rather negative in that equilibrium under majority rule would seem to be an almost nonexistent phenomenon. In view of this, a review of the concept of equilibrium used, and an inquiry into whether or not any important ingredients were left out should be useful.

Of great importance is the process by which motions are proposed. The decision process *itself* may dictate that some motions cannot be proposed. Often, before a motion can be voted on by a decision body, it must pass through a subcommittee. Unless proposals and amendments can be made from the floor, the subcommittee determines what motions can be voted on. In such cases, the equilibrium conditions would apply only to members of this committee. When the members of such a committee are in equilibrium, the complete voting body is in equilibrium regardless of the preferences of those not on the subcommittee.

This brings up an additional observation on coalitions. Where there exists a majority coalition which, itself, operates under a rule of unanimity, condition (2), when satisfied for the members of this coalition, defines the equilibrium points. If there exists a constraint, the traditional requirements on utility functions are sufficient to assure the existence of an equilibrium in such cases.

A third observation pertains to the role of time in the analysis. Certainly there may be great costs of indecision—if nothing else, simply sitting. As time passes, the utility functions as reflected in the commodity space may tend to "flatten." That is, with time, the marginal utility associated with an additional unit of a variable may tend to diminish. The points of maximum may extend to a neighborhood around the original maximum and the "contract curves" may tend to become "broad" as the time taken to reach a decision increases. This type of phenomenon would certainly tend to increase the possibility of the existence of an equilibrium.

Associated with the possibility of a change in evaluations through time is the possibility of a change in preference due to the decision process itself. The exchange of information associated with any decision process may serve actually to change the utility functions. A "persuasive" individual may be successful in changing the utility functions of others so that all of the orderings, in the end, resemble his. Or, the process may tend to cause all utility functions to change to a "similar" ordering but one which is not "similar" to any of the original functions.

The model outlined here lends itself to the possible testing of all of these speculations. If the original utility functions are known, the contract curves can be described as a system of equations and solved (at least theoretically) for an equilibrium. In the absence of such detailed information one might assume that the contract curves are linear. The only information then needed would be the points of maximum for the various individuals (or constrained maximums as the case may be). The contract curves would then be the curves L of the form

$$L = tC + A \qquad 0 \leq t \leq 1$$

where

$$B - A = C$$

assuming B is the maximum for individual B and A is the maximum for individual A.

Having obtained this information the experimenter can estimate the equilibrium. The decision process can then be observed. The resulting equilibrium can be compared with the "theoretical" equilibrium and the final utility functions compared with the originals. The path to equilibrium can be observed along with associated strategic behavior, etc. It may be the case that strategic considerations influence individuals to vote against some motion which would increase their utility. Such observations could, in principle, lead to the identification of variables which systematically contribute to the final outcome of group decisions.

Mathematical Appendix

The proofs for the various statements and conditions given in the text are presented below. The propositions given below follow in the same order as the corresponding descriptions in the text.

Definitions

A is an $m \times n$ (m, n finite and m an odd number) matrix.

(m) is the set of vectors which serve as rows of A. The elements of (m) are $(a_1, \cdots, a_i, \cdots, a_m)$.

M is an $(m+1)/2 \times n$ submatrix of A. (M) is the set of all M that can be formed from A.

A^* is a $r \times n$ submatrix of $A (r = m-1)$.

(r) is the set of vectors which serve as rows of A^*.

In general, except for specifically defined submatrices such as those above, if (ξ) is a set of row vectors of some matrix with ξ elements, then $[\xi]$ is the submatrix with the ξ members of (ξ) as rows.

For *any* matrix $[\xi]$ and vector b the notation $[\xi]\ b > 0$ means that the inner product of b with every row of $[\xi]$ is strictly greater than zero. The complement of a set (ξ) is denoted as $(\bar{\xi})$. E_n denotes Euclidean n-space. The notation ab represents an inner product. Define A to be an $\hat{A}$ if and only if for any $b \in E_n \exists M \in (M)$ such that $Mb \leq 0$.

Lemma 1. A is an $\hat{A}$ if and only if for *no* $M \in (M)$ there is a $b \in E_n$ such that $Mb > 0$.

Proof. Examine first the "if" part. Assume that for $M_i \exists b_i$ such that $M_i b_i > 0$. Since M_i has $(m+1)/2$ rows, there are at most $m-(m+1)/2 = (m-1)/2$ remaining rows of A for which $ab_i \leq 0$ so A cannot be $\hat{A}$.

Assume that for $b_i \not\exists$ an $M \in (M)$ such that $Mb_i \leq 0$. At most, there could be $(m-1)/2$ rows of A such that $ab_i \leq 0$ is the case. For the remaining $(m+1)/2$ rows—call them M_i—it must be the case that $ab_i > 0$. Thus, for M_i there exists a solution to $M_i b > 0$.

Lemma 2. If A is an $\hat{A}$, then for every $b \in E_n \exists$ an $a \in (m)$ such that $ab = 0$.

PROOF. Assume A is an $\hat{A}$. Assume that for b_i there does not exist an $a \in (m)$ such that $ab_i = 0$. By the definition of $\hat{A} \exists$ some $M_j \in (M)$ such that $M_j b_i < 0$. But, by letting $b_k = -b_i$, we find $M_j b_k > 0$, thus contradicting the assumption that A is an $\hat{A}$.

Theorem 1. If A is an $\hat{A}$, then there exists at least one $a_i \in (m)$ such that $a_i = 0$.

PROOF. For any $b \in E_n$ there must exist an orthogonal row of A by Lemma 2. To prove the theorem, it is sufficient to show that if at least one row of A is not the zero vector, then there exists a solution, say b^*, to $Ab \neq 0$. This will be shown by induction.

Assume A has only one column, no element of which is zero. Then any non-zero scalar satisfies the requirements of b^*.

Assume A has n columns. Assume further that there exists a solution, say b, to

$$\sum_{j=1}^{n-1} a_{ij} b_j = \alpha_i \neq 0 \qquad i = 1, \cdots, m.$$

Now, if there exists b_n such that $a_{in} b_n \rightarrow -\alpha_i$ for $i = 1, \cdots, m$, we are finished. Certainly such a number exists. Just choose b_n not equal to any of the numbers

$$\left(-\frac{\alpha_1}{a_{1n}}, \ldots, -\frac{\alpha_n}{a_{nn}}\right).$$

By the theorem, at least one row of A must be the zero vector. The reduced matrix A^* is formed from A by the elimination of a row which is the zero vector. The remaining discussion pertains to A^* which has $r=m-1$ rows called the set (r).

Lemma 3. The equation

$$[\xi]x > 0$$

has a solution x if and only if the equation

$$y[\xi] = 0$$

has no semipositive solution y.[10]

Corollary. The equation

$$[\xi]x < 0$$

has a solution, x, if and only if the equation

$$y[\xi] < 0$$

has no semipositive solution y.

Proof. It is sufficient to show that $[\xi]x<0$ has a solution if and only if $[\xi]x>0$ has a solution. Suppose x is a solution to $[\xi]x>0$. Then $x'=-x$ is a solution to $[\xi]x'<0$. If x is a solution to $[\xi]x<0$, then $x'=-x$ is a solution to $[\xi]x'>0$.

Lemma 4. If A is an $\hat{A}$ then for any $b\in E_n$ such that $a_ib_i=0$ for some non-zero $a_i\in(r)$, then there exists at least one $a_j\in(r)$, where $i\neq j$, such that $a_jb_i=0$.

PROOF. Let A be an $\hat{A}$ and assume that for b_i, $ab_i=0$ for one and only one non-zero $a\in(r)$. Since A is an $\hat{A}$, we know the following:

$ab_i=0$ for one and only one non-zero $a\in(r)$, by assumption. Call it a_i.

$ab_i>0$ for $\dfrac{r}{2}$ rows of A^*. Call them the set (P).

$ab_i<0$ for $\dfrac{r}{2}-1$ rows of A^*. Call them the set (D).

The last two statements are valid because otherwise either b_i or $-b_i$ would be a solution to $Mb>0$ contrary to the assumption that A is an $\hat{A}$. It makes no difference whether (P) or (D) contains $r/2$ elements since (D) becomes (P) for $-b_i$. By Lemma 3, there exists no semipositive solution to

$$y[P] = 0.$$

[10] For the proof, see [3, p. 48]. "Semipositive" is taken to mean that $y_i\geq 0$ and not all $y_i=0$.

This is forced by the definition of (P). There also exists no semipositive solution to

$$(1) \qquad y\begin{bmatrix} P \\ a_i \end{bmatrix} = 0.$$

Assume that there does exist a semipositive solution to (1). This implies that $-a_i$ can be expressed as a positive combination of some of the elements of (P). Assume them to be the first k elements. That is

$$-a_i = \sum_{j=1}^{k} y_j a_j \quad \text{where} \quad y_j > 0.$$

Multiplying by b_i, we find

$$0 = -a_i b_i = y_1 a_1 b_i + \cdots + y_k a_k b_i \qquad y_i > 0.$$

But, this contradicts the fact that $ab_i > 0$ for all $a \in (P)$. Because (1) has no solution, by Lemma 3 there must exist a solution to

$$(2) \qquad \begin{bmatrix} P \\ a_i \end{bmatrix} \qquad b > 0$$

and, since there are $(m+1)/2$ rows of the matrix in (2), A cannot be an $\hat{A}$. The lemma is proved.

Lemma 5. If A is an $\hat{A}$, then for each row vector a_i of A^* there exists at least one row vector a_j, $j \neq i$, of A^* and scalars α_i and α_j (not both zero) such that $\alpha_j a_j + \alpha_i a_i = 0$.

PROOF. If some $a \in (r)$ is the zero vector, the condition of the lemma is satisfied. If no $a \in (r)$ is the zero, then any b such that $a_i b = 0$ must be orthogonal to at least one other row of A^*. The finite condition on the rows forces there to be at least one row, say a_j, which is orthogonal to every vector b which is orthogonal to a_i. This means a_i and a_j are parallel and the lemma is proved. The formal proof follows closely the proof of Theorem 1.

Theorem 2. If one and only one row of A is the zero vector, then A is an $\hat{A}$ if and only if the rows of A^* can be partitioned into two element sets such that there exists a solution to

$$y_i a_i + y_j a_j = 0 \qquad y_i, y_j > 0.$$

PROOF. By Lemma 5, if A is an $\hat{A}$, then the rows of A^* can be divided into sets which are parallel. It will be shown that each such set contains an even number of vectors and that half lie in one direction and the other half lie in the opposite direction. This is simply another way of stating the theorem.

Choose any row of A^*, say a_i, and divide all rows of A^* into the sets with respect to a_i as defined below.

s = the number of vectors in a set (s) where $(s) = \{a_j \in (r) \mid \alpha_i a_i + \alpha_j a_j = 0$

has a solution α_i and $\alpha_j\}$. Note that (s) contains a_i. Recall also that by assumption, (r) contains no zero.

q= the number of vectors in a set (q) where $(q)=\{a_j \in (s) \mid a_j=\lambda a_i$ has a solution $\lambda>0\}$. Note that (q) contains a_i.

l= the number of vectors in a set (l) where $(l)=\{a\in(s) \mid a\notin(q)\}$. Notice that $s-q=l$.

b_i= some vector in E_n such that $ab_i=0$ for all $a\in(s)$ and $ab_i\neq 0$ for all $a\in(r)\cap(\bar{s})$.

p= the number of vectors in a set (p) where $(p)=\{a\in(r)\cap(\bar{s}) \mid ab_i>0\}$.

d= the number of vectors in a set (d) where $(d)=\{a\in(r)\cap(\bar{s}) \mid ab_i<0\}$.

Observe that

$$r = p + d + l + q.$$

By Lemma 3 and its corollary there exists no semipositive solution, y, to

$$y[p] = 0 \quad \text{or} \quad y[q] = 0.$$

Also, there exists no semipositive solution, y, to

$$y\begin{bmatrix} p \\ q \end{bmatrix} = 0. \tag{3}$$

A solution to (3) would imply that the negative of some member of (q) can be expressed as a positive combination of a subset of (p) or vice versa. Assume $a_q\in(q)$ can be expressed as a positive combination of the first k elements of (p), i.e.,

$$-a_q = y_1a_1 + \cdots + y_ka_k \qquad y_i > 0.$$

Multiply by b_i and obtain

$$0 = -a_qb_i = y_1a_1b_i + \cdots + y_ka_kb_i \qquad y_i > 0.$$

But, by the definition of (p), all members on the right of the equation are positive thus establishing a contradiction. Since there is no semi-positive solution to (3), there must exist a solution to

$$\begin{bmatrix} p \\ q \end{bmatrix} \quad b > 0. \tag{4}$$

If A is to be an $\hat{A}$, (4) dictates that

$$p + q \leq \frac{r}{2}\cdot \tag{5}$$

Otherwise, $\begin{bmatrix} p \\ q \end{bmatrix}$ would be an M for which there is a solution to $Mb>0$

thus violating Lemma 1. By the same reasoning, we know that

$$p + l \leq \frac{r}{2} \tag{6}$$

and, since (d) becomes (p) for $-b_i$, the same argument gives

$$d + l \leq \frac{r}{2} \tag{7}$$

$$d + q \leq \frac{r}{2}\cdot \tag{8}$$

Recalling that $r = p + d + l + q$ and solving equations 5 through 8, one can derive

$$q \leq l \quad \text{and} \tag{9}$$

$$l \leq q. \tag{10}$$

Therefore, $l = q$ and the "only if" part of the theorem is proved.

Now the "if" part. Assume the conditions of the theorem are satisfied. Choose any $b_i \in E_n \cdot ab_i = 0$ for the zero vector. For every $a \in ab_i > 0$, there is an $a \in ab_i \leq 0$. Therefore $ab_i \leq 0$ for $(m+1)/2$ elements of (m) and A is an $\hat{A}$.

The theorems below are related to the discussion contained in Section Three of the text. Recall that the discussion there is about majority decisions which must satisfy a single constraint such as

$$\sum_{i=1}^{n} p_i x_i \leq I.$$

It is assumed that the "existing position" is one which satisfies

$$\sum_{i=1}^{k} p_i x_i = I.$$ [11]

The following definitions are needed.

Let C be a specific, non-zero vector (the negative of the gradient of the constraint).

$$G = \{b \in E_n \mid b \in J \cup K\}$$
$$J = \{b \in E_n \mid Cb > 0\}$$
$$K = \{b \in E_n \mid Cb = 0\}.$$

Let A be an $\hat{A}$ if and only if for every $b \in G$ there exists an $M \in (M)$ such that $Mb \leq 0$.

[11] See the discussion in the text.

Lemma 6. A is an $\hat{A}$ if and only if for *no* $M \in (M)$ does there exist a $b \in G$ which is a solution to $Mb > 0$. (proof omitted)

Lemma 7. If A is an $\hat{A}$, then there must exist some $a \in (m)$ and a set of scalars (α_i, a_c) with at least one $\alpha \neq 0$ such that

$$\alpha_i a + \alpha_c C = 0.$$

PROOF. Assume no $a \in (m)$ satisfies the conditions of Lemma 7. Then there exists $b_i \in K$ such that $ab = 0$ for *no* $a \in (m)$. The elements of (m) can be partitioned into two sets according to whether $ab_i > 0$ or $ab_i < 0$. One of these sets must contain at least $(m+1)/2$ elements. Since these sets reverse positions for $-b_i$, A cannot be an $\hat{A}$ by Lemma 6.

Theorem 3. If A is an $\hat{A}$, then there exists some $a \in (m)$ such that a solution exists to

$$y_i a + y_c C = 0 \qquad y_i > 0,\ y_c \geq 0.$$

PROOF. The following definitions are needed.

H = the number of elements in the set (H) where $(H) = \{a \in (m) \mid \alpha_i a + \alpha_c C = 0 \text{ not all } \alpha = 0\}$.

By Lemma 7, the set (H) is not empty. If a member of (H) is the zero vector, it also satisfies the conditions of the theorem. So only the case where (H) contains no zero vector need be examined.

Assume A to be an $\hat{A}$ and assume that no member of (H) satisfies the condition of the theorem.

b_i = some $b \in K$ such that $ab_i \neq 0$ for all $a \in (\overline{H})$.

P = the number of elements in (P) where $(P) = \{a \in (\overline{H}) \mid ab_i > 0\}$.

D = the number of elements in (D) where $(D) = \{a \in (\overline{H}) \mid ab_i < 0\}$. Note that $P + D + H = m$.

If A is an $\hat{A}$, then

$$P < \frac{m+1}{2} \quad \text{and} \quad D < \frac{m+1}{2} \cdot$$

Otherwise either $[P]b$ or $[D](-b)$ would satisfy $Mb > 0$ contrary to Lemma 6.

Now, there can exist no semipositive solution to

$$y \begin{bmatrix} P \\ H \\ C \end{bmatrix} = 0.$$

A solution to this implies that any $a_k \in (H) \cup C$ with $y_k \neq 0$ can be expressed as

$$\sum y_j a_j = -a_k \qquad y_j \geq 0.$$

Multiply this equation by b_i and observe

$$\sum_{j=1}^{P} y_j a_j b_i = -a_k b_i = 0 \qquad y_j \geq 0$$

which contradicts the fact that $a_j b_i > 0$ for $j = 1, \cdots, P$.

So, by Lemma 3, there does exist a solution to

$$\begin{bmatrix} P \\ H \\ C \end{bmatrix} \qquad b > 0.$$

The same argument shows there is a solution to

$$\begin{bmatrix} D \\ H \\ C \end{bmatrix} \qquad b > 0.$$

But, since $P+D+H=m$, either $P+H \geq (m+1)/2$ or $D+H \geq (m+1)/2$ so either one or the other satisfies $Mb>0$. By Lemma 6, A cannot be an $\hat{A}$.

If A is an $\hat{A}$ at least one element of (m) must satisfy the condition of Theorem 3. Elimination of this element yields a set called (r) with $r=m-1$ elements.

Lemma 8. If A is an $\hat{A}$ and if the condition of Theorem 3 is satisfied by one and only one row of A, then for *every* $a_i \in (r)$ there exists at least one $a_j \in (r)$, $i \neq j$, such that a solution exists to

$$\alpha_i a_i + \alpha_j a_j + \alpha_c C = 0 \qquad \alpha_i, \alpha_j \neq 0.$$

PROOF. It can be shown that if the condition of this Lemma is *not* satisfied for some row, say a_i, then there exists a $b \in K$ such that $a_i b = 0$ and $a_j b \neq 0$ for all $j \in (r)$, $j \neq i$.[12]

Assume A is an $\hat{A}$, no $a \in (r)$ satisfies the condition of Theorem 3, and that for a_i there exists a $b \in K$, say b_i, such that $a_i\ b_i = 0$ and $a_j b_i \neq 0$ for all $a_j \in (r)$, $i \neq j$. We let

P = the number of elements in a set (P) where $(P) = \{a \in (r) \mid ab_i > 0\}$

D = the number of elements in a set (D) where $(D) = \{a \in (r) \mid ab_i < 0\}$.

If A is an $\hat{A}$ then one set contains $r/2$ elements and the other contains $r/2-1$. Otherwise, either b_i or $-b_i$ would be a solution to $Mb>0$. Assume $P=r/2$. By the same argument used previously there exists no semipositive solution to

$$y \begin{bmatrix} P \\ a_i \\ C \end{bmatrix} = 0.$$

[12] Use corollary to Theorem 2.3 in [3, p. 37].

Thus, by Lemma 3, there exists a solution to

$$\begin{bmatrix} P \\ a_i \\ C \end{bmatrix} \qquad b > 0.$$

But the set $(P) \cup a_i$ contains $(m+1)/2$ elements and is therefore an M. So A cannot be an $\hat{A}$. Recalling that (D) becomes (P) for $-b_i$ proves the Lemma.

Lemma 9. If A is an $\hat{A}$ and if the condition of Theorem 3 is satisfied by one and only one row of A, then the elements of (r) can be partitioned into mutually exclusive sets each containing two elements such that there exists a solution to

$$\alpha_i a_i + \alpha_j a_j + \alpha_c C = 0 \qquad \alpha_i, \alpha_j \neq 0.$$

PROOF. Assume A is an $\hat{A}$ and that the condition of Theorem 3 is satisfied by one and only one row of A. Choose any member of (r), say a_i, and partition the members of (r) with respect to a_i as follows.

H = the number of elements in a set (H) where $(H) = \{a_j \in (r) \mid \text{a solution exists to } \alpha_i a_i + \alpha_j a_j + \alpha_c C = 0,\ \alpha_i, \alpha_j \neq 0\}$. Notice that $a_i \in (H)$ and by Lemma 8, (H) contains more than one element.

$b_i = \{\text{some } b \in K \mid \text{for all } a \in (H),\ ab_i = 0 \text{ and for all } a \in (r) \cap (\overline{H}),\ ab_i \neq 0\}$.
$b_j = \{\text{some } b \in K \mid \text{for all } a \in (H),\ ab_j \neq 0\}$.[13]
P = the number of elements in a set (P) where $(P) = \{a \in (r) \cap (\overline{H}) \mid ab_i > 0\}$.
D = the number of elements in a set (D) where $(D) = \{a \in (r) \cap (\overline{H}) \mid ab_i < 0\}$.
Q = the number of elements in a set (Q) where $(Q) = \{a \in (H) \mid ab_j > 0\}$.
L = the number of elements in a set (L) where $(L) = \{a \in (H) \mid ab_j < 0\}$.

Observe that

$$r = Q + L + P + D.$$

By arguments used previously, there exists no semipositive solution to

$$y \begin{bmatrix} P \\ Q \\ C \end{bmatrix} = 0.$$

So, if A is to be an $\hat{A}$

$$P + Q \leq \frac{r}{2} \cdot \tag{11}$$

[13] The existence of such a b_j involves an assumption that $\alpha_i a + \alpha_i C = 0$ has no solution for any $a \epsilon (H)$. No confusion should result since a_i could be so chosen.

The same argument gives

$$P + L \leq \frac{r}{2} \cdot \tag{12}$$

Since (D) becomes (P) for $-b_i$, we have

$$D + L \leq \frac{r}{2} \tag{13}$$

$$D + Q \leq \frac{r}{2} \cdot \tag{14}$$

Recalling that $r = Q + L + P + D$, it can be shown that

$$L = Q. \tag{15}$$

Since $L + Q = H$, the lemma is proved.

Theorem 4. If the condition of Theorem 3 is satisfied by one and only one row of A, A is an $\hat{A}$ if and only if the elements of (r) can be partitioned into pairs such that there exists a solution to

$$y_i a_i + y_j a_j + y_c C = 0 \qquad \begin{matrix} y_i, y_j > 0 \\ y_c \geq 0. \end{matrix}$$

Proof. Choose some element of (r), say a_k. The elements of (r) can be partitioned with respect to a_k as follows.

$H =$ the number of elements in a set (H) where $(H) = \{a_i \in (r) \mid$ a solution exists to $\alpha_i a_i + \alpha_k a_k + \alpha_c C = 0$ $\alpha_i,\ \alpha_k \neq 0\}$.

By Lemma 8 and Lemma 9, the number of elements in (H) is greater than zero and even. Notice that $a_k \in (H)$.

Now partition (H) into pairs such that for the maximum possible number of pairs a solution exists to

$$y_i a_i + y_j a_j + y_c C = 0 \qquad y_i, y_j > 0,\ y_c \geq 0.$$

Call the set of pairs for which this condition is satisfied $(H_p) \cdot (H_p)$ is a subset of (H) and contains H_p (an even number) elements.

$H_n =$ the (even) number of elements in (H_n) where $(H_n) = (H) \cap (\overline{H}_p)$.

$b_i = \{$some $b \in K \mid ab_i = 0$ for all $a \in H$ and $ab_i \neq 0$ for all $a \in (r) \cap (\overline{H})\}$.

$P =$ the number of elements in a set (P) where $(P) = \{a \in (r) \cap (\overline{H}) \mid ab_i > 0\}$.

$D =$ the number of elements in a set (D) where $(D) = \{a \in (r) \cap (\overline{H}) \mid ab_i < 0\}$.

There exists no semipositive solution to $y \begin{bmatrix} P \\ H_n \\ C \end{bmatrix} = 0$. This can be shown by the method used previously. By eliminating the proper elements ($H_p/2$ in number) from (H_p) a set (H_p') containing $H_p/2$ elements can be formed such that there exists no semipositive solution to

$$y\begin{bmatrix} P \\ H_n \\ H_p' \\ C \end{bmatrix} = 0.$$

Therefore, if A is an $\hat{A}$

$$P + \frac{H_p}{2} + H_n \leq \frac{r}{2} \cdot$$

The same argument gives

$$D + \frac{H_p}{2} + H_n \leq \frac{r}{2} \cdot$$

By noting that $r = P + D + H_p + H_n$, it can be shown that

$$H_n \leq 0.$$

The "only if" part is proved. Proof of the "if" is omitted.

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