J. Soc. Indust. Appl. Math.
Vol. 10, No. 4, December, 1962
Printed in U.S.A.

ALTERNATING DIRECTION METHODS FOR HYPERBOLIC DIFFERENTIAL EQUATIONS*

MILTON LEES†

1. Introduction. The difference equation

$$u_{t\bar{t}} = u_{x\bar{x}} + \eta k^2 u_{x\bar{x}t\bar{t}}, \tag{1.1}$$

depending on the nonnegative parameter η, was introduced by von Neumann (cf. [1]) for the numerical solution of boundary value problems for the one-dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2}.$$

In (1.1), the subscripts t and x denote forward difference quotients, $\bar{t}$ and $\bar{x}$ denote backward difference quotients, and k denotes the time step. Except when $\eta = 0$, when it reduces to the classical *explicit* difference equation studied by Courant, Friedrichs and Lewy [2], (1.1) is an *implicit* difference equation, whose solution at each time step is obtained by solving a tridiagonal system of linear algebraic equations. Using Fourier methods, von Neumann proved that the difference equation (1.1) is unconditionally stable if $4\eta > 1$ and is conditionally stable if $4\eta \leqq 1$, the stability condition in the latter case being $kh^{-1} \leqq (1 - 4\eta)^{-1/2}$.

Recently, Friberg [3] and Lees [4] generalized this result to quasi-linear hyperbolic equations of the form

$$\frac{\partial^2 w}{\partial t^2} = a(x, t)\frac{\partial^2 w}{\partial x^2} + F\left(x, t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial t}\right).$$

It is not difficult to show that the results of Friberg and Lees can be extended to cover the von Neumann type difference approximation to certain multidimensional hyperbolic differential equations. But in this case, the implicit equations that arise are no longer tridiagonal, and, consequently, are much more difficult to solve.

The purpose of this paper is to describe and analyze two difference methods of the von Neumann type for the numerical solution of certain

* Received by the editors August 21, 1961. The work presented in this paper was supported by the AEC Computing and Applied Mathematics Center, Courant Institute of Mathematical Sciences, New York University, under Contract AT(30-1)-1480 with the U. S. Atomic Energy Commission.

† Mathematics Department, California Institute of Technology, Pasadena, California.

multidimensional hyperbolic differential equations for which only tridiagonal systems need be solved at each time step. Our difference approximations are derived by applying an alternating direction procedure (cf. Douglas and Rachford [5]) to the standard von Neumann difference equation. Using the methods of [4] and [6]—the energy method—we shall prove that these modified von Neumann type difference equations are unconditionally stable if $4\eta > 1$.

2. Derivation of the difference methods. Let G be a bounded, open subset of the xy-plane with boundary ∂G and closure $\bar{G}$. We consider, first, an initial-boundary value problem for the simplest two-dimensional wave equation

$$Lw = \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 0, \qquad \text{in} \quad G \times [0, T]. \tag{2.1}$$

Initially, $w(0)$ and $\frac{\partial w}{\partial t}(0)$ are prescribed on G, and $w(t)$ is prescribed on ∂G for all $t \in [0, T]$. We assume that w exists and belongs to C^4 in $\bar{G} \times [0, T]$.

We denote by ℓ the (uniform) *lattice* in the xy-plane, composed of the nodes (ih, jh), where $h > 0$ is the *lattice spacing* and i and j are integers, positive, negative or zero. Let $\bar{G}_h = \ell \cap \bar{G}$; in the usual way [6], we decompose $\bar{G}_h$ into two disjoint subsets: ∂G_h, the *boundary nodes* of $\bar{G}_h$, and G_h, the *interior nodes* of $\bar{G}_h$. For simplicity, we assume that $\partial G_h \subset \partial G$ and that h is so small that G_h is nonempty.

Let $M \subset \ell$; the collection of all real-valued functions defined on ℓ that vanish identically on $\ell - M$ will be denoted by $\mathcal{C}(M)$. Clearly, $\mathcal{C}(M)$ is a real linear function space for the usual operations, and, in particular, $\mathcal{C}(G_h)$ is finite-dimensional with dimension equal to the number of nodes in G_h.

Finally, for $n = 0, 1, \cdots, N$, $Nk = T$, we define $[nk, T]_k$ to be the set $\{t \mid t = nk, (n+1)k, \cdots, Nk\}$.

A function $t \to u(t)$ defined on $[0, T]_k$ with range in the linear space $\mathcal{C}(\bar{G}_h)$ will be called *admissible* if

$$u(0) = w(0), \qquad \text{on } G_h,$$

$$u(k) = w(0) + k\frac{\partial w}{\partial t}(0) + \frac{k^2}{2}\left(\frac{\partial^2 w}{\partial x^2}(0) + \frac{\partial^2 w}{\partial y^2}(0)\right) \qquad \text{on } G_h,$$

and $u = w$ on $\partial G_h \times [0, T]_k$.

The solution w will be approximated by an admissible function $t \to u(t)$ defined recursively as follows: If $u(t)$ has been defined, we construct

$u^*(t) \in \mathcal{C}(\bar{G}_h)$ by setting $u^*(t) = w(t)$ on $\partial G_h \times [t, t]_k$ and

$$\begin{aligned} u^*(t) - 2u(t) + u(t-k) \\ = k^2[\eta u^*_{x\bar{x}}(t) + (1-2\eta)u_{x\bar{x}}(t) + \eta u_{x\bar{x}}(t-k)] \\ + k^2[2\eta u_{y\bar{y}}(t-k) + (1-2\eta)u_{y\bar{y}}(t)] \end{aligned} \tag{2.2}$$

in $G_h \times [t, t]_k$. It is clear that $u^*(t)$ is determined uniquely, by solving approximately $\sqrt{p}$ tridiagonal systems of linear algebraic equations, where p is the dimension of $\mathcal{C}(G_h)$; see, for example, [5] and [6]. Now we determine $u(t+k)$ as a solution of the system

$$u(t+k) - u^*(t) = k^2\eta[u_{y\bar{y}}(t+k) - u_{y\bar{y}}(t-k)] \tag{2.3}$$

in $G_h \times [t+k, t+k]_k$. Again, $u(t+k)$ is determined uniquely, by solving approximately $\sqrt{p}$ tridiagonal linear systems. Hence, the admissible function $t \to u(t)$ is determined uniquely, by induction.

For the stability analysis of this difference method we eliminate the auxiliary function $u^*(t)$ between equations (2.2) and (2.3). A straightforward calculation shows that u is a solution of the difference equation

$$L_{\eta,h}[u] = u_{t\bar{t}} - \Delta_h u - \eta k^2 \Delta_h u_{t\bar{t}} + 2k^3\eta^2 R_h[u_{\hat{t}}] = 0 \tag{2.4}$$

in $G_h \times [k, T-k]_k$, where the difference operators Δ_h and R_h are defined as follows:

$$\Delta_h u = u_{x\bar{x}} + u_{y\bar{y}},$$
$$R_h[u] = u_{x\bar{x}y\bar{y}},$$

and $u_{\hat{t}}$ is the centered difference quotient of u, i.e.,

$$u_{\hat{t}} = \tfrac{1}{2}(u_t + u_{\bar{t}}).$$

It is apparent from (2.4) that the difference operator $L_{\eta,h}$ is consistent with the wave operator L; that is, for all $\eta \geqq 0$ and all $(x, y, t) \in G_h \times [k, T-k]_k$,

$$|Lv - L_{\eta,h}[v]| = O(k^2 + h^2) \tag{2.5}$$

for any sufficiently smooth function v on $\bar{G} \times [0, T]$.

Remark. If the term $2k^3\eta^2 R_h[u_t]$ is deleted from (2.4), what remains is the standard, two-dimensional analogue of the von Neumann difference equation (1.1).

In the second difference method for the approximation of w, we replace (2.2) and (2.3) by

$$\begin{aligned} u^*(t) - 2u(t) + u(t-k) \\ = k^2[\eta u^*_{x\bar{x}}(t) + (1-2\eta)u_{x\bar{x}}(t) + \eta u_{x\bar{x}}(t-k)] + k^2 u_{y\bar{y}}(t) \end{aligned} \tag{2.6}$$

$$\begin{aligned} u(t+k) - 2u(t) + u(t-k) \\ = k^2[\eta u^*_{x\bar{x}}(t) + (1-2\eta)u_{x\bar{x}}(t) + \eta u_{x\bar{x}}(t-k)] \\ + k^2[\eta u_{y\bar{y}}(t+k) + (1-2\eta)u_{y\bar{y}}(t) + \eta u_{y\bar{y}}(t-k)] \end{aligned} \tag{2.7}$$

The other conditions determining u and u^* remain unchanged. When u^* is eliminated between (2.6) and (2.7), the following difference equation is seen to be satisfied by u in $G_h \times [k, T - k]_k$:

$$L'_{\eta,h}[u] = u_{t\bar{t}} - \Delta_h u - k^2\eta\Delta_h u_{t\bar{t}} + k^4\eta^2 R_h[u_{t\bar{t}}] = 0 \tag{2.8}$$

Equation (2.8) is also a perturbation of the standard two-dimensional analogue of the von Neumann difference equation (1.1).

3. Stability. In this section we prove that the difference operator $L_{\eta,h}$ is *unconditionally stable* if $4\eta > 1$. We will establish this result by showing that $L_{\eta,h}$ satisfies an energy inequality if $4\eta > 1$. A similar result can be established for the difference operator $L'_{\eta,h}$, but we do not give the details.

First, we require some notation. We endow the linear space $\mathcal{C}(G_h)$ with a real inner product (u, v), defined as follows:

$$(u, v) = h^2 \sum_{(x,y)\in\ell} u(x, y)v(x, y).$$

The norm $(u, u)^{\frac{1}{2}}$ induced on $\mathcal{C}(G_h)$ by this inner product will be denoted by $| u |_0$. Another real inner product for $\mathcal{C}(G_h)$ is given by the formula

$$(u, v)_1 = (u_{\bar{x}}, v_{\bar{x}}) + (u_{\bar{y}}, v_{\bar{y}}),$$

and the norm $(u, u)_1^{\frac{1}{2}}$ induced on $\mathcal{C}(G_h)$ by this inner product will be denoted by $| u |_1$. Finally, if $t \to u(t)$ is a function from $[0, T]_k$ into $\mathcal{C}(G_h)$, then we set

$$\| u(t) \| = \{| u_{\bar{t}}(t) |_0^2 + \tfrac{1}{2}[| u(t) |_1^2 + | u(t - k) |_1^2]\}^{\frac{1}{2}}.$$

We now give the main result of this paper.

THEOREM. *If $t \to u(t)$ is a function from $[0, T]_k$ into $\mathcal{C}(G_h)$ and $4\eta > 1$, then there exists $B(\eta, T) > 0$ such that*

$$\| u(t) \|^2 \leqq B[\| u(k) \|^2 + k\textstyle\sum_{\rho=k}^{t} | \chi L_{\eta,h}[u(\rho)] |_0^2] \tag{3.1}$$

where χ is the characteristic function of G_h[1].

We preface the proof of the theorem with a lemma, proved in [4] and [6].

LEMMA 1. *If $t \to u(t)$ is a function from $[0, T]_k$ into $\mathcal{C}(G_h)$, then*

$$2(u_{\hat{t}}, u_{t\bar{t}}) = (| u_t |_0^2)_{\bar{t}},$$

$$2(u_{\hat{t}}, \Delta_h u) = -(| u |_1^2)_{\hat{t}} + \frac{k^2}{2}(| u_t |_1^2)_{\bar{t}},$$

$$2(u_{\hat{t}}, \Delta_h u_{t\bar{t}}) = -(| u_t |_1^2)_{\bar{t}},$$

$$(u_{\hat{t}}, R_h[u_{\hat{t}}]) = | u_{\bar{x}\bar{y}\hat{t}} |_0^2.$$

[1] $\chi(x, y) = 1$ if $(x, y) \in G_h$, and $\chi(x, y) = 0$ if $(x, y) \notin G_h$.

Proof of the theorem. Since $u(t) \in \mathcal{C}(G_h)$, we have that

$$u_{\hat{t}}(u_{t\bar{t}} - \Delta_h u - \eta k^2 \Delta_h u_{t\bar{t}} + 2k^3\eta^2 R_h[u_{\hat{t}}]) = u_{\hat{t}} L_{\eta,h}[u],$$

valid for all $(x, y, t) \in \ell \times [k, T - k]_k$. Hence

$$(u_{\hat{t}}, u_{t\bar{t}}) - (u_{\hat{t}}, \Delta_h u) - \eta k^2 (u_{\hat{t}}, \Delta_h u_{t\bar{t}}) + 2k^3\eta^2 (u_{\hat{t}}, R_h[u_{\hat{t}}]) = (u_{\hat{t}}, L_{\eta,h}[u]).$$

From this identity and Lemma 1 it follows that

$$\{| u_t |_0^2 + k^2(\eta - \tfrac{1}{2})| u_t |_1^2\}_{\bar{t}} + (| u |_1^2)_{\hat{t}} + 4k^2\eta^2 | u_{\bar{x}\hat{y}\hat{t}} |_0^2 = 2(u_{\hat{t}}, L_{\eta,h}[u]). \tag{3.2}$$

Multiplying (3.2) through by k, dropping the fourth term on the left, and summing the resulting inequalities from k to $t - k$, we find that

$$\begin{aligned} \| u(t) \|^2 + k^2(\eta - \tfrac{1}{2})| u_{\bar{t}}(t) |_1^2 &\leqq \| u(k) \|^2 \\ &+ k^2(\eta - \tfrac{1}{2})| u_{\bar{t}}(k) |_1^2 + 2k \textstyle\sum_{\rho=k}^{t-k} (u_{\hat{t}}, L_{\eta,h}[u(\rho)]). \end{aligned} \tag{3.3}$$

But since

$$k^2| u_{\bar{t}}(t) |_1^2 = | u(t) - u(t - k) |_1^2 = | u(t) |_1^2 + | u(t - k) |_1^2 - 2(u(t), u(t - k))_1,$$

the inequality (3.3) becomes

$$\begin{aligned} &| u_{\bar{t}}(t) |_0^2 + \eta[| u(t) |_1^2 + | u(t - k) |_1^2] \\ &\quad - (2\eta - 1)(u(t), u(t - k))_1 \leqq | u_{\bar{t}}(k) |_0^2 + \eta[| u(k) |_1^2 \\ &\quad + | u(0) |_1^2] - (2\eta - 1)(u(k), u(0))_1 \\ &\quad + 2k \textstyle\sum_{\rho=k}^{t-k} (u_{\hat{t}}(\rho), L_{\eta,h}[u(\rho)]). \end{aligned} \tag{3.4}$$

Now consider the quadratic form

$$Q(r, s) = \eta r^2 + (2\eta - 1)rs + \eta s^2,$$

which, in view of our assumption that $4\eta > 1$, is positive definite; moreover, it is easy to see that

$$\mu_0(r^2 + s^2) \leqq Q(r, s) \leqq \mu_1(r^2 + s^2),$$

where

$$2\mu_0 = \min (1, 4\eta - 1) \quad \text{and} \quad 2\mu_1 = \max (1, 4\eta - 1).$$

We conclude now from this and (3.4) that

$$\| u(t) \|^2 \leqq B_1 \| u(k) \|^2 + 2B_2 k \textstyle\sum_{\rho=k}^{t-k} (u_{\hat{t}}(\rho), L_{\eta,h}[u(\rho)]), \tag{3.5}$$

where

$$B_1 = \max(1, \mu_1)/\min(1, \mu_0)$$

and

$$B_2 = 1/\min(1, \mu_0).$$

Using the definition of $u_{\hat{t}}$ and Schwarz's inequality, we find that

$$(u_{\hat{t}}, L_{\eta,h}[u]) \leqq |u_{\hat{t}}|_0 |\chi L_{\eta,h}[u]|_0 \leqq \tfrac{1}{2}[|u_t|_0 + |u_{\bar{t}}|_0] |\chi L_{\eta,h}[u]|_0 .$$

Therefore, being a little extravagant, we obtain from (3.5) that

$$\|u(t)\|^2 \leqq B_1 \|u(k)\|^2 + 2B_2 k \textstyle\sum_{\rho=k}^{t} \|u(\rho)\| |\chi L_{\eta,h}[u(\rho)]|_0 . \tag{3.6}$$

To complete the proof we require the following lemma:

LEMMA 2. *Let f and g be nonnegative functions defined on $[0, T]_k$. If c_1, $c_2 \geqq 0$ and*

$$f^2(t) \leqq c_1 + 2c_2 k \textstyle\sum_{\rho=k}^{t} f(\rho) g(\rho),$$

then

$$f^2(t) \leqq (1 + 2c_2 T) c_1 + (1 + 4c_2 T^2) c_2^2 k \textstyle\sum_{\rho=k}^{t} g^2(\rho).$$

For any $t_1 \in [k, t]_k \subset [0, T]_k$, we have

$$f^2(t_1) \leqq c_1 + 2c_2 k \textstyle\sum_{\rho=k}^{t} f(\rho) g(\rho).$$

Hence, by the generalized arithmetic-geometric mean inequality,

$$\begin{aligned} k \textstyle\sum_{t_1=k}^{t} f^2(t_1) &\leqq c_1 t + 2c_2 t k \textstyle\sum_{\rho=k}^{t} f(\rho) g(\rho) \\ &\leqq c_1 t + c_2 t \epsilon k \textstyle\sum_{\rho=k}^{t} f^2(\rho) + c_2 t \epsilon^{-1} k \textstyle\sum_{\rho=k}^{t} g^2(\rho), \end{aligned}$$

where $\epsilon > 0$ is arbitrary. Now choose ϵ so that $2c_2 t\epsilon = 1$. Then

$$k \textstyle\sum_{\rho=k}^{t} f^2(\rho) \leqq 2c_1 T + 4c_2^2 T^2 k \textstyle\sum_{\rho=k}^{t} g^2(\rho). \tag{3.7}$$

But, we also have that

$$f^2(t) \leqq c_1 + c_2 k \textstyle\sum_{\rho=k}^{t} f^2(\rho) + c_2 k \textstyle\sum_{\rho=k}^{t} g^2(\rho),$$

and the lemma follows by replacing the second term on the right by its upper bound, given by (3.7).

Returning to the proof of the energy inequality, we take $f(t) = \|u(t)\|$, $g(t) = |\chi L_{\eta,h}[u(t)]|_0$, $c_1 = B_1 \|u(k)\|$, and $c_2 = 2B_2$, and the inequality (3.1) follows from Lemma 2 and (3.7).

If the energy inequality is applied to the error function $w - u$, one deduces in the usual way from (2.5) that

$$\|w(t) - u(t)\| = O(k^2 + h^2).$$

We mention, finally, that the foregoing results can be generalized to hyperbolic differential equations of the form

$$\frac{\partial^2 w}{\partial t^2} = a(x, y, t)\frac{\partial^2 w}{\partial x^2} + b(x, y, t)\frac{\partial^2 w}{\partial y^2} + F\left(x, y, t, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial t}\right).$$

as well as higher dimensional equations (cf. [4]).

REFERENCES

[1] G. O'Brien, M. A. Hyman and S. Kaplan, *A study of the numerical solution of partial differential equations*, J. Math. Phys., 29 (1951), pp. 223–251.

[2] R. Courant, K. O. Friedrichs and H. Lewy, *Uber die Partiellen Differenzengleichungen der Mathematischen Physik*, Math. Ann. 100 (1928), pp. 32–74.

[3] J. Friberg, *Conditionally stable difference approximations for the wave operator*, Nordisk Tid. Inform. Behand., 1 (1961), pp. 69–86.

[4] M. Lees, *von Neumann difference approximation to hyperbolic equations*, Pacific J. Math., 10 (1960), pp. 213–222.

[5] J. Douglas, Jr., and H. H. Rachford, Jr., *On the numerical solution of heat conduction problems in two and three space variables*, Trans. Amer. Math. Soc., 82 (1956), pp. 421–439.

[6] M. Lees, *Alternating direction and semi-explicit difference methods for parabolic differential equations*, Technical report, New York University, Institute of Mathematical Sciences, NYO-9494 (1961).